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GOLDBACH NUMBERS IN SPARSE SEQUENCES

by J. BRÜDERN and A. PERELLI

1. Introduction.

An integer which can be written as the sum of two odd primes is called a Goldbach number, in remembrance of Goldbach's famous letter to Euler, dated 1742, where the belief was expressed that all even natural numbers exceeding 4 are of this form. The question is still undecided, and even counting possible exceptions is a difficult problem. Hardy & Littlewood [HL] assumed the truth of the Riemann Hypothesis for all Dirichlet L-functions and were then able to estimate successfully the number $E(N)$ of all even integers $n \leq N$ which are *not* the sum of two primes. Their result was $E(N) \ll N^{\frac{1}{2}+\epsilon}$, and at the moment there seems to be no argument available which would produce bounds for $E(N)$ breaking the barrier $\sqrt{N}$, unless one is prepared to introduce further hypotheses which go well beyond the scope of the Riemann Hypothesis (see Goldston [G] and Languasco & Perelli [LP]). The best unconditional estimates, of the form $E(N) \ll N^{1-\delta}$, derive from work of Montgomery & Vaughan [MV]; here $\delta > 0$ is rather small.

One can set up a more sensitive test for the distribution of possible exceptions by choosing a thin subset $\mathcal{V}$ of the even integers, and then try to establish that almost all elements of $\mathcal{V}$ are Goldbach numbers. In this spirit, Perelli [P] has checked the values of a fixed integer polynomial, satisfying some natural arithmetical conditions. There is also a vast literature concerned with short intervals. We only mention the current record due to Baker, Harman & Pintz [BHP]: if $\theta > \dfrac{11}{160}$ then almost all even integers in the interval $[x, x + x^{\theta}]$ are Goldbach numbers.

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In a precursor to this paper [BP], we have described a method which has the potential to yield almost-all results for Goldbach numbers in sequences rather thinner than the values taken by any polynomial. We shall now refine these ideas and prove the following result.

THEOREM 1. — *Let* $1 < \gamma < \frac{3}{2}$. *Let* $E(N)$ *be the number of all natural numbers* $n \le N$ *for which the inequality*

$$|p_1 + p_2 - \exp\Big((\log n)^\gamma\Big)| < 1 \tag{1}$$

has no solution in odd primes p_1, p_2. *Then, there exists a constant* $c > 0$ *such that*

$$E(N) \ll N \exp\Big(- c(\log N)^{3-2\gamma}\Big).$$

We note that c and the implicit constant are effective; our argument will, in particular, be free of any use of the Siegel-Walfisz theorem.

Throughout this paper, we shall write

$$F(x) = F_\gamma(x) = \exp\Big((\log x)^\gamma\Big).$$

For $n \in \mathbf{N}$, let $\nu_n = [F(n)]$ for convenience of notation. One of the two integers $\nu_n, \nu_n + 1$ next to $F(n)$ is even, and this number is denoted by $\tilde{\nu}_n$. Theorem 1 implies, in particular, that almost all numbers $\tilde{\nu}_n$ are Goldbach numbers.

By the methods of [BP], the estimation of the exceptional set is linked with a study of the exponential sum formed with the numbers $F(n)$. Our basic auxiliary result is Theorem 2 below, which might be of some independent interest.

THEOREM 2. — *Let* $1 < \gamma < \frac{3}{2}$. *Given real numbers* $0 < c < 1$ *and* $0 < C < \gamma - 1$, *there exists a constant* $\kappa > 0$ *such that uniformly in* $F(2N)^{-c} \le |\alpha| \le F(N)^C$ *one has*

$$\sum_{N \le n \le 2N} e(\alpha F(n)) \ll N \exp(-\kappa(\log N)^{3-2\gamma}).$$

A very similar result occurs as Theorem 2 in Karacuba [K], but with a much more restricted range of uniformity. Since Karacuba's argument is rather sketchy and it is not immediately clear that values of $|\alpha|$ as small as $F(N)^{\epsilon-1}$ are covered by his method, we have felt a need to present a detailed proof, in §8.

In §3 the principal ideas of [BP] will be recalled. We shall count the solutions of (1) by a Fourier transform method. An application of the Poisson summation formula will reduce a relevant integral to more familiar Fourier coefficients over major and minor arcs in the Hardy-Littlewood treatment of Goldbach's problem. The minor arcs are readily dispensed with by the methods of [BP] in conjunction with Theorem 2 and Vinogradov's estimate for exponential sums over primes.

The major arcs will be dealt with in §4. Based on the subtle approach of Montgomery & Vaughan [MV], there is a ramification in the argument due to the possible existence of exceptional zeros of L-functions. The treatment of Montgomery & Vaughan applies immediately in the new context if such zeros do not exist, but if they do then Montgomery & Vaughan use an averaging procedure, and it is this part of their method which we will have to reexamine. Roughly speaking, we will have to show that averaging over the thin set $\tilde{\nu}_n$ is enough. There is also the unconventional problem that an upper bound is needed for the major arc contribution in the binary Goldbach problem for an odd number. It is, however, not appropriate to comment on this matter at the present stage.

2. Notation.

Some parts of this analysis are heavily dependent on the work of Montgomery & Vaughan [MV], and for ease of reference we have tried to use their notation whenever possible. Most of our notation is standard and should be clear from the context. Implicit constants in the O and $\ll$ symbols will depend on γ and on various other constants introduced later, for which we use c, C or δ, but they will never be dependent on P, X, N and Δ. The parameters P, N and X are always assumed to be sufficiently large, and will ultimately be related via $P = X^{6\delta}$ and $X = F(N)$, although this is not necessary in parts of the paper. The parameter Δ will take different values (in terms of X) in different contexts.

3. The Fourier transform method.

In the next two sections we prove Theorem 1. Our first lemma is an abstract version of [BP], Lemma 1.

LEMMA 1. — *Let* $h : \mathbf{R} \to \mathbf{C}$ *be a function of period* 1 *which is*

integrable on $[0,1]$, *and let*

$$\check{h}(n) = \int_0^1 h(\alpha)e(-\alpha n)\mathrm{d}\alpha.$$

Let $K \in L^1(\mathbf{R})$ *be continuous, and such that its Fourier transform*

$$\hat{K}(\alpha) = \int_{-\infty}^{\infty} K(\beta)e(-\alpha\beta)\mathrm{d}\beta$$

has support in $[-1,1]$. *Then, for any* $v \in \mathbf{R}$,

$$\int_{-\infty}^{\infty} h(\alpha)K(\alpha)e(-\alpha v)\mathrm{d}\alpha = \hat{K}(\{v\})\check{h}([v]) + \hat{K}(\{v\}-1)\check{h}([v]+1).$$

Proof. — The left hand side of the proposed identity equals

$$\sum_{n=-\infty}^{\infty} \int_n^{n+1} h(\alpha)K(\alpha)e(-\alpha v)\mathrm{d}\alpha = \int_0^1 h(\alpha) \sum_{n=-\infty}^{\infty} K(\alpha+n)e(-v(\alpha+n))\mathrm{d}\alpha.$$

The Fourier transform of $G(x) = e(-v(\alpha+x))K(\alpha+x)$ is $\hat{G}(x) = e(\alpha x)$ $\hat{K}(v+x)$. Hence, by the Poisson summation formula,

$$\sum_{n=-\infty}^{\infty} e(-v(\alpha+n))K(\alpha+n) = e(-\alpha[v])\hat{K}(\{v\}) + e(-\alpha[v]-\alpha)\hat{K}(\{v\}-1),$$

and the lemma follows immediately.

We now take $0 < \Delta < \frac{1}{4}$, to be determined later, and define the weight function

$$w(\alpha) = \begin{cases} 1 & \text{if} \quad |\alpha| \le 1-\Delta \\ 0 & \text{if} \quad |\alpha| \ge 1 \\ \frac{1-|\alpha|}{\Delta} & \text{if} \quad 1-\Delta < |\alpha| < 1. \end{cases} \tag{2}$$

Then $w(\alpha)$ is continuous and even. We define $K(\alpha) = \hat{w}(\alpha)$ and note that $K(\alpha)$ is infinitely often differentiable and even; Fourier's inversion formula shows $\hat{K}(\alpha) = w(\alpha)$. To write $K(\alpha)$ in closed form, we put $\Upsilon(\alpha) = \max(0, 1-|\alpha|)$ and note that

$$w(\alpha) = \frac{1}{\Delta}\Upsilon(\alpha) - \left(\frac{1}{\Delta} - 1\right)\Upsilon\left(\frac{\alpha}{1-\Delta}\right).$$

Since $\hat{\Upsilon}(\alpha) = \left(\dfrac{\sin\pi\alpha}{\pi\alpha}\right)^2$ we have

$$K(\alpha) = \frac{1}{\Delta}\left(\frac{\sin\pi\alpha}{\pi\alpha}\right)^2 - \frac{1}{\Delta}(1-\Delta)^2\left(\frac{\sin\pi(1-\Delta)\alpha}{\pi(1-\Delta)\alpha}\right)^2. \tag{3}$$

For later use, we note the simple bound

$$K(\alpha) \ll \min(1, \Delta^{-1}|\alpha|^{-2}) \tag{4}$$

implied by (3).

