TANYA BANDMAN
GERD DETHLOFF

Estimates of the number of rational mappings from a fixed variety to varieties of general type


<http://www.numdam.org/item?id=AIF_1997__47_3_801_0>
ESTIMATES OF THE NUMBER OF RATIONAL MAPPINGS FROM A FIXED VARIETY TO VARIETIES OF GENERAL TYPE

by T. BANDMAN and G. DETHLOFF

0. Introduction.

Let $X$ and $Y$ be algebraic varieties, i.e. complete integral schemes over a field of characteristic zero. Denote by $R(X,Y)$ the set of dominant rational maps $f : X \to Y$. Then the classical theorems of de Franchis [Fran] and Severi (cf. [Sam]) can be stated as follows:

**Theorem 0.1.**

a) *(de Franchis)*: For any Riemann surface $X$ and any hyperbolic Riemann surface $Y$ the set $R(X,Y)$ is finite. Furthermore, there exists an upper bound for $\#R(X,Y)$ only in terms of $X$.

b) *(Severi)*: For a fixed algebraic variety $X$ there exist only finitely many hyperbolic Riemann surfaces $Y$ such that $R(X,Y)$ is nonempty.

S. Kobayashi and T. Ochiai [KobOch] prove the following generalization of the de Franchis Theorem: If $X$ is a Moishezon space and $Y$ a compact complex space of general type, then the set of surjective meromorphic maps from $X$ to $Y$ is finite. Other generalizations can be found...
in [DesMen1], [Nog], [Suz], and in the survey [Zai Lin]. Generalizations of
the second part of de Franchis’ Theorem were given in [Ban1], [Ban2] and
[BanMar]. In the latter paper it is proved that for complex projective vari-
eties \( X \) and \( Y \) with only canonical singularities and nef and big canonical
classes \( K_X \) and \( K_Y \) respectively the number \( \#R(X,Y) \) can be bounded
in terms of the selfintersection \( K_X^3 \) of the canonical class of \( X \) and of the
indices of \( X \) and \( Y \). This bound is not effective. Effective bounds are known
only if the varieties \( Y \) are curves or surfaces ([Kan], [HowSom2], [Tsa3]).

Section 1 of this paper contains an effective estimate of the number
of mappings in \( R(X,Y) \), provided that both varieties \( X \) and \( Y \) are smooth
projective with ample canonical bundles \( K_X, K_Y \) respectively. This bound
has the form \( \{A \cdot K_X^n\}^{B \cdot K_Y^2} \), where \( n = \dim \mathbb{C} \cdot X \), \( K_X \) is the canonical
bundle of \( X \) and \( A, B \) are some constants, depending only on \( n \).

This bound seems to be very big. But it is known that the bound
cannot be polynomial in \( K_X^n \) ([Kan]). Moreover, even for the case of curves
of genus 5 the best bound in [Kan] is of order \( \exp(30) \).

The idea to obtain this bound was used in ([HowSom1]) for proving
finiteness of the automorphism group of a projective variety with ample
canonical bundle. It could not be made effective at that time, as no
effective variants of the Big Matsusaka theorem were available. Moreover,
exponential bounds for the number of automorphisms are not interesting,
as they should be linear in \( K_X^n \) ([Sza]).

In Sections 2, 3 and 4 we generalize Severi’s result (Theorem 0.1(b))
to higher dimensions.

Denote by \( \mathcal{F}(X) \) the set of pairs \( (Y, f) \), where \( Y \) is of general type
and \( f \in R(X,Y) \). Let \( \mathcal{F}_m(X) \subseteq \mathcal{F}(X) \) the subset of those pairs \( (Y, f) \)
for which the \( m \)-th pluricanonical mapping of a desingularization of \( Y \)
is birational onto its image. Consider the equivalence relations on \( \mathcal{F} \) and \( \mathcal{F}_m \):
\( (f : X \to Y) \sim (f_1 : X \to Y_1) \) iff \( b \circ f = f_1 \), where \( b \in \text{Bir}(Y, Y_1) \). The
elements of \( \mathcal{F}(X)/\sim \) we call targets.

The following conjecture is stated by Maehara ([Mae3]) as Iitaka’s
Conjecture based on Severi’s Theorem:

**Conjecture 0.2.** — *The set \( \mathcal{F}(X)/\sim \) of targets is a finite set.*

Maehara proved in Proposition 6.5 in [Mae2] that in characteristic zero
\( \mathcal{F}_m(X)/\sim \) is finite for all \( m \). In particular the Conjecture is valid for
In Section 3 we prove Conjecture 0.2 for the case that the targets are complex threefolds (Theorem 3.1).

For the proof we use the following theorem of Luo [Luo1], [Det]:

**Theorem 0.3.** — Consider the set of smooth threefolds $Y$ of general type, and denote by $\chi(Y, \mathcal{O}_Y)$ the holomorphic Euler characteristic. Then for any fixed $\chi = \chi(Y, \mathcal{O}_Y)$, there is a universal integer $m'$ such that $h^0(Y, \mathcal{O}_Y(m'K_Y)) \geq 2$. Furthermore, there is a universal integer $m$ such that the $m$-th pluricanonical map $\Phi_{mK} : Y \to \Phi_{mK}(Y)$ maps birationally onto its image.

In Section 4 the domain is a threefold of general type. In this case we show (Theorem 4.1) that there is a bound for the number of targets $\#(\mathcal{F}(X)/\sim)$, which depends only on the selfintersection $K_X^3$ and the index $r_{X_e}$ of the canonical model $X_e$ of $X$.