We are now in a position to embark on the proof of Theorem 1. Let $0 < \delta < \frac{1}{100}$ be another parameter to be chosen later. Put $P = X^{6\delta}$. For a real number v consider the weighted counting function

$$r(v) = r(v; X) = \sum_{P<p_1,p_2\le X} \log p_1 \log p_2 \; w(p_1 + p_2 - v). \tag{5}$$

Roughly speaking, one expects that if $\frac{1}{2}X < v \le X$ and v is not too close to an integer, then $r(v) \gg X$, and we shall now prove this *on average* for $v = F(n)$.

With $K(\alpha)$ given by (3) and

$$S(\alpha) = \sum_{P<p\le X} \log p \; e(\alpha p)$$

we can write (5) as a Fourier integral,

$$r(v) = \int_{-\infty}^{\infty} S(\alpha)^2 K(\alpha) e(-\alpha v) \mathrm{d}\alpha. \tag{6}$$

We dissect the real line into the *major arcs* $\mathfrak{N}$, defined as the union of all intervals $|q\alpha - a| \le PX^{-1}$ with $(a, q) = 1$, $a \in \mathbf{Z}$ and $1 \le q \le P$, and the *minor arcs* $\mathfrak{n} = \mathbf{R} \setminus \mathfrak{N}$. We also write $\mathfrak{M} = \mathfrak{N} \cap [0, 1]$.

To prepare for the treatment of the major arcs in the next section we introduce, for $m \in \mathbf{Z}$,

$$R_1(m) = \int_{\mathfrak{M}} S(\alpha)^2 e(-\alpha m) \mathrm{d}\alpha \tag{7}$$

which coincides exactly with the quantity $R_1(m)$ used by Montgomery & Vaughan [MV]. By Lemma 1 with $h(\alpha) = S(\alpha)^2$ for $\alpha \in \mathfrak{N}$ and $h(\alpha) = 0$ otherwise, we have

$$\int_{\mathfrak{N}} S(\alpha)^2 K(\alpha) e(-\alpha v) \mathrm{d}\alpha = w(\{v\}) R_1([v]) + w(\{v\} - 1) R_1([v] + 1). \tag{8}$$

One of the numbers $[v]$ and $[v] + 1$ is even, and for even m one may expect that $R_1(m) \gg m$. It is therefore plausible to hope that the left hand side of (8) is large. We shall make this precise in the next section.

4. The major arcs.

In this section we try to evaluate $R_1(m)$, defined by (7), as precisely as possible. Our argument heavily borrows from Montgomery & Vaughan [MV]. It will be convenient to recall the concept of an exceptional character: there exists a constant $c > 0$ such that

$$L(\sigma, \chi) \neq 0 \quad \text{for} \quad \sigma \geq 1 - \frac{c}{\log P} \tag{9}$$

for all primitive characters χ of modulus $q \leq P$, with the possible exception of at most one real primitive character, called the *exceptional character*. We write $\tilde{\chi}$ for the exceptional character (if it exists) and $\tilde{r}$ for its modulus; $\tilde{\beta}$ is the unique zero of $L(s, \tilde{\chi})$ violating (9), which satisfies

$$\tilde{r}^{-\frac{1}{2}} (\log \tilde{r})^{-2} \ll 1 - \tilde{\beta} \leq \frac{c}{\log P}. \tag{10}$$

For all this see Davenport [D], chapter 14, for example.

Next we introduce the *singular series*

$$\mathfrak{S}(m) = \prod_{p \nmid m} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p \mid m} \left(1 + \frac{1}{p-1}\right) \tag{11}$$

and can then state

LEMMA 2. — *There exists an absolute constant $c_1 > 0$ such that if the exceptional character does not exist one has*

$$R_1(m) = m\mathfrak{S}(m) + O\Big(\frac{m}{\varphi(m)} X \exp\Big(-c_1 \frac{\log X}{\log P}\Big)\Big)$$

whenever $m \leq X$.

Proof. — Combine (6.17) and (7.1) of Montgomery & Vaughan [MV].

The argument is more subtle if the exceptional character exists, as we now suppose. We follow through the argument of Montgomery & Vaughan [MV] leading to their important formula (6.17), but avoid divisor estimates producing an X^ϵ (these occur only in (6.19) and the formula thereafter). A careful examination of their work then produces

LEMMA 3. — *There exists an absolute constant $c_2 > 0$ such that if the exceptional character exists one has, for $m \leq X$,*

$$\begin{aligned} R_1(m) = {} & m\mathfrak{S}(m) + \tilde{I}(m)\tilde{\mathfrak{S}}(m) + X\Gamma_1(m) + X\Gamma_2(m) \\ & + O\left(\frac{\tilde{\chi}(m)^2 \tilde{r} m X}{\varphi(\tilde{r})^2 \varphi(m)}\right) + O\left(\frac{mX}{\varphi(m)}(1-\tilde{\beta}) \exp\left(-c_2 \frac{\log X}{\log P}\right) \log P\right), \end{aligned}$$

where

$$\tilde{\mathfrak{S}}(m) = \tilde{\chi}(-1)\mu\Big(\frac{\tilde{r}}{(\tilde{r},m)}\Big)\frac{\tilde{r}}{\varphi(\tilde{r})\varphi(\frac{\tilde{r}}{(\tilde{r},m)})}\prod_{p\nmid \tilde{r}m}\Big(1-\frac{1}{(p-1)^2}\Big)\prod_{\substack{p\nmid \tilde{r}\\ p\mid m}}\Big(1+\frac{1}{p-1}\Big)$$

(12)

$$\tilde{I}(m) = \sum_{P<k<m-P}(k(m-k))^{\tilde{\beta}-1} \tag{13}$$

$$\Gamma_1(m) \ll \frac{\tilde{r}}{P}\sum_{\substack{q\le P\\ q\equiv 0 \,(\mathrm{mod}\ \tilde{r})}}\frac{q}{\varphi(q)\varphi(\frac{q}{(m,q)})} \tag{14}$$

$$\Gamma_2(m) \ll \tilde{r}\sum_{\substack{q>P\\ q\equiv 0 \,(\mathrm{mod}\ \tilde{r})}}\frac{1}{\varphi(q)\varphi(\frac{q}{(m,q)})}. \tag{15}$$

This is (6.17) of Montgomery & Vaughan [MV], with the error in (6.19) and thereafter kept explicit, followed by an appeal to their (7.1) to estimate W.

We now have to show that $\Gamma_1(m)$ and $\Gamma_2(m)$ are small, at least for almost all $m = \nu_n$. Our approach to this problem rests on the following consequence of Theorem 2.

LEMMA 4. — *Let κ be as in Theorem 2, and suppose that $0 < \tilde{\kappa} < \kappa$. Then, uniformly for $F(N)^{-\frac{1}{2}} < |\alpha| \le 1$ and $0 \le z_2 < z_1 \le 1$, one has*

$$\#\Big\{\frac{1}{2}N < n \le N : z_2 \le \{\alpha F(n)\} < z_1\Big\} = \frac{1}{2}(z_1 - z_2)N + O(N\exp(-\tilde{\kappa}(\log N)^{3-2\gamma}).$$

Proof. — For the rest of this paper we shall write

$$\mathcal{E} = \exp(\tilde{\kappa}(\log N)^{3-2\gamma}).$$

In Theorem 2 take $c = \frac{1}{2}$ and apply the standard argument of Vinogradov [Vi], chapter 5. We may omit the details. For a very simple variant which quickly yields the upper bound $\ll (z_1 - z_2)N + N\mathcal{E}^{-1}$ (which suffices below) see [BP].

Recall the notation $\nu_n = [F(n)]$ introduced in §1, and note that $\nu_n \equiv a \pmod d$ holds if and only if $\left\{\frac{a}{d}\right\} \le \left\{\frac{F(n)}{d}\right\} < \left\{\frac{a+1}{d}\right\}$. From Lemma 4 we infer that

$$\#\left\{\frac{1}{2}N < n \le N : \nu_n \equiv a \pmod d\right\} = \frac{N}{2d} + O(N\mathcal{E}^{-1}) \tag{16}$$

is valid uniformly for $a \in \mathbf{Z}$ and $1 \le d \le F(N)^{\frac{1}{2}}$.