The proof is based on the fact, due to Kollar [Kol1], that canonical threefolds with fixed Hilbert polynomial form a bounded family. Using semicontinuity theorems for the dimensions of cohomology groups we get estimates of the holomorphic Euler characteristics of the targets. Then we show that the graphs of maps under consideration, which map from canonical threefolds $X$ with fixed index $r_X$ and fixed $K_X^3$, form a finite number of algebraic families. The number of targets is bounded by the number of irreducible components of the members of these families.

In Section 5 we return to generalizations of de Franchis’ result (Theorem 0.1(a)). Consider a threefold $X$ of general type. We prove (Theorem 5.1) that there exists a bound for $\#(R(X,Y))$, depending only on $X$. Namely, it depends on the selfintersection $K_X^3$ and on the index $r_{X_e}$ of the canonical model $X_e$ of $X$.

In the review of Sh. Kobayashi ([Kob], problem D3) the question is raised if for a compact complex space $X$ and a hyperbolic compact complex space $Y$ the number of surjective meromorphic maps from $X$ to $Y$ can be bounded only in terms of $X$. Theorem 5.1 is an answer to this question for threefolds of general type.

Further on all the varieties are complex; we do not make difference between line bundles, divisor classes and the divisors themselves, if no confusion may arise. We fix resp. recall the following notations, which are used in the paper:
$X, Y$ – complex varieties;

$\mathcal{R}(X, Y)$ – the set of rational dominant maps from $X$ to $Y$;

$\mathcal{F}(X)$ – the set of pairs $(Y, f)$, where $Y$ is of general type and $f \in \mathcal{R}(X, Y)$;

$\mathcal{F}_m(X)$ – the subset of those pairs $(Y, f)$ for which the $m$-th pluricanonical mapping of a desingularization of $Y$ is birational onto its image;

$(f : X \to Y) \sim (f_1 : X \to Y_1)$ iff $b \circ f = f_1$, where $b \in \text{Bir}(Y, Y_1)$;

$K_X$ – the canonical sheaf of a variety $X$ with at most canonical singularities;

$K^n_X$ – the $n$-times self-intersection of the class $K_X$, where $n = \dim \mathbb{C} X$;

$c_i(X)$ – the $i$th Chern class of the variety $X$;

$H^i(X, D) = H^i(X, \mathcal{O}_X(D)); h^i(X, D) = \dim \mathbb{C} H^i(X, D)$;

$\chi(X, D) = \sum_{i=1}^{n} (-1)^i h^i(X, D)$;

$X_c$ – the canonical model of a variety $X$ of general type of $\dim \mathbb{C} X \leq 3$;

$r_X$ – the index of a variety $X$ with at most canonical singularities;

$\Phi_{mK_Y}$ – the $m$-th pluricanonical map from a variety $Y$ with at most canonical singularities.

1. Effective estimates of $\mathcal{R}(X, Y)$ for smooth manifolds $X, Y$ with ample canonical bundles.

The main theorem of this section is Theorem 1.6 below. It provides an effective estimate for $\#\mathcal{R}(X, Y)$ if $X, Y$ are smooth manifolds with ample canonical divisors.

We first recall some notations and facts about duality:

a) A subspace $E \subset \mathbb{P}^N$ is called linear if it is the projectivization of a linear subspace $E^a \subset \mathbb{C}^{N+1}$. Let $\rho : \mathbb{C}^{N+1} \to (\mathbb{C}^{N+1})^*$ denote the canonical isomorphism between $\mathbb{C}^{N+1}$ and the space $(\mathbb{C}^{N+1})^*$ of linear functionals on it, which is given by the standard hermitian product on $\mathbb{C}^{N+1}$. We denote by $(\mathbb{P}^N)^*$ resp. $E^*$ the projectivizations of $(\mathbb{C}^{N+1})^*$ resp. of $\rho(E^a) \subset (\mathbb{C}^{N+1})^*$, and call them the conjugate spaces to $\mathbb{P}^N$ resp. $E$. (We don’t use the word ‘dual’ here in order not to have confusion with the notion of a dual variety which is defined below.)
b) We call a rational mapping $L : \mathbb{P}^N \to \mathbb{P}^M$ linear if it is the projectivization of a linear map $L^a : \mathbb{C}^{N+1} \to \mathbb{C}^{M+1}$. The projectivization of the induced map $(L^a)^* : (\mathbb{C}^{M+1})^* \to (\mathbb{C}^{N+1})^*$ is denoted by $L^*$ and called the dual map to $L$.

c) Let $Z$ be an $n$-dimensional projective variety embedded into the projective space $\mathbb{P}^N$. In any non-singular point $z \in Z$ the projective tangent plane $T_z$ is well defined. In the conjugate projective space $(\mathbb{P}^N)^*$ we consider the set $Z_0^V$ of all points $y \in (\mathbb{P}^N)^*$ such that the corresponding hyperplane $H_y \subset \mathbb{P}^N$ contains the tangent plane $T_z$ to some nonsingular point $z \in Z$. We define the dual variety $Z^V$ of the variety $Z$ to be the closure of $Z_0^V$ in the Zarisky topology.

d) The dual varieties have the following fundamental properties ([DelKat]):

1) If $Z$ is nonsingular and Kodaira dimension $k(Z) > -\infty$, then $Z^V$ is irreducible and codim $Z^V = 1$.

2) Moreover, if $L$ is a hyperplane section of $Z \subset \mathbb{P}^N$, the degree $\deg Z^V$ of the variety $Z^V$ may be computed by the Chern classes:

$$\deg Z^V = \sum_{i=0}^{n} (-1)^{n+i}(1+i)c_i^1(L)c_{n-i}(Z),$$

where $c_i^1(L)$ is the first Chern class of the line bundle corresponding to $L$.