LEMMA 5. — *Suppose that* $X = F(N)$. *With* $\Gamma_j(m)$ *satisfying* (14) *and* (15), *we have for* $j = 1, 2$

$$\sum_{\frac{1}{2}N<n\le N} (|\Gamma_j(\nu_n)| + |\Gamma_j(\nu_n+1)|) \ll N\mathcal{E}^{-1/2}.$$

Proof. — We use the elementary inequality $\varphi(q) \gg q(\log\log q)^{-1}$. By (14),

$$\begin{aligned}\sum_{\frac{1}{2}N<n\le N} |\Gamma_1(\nu_n)| &\ll \frac{\tilde r}{P}(\log\log X)^2 \sum_{\frac{1}{2}N<n\le N} \sum_{\substack{q\le P\\ q\equiv 0 \pmod{\tilde r}}} \frac{(\nu_n, q)}{q}\\ &\ll P^{-1}(\log\log X)^2 \sum_{\frac{1}{2}N<n\le N} \sum_{q\le P/\tilde r} \frac{(\nu_n, q\tilde r)}{q}.\end{aligned}$$

A routine transformation in conjunction with Lemma 4 shows that

$$\begin{aligned}\sum_{\frac{1}{2}N<n\le N} \sum_{q\le P/\tilde r} \frac{(\nu_n, q\tilde r)}{q} &= \sum_{q\le P/\tilde r} \sum_{d\mid \tilde r q} \frac{d}{q} \sum_{\substack{\frac{1}{2}N<n\le N\\ \nu_n\equiv 0 \pmod d}} 1\\ &\ll \sum_{q\le P/\tilde r} \sum_{d\mid \tilde r q} \frac{d}{q}\left(\frac{N}{d} + N\mathcal{E}^{-1}\right)\\ &\ll N\sum_{q\le P/\tilde r} q^{-1}(d(\tilde r q) + \sigma(\tilde r q)\mathcal{E}^{-1}).\end{aligned}$$

Now use $d(\tilde r q) \le d(\tilde r)d(q)$ and similarly for $\sigma(\tilde r q)$. Well-known divisor sum estimates then bound the previous sum as

$$\begin{aligned}&\ll N\left(d(\tilde r)\sum_{q\le P}\frac{d(q)}{q} + \sigma(\tilde r)\mathcal{E}^{-1}\sum_{q\le P/\tilde r}\frac{\sigma(q)}{q}\right)\\ &\ll N\tilde r^{\epsilon}(\log P)^2 + N\mathcal{E}^{-1}P\frac{\sigma(\tilde r)}{\tilde r}.\end{aligned}$$

But $\sigma(\tilde{r}) \ll \tilde{r} \log\log \tilde{r}$. Since $\tilde{r} \le P$ we deduce

$$\sum_{\frac{1}{2}N<n\le N} |\Gamma_1(\nu_n)| \ll NP^{\epsilon-1} + N\mathcal{E}^{-1}(\log\log X)^3$$

which is more than required. The argument is exactly the same if ν_n is replaced by $\nu_n + 1$.

The estimation of Γ_2 requires a little more care. We begin as before and find that

$$\begin{aligned}\sum_{\frac{1}{2}N<n\le N} |\Gamma_2(\nu_n)| &\ll \tilde{r}(\log\log X)^2 \sum_{\frac{1}{2}N<n\le N} \sum_{\substack{q>P \\ q\equiv 0 \pmod{\tilde{r}}}} \frac{(\nu_n, q)}{q^2} \\ &\ll \tilde{r}^{-1}(\log\log X)^2 \sum_{q>P/\tilde{r}} \sum_{\frac{1}{2}N<n\le N} \frac{(\nu_n, q\tilde{r})}{q^2} \\ &\ll \tilde{r}^{-1}(\log\log X)^2(Z_1 + Z_2), \qquad (17)\end{aligned}$$

where

$$Z_1 = \sum_{P/\tilde{r}<q\le X^{1/3}} \sum_{\frac{1}{2}N<n\le N} \frac{(\nu_n, q\tilde{r})}{q^2}, \quad Z_2 = \sum_{\frac{1}{2}N<n\le N} \sum_{q>X^{1/3}} \frac{(\nu_n, q\tilde{r})}{q^2}.$$

For notational convenience, we temporarily write $Y = X^{1/3}$. We estimate Z_1 in the same way as we treated Γ_1; this gives

$$\begin{aligned}Z_1 &\le \sum_{P/\tilde{r}<q\le Y} \sum_{d|q\tilde{r}} \frac{d}{q^2} \sum_{\substack{\frac{1}{2}N<n\le N \\ \nu_n\equiv 0 \pmod{d}}} 1 \\ &\ll N \sum_{P/\tilde{r}<q\le Y} \left(\frac{d(q\tilde{r})}{q^2} + \mathcal{E}^{-1}\frac{\sigma(q\tilde{r})}{q^2} \right).\end{aligned}$$

(Note that $Y\tilde{r} \le YP < \sqrt{X}$, so that Lemma 4 is applicable for all relevant d.) Proceeding as with the estimation of Γ_1 we now find that

$$Z_1 \ll N(d(\tilde{r})\tilde{r}P^{-1} + \mathcal{E}^{-1}\sigma(\tilde{r}))\log X. \qquad (18)$$

Turning our attention to Z_2, we first use $(\nu_n, q\tilde{r}) \le (\nu_n, q)(\nu_n, \tilde{r})$ and observe that for any $m \in \mathbf{N}$ one has

$$\begin{aligned}\sum_{q>Y} \frac{(m,q)}{q^2} &= \sum_{d|m} d \sum_{\substack{q>Y \\ q\equiv 0 \pmod{d}}} q^{-2} = \sum_{d|m} d^{-1} \sum_{q>Y/d} q^{-2} \\ &\ll \sum_{\substack{d|m \\ d\le Y}} Y^{-1} + \sum_{\substack{d|m \\ d>Y}} d^{-1} \ll Y^{-1}d(m).\end{aligned}$$

Trivial bounds then immediately yield

$$Z_2 \ll Y^{-1} \sum_{\frac{1}{2}N<n\le N} (\nu_n, \tilde{r}) d(\nu_n) \ll X^{\epsilon-\frac{1}{3}} N. \tag{19}$$

From (17), (18) and (19) we obtain

$$\sum_{\frac{1}{2}N<n\le N} |\Gamma_2(\nu_n)| \ll N\mathcal{E}^{-1/2},$$

and as before, the same bound holds if ν_n is replaced by $\nu_n + 1$. The proof of Lemma 5 is complete.

To extract a lower bound for the major arcs contribution, we also require an upper bound for $R_1(m)$ when m is odd, which is superior to what is obtainable from the previous lemmata. We can supply the required estimate only on average. With this mind, we write ν_n° for the unique odd integer from the set $\{\nu_n, \nu_n + 1\}$, and recall that $\tilde{\nu}_n$ is the even integer in this set. In this notation, our result is

LEMMA 6. — *For all but $O(N\mathcal{E}^{-1/12})$ values of $n \in \left[\frac{1}{2}N, N\right]$ one has*

$$R_1(\nu_n^\circ) \ll X\mathcal{E}^{-1/8}.$$

We postpone the proof to the next section; it will transpire later that Lemma 6 is more like a minor arc estimate. We can now deduce the important

LEMMA 7. — *Suppose that $\delta > 0$ is sufficiently small. Then, for all but $O(N\mathcal{E}^{-1/12})$ values of n with $\frac{1}{2}X < F(n) \le X$ one has*

$$R_1(\nu_n) + R_1(\nu_n + 1) \gg X\mathcal{E}^{-1/10}.$$

Proof. — In view of Lemma 6 it suffices to show that $R_1(\tilde{\nu}_n) \gg X\mathcal{E}^{-1/10}$ for all but $O(N\mathcal{E}^{-1/12})$ values of n in the relevant range. We consider various cases.

(i) First suppose that the exceptional character does not exist. Then, by Lemma 2,

$$R_1(\tilde{\nu}_n) \ge \tilde{\nu}_n \mathfrak{S}(\tilde{\nu}_n) + O\left(Xe^{-c_1/6\delta} \frac{\tilde{\nu}_n}{\varphi(\tilde{\nu}_n)}\right);$$

but

$$\mathfrak{S}(m) \gg \frac{m}{\varphi(m)} \quad \text{for} \quad m \equiv 0 \,(\mathrm{mod}\, 2),$$

and hence $R_1(\tilde{\nu}_n) \gg X$ for all n with $\frac{1}{2}X < \tilde{\nu}_n \le X$.

(ii) We may now assume that the exceptional character exists. Moreover, by Lemma 5 we may also assume that

$$|\Gamma_j(\tilde{\nu}_n)| \le \mathcal{E}^{-1/9} \quad (j = 1, 2)$$

since the number of n violating this is acceptable. By Lemma 3 we then infer that

$$R_1(\tilde{\nu}_n) = \tilde{\nu}_n \mathfrak{S}(\tilde{\nu}_n) + \tilde{I}(\tilde{\nu}_n)\tilde{\mathfrak{S}}(\tilde{\nu}_n) + O\Big(X\mathcal{E}^{-1/9}\Big)$$

$$+O\Big(X\tilde{\chi}(\tilde{\nu}_n)^2 \frac{\tilde{r}}{\varphi(\tilde{r})^2}\frac{\tilde{\nu}_n}{\varphi(\tilde{\nu}_n)}\Big) + O\Big(X\frac{\tilde{\nu}_n}{\varphi(\tilde{\nu}_n)}(1-\tilde{\beta})e^{-c_2/6\delta}\log P\Big). \tag{20}$$

The argument now bifurcates again.