3) $Z^{VV} = Z$.

Let $X$ and $Y$ be two smooth projective $n$-dimensional varieties with Kodaira dimension bigger than infinity, and let $E$ and $F$ be very ample line bundles on $X$ and $Y$ respectively. Then the varieties $X$ and $Y$ are canonically embedded into the projectivizations of the conjugate spaces to $H^0(X, E)$ and to $H^0(Y, F)$ respectively, which we denote by $\mathbb{P}^N$ and $\mathbb{P}^M$.

Let $f : X \to Y$ be a rational dominant mapping, and let $\Psi : H^0(Y, F) \to H^0(X, E)$ be an injective linear map. We call $f$ to be induced by the map $\Psi$ if the projectivization of the dual map $\Psi^*$, restricted to $X$, is the map $f$. Denote by $R(X, E, Y, F)$ the set

$$R(X, E, Y, F) = \{ f \in R(X, Y) : f \text{ is induced by an injective linear map } \Psi : H^0(Y, F) \to H^0(X, E) \}. $$
PROPOSITION 1.1. — If the set \( R(X, E, Y, F) \) is finite, we have:
\[
\#R(X, E, Y, F) \leq m(E)^{\psi(E)}
\]
where
\[
m(E) = \sum_{i=0}^{n} (-1)^{n+i}(1+i)c_i^1(E)c_{n-i}(X),
\]
\[
\psi(E) = (h^0(X, E))^2 - 1.
\]

Before we start with the proof of Proposition 1.1, we need two lemmas.

LEMMA 1.2. — Denote by \( G \) the set of all linear injections \( A : \mathbb{P}^M \to \mathbb{P}^N \) such that \( A(Y^V) \subset X^V \). Then \( G \) is a quasiprojective subset of \( \mathbb{P}^K \), \( K = (N+1)(M+1) - 1 \), of degree

(2) \[
\deg G \leq (\deg X^V)^K.
\]

Proof of Lemma 1.2. — Any element of \( G \) is defined by a \( (N+1)(M+1) \) matrix \( A \), and its components \( (a_{ij}) \) may be considered as its coordinates in the projective space \( \mathbb{P}^K \), \( K = (N+1)(M+1) - 1 \). Since codim \( X^V = 1 \), it is defined in \( (\mathbb{P}^N)^* \) by a single equation \( F(z_0, \ldots, z_N) = 0 \) with \( \deg F = \deg X^V \). If \( y \in Y^V \) we have \( A(y) \in X^V \), and so \( F(Ay) = 0 \). For a fixed point \( y \) and a fixed polynomial \( F \) this is an equation for the coordinates \( a_{ij} \) in the space \( \mathbb{P}^K \).

This means that for any finite sequence of points \( y_1, \ldots, y_r, y_t \in Y^V \), the set \( G \) is contained in the algebraic set \( G^{(r)} \), defined by equations
\[
F(Ay_1) = 0
\]
\[
\vdots
\]
\[
F(Ay_r) = 0
\]
in the space \( \mathbb{P}^K \).

Choose any point \( y_1 \in Y^V \). Suppose that \( \overline{G} \neq G^{(1)} \), where \( \overline{G} \) denotes the Zariski closure of \( G \) in \( \mathbb{P}^K \). Then there exists a point \( y_2 \) such that for some \( A \in G^{(1)} \)
\[
F(Ay_2) \neq 0.
\]
Define the set \( G^{(2)} \) by the pair \( y_1, y_2 \). It follows that \( G^{(2)} \subset G^{(1)} \), and for some component \( C \) of \( G^{(1)} \) all components of \( G^{(2)} \) which lie in \( C \) (if there are any at all) are of smaller dimension than \( C \). After performing a finite
number of such steps we get a set \( y_1, y_2, \ldots, y_r \), such that \( G^{(r)} = \overline{G} \). Hence, \( G \) can be defined in \( \mathbb{P}^K \) by equations of degree \( \deg F = \deg X^V \) only. Now, the inequality

\[
\deg G \leq (\deg X^V)^K.
\]

follows from the

**Sublemma (the analogue of the Bezout Theorem).** — Let \( X \subset \mathbb{P}^n \) be an irreducible variety, \( \dim_X X = i, \deg X = a \). Let \( F_1, \ldots F_s \) be homogeneous polynomials of degree \( d \) and \( X_s = \{ z \in \mathbb{P}^n : F_1(z) = F_2(z) = \ldots = F_s(z) = 0 \} \). Assume that \( X \cap X_s = \bigcup_{j=1}^N B_j \) is a union of irreducible components \( B_j \). Then

\[
\deg(X \cap X_s) = \sum_j \deg B_j \leq ad^i.
\]

**Proof of the Sublemma.** — We perform induction by \( i = \dim X \). If \( i = 1 \), there are two possibilities:

1. \( F_k \big|_X \neq 0 \) for all \( k = 0, \ldots, s \); then \( X = B_1, N = 1 \), and \( \deg B_1 = \deg X = a \).
2. \( F_1 \big|_X \neq 0 \). Then \( X \cap X_s \subset X \cap X_1 \) is a finite number \( T \) of points and

\[
T \leq \deg(X \cap X_1) \leq ad.
\]

Assume that the fact is true for every \( i < m \). If \( F_k \big|_X \neq 0 \) for all \( k = 1, \ldots, s \), then \( N = 1, X = B_1 \) and \( \deg B_1 = a \). If \( F_s \big|_X \neq 0 \), then \( X \cap \{ F_s = 0 \} = \bigcup A_q \) is a union of irreducible components \( A_q \) such that \( n_q = \dim A_q < m \) and

\[
\sum_q \deg A_q \leq ad
\]

(see, for example, [Har], Th.7., ch. 1). Let \( A_q \cap X_{s-1} = \bigcup B_q^r \). Since

\[
\bigcup_{q,r} B_q^r = \left( \bigcup_q A_q \right) \cap X_{s-1} = X \cap X_s = \bigcup_j B_j,
\]

and all \( B_q^r \) and \( B_j \) are irreducible, we obtain that for any \( j \) there are numbers \((q,r)\) such that \( B_j = B_q^r \). Thus

\[
\sum_j \deg B_j \leq \sum_{q,r} \deg B_q^r.
\]
By induction assumption
\[ \sum_r \deg B^r_q \leq \deg A_q d^{m_q} \leq \deg A_q d^{m-1}. \]
Summation over \( q \) provides the desired inequality:
\[ \sum_j \sum_r \deg B^r_j \leq \sum_q \sum_r \deg B^r_q \leq \sum_q \deg A_q d^{m-1} \leq d^{m-1} \sum_q \deg A_q \leq ad^m. \]

□

Let \( G = \bigcup_i G_i \) be the decomposition of \( G \) in irreducible components \( G_i \).

**Lemma 1.3.** — Suppose that the points \( t_1, t_2 \in G_i \) define linear maps \( A_j, j = 1, 2 \), which are dual to linear projections \( A^*_j : \mathbb{P}^N \to \mathbb{P}^M \) satisfying \( A^*_j(X) = Y \) (i.e. \( f_j := A^*_j|_X \in R(X, E, Y, F) \)). Then \( f_1 = f_2 \).

**Proof of Lemma 1.3.** — From now on we fix a basis in \( \mathbb{P}^N \) and \( \mathbb{P}^M \). Let \( t \in G_i \) and let \( A_t \) be a linear embedding \( A_t : (\mathbb{P}^M)^* \to (\mathbb{P}^N)^* \) corresponding to a point \( t \).

Consider the following diagram:

\[
\begin{align*}
X & \subset \mathbb{P}^N \cong (\mathbb{P}^N)^* \subset X^V \\
\downarrow A^*_t \big|_X & \quad \downarrow A^*_t \\
X_t & \subset L_t \cong E_t \subset X^V \cap E_t \\
\vdots \quad \tau_t & \quad \vdots \quad \tau_t \quad \uparrow A_t \\
Y & \subset \mathbb{P}^M \cong (\mathbb{P}^M)^* \subset Y^V
\end{align*}
\]

In this diagram:

1) \( A_t \) is a linear embedding of \( (\mathbb{P}^M)^* \) into \( (\mathbb{P}^N)^* \).

2) \( E_t = A_t((\mathbb{P}^M)^*) \) is a linear subspace of \( (\mathbb{P}^N)^* \), which is isomorphic to \( (\mathbb{P}^M)^* \).

3) The dual map \( A^*_t : \mathbb{P}^N \to \mathbb{P}^M \) is the composition of a projection of \( \mathbb{P}^N \) onto the subspace \( L_t \subset \mathbb{P}^N \), which is dual to \( E_t \), and of an isomorphism
\( \tau_t : L_t \to \mathbb{P}^M \) (recall that we have chosen a basis in \( \mathbb{P}^N \) and a dual basis in \((\mathbb{P}^N)^*\)). The projection we again denote by \( A^*_t \).

4) \( X_t = A^*_t(X) \subset L_t \).

Now the linear map \( A_t \) induces a dominant rational map \( f_t : X \to Y \), iff \( \tau_t(X_t) = Y \).

In order to proceed with the proof, the following two claims are needed.

**Claim 1.** — Let \( R_i(x, t), i = 1, \ldots, l \), be a finite number of polynomials in the variable \( x \in \mathbb{P}^k \) with coefficients which are polynomials in the variable \( t \in T \), where \( T \) is an irreducible projective variety. Let \( V_i = \{ x \in \mathbb{P}^k : R_1(x, t) = \ldots = R_l(x, t) = 0 \} \). Assume that for some point \( t_0 \in T \),

\[
\begin{align*}
    r(x, t_0) = \text{rank} \left\{ \frac{\partial R_i}{\partial x_j}(x, t_0) \right\} = k 
\end{align*}
\]

for all \( x \in V_{t_0} \). Then the set of points \( t \in T \), such that \( r < k \) for some \( x \in V_t \), is proper and closed in \( T \).

**Proof.** — Consider the sets \( A = \{ (x, t) \in \mathbb{P}^k \times T : r(x, t) < k \} \) and \( B = \{ (x, t) \in \mathbb{P}^k \times T : x \in V_i \} \). Since \( A \cap B \) is Zariski closed in \( \mathbb{P}^k \times T \), its projection to \( T \) is Zariski closed in \( T \). Since there is at least one point \( t_0 \), which does not belong to the image of this projection, it has to be a proper Zariski closed subset. □

By the assumption \( G_i \) has two points \( t_1, t_2 \), defining the maps \( f_1, f_2 \in R(X, E, Y, F) \).

**Claim 2.** — Let \( T' \subset G_i \) be the set of points \( t \in G_i \), for which the image \( X_t \) of the projection of \( X \) into \( L_t \) is smooth and of the same dimension as \( X \). Then \( T' \) is Zarisky open and contains \( t_1, t_2 \).