(ii a) Suppose that $(\tilde{\nu}_n, \tilde{r}) = 1$. Then, by (12) and (13),

$$\tilde{\mathfrak{S}}(\tilde{\nu}_n) \ll \frac{\tilde{\nu}_n}{\varphi(\tilde{\nu}_n)}\frac{\tilde{r}}{\varphi(\tilde{r})^2}$$

and the bound $|\tilde{I}(\tilde{\nu}_n)| \le \tilde{\nu}_n$ is trivial. The term $\tilde{I}(\tilde{\nu}_n)\tilde{\mathfrak{S}}(\tilde{\nu}_n)$ can therefore be absorbed into the error term in (20). Moreover, (10) shows that $\tilde{r} \gg (\log P)^{3/2}$ from which we see that $\dfrac{\tilde{r}}{\varphi(\tilde{r})^2} \ll (\log P)^{-1}$. We can now argue as in case (i) to show that $R_1(\tilde{\nu}_n) \gg X$ for all n under consideration.

(ii b) We may now assume that $(\tilde{\nu}_n, \tilde{r}) > 1$. Then $\tilde{\chi}(\tilde{\nu}_n) = 0$, and the middle error term in (20) vanishes. Now, by (11) and (12) we find (see [MV], (8.5))

$$|\tilde{\mathfrak{S}}(\tilde{\nu}_n)| \le \mathfrak{S}(\tilde{\nu}_n) \prod_{p|\tilde{r}, p\nmid\tilde{\nu}_n, p>3} \frac{1}{p-2}. \tag{21}$$

If the product is non-empty, then $|\tilde{\mathfrak{S}}(\tilde{\nu}_n)| \le \dfrac{1}{3}\mathfrak{S}(\tilde{\nu}_n)$, and we recall that $|\tilde{I}(\tilde{\nu}_n)| \le \tilde{\nu}_n$. Hence

$$\tilde{\nu}_n\mathfrak{S}(\tilde{\nu}_n) + \tilde{I}(\tilde{\nu}_n)\tilde{\mathfrak{S}}(\tilde{\nu}_n) \ge \frac{2}{3}\tilde{\nu}_n\mathfrak{S}(\tilde{\nu}_n)$$

and, as before, (20) shows that $R_1(\tilde{\nu}_n) \gg X$ for all n under consideration.

(ii c) We may now suppose that the product in (21) is empty. In this case we have $p|\tilde{\nu}_n$ for any prime divisor $p > 3$ of $\tilde{r}$, and $\tilde{r}$ is the modulus of a real primitive character. Hence $\tilde{r} = 2^k u$ where u is odd and squarefree, and $k \le 3$. Thus, if the product (21) is empty, we must have $(\tilde{\nu}_n, \tilde{r}) \ge \dfrac{1}{24}\tilde{r}$. With the aim of showing that this happens only for very few n, we estimate the sum

$$\sum_r = \sum_{\frac{1}{2}N<n\le N} (r, \nu_n) = \sum_{\substack{d|r \\ d\le N}} \sum_{\substack{\frac{1}{2}N<n\le N \\ \nu_n\equiv 0 \,(\mathrm{mod}\ d)}} d.$$

By (16),

$$\sum_r \ll \sum_{\substack{d|r \\ d\le N}} d\Big(\frac{N}{d} + N\mathcal{E}^{-1}\Big) \ll d(r)N + \sigma(r)N\mathcal{E}^{-1}$$

and hence

$$(22)\qquad \#\Big\{\frac{1}{2}N < n \le N:\ (\nu_n, \tilde r) \ge \frac{\tilde r}{24}\Big\} \ll \frac{d(\tilde r)}{\tilde r}N + \frac{\sigma(\tilde r)}{\tilde r}N\mathcal{E}^{-1}.$$

If $\tilde r > \mathcal{E}^{1/7}$, say, then it follows from (22) that the product in (21) is empty only for acceptably few values of n.

(ii d) We are now reduced to the case where $\tilde r \le \mathcal{E}^{1/7}$ and where the product in (21) is empty. In this case, we note that $\tilde I(m) \le m^{\tilde\beta}$ whence

$$\tilde\nu_n \mathfrak{S}(\tilde\nu_n) + \tilde I(\tilde\nu_n)\tilde{\mathfrak{S}}(\tilde\nu_n) \ge \mathfrak{S}(\tilde\nu_n)(\tilde\nu_n - \tilde\nu_n^{\tilde\beta}) \gg (1-\tilde\beta)\tilde\nu_n(\log P)\frac{\tilde\nu_n}{\varphi(\tilde\nu_n)}$$

(compare with (6.21) of [MV]). From (20) we now infer that for $\frac{1}{2}X < \nu_n \le X$ one has

$$R_1(\tilde\nu_n) \gg (1-\tilde\beta)(\log P)X\frac{\tilde\nu_n}{\varphi(\tilde\nu_n)} + O(X\mathcal{E}^{-1/9}) + O\Big(Xe^{-c_2/6\delta}(\log P)(1-\tilde\beta)\frac{\tilde\nu_n}{\varphi(\tilde\nu_n)}\Big).$$

For sufficiently small δ the last error term will not exceed $\frac{1}{2}$ times the leading term. Moreover, by (10),

$$1-\tilde\beta \gg \tilde r^{-1/2}(\log\tilde r)^{-2} \gg \mathcal{E}^{-1/14}(\log N)^{-2(3-2\gamma)}.$$

It follows that $R_1(\tilde\nu_n) \gg X\mathcal{E}^{-1/10}$, as required. This completes the proof of Lemma 7.

We are now able to estimate the major arc contribution to (6). In (8) we take $v = F(n)$. Then, if

$$(23)\qquad \Delta \le \{F(n)\} \le 1-\Delta,$$

we deduce from (8) and (2) that

$$\int_{\mathfrak{N}} S(\alpha)^2 K(\alpha) e(-\alpha F(n))\,d\alpha = R_1(\nu_n) + R_1(\nu_n+1),$$

and we can apply Lemma 7. However, if n violates (23), then $\{F(n)\} \le \Delta$ or $\{F(n)\} \ge 1-\Delta$. By Lemma 4, the number of such n does not exceed $O(N\Delta + N\mathcal{E}^{-1})$. We can now conclude as follows:

LEMMA 8. — *Suppose that $\delta > 0$ is sufficiently small, and that*

$$\Delta = \mathcal{E}^{-1/8}.$$

Then, for all but $O(N\mathcal{E}^{-1/12})$ values of n with $\frac{1}{2}X < \nu_n \leq X$ one has

$$\int_{\mathfrak{N}} S(\alpha)^2 K(\alpha) e(-\alpha F(n))\, d\alpha \gg X\mathcal{E}^{-1/10}.$$

5. The minor arcs.

In this section we complement the results of Lemma 8 with an upper bound for the contribution from the minor arcs. The argument is essentially the same as in [BP] but we provide the details for completeness. The first step is to reduce to a finite interval. By (4), for $k \in \mathbf{N}$ we have

$$\int_{k<|\alpha|\leq k+1} |S(\alpha)^2 K(\alpha)|\, d\alpha \ll \Delta^{-1} \int_k^{k+1} \frac{|S(\alpha)|^2}{|\alpha|^2}\, d\alpha \ll \Delta^{-1} k^{-2} X \log X.$$

We sum over $k \geq P$. Then, writing

$$\mathfrak{n}_1 = \mathfrak{n} \cap [-P, P]$$

we deduce that

$$\int_{\mathfrak{n}} S(\alpha)^2 K(\alpha) e(-\alpha v)\, d\alpha = \int_{\mathfrak{n}_1} S(\alpha)^2 K(\alpha) e(-\alpha v)\, d\alpha + O\Big(\frac{X \log X}{\Delta P}\Big) \tag{24}$$

for any $v \in \mathbf{R}$.

We now consider

$$\Xi = \sum_{\frac{1}{2}N<n\leq N} \Big| \int_{\mathfrak{n}_1} S(\alpha)^2 K(\alpha) e(-\alpha F(n))\, d\alpha \Big|^2. \tag{25}$$

Squaring out brings in the sum

$$\Phi(\alpha) = \sum_{\frac{1}{2}N<n\leq N} e(\alpha F(n))$$

and yields

$$\Xi = \int_{\mathfrak{n}_1} \int_{\mathfrak{n}_1} S(\alpha)^2 S(-\beta)^2 \Phi(\alpha - \beta) K(\alpha) K(-\beta)\, d\alpha\, d\beta. \tag{26}$$

In Theorem 2 we take $c = 1 - 3\delta$. Then, for all $(\alpha, \beta) \in \mathfrak{n}_1^2$ with $|\alpha - \beta| > X^{3\delta-1}$ we have $\Phi(\beta - \alpha) \ll N\mathcal{E}^{-1}$. Therefore, this set contributes to (26) at most

$$\ll N\mathcal{E}^{-1}\Big(\int_{-\infty}^{\infty} |S(\alpha)^2 K(\alpha)|\, d\alpha\Big)^2 \ll NX^2(\log X)^2\mathcal{E}^{-1}.$$

On the complementary set we have $|\alpha - \beta| \le X^{3\delta-1}$. By a change of variable we see that this contribution to (26) is bounded by

$$N\int_{\mathfrak{n}_1} |S(\alpha)^2 K(\alpha)|\, d\alpha \sup_{\alpha} \int_{(\alpha - X^{3\delta-1}, \alpha + X^{3\delta-1}) \cap \mathfrak{n}_1} |S(\beta)|^2\, d\beta; \tag{27}$$

here we have estimated $\Phi(\alpha - \beta)$ and $K(\beta)$ trivially. By Vinogradov's estimate for trigonometric sums over primes, $|S(\beta)|^2 \ll X^2 P^{-1}(\log X)^8$ uniformly for $\beta \in \mathfrak{n}_1$ (see Vaughan [Va], Theorem 3.1), and hence (27) is

$$\ll N\Big(\int_{-\infty}^{\infty} |S(\alpha)^2 K(\alpha)|\, d\alpha\Big)(X^2 P^{-1}(\log X)^8) X^{3\delta-1} \ll NX^{2-2\delta}.$$

This shows that $\Xi \ll NX^2(\log X)^2\mathcal{E}^{-1}$. From (24) and (25) we finally deduce via a standard argument that the inequality

$$\Big|\int_{\mathfrak{n}} S(\alpha)^2 K(\alpha) e(-\alpha F(n))\, d\alpha\Big| \le X\mathcal{E}^{-1/3}$$

holds for all but $O(N\mathcal{E}^{-1/4})$ values of $n \in \Big[\frac{1}{2}N, N\Big]$. When combined with Lemma 8 and (6), it follows that $r(F(n)) > 0$ for all but $O(N\mathcal{E}^{-1/12})$ values of n with $\frac{1}{2}X < \nu_n \le X$. A routine dyadic dissection argument completes the proof of Theorem 1.

6. Proof of Lemma 6.

The basic observation is that for odd m one has

$$\int_0^1 S(\alpha)^2 e(-\alpha m)\, d\alpha = 0$$

so that

$$R_1(m) = -\int_{\mathfrak{m}} S(\alpha)^2 e(-\alpha m)\, d\alpha = -R_2(m), \tag{28}$$

say, where $\mathfrak{m} = \mathfrak{n} \cap [0, 1]$.

Now let

$$\mathcal{N}_{\mathrm{e}} = \left\{n: \frac{1}{2}N < n \le N, \nu_n \equiv 0 \,(\mathrm{mod}\ 2)\right\},$$
$$\mathcal{N}_{\mathrm{o}} = \left\{n: \frac{1}{2}N < n \le N, \nu_n \equiv 1 \,(\mathrm{mod}\ 2)\right\}.$$

We proceed to show that

$$\sum_{n\in\mathcal{N}_{\mathrm{e}}} |R_2(\nu_n+1)|^2 \ll NX^2\mathcal{E}^{-1/3} \tag{29}$$

$$\sum_{n\in\mathcal{N}_{\mathrm{o}}} |R_2(\nu_n)|^2 \ll NX^2\mathcal{E}^{-1/3}. \tag{30}$$

But $\nu_n + 1 = \nu_n^{\circ}$ when $n \in \mathcal{N}_{\mathrm{e}}$, and $\nu_n = \nu_n^{\circ}$ when $\mathcal{N}_{\mathrm{o}}$. Therefore, by (28), (29) and (30) we get

$$\sum_{\frac{1}{2}N<n\le N} |R_1(\nu_n^{\circ})|^2 \ll NX^2\mathcal{E}^{-1/3},$$

and Lemma 6 follows easily.

It remains to establish (29) and (30). We shall concentrate on the "even case" (29), the alterations for the cognate (30) will only be indicated. We shall again use Lemma 1, with a different kernel. Let $w(\alpha)$ be the function defined in (2), and put

$$w_{\mathrm{e}}(\alpha) = w(2\alpha+1). \tag{31}$$

Then $w_{\mathrm{e}}(\alpha) = 0$ for $\alpha \le -1$ or $\alpha \ge 0$, $w_{\mathrm{e}}(\alpha) = 1$ for $\frac{1}{2}\Delta - 1 \le \alpha \le -\frac{\Delta}{2}$, and $w_{\mathrm{e}}(\alpha)$ is continuous and linear in the intervals $\left[-\frac{\Delta}{2}, 0\right], \left[-1, \frac{\Delta}{2}-1\right]$. Its inverse Fourier transform is

$$K_{\mathrm{e}}(\alpha) = \int_{-\infty}^{\infty} w_{\mathrm{e}}(\beta)e(\beta\alpha)\,d\beta = \frac{1}{2}\int_{-\infty}^{\infty} w(\xi)e\left(\frac{1}{2}(\xi-1)\alpha\right)d\xi$$
$$= \frac{1}{2}e\left(-\frac{\alpha}{2}\right)K\left(\frac{\alpha}{2}\right)$$

with $K(\alpha)$ given by (3). Hence (4) is valid for $K_{\mathrm{e}}(\alpha)$ in place of $K(\alpha)$.

In Lemma 1 we take $h(\alpha) = S(\alpha)^2$ when $\alpha \in \mathfrak{n}$, and $h(\alpha) = 0$ otherwise. Then, since $\hat{K}_{\mathrm{e}}(\alpha) = w_{\mathrm{e}}(\alpha)$ by construction, we have

$$\int_{\mathfrak{n}} S(\alpha)^2 K_{\mathrm{e}}(\alpha)e(-\alpha F(n))\,d\alpha = w_{\mathrm{e}}(\{F(n)\}-1)R_2(\nu_n+1).$$

Summing over n we find that

$$(32)\quad \sum_{n\in\mathcal{N}_e} |R_2(\nu_n+1)|^2 = \sum_{n\in\mathcal{N}_e} \Big| \int_{\mathfrak{n}} S(\alpha)^2 K_e(\alpha) e(-\alpha F(n))\, d\alpha \Big|^2 + O(R)$$

where R is the contribution from n with $0 < w_e(\{F(n)\} - 1) < 1$. This happens only for $0 < \{F(n)\} < \frac{\Delta}{2}$ or $1 - \frac{\Delta}{2} < \{F(n)\} < 1$, and by Lemma 4 the number of such n is $O(N\Delta + N\mathcal{E}^{-1})$. Combined with the trivial bound

$$|R_2(m)| \le \int_0^1 |S(\alpha)|^2\, d\alpha \ll X \log X$$

this yields

$$(33)\quad R \ll (N\Delta + N\mathcal{E}^{-1}) X^2 (\log X)^2.$$

We note that Δ is again freely at our disposal, and we now take

$$(34)\quad \Delta = \mathcal{E}^{-1/3}.$$

Then (33) is certainly acceptable.

As in §5 we now remove a tail from $\mathfrak{n}$. We remarked earlier that (4) holds with $K_e(\alpha)$ in place of $K(\alpha)$. Arguing as in §5 we find that

$$\int_Y^\infty |S(\alpha)^2 K_e(\alpha)|\, d\alpha \ll X \log X (Y\Delta)^{-1}.$$

Therefore, if $\mathfrak{n}_2 = \mathfrak{n} \cap [-Y, Y]$, we have

$$\Big| \int_{\mathfrak{n}} S(\alpha)^2 K_e(\alpha) e(-\alpha F(n))\, d\alpha \Big|^2 \ll \Big| \int_{\mathfrak{n}_2} S(\alpha)^2 K_e(\alpha) e(-\alpha F(n))\, d\alpha \Big|^2 + X^2 (\log X)^2 (Y\Delta)^{-2}.$$

We take $Y = \mathcal{E}^{1/2}$. Then, by (34), (33) and (32),

$$(35)\quad \sum_{n\in\mathcal{N}_e} |R_2(\nu_n+1)|^2 \le \sum_{n\in\mathcal{N}_e} \Big| \int_{\mathfrak{n}_2} S(\alpha)^2 K_e(\alpha) e(-\alpha F(n))\, d\alpha \Big|^2 + N X^2 \mathcal{E}^{\epsilon - \frac{1}{3}}.$$

Squaring out now brings in the exponential sum

$$(36)\quad \Phi_e(\alpha) = \sum_{n\in\mathcal{N}_e} e(\alpha F(n)),$$

and further progress with (35) will depend on a successful estimate of this sum.