**Proof.** — We apply Claim 1 with \( T = G_i, \mathbb{P}^k = L_t \) and where the \( R_i(x, t), i = 1, \ldots, l \) are the resultants of the polynomial equations of \( X \) in \( \mathbb{P}^N \). Then \( V_i = X_t \) is smooth in the point \( x \) iff the rank \( r(x, t) \) is maximal. Since \( t_1, t_2 \) define the maps \( f_1, f_2 \in R(X, E, Y, F) \), the varieties \( X_t \) are isomorphic to \( Y \) through the maps \( \tau_t \). Especially \( X_{t_1}, X_{t_2} \) are smooth and of the same dimension as \( X \). Now Claim 2 follows from Claim 1. □

Using Claim 2, we will show that all \( t \in T' \) correspond to maps \( f_t \in R(X, E, Y, F) \), and moreover, that \( f_t \) does not depend on the parameter \( t \in T' \). Especially, we get \( f_{t_1} = f_{t_2} \).
Since $X_t$ is isomorphic to $Y$, we get for the Kodaira dimensions $k(X_t) = k(Y) > -\infty$. Applying the invariance of plurigenera to the algebraic family with base $T'$ and fiber $X_t$ ([Har], 9.13, ch.3), we get that $k(X_t) > -\infty$ for all $t \in T'$. Then $X'_t$ has to be an irreducible hypersurface in $E_t$, contained in $X^V \cap E_t$. Thus $X^V \cap E_t$ contains an irreducible component $C_t$ such that $C_t = X'_t$, and $C'_t = X^V'_t = X_t$. Let $B_{t,i}$ be other irreducible components of $X^V \cap E_t$. Then $B_{t,i} \subset X_t$. Since $X_t$ is irreducible, we get $\dim B_{t,i} < \dim X_t$. Thus, the intersection $X^V \cap E_t$ is a union of irreducible components $B_{t,i}$ and $C_t$, such that

a) the components $B_{t,i} \subset E_t$ are dual to some subsets of $X_t$ (actually, the image of singular points of the projection of $X$ to $X_t$) of dimension less than $n = \dim_C X$;

b) the component $C_t \subset E_t$ is the only one which has $n$-dimensional dual, and $C'_t = X_t$.

For any point $t \in T'$ the variety $A_t(Y^V)$ is isomorphic to $Y^V$. Thus $\{A_t(Y^V)\}^V$ is isomorphic to $Y^{VV} = Y$ and, hence, it is $n$-dimensional. On the other hand, $A_t(Y^V)$ is contained in $X^V \cap E_t$ and is irreducible and of the same dimension as $X^V \cap E_t$. Hence, $A_t(Y^V) = C_t = X_t$.

That means that $A_t$ is an isomorphism between $(\mathbb{P}^M)^* \times E_t$ such that $A_t(Y^V) = X'_t \subset E_t$. Then the dual isomorphism $\tau_t = A_t^* \mid_{L_t} : L_t \to \mathbb{P}^M$ maps $(X_t)^{VV} = X_t$ onto $Y^{VV} = Y$ (see [HowSom1]). It follows that the map $f_t = \tau_t \circ A_t^* \mid_X$ belongs to $R(X,E,Y,F)$. Since the latter set is finite and the family of the projections $A_t^*$, which give $f_t : X \to Y$, varies continuously over $T'$, the map $f_t$ does not depend on $t$.

**Proof of Proposition 1.1.** — A mapping $f \in R(X,E,Y,F)$ is, by definition, induced by a linear injection $\Psi : H^0(Y,F) \to H^0(X,E)$. More precisely, if we denote by $\tilde{f} : \mathbb{P}^N \to \mathbb{P}^M$ the projectivization of the dual map $\Psi^*$ to $\Psi$, then $f = \tilde{f} \mid_X$.

Using the surjectivity of the maps $f : X \to Y$ and $\tilde{f} : \mathbb{P}^N \to \mathbb{P}^M$, an easy computation yields that the dual map $\tilde{f}^* : (\mathbb{P}^M)^* \to (\mathbb{P}^N)^*$ maps $Y^V$ into $X^V$.

By Lemma 1.3 the number of mappings in $R(X,E,Y,F)$ is, at most, the number of irreducible components in $G$, which obviously does not exceed the sum of degrees of these components. Since the mappings in $R(X,E,Y,F)$ are induced by injections of $H^0(Y,F)$ into $H^0(X,E)$, we
have
\[ M = h^0(Y, F) - 1 \leq h^0(X, E) - 1 = N. \]

By the formula (1) we get, for \( \deg X^V \):
\[ \deg X^V = \sum_{i=0}^{n} (-1)^{n+i}(1 + i)c_i^1(E) \cdot c_{n-i}(X). \]

To obtain the statement of Proposition 1.1 it suffices to insert these values into equation (2).

Using Proposition 1.1 we obtain the following

**Theorem 1.6.** — Let \( X, Y \) be two smooth complex projective varieties with ample canonical bundles \( K_X \) and \( K_Y \). Let \( R(X, Y) \) be the set of dominant rational maps \( f : X \rightarrow Y \), and let the divisors \( sK_X, sK_Y \) be very ample (for example \( s \) may be \( 2 + 12g^1_X([Dem]) \)). Then
\[ \#R(X, Y) \leq \left[ (-1)^n \sum_{i=0}^{n} (1 + i)s^i c_i^1(X)c_{n-i}(X) \right] \left\{ K_X^n \left( \frac{s^n}{n!} - \frac{s^{n-1}}{2(n-1)!} + q_{n-2}(s) \right) \right\}^2, \]

where, for each \( n, q_{n-2}(s) \) is a universal polynomial of degree \( n - 2 \).

**Proof of Theorem 1.6.** — Let \( E = sK_X, F = sK_Y \) and \( f \in R(X, E, Y, F) \). If \( f^* \) here denotes the pull back of pluricanonical forms by the rational map \( f \), we get, by [litr], Theorem 5.3, that \( f^* : H^0(Y, msK_Y) \rightarrow H^0(X, msK_X) \) is an injective linear map for any \( m \in \mathbb{N} \). (Since the divisors \( E \) and \( F \) are very ample, any rational map \( f \in R(X, Y) \) is even regular ([Ban1]), but we don’t need this fact here.) It is easy to see that the map \( f \) is induced by the linear map \( f^* \). That is why \( R(X, Y) = R(X, sK_X, Y, sK_Y) \). Since this set is finite ([KobOch]), we can apply Proposition 1.1:
\[ \#R(X, Y) = \#R(X, sK_X, Y, sK_Y) \leq \left[ \sum_{i=0}^{n} (-1)^{n+i}(1 + i)s^i c_i^1(K_X)c_{n-i}(X) \right] h^0(X, sK_X)^2 - 1 \]

In this expression we substitute \( -c_1(K_X) \) by \( c_1(X) \) (these numbers are equal). Further, by the Riemann-Roch Theorem and the Vanishing Theorem for ample line bundles
\[ h^0(X, sK_X) = \chi(X, sK_X) = K_X^n \left[ \frac{s^n}{n!} - \frac{s^{n-1}}{2(n-1)!} \right] + P(s), \]
where $P(s) = \sum_{i=0}^{n-2} \alpha_i s^i$ is a polynomial of degree $n-2$ in $s$, the coefficients of which are linear combinations of monomials of the form $c_i(X) = c_i(X)\ldots c_i(X)$, $i_1 + \cdots + i_k = n$.

According to ([FulLaz], [CatSch]), there exist universal constants $D_I$, depending only on $n$, such that

$$|c_i(X)| \leq D_I K^n_X.$$

It follows, that there are other universal constants $\tilde{D}_i$, $i = 0,\ldots, n-2$, which depend only on $n$, such that

$$|P(s)| \leq \sum_{i=0}^{n-2} |\alpha_i| s^i \leq K^n_X \cdot \sum_{i=0}^{n-2} \tilde{D}_i \cdot s^i.$$

Hence, it is possible to choose

$$q_{n-2}(s) = \sum_{i=0}^{n-2} \tilde{D}_i s^i.$$

2. Effective estimates for pluricanonical embeddings for threefolds.

This section is motivated by the following

**Question 2.1.** — Let $Y$ be a smooth projective manifold of dimension $n$ which is of general type. Does there exist an integer $m$, depending only on $n$, such that the $m$-th pluricanonical map $\Phi_{mK_Y} : Y \to \Phi_{mK_Y}(Y)$ is birational onto its image?

It is well known (cf. [BPV]) that for curves we can choose $m = 3$, and for surfaces we can choose $m = 5$. Luo conjectured in [Luo1], [Luo2] that for the case of threefolds the answer to the question should also be affirmative. In these two papers, he proves his conjecture in ‘almost all’ possible cases. Especially he shows Theorem 0.3 (cf. Theorem 5.1, Corollary 5.3 of [Luo1]).

When the second named author gave a proof of Conjecture 0.2 for threefolds (cf. [Det]) he was not aware of the papers [Luo1] and [Luo2] of Luo. So he independently gave a proof of Theorem 0.3, using however the same basic idea (apparently both proofs were motivated by the paper [Flet] of Fletcher). Since the proof given in [Det] seems to use the basic
idea in a shorter way and, moreover, easily gives effective bounds, we want to include it here. More precisely we prove the following statement:

**Theorem 2.2.** — Let $C$ be a positive integer. Define $R = \text{lcm}(2, 3, ..., 26C - 1)$ and $m = \text{lcm}(4R + 3, 143C + 5)$. Let $Y$ be any smooth projective threefold of general type for which $\chi(Y, \mathcal{O}_Y) \leq C$ holds. Then $\Phi_{mK_Y} : Y \to \Phi_{mK_Y}(Y)$ is birational onto its image.

For the convenience of the reader and to fix further notations we recall some facts on which the proof is built.

We need the Plurigenus Formula due to Barlow, Fletcher and Reid (cf. [Flet], [Rei2], see also [KolMor], p. 666 for the last part):

**Theorem 2.3.** — Let $Y$ be a projective threefold with only canonical singularities. Then

$$
\chi(Y, mK_Y) = \frac{1}{12}(2m - 1)m(m - 1)K_Y^3 - (2m - 1)\chi(Y, \mathcal{O}_Y) + \sum_Q l(Q, m)
$$

with

$$
l(Q, m) = \sum_{k=1}^{m-1} \frac{\overline{b}k(r - \overline{b}k)}{2r} = \frac{r^2 - 1}{12}(m - \overline{m}) + \sum_{k=1}^{\overline{m}-1} \frac{\overline{b}k(r - \overline{b}k)}{2r}.
$$

Here the summation takes place over a basket of singularities $Q$ of type $\frac{1}{r}(a, -a, 1)$. $\overline{j}$ denotes the smallest nonnegative residue of $j$ modulo $r$, and $b$ is chosen such that $ab = 1$.

Furthermore,

$$\text{index } (Y) = \text{lcm}\{r = r(Q) : Q \in \text{basket}\}.$$

Hanamura ([Han]) proves:

**Theorem 2.4.** — Let $Y$ be a smooth projective threefold of general type, which has a minimal or canonical model of index $r$. Then for any $m \geq m_0$ the $m$-th pluricanonical map is birational onto its image, where

- $m_0 = 4r + 5$ for $1 \leq r \leq 2$
- $m_0 = 4r + 4$ for $3 \leq r \leq 5$
- $m_0 = 4r + 3$ for $r \geq 6$.