LEMMA 9. — *Let $\alpha \in \mathbf{R}$ with $|\alpha| \le X^\delta$. Suppose that $|\Phi_e(\alpha)| > N\mathcal{E}^{-1/3}$. Then there is an integer h with $|h| \le \mathcal{E}^{2/3}$ and $\left|\alpha - \frac{1}{2}h\right| \le X^{\delta-1}$.*

We prove Lemma 9 in the next section, and proceed directly with (35), arguing as in §5. Squaring out yields

$$\int_{\mathfrak{n}_2}\int_{\mathfrak{n}_2} S(\alpha)^2 S(-\beta)^2 K_{\mathrm{e}}(\alpha)K_{\mathrm{e}}(-\beta)\Phi_{\mathrm{e}}(\beta-\alpha)\,d\alpha\,d\beta, \tag{37}$$

and we begin by considering the subset of all $(\alpha,\beta)\in\mathfrak{n}_2^2$ with $|\Phi_{\mathrm{e}}(\beta-\alpha)|\le N\mathcal{E}^{-2/3}$. The contribution of this set to (37) is

$$\ll N\mathcal{E}^{-2/3}\Big(\int_{\mathfrak{n}_2}|S(\alpha)^2K_{\mathrm{e}}(\alpha)|\,d\alpha\Big)^2,$$

and since $|K_{\mathrm{e}}(\alpha)|\le\frac{1}{2}\left|K\left(\frac{\alpha}{2}\right)\right|$ we have

$$\int_{\mathfrak{n}_2}|S(\alpha)^2K_{\mathrm{e}}(\alpha)|\,d\alpha\ll\int_{-\infty}^{\infty}|S(2\alpha)^2K(\alpha)|\,d\alpha\ll X\log X. \tag{38}$$

Therefore, the total contribution of pairs $(\alpha,\beta)\in\mathfrak{n}_2^2$ with $|\Phi_{\mathrm{e}}(\beta-\alpha)|\le N\mathcal{E}^{-2/3}$ is

$$\ll NX^2\mathcal{E}^{-2/3}(\log X)^2,$$

which is acceptable.

It remains to consider the set where $|\Phi_{\mathrm{e}}(\beta-\alpha)|>N\mathcal{E}^{-2/3}$. Here we estimate $\Phi_{\mathrm{e}}(\beta-\alpha)$ trivially, but by Lemma 9 we can deduce that the total contribution to (37) of the set in question is

$$\ll N\int_{\mathfrak{n}_2}|S(\alpha)^2K_{\mathrm{e}}(\alpha)|\,d\alpha\Big(\sum_{|h|\le\mathcal{E}^{2/3}}1\Big)\sup_{\alpha,h}\int_{(\alpha+\frac{1}{2}h-X^{\delta-1},\alpha+\frac{1}{2}h+X^{\delta-1})\cap\mathfrak{n}_2}|S(\beta)|^2\,d\beta. \tag{39}$$

Again, by Vinogradov's estimate we have $S(\beta)\ll XP^{-\frac{1}{2}}(\log X)^4$ uniformly for $\beta\in\mathfrak{n}_2$, hence by (38) we have that (39) is

$$\ll NX\log X\mathcal{E}^{2/3}X^{\delta-1}X^2P^{-1}(\log X)^8\ll X^{2-\delta}N.$$

Collecting these results, we see that (29) follows from (35). To deduce (30) one merely has to replace $w_{\mathrm{e}}(\alpha)$ with $w_{\mathrm{o}}(\alpha)=w_{\mathrm{e}}(-\alpha)$ in the above argument.

7. Another exponential sum.

It remains to establish Lemma 9. We will remove the summation condition $\nu_n\equiv 0\,(\mathrm{mod}\,2)$ implicit in (36) by a Fourier technique. We again use the function $w_{\mathrm{e}}(\alpha)$ from the previous section.

The function

$$\psi(\alpha) = \sum_{k \in \mathbf{Z}} w_{\mathrm{e}}(k - 2\alpha)$$

is continuous, of period 1 and vanishes on $[\frac{1}{2}, 1]$. We write

$$\psi(\alpha) = \sum_{h \in \mathbf{Z}} c_h e(h\alpha),$$

and by the Poisson summation formula, (4) and $|K_{\mathrm{e}}(\alpha)| \le \frac{1}{2}\left|K\left(\frac{\alpha}{2}\right)\right|$ we see that

$$c_h \ll \min(1, \Delta^{-1}h^{-2}). \tag{40}$$

Here we take $\Delta = \mathcal{E}^{-1/3}$, and recall that $\nu_n \equiv 0 \,(\mathrm{mod}\, 2)$ if and only if $\left\{\frac{F(n)}{2}\right\} \le \frac{1}{2}$. By Lemma 4 and (31) it follows that

$$\begin{aligned} \Phi_{\mathrm{e}}(\alpha) &= \sum_{\frac{1}{2}N < n \le N} \psi\Big(\frac{F(n)}{2}\Big) e(\alpha F(n)) + O(N\mathcal{E}^{-1/3}) \\ &= \sum_{h \in \mathbf{Z}} c_h \Phi(\alpha + \frac{1}{2}h) + O(N\mathcal{E}^{-1/3}). \end{aligned} \tag{41}$$

By (40) and the trivial bound $|\Phi(\alpha)| \le N$ we can remove from (41) all terms with $|h| \ge \Delta^{-1}\mathcal{E}^{1/3} = \mathcal{E}^{2/3}$. Now, if $|\alpha + \frac{1}{2}h| > X^{\delta - 1}$ for all $|h| \le \mathcal{E}^{2/3}$, then $|\Phi(\alpha + \frac{1}{2}h)| \ll N\mathcal{E}^{-1}$ for all h in this interval, by Theorem 2. From (40) we then find that

$$\Phi_{\mathrm{e}}(\alpha) \ll N\mathcal{E}^{-1}\Delta^{-1} + N\mathcal{E}^{-1/3},$$

contrary to the hypothesis in Lemma 9. This completes the proof.

8. Proof of Theorem 2.

In this section we deduce Theorem 2 from the following exponential sum estimate due to Karacuba [K].

LEMMA 10. — *For given real numbers θ, Θ, Θ_0 and δ satisfying the inequalities*

$$0 < \delta < 1 \quad \text{and} \quad 0 < \Theta < \theta < \Theta_0 < 1,$$

let $\mathcal{F} = \mathcal{F}(N, \theta, \Theta, \Theta_0, \delta)$ be the set of all functions $f(x)$ for which there exists a $k \ge 3$ such that $f(x)$ is k-times differentiable on $[N, 2N]$,

$$\left|\frac{f^{(k)}(x)}{k!}\right| \le N^{-\Theta_0 k} \quad (N \le x \le 2N)$$

and there exists a set $\mathcal{S} \subset \{1, 2, \dots, k-1\}$ *with* $\#\mathcal{S} \geq \delta k$ *and*

$$N^{-\theta s} \leq \left| \frac{f^{(s)}(x)}{s!} \right| \leq N^{-\Theta s} \quad (N \leq x \leq 2N)$$

for all $s \in \mathcal{S}$. *Then there exist constants* $B, c > 0$, *depending only on* θ, Θ, Θ_0 *and* δ, *such that for any* $f \in \mathcal{F}$ *one has*

$$\left| \sum_{N < n \leq 2N} e(f(n)) \right| \leq BN^{1 - \frac{c}{k^2}}.$$

Proof. — This is Theorem 1 of Karacuba [K], with a change of notation (our k is Karacuba's $n+1$, $c_0 = \delta, c_1 = \Theta_0$ etc.).

It transpires that precise information is required about the derivatives of the function $F(x) = \exp((\log x)^\gamma)$. It is, in fact, quite straightforward to determine the asymptotic behaviour of the s-th derivative $F^{(s)}(x)$ of $F(x)$ as x tends to infinity, for fixed s. However, we will be forced to use Karacuba's estimate in the case where k tends to infinity with N, and must therefore supply estimates for $F^{(s)}(x)$ which are explicit in both x and s. It is this uniformity problem which is responsible for most of the complication below.

We now proceed to examine in detail the function $F^{(s)}(x)$. Most of the following is elementary calculus, but the situation is sufficiently tricky to justify a fairly detailed exposition. We begin with a preliminary analysis of the factorial polynomials defined by $q_0(x) = 1$ and, for $s \geq 1$, by

$$q_s(x) = x(x-1)\dots(x-s+1). \tag{42}$$

For $1 \leq t \leq s$ one has

$$q_s^{(t)}(x) = \sum_{i_1, \dots, i_t} \prod_{j \in \mathcal{J}(i_1, \dots, i_t)} (x - j) \tag{43}$$

where the sum is over all ordered t-tuples $(i_1, \dots, i_t)$ with all entries distinct and $0 \leq i_j \leq s-1$, and where $\mathcal{J}(i_1, \dots, i_t)$ is the complement of $\{i_1, \dots, i_t\}$ in $\{0, 1, \dots, s-1\}$. This is readily established by induction on t, using Leibniz's rule. Here the number of terms in the sum over i_j is $s(s-1)\dots(s-t+1) = q_t(s)$, and for $x \geq s$ we always have

$$\prod_{j \in \mathcal{J}(i_1, \dots i_t)} (x - j) \leq x(x-1)\dots(x-(s-t)+1) = q_{s-t}(x).$$

From (43) we now deduce that for $0 \leq t \leq s$ and $x \geq s$ one has

$$0 \leq q_s^{(t)}(x) \leq q_t(s) q_{s-t}(x) \leq s^t q_{s-t}(x) \tag{44}$$

(we proved this only when $t \geq 1$, but for $q_s^{(0)} = q_s$ this is obvious from (42)). The upper bound in (44) is almost sharp.