In the last step of the proof we use the following theorem of Kollár (Corollary 4.8 in [Kol2]):
THEOREM 2.5. — Assume that for a smooth projective complex threefold \( Y \) of general type we have \( h^0(Y, tK_Y) \geq 2 \). Then the \((11l + 5)\)-th pluricanonical map is birational onto its image.

For estimating from below the terms \( l(Q, m) \) in the Plurigenus Formula, we need two propositions due to Fletcher [Flet]. In these propositions \( [s] \) denotes the integral part of \( s \in \mathbb{R} \).

PROPOSITION 2.6.

\[
l \left( \frac{1}{r}(1, -1, 1), m \right) = \frac{m(m - 1)(3r + 1 - 2m)}{12r} + \frac{r^2 - 1}{12} \left[ \frac{m}{r} \right].
\]

PROPOSITION 2.7. — For \( \alpha, \beta \in \mathbb{Z} \) with \( 0 \leq \beta \leq \alpha \) and for all \( m \leq \lfloor (\alpha + 1)/2 \rfloor \), the following holds:

\[
l \left( \frac{1}{\alpha}(a, -a, 1), m \right) \geq l \left( \frac{1}{\beta}(1, -1, 1), m \right).
\]

The basic idea of the proof is the following: We look at the canonical model of the threefold \( Y \), which exists by the famous result of Mori [Mor], combined with results of Fujita [Fuj], Benveniste [Ben] and Kawamata [Kaw]. If the index of the canonical model is small, we can finish the proof by using Hanamura’s Theorem. If the index is big, we use the Plurigenus Formula due to Barlow, Fletcher and Reid to show that for some \( m \) we have \( h^0(Y, mK_Y) \geq 2 \), and finish the proof by using Kollár’s theorem.

Proof of Theorem 2.2. — We first observe that by a theorem due to Elkik [Elk] and Flenner [Flen] (cf. [Rei2], p. 363), canonical singularities are rational singularities. Hence, by the degeneration of the Leray spectral sequence we have

\[
\chi(Y, \mathcal{O}_Y) = \chi(Y_c, \mathcal{O}_{Y_c}).
\]

If the index of \( Y_c \) divides \( R \), we apply Hanamura’s Theorem and get that \( \Phi_{(4R+3)K_Y} \) embeds birationally. Hence, we may assume that the index does not divide \( R \). Then in the Plurigenus Formula we necessarily have at least one singularity \( Q \) in the basket of singularities which is of the type \( \frac{1}{r}(a, -a, 1) \) with \( r \geq 26C \). Applying a vanishing theorem for ample sheaves (cf. Theorem 4.1 in [Flet]), the fact that \( K_{Y_c}^3 > 0 \) (since \( K_{Y_c} \) is an ample
Q-divisor) and finally Propositions 2.6 and 2.7 of Fletcher, we obtain:

\[ h^0(Y, (13C)K_Y) \]
\[ = \chi(Y, (13C)K_Y) \]
\[ \geq (1 - 26C)\chi(Y, \mathcal{O}_Y) + \sum_{Q \in \text{basket}} l(Q, 13C) \]
\[ \geq (1 - 26C)C + l(\tilde{Q}, 13C) \]
\[ \geq (1 - 26C)C + \frac{l}{26C}(1, -1, 1), 13C) \]
\[ = (1 - 26C)C + \frac{13C(13C - 1)(78C + 1 - 26C)}{312C} \]
\[ = \frac{52C^2 - 15C - 1}{24} \geq \frac{36}{24} = 1.5. \]

The last inequality is true since \( C \geq 1 \). Since \( h^0(Y, (13C)K_Y) \) is an integer, it has to be at least 2. From the definition of canonical singularities it easily follows (cf. e.g. [Rei1], p. 277, [Rei2], p. 355 or [Flet], p. 225) that \( h^0(Y, (13C)K_Y) \geq 2 \). Now we can finish the proof by applying Theorem 2.5 due to Kollár.

\[ \square \]

Despite the fact that our \( m = m(C) \) is explicit, it is so huge that it is only of theoretical interest. For example for \( C = 1 \) one can choose \( m = 269 \) ([Flet]), but for \( C = 1 \) our \( m \) is already of the size \( 10^{13} \). Moreover, for all examples of threefolds of general type which are known so far, any \( m \geq 7 \) works. So we guess there should exist a bound which is independent of the size of the holomorphic Euler characteristic.

### 3. Iitaka-Severi’s conjecture for threefolds.

The claim of this section is the following

**Theorem 3.1.** — Let \( X \) be a fixed complex variety. Then the set of targets \( \mathcal{F}(X)/\sim \) with \( \dim Y \leq 3 \) is a finite set.

By Proposition 6.5 of Maehara [Mae2], it is sufficient to show the following: There exists a natural number \( m \), only depending on \( X \), such that \( \mathcal{F}(X) \subset \mathcal{F}_m(X) \) for varieties \( Y \) with \( \dim Y \leq 3 \). Since we prove finiteness only up to birational equivalence, we may assume, without loss of generality, that \( X \) and all \( Y \) in Theorem 3.1 are nonsingular projective varieties. This is by virtue of Hironaka’s resolution theorem [Hir], cf. also
Hence, using Theorem 2.2 or Theorem 0.3 of Luo we get Theorem 3.1 as a consequence of the following:

**Proposition 3.2.** — Let $X$ be a fixed smooth projective variety and $f : X \to Y$ a dominant rational map to another smooth projective variety $Y$ with $\dim \mathbb{C} Y = n$. Then we have

$$\chi(Y, \mathcal{O}_Y) \leq \sum_{\{i \mid 2i \leq n\}} h^{2i}(X, \mathcal{O}_X).$$