LEMMA 11. — *Let $\gamma > 0$ and $F(x) = \exp((\log x)^\gamma)$. Then, for any $s \in \mathbf{N}$ there are polynomials $Q_{s,t}$ $(0 \leq t \leq s-1)$ with real coefficients, of degree at most $s-t$ and with no constant term, such that*

$$F^{(s)}(x) = \frac{F(x)}{x^s} \sum_{t=0}^{s-1} (\log x)^{-t} Q_{s,t}(\gamma(\log x)^{\gamma-1}).$$

These polynomials satisfy the recursion relations

$$Q_{s,0}(\xi) = q_s(\xi), \quad Q_{s+1,s}(\xi) = q_s(\gamma-1)\xi \quad (s \geq 1) \tag{45}$$

and, for $1 \leq t \leq s-1$,

$$Q_{s+1,t}(\xi) = (\xi - s)Q_{s,t}(\xi) + (\gamma-1)\xi Q'_{s,t-1}(\xi) - (t-1)Q_{s,t-1}(\xi). \tag{46}$$

Note that this is some sort of asymptotic expansion. We will see later that terms with $t \geq 1$ have a negligible effect. Before we embark on the proof, we introduce some useful abbreviations which will be valid for the rest of the paper. Let

$$L = \log x, \quad \nu = \gamma - 1, \quad \xi = \xi(x) = \gamma L^\nu.$$

Proof. — We have

$$F'(x) = \frac{F(x)}{x} \gamma(\log x)^{\gamma-1} = \frac{F(x)}{x} \xi(x) \tag{47}$$

which proves the lemma when $s = 1$, with $Q_{1,0}(\xi) = \xi$.

Now suppose the lemma holds for s. Then, by (47)

$$F^{(s+1)}(x) = \frac{d}{dx}\Big(\frac{F(x)}{x^s} \sum_{t=0}^{s-1} L^{-t} Q_{s,t}(\xi)\Big) = \frac{F(x)}{x^{s+1}}(\xi - s) \sum_{t=0}^{s-1} L^{-t} Q_{s,t}(\xi)$$
$$+ \frac{F(x)}{x^s} \sum_{t=0}^{s-1} \Big(L^{-t} Q'_{s,t}(\xi)\xi'(x) - tL^{-t-1}x^{-1}Q_{s,t}(\xi)\Big)$$
$$= \frac{F(x)}{x^{s+1}}\Big(\sum_{t=0}^{s} L^{-t}(\xi - s)Q_{s,t}(\xi) + \sum_{\tau=1}^{s} L^{-\tau}(\nu\xi Q'_{s,\tau-1}(\xi) - (\tau-1)Q_{s,\tau-1}(\xi)\Big).$$

This establishes the formula for $F^{(s+1)}(x)$ and (46). Moreover, we see that

$$Q_{s+1,0}(\xi) = (\xi - s)Q_{s,0}(\xi), \tag{48}$$

$$Q_{s+1,s}(\xi) = \nu\xi Q'_{s,s-1}(\xi) - (s-1)Q_{s,s-1}(\xi). \tag{49}$$

The first identity in (45) is immediate from (48) and $Q_{1,0}(\xi) = \xi$. To verify the second identity in (45), we use (49) to confirm that $Q_{2,1}(\xi) = \nu\xi$, which is the identity in question for $s = 1$. Now, if the second identity in (45) holds for $s-1$ in place of s, then (49) yields

$$Q_{s+1,s}(\xi) = \nu\xi q_{s-1}(\nu) - (s-1)q_{s-1}(\nu)\xi = \xi q_s(\nu),$$

as required.

LEMMA 12. — *Let $s \in \mathbf{N}$, $0 \le t \le s-1$ and $j \ge 0$ with $t+j \le s$. Then, for $1 < \gamma < 2$ and $\xi \ge s$ we have*

$$|Q^{(j)}_{s,t}(\xi)| \le 2^{2t+j} q_{s-t-j}(\xi) \tag{50}$$

and

$$Q^{(s-t+1)}_{s,t} = 0. \tag{51}$$

Proof. — Note that (51) is trivial since $Q_{s,t}$ is a polynomial of degree at most $s-t$.

The proof of (50) is by nested induction. First suppose that $t = 0$. Then, by (45) and (44) we immediately have (50) for all possible values of s and j.

We now proceed by induction on t, and assume that (50) holds with t replaced by $t-1$, and all possible values of s and j. We have to verify (50) for all $s \ge t+1$, and all $0 \le j \le s-t$. First take $s = t+1$. Then, by the second identity in (45) (with $s = t$), the required estimate in (50) follows from the obvious inequality $|q_t(\nu)| \le (t+1)^{2t}$ (note that $|\nu| \le 1$ so that $|q_t(\nu)| \le t!$ which is rather sharper). This verifies (50) for $s = t+1$, and all j (note that only $j = 0$ and $j = 1$ are of interest).

With the value of t fixed, we now induct on s. We suppose that (50) is known for s (and all j), and proceed to verify (50) for $s+1$ (and all j). Since $s \ge t+1$, we can use the recursion formula (46). We differentiate (46) j times, using (51) whenever necessary and the obvious formula

$$\frac{d^j}{d\xi^j}(\xi - a)g(\xi) = (\xi - a)g^{(j)}(\xi) + kg^{(j-1)}(\xi),$$

to deduce that

$$\begin{aligned} Q^{(j)}_{s+1,t}(\xi) = (\xi - s)Q^{(j)}_{s,t}(\xi) + jQ^{(j-1)}_{s,t}(\xi) + \nu\xi Q^{(j+1)}_{s,t-1} \\ +j\nu Q^{(j)}_{s,t-1}(\xi) - (t-1)Q^{(j)}_{s,t-1}(\xi). \end{aligned} \tag{52}$$

We now begin with $j = 0$. Then, for $\xi \geq s + 1$, we deduce from (46) and the induction hypothesis (which applies with $t - 1$ for all s and j)

$$\begin{aligned}
&|Q_{s+1,t}(\xi)| \leq (\xi - s)|Q_{s,t}(\xi)| + \xi|Q'_{s,t-1}(\xi)| + (t-1)|Q_{s,t-1}(\xi)| \\
&\leq (\xi - s)s^{2t}q_{s-t}(\xi) + \xi s^{2t-1}q_{s-t}(\xi) + (t-1)s^{2t-2}q_{s+1-t}(\xi) \\
&= ((\xi - s + t) - t)s^{2t}q_{s-t}(\xi) + ((\xi - s + t) + s - t)s^{2t-1}q_{s-t}(\xi) \\
&+(t-1)s^{2t-2}q_{s+1-t}(\xi) \\
&= (s^{2t} + s^{2t-1} + (t-1)s^{2t-2})q_{s+1-t}(\xi) + (s^{2t} - ts^{2t} - ts^{2t-1})q_{s-t}(\xi).
\end{aligned}$$

Here we used $(\xi - s + t)q_{s-t}(\xi) = q_{s+1-t}(\xi)$. Now, for $t \geq 1$,

$$s^{2t} + s^{2t-1} + (t-1)s^{2t-2} \leq (s+1)^{2t}, \quad s^{2t} - ts^{2t} - ts^{2t-1} < 0,$$

and (50) follows for $s + 1$ in place of s when $j = 0$.

For $j > 0$ we can use (??), and the same reasoning produces

$$\begin{aligned}
&|Q^{(j)}_{s+1,t}(\xi)| \leq (\xi - s)|Q^{(j)}_{s,t}(\xi)| + j|Q^{(j-1)}_{s,t}(\xi)| + \xi|Q^{(j+1)}_{s,t-1}(\xi)| + j|Q^{(j)}_{s,t-1}(\xi)| \\
&+(t-1)|Q^{(j)}_{s,t-1}(\xi)| \leq (\xi - s)s^{2t+j}q_{s-t-j}(\xi) + js^{2t+j-1}q_{s-t-j+1}(\xi) \\
&+\xi s^{2t+j-1}q_{s-t-j}(\xi) + (j+t-1)s^{2t-2+j}q_{s+1-t-j}(\xi).
\end{aligned}$$

As before, we use brute force to generate terms with $q_{s+1-t-j}(\xi)$. This yields

$$|Q^{(j)}_{s+1,t}(\xi)| \leq B_1 q_{s+1-t-j}(\xi) + B_2 q_{s-t-j}(\xi).$$

The coefficient of $q_{s+1-t-j}(\xi)$ is

$$B_1 = s^{2t+j} + (j+1)s^{2t+j-1} + (j+t-1)s^{2t+j-2} \leq (s+1)^{2t+j},$$

and the coefficient of $q_{s-t-j}(\xi)$ is

$$B_2 = s^{2t+j} - (t+j)(s^{2t+j} + s^{2t+j-1}) < 0.$$

This establishes (50) for $s + 1$ in place of s, for all $j \geq 1$. This completes the inductions on s and t.

We can now deduce a precise asymptotic formula for $F^{(s)}(x)$. In Lemma 11 we single out the term with $t = 0$ to obtain

$$F^{(s)}(x) = \frac{F(x)}{x^s}(q_s(\xi) + E(x))$$

where

$$E(x) = \sum_{t=1}^{s-1} L^{-t} Q_{s,t}(\xi).$$

We bound $E(x)$ with the aid of Lemma 12. When $\xi \geq s$ this yields

$$|E(x)| \leq \sum_{t=1}^{s-1} \Big(\frac{s^2}{L}\Big)^t q_{s-t}(\xi) \leq q_{s-1}(\xi) \sum_{t=1}^{s-1} \Big(\frac{s^2}{L}\Big)^t$$

(note that $\xi \geq s$ is needed in Lemma 12 as well as to arrange that $q_{s-t}(\xi) \leq q_{s-1}(\xi)$ for $1 \leq t \leq s-1$). If we also suppose that $s^2 \leq \frac{1}{2}L$, we can sum the geometric series and may conclude as follows

LEMMA 13. — *Let $1 < \gamma < 2$ and $s \in \mathbf{N}$. Then*

$$F^{(s)}(x) = \frac{F(x)}{x^s}(q_s(\xi) + E(x))$$

where, for $\gamma(\log x)^{\gamma-1} \geq s$ and $s^2 \leq \frac{1}{2}\log x$, the function $E(x)$ satisfies

$$|E(x)| \leq \frac{2s^2}{\log x} q_{s-1}(\gamma(\log x)^{\gamma-1}).$$

One comment might be useful. When $\gamma < \frac{3}{2}$ the condition $s \leq \xi$ implies $s^2 \leq \gamma^2(\log x)^{2\gamma-2} \leq \frac{1}{4}\log x$ for $x \geq x_0(\gamma)$, so in this range for γ only the condition $\xi \geq s$ is relevant in Lemma 13. Moreover, still subject to $\xi \geq s$, we then find

$$F^{(s)}(x) \geq \frac{F(x)}{x^s}\Big(q_s(\xi) - \frac{2s^2}{\log x} q_{s-1}(\xi)\Big) \geq \frac{F(x)}{x^s} q_{s-1}(\xi)\Big(\xi - s + \frac{1}{2}\Big).$$

The factor $q_{s-1}(\xi)(\xi - s + \frac{1}{2})$ is an increasing function for $\xi \geq s$, and takes the value $\frac{1}{2}s!$ at $\xi = s$. This proves the following

LEMMA 14. — *Let $1 < \gamma < \frac{3}{2}$. Then there is a constant $x_0 = x_0(\gamma)$ such that for all $s \in \mathbf{N}$ and all $x \geq x_0$ with $\gamma(\log x)^{\gamma-1} \geq s$ one has*

$$\frac{F^{(s)}(x)}{s!} \geq \frac{F(x)}{2x^s}.$$

It would be easy to establish a complementary upper bound for $F^{(s)}(x)$ from the above results. However, such a bound would be subject to $\xi \geq s$ which is inappropriate for our later purposes. Although it would be possible to provide the required estimate by a variant of the previous argument, it is much simpler to estimate $F^{(s)}(x)$ by Cauchy's integral formula.

LEMMA 15. — *Let $\gamma > 1$ and $N \geq 10$. Then, for all $s \in \mathbf{N}$ and $N < x \leq 2N$ we have*

$$\Big|\frac{F^{(s)}(x)}{s!}\Big| \leq 2\Big(\frac{2}{N}\Big)^s F(3N).$$

Proof. — The function $F(z) = \exp((\log z)^\gamma)$ is a holomorphic function of $z = x + iy$ in the half-plane $x > 1$, defined by analytic continuation of the real function. Let $\mathcal{C}$ be the oval in the complex plane, consisting of all points of exact distance $\frac{1}{2}N$ from the real interval $[N, 2N]$. Then, for $N \le x \le 2N$, by Cauchy's formula

$$\frac{F^{(s)}(x)}{s!} = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{F(\zeta)}{(\zeta - x)^{s+1}}\, d\zeta.$$

Since $|\zeta - x| \ge \frac{1}{2}N$ for all $\zeta \in \mathcal{C}$, and the length of the oval is $(\pi + 2)N$, we have

$$\frac{F^{(s)}(x)}{s!} \le 2\Big(\frac{2}{N}\Big)^s \max_{z\in\mathcal{C}} |F(z)|.$$

It is readily confirmed that

$$|F(z)| \le \exp(|\log z|^\gamma)$$

holds for all $z \in \mathcal{C}$. Now write $z = re^{i\phi}$. For $z \in \mathcal{C}$ one has $|\phi| \le \frac{\pi}{6}$, whence $|\log z| \le \log r + \frac{\pi}{6} \le \log(3N)$. Consequently, $|F(z)| \le F(3N)$ for $z \in \mathcal{C}$, and the lemma follows.

We now establish Theorem 2 by an appeal to Lemma 10. The following result will be useful.

LEMMA 16. — *Let $\alpha > 0$, $0 < \vartheta < 1$ and write $\alpha = N^{-A}$.*

(a) *Let s be a natural number satisfying*

$$s(1 - \vartheta) + A \ge \frac{(s+1)\log 2}{\log N} + \frac{(\log 3N)^\gamma}{\log N}. \tag{53}$$

Then

$$\Big|\frac{\alpha F^{(s)}(x)}{s!}\Big| \le N^{-\vartheta s} \quad (N < x \le 2N).$$

(b) *Let s be a natural number satisfying $s \le \gamma(\log N)^{\gamma-1}$ and*

$$s(1 - \vartheta) + A \le (\log N)^{\gamma-1} - \frac{\log 2}{\log N}. \tag{54}$$

Then

$$\Big|\frac{\alpha F^{(s)}(x)}{s!}\Big| \ge N^{-\vartheta s} \quad (N < x \le 2N).$$

Proof. — For (a), use Lemma 15. Then, it suffices to verify the inequality

$$2\alpha F(3N)\Big(\frac{2}{N}\Big)^s \le N^{-\vartheta s}$$

which after taking logarithms is seen to be equivalent with (53). For (b), use Lemma 14 and note that $x^{-s}F(x)$ is increasing for $x > N$, since $s \leq \gamma(\log N)^{\gamma-1}$. Therefore, it suffices to show that

$$\frac{\alpha F(N)}{2N^s} \geq N^{-\vartheta s}$$

which is equivalent with (54).

To prove Theorem 2 it will suffice to consider $\alpha > 0$. Choose θ with $0 < \theta < \frac{1}{2}$ and $C < \gamma(1-\theta)$, which is always possible. We now proceed to show that the function $\alpha F(x)$ is in $\mathcal{F}(\theta, \frac{1}{2}\theta, \frac{3}{4}, \delta)$, where

$$\delta = \frac{1}{20}(1-c)\Big(\frac{1}{1-\theta} - \frac{1}{1-\frac{1}{2}\theta}\Big),$$

for all $\alpha \in [F(2N)^{-c}, F(N)^{C}]$. This will establish Theorem 2.

We confirm the hypotheses of Lemma 10 with $\Theta_0 = \frac{3}{4}$, $\Theta = \frac{1}{2}\theta$ and $k = [9(\log N)^{\gamma-1}] + 1$. With $\alpha = N^{-A}$ we have

$$-C(\log N)^{\gamma-1} \leq A \leq c\frac{(\log 2N)^{\gamma}}{\log N}. \tag{55}$$

Take $\vartheta = \frac{3}{4}$ and $s = k$ in Lemma 16 (a) to verify $\Big|\frac{\alpha F^{(k)}(x)}{k!}\Big| \leq N^{-\frac{3}{4}k}$, as required.

We now wish to determine the values of s for which we have $N^{-\theta s} \leq \Big|\frac{\alpha F^{(s)}(x)}{s!}\Big| \leq N^{-\frac{1}{2}\theta s}$, for $N < x \leq 2N$. Suppose for the moment that $s \leq \gamma(\log N)^{\gamma-1}$. Then the required inequalities will follow from Lemma 16 provided we have

$$\Big(1-\frac{1}{2}\theta-\frac{\log 2}{\log N}\Big)^{-1}\Big(\frac{(\log 3N)^{\gamma}}{\log N}-A\Big) \leq s \leq \frac{1}{1-\theta}\Big((\log N)^{\gamma-1}-A-\frac{\log 2}{\log N}\Big). \tag{56}$$

Note that the choice for θ implies that all s satisfying (56) will automatically satisfy $s \leq \gamma(\log N)^{\gamma-1} < k$; this follows from (55). Therefore, in Lemma 10 we can take $\mathcal{S}$ as the set of all s satisfying (56). It is then readily seen from (56) and (55) that $\#\mathcal{S} \geq \delta k$, at least for large N. All the conditions of Lemma 10 are satisfied, and the proof of Theorem 2 is complete.

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