**Proof of Proposition 3.3.** — First we obtain, by Hodge theory on compact Kähler manifolds (cf. [GriHar], or [Iit], p.199)

$$h^i(Y, \mathcal{O}_Y) = h^0(Y, \Omega_Y^i),$$

where $i = 1, \ldots, n$. The same kind of equalities hold for $X$. Now by [Iit], Theorem 5.3, we obtain that

$$h^0(Y, \Omega_Y^i) \leq h^0(X, \Omega_X^i),$$

where again $i = 1, \ldots, n$. Hence, we can conclude:

$$\chi(Y, \mathcal{O}_Y) \leq \sum_{\{i \mid 2i \leq n\}} h^{2i}(Y, \mathcal{O}_Y) = \sum_{\{i \mid 2i \leq n\}} h^0(Y, \Omega_Y^{2i}) \leq \sum_{\{i \mid 2i \leq n\}} h^0(X, \Omega_X^{2i}) = \sum_{\{i \mid 2i \leq n\}} h^{2i}(X, \mathcal{O}_X).$$

4. On the number of targets.

Let $X$ be a smooth threefold of general type, and define $r = r_X$, $k = K^3_X$.

**Theorem 4.1.** — There exists a universal constant $C(r, k)$, depending only on $r$ and $k$, such that

$$\#(\mathcal{F}(X)/\sim) \leq C(r, k).$$
THEOREM 4.2. — There exists a universal constant $C'(r, k)$, depending only on $r$ and $k$, such that if $Y$ is a smooth threefold and $R(X, Y) \neq \emptyset$, then

$$r_Y \leq C'(r, k).$$

The rest of this section deals with the proof of Theorem 4.1 and Theorem 4.2, which we prove simultaneously. We fix positive integers $r$ and $k$. Denote by $X(r, k)$ the set of threefolds $X_c$ with only canonical singularities and ample canonical sheaves $K_X$, which satisfy $r_X = r$, $K_X^3 = k$. Let $X$ be a smooth threefold such that $X_c \in X(r, k)$.

a) In this part of the proof we only consider targets $((Y, f)/\sim) \in (\mathcal{F}(X)/\sim)$ with $\dim \mathcal{C} Y = 3$.

Due to Theorem 2.4 of Hanamura, the map

$$\Phi_{9rK_X} : X \rightarrow \mathbb{P}^N$$

is birational onto its image, where, by ([MatMum])

$$N = h^0(X, 9rK_X) - 1 \leq 9^3r^3k + 3.$$ 

Moreover, by [BanMar], Lemma 1 (cf. also Proposition 2, part 2), the degree $d_X$ of the image $X' = \Phi_{9rK_X}(X)$ has the bound

$$d_X \leq 9^3r^3k.$$ 

Let $Y$ be a smooth threefold of general type with $R(X, Y) \neq \emptyset$.

PROPOSITION 4.3. — There exists a universal constant $C_1(r, k)$, depending only on $r$ and $k$, such that we have

$$\chi(Y, \mathcal{O}_Y) \leq C_1(r, k).$$ 

Proof of Proposition 4.3. — By Proposition 3.2 we have

$$\chi(Y, \mathcal{O}_Y) \leq h^2(X, \mathcal{O}_X) + 1.$$ 

In the Hilbert polynomials $\chi(X_c, mrK_X)$ the expressions $l(Q, rm)$, cf. Theorem 2.3, are linear in $m$, and so the two highest coefficients of the polynomial $\chi(X_c, mrK_X)$ in the variable $m$ only depend on $r^3k$. But then by Theorem 2.1.3 of Kollár [Kol1], the family of the $(X_c, rK_X)$, where $X_c \in X(r, k)$, is a bounded family. That means there exists a morphism
$\pi : \mathcal{X} \to S$ between (not necessarily complete) varieties $\mathcal{X}$ and $S$ and a $\pi$-ample Cartier divisor $D$ on $\mathcal{X}$ such that every $(\mathcal{X}_c, rK_{\mathcal{X}_c})$ is isomorphic to $(\pi^{-1}(s), D|_{\pi^{-1}(s)})$ for some $s \in S$. So it is sufficient to prove that there exists a constant $C_0$ which satisfies: For all $s \in S$ and for some desingularization $X(s)$ of $\pi^{-1}(s)$ we have $h^2(\pi^{-1}(s), \mathcal{O}_{X(s)}) \leq C_0$.

This is shown by using first generic uniform desingularization of the family $\pi : \mathcal{X} \to S$ (cf. [Hir], [BinFle]), and afterwards a semi-continuity theorem (cf. [Gro], [Gra]). By applying generic uniform desingularization and induction on the dimension there exist finitely many subvarieties $S_i, i = 1, \ldots, l$, which cover $S$, and morphisms $\Psi_i : \mathcal{Y}_i \to S_i$ between varieties $\mathcal{Y}_i$ and $S_i$ which desingularize $\mathcal{X}_i := \pi^{-1}(S_i)$ fiberwise, i.e., there exist morphisms $\Phi_i : \mathcal{Y}_i \to \mathcal{X}_i$ over $S_i$ such that for any $s \in S_i$ the map $\Phi_i : \Psi_i^{-1}(s) \to \pi^{-1}(s)$ is a desingularization.

Using semi-continuity for the families $\Phi_i : \mathcal{Y}_i \to S_i$, we obtain finitely many subvarieties $S_{ij}, j = 1, \ldots, l_i$ of $S_i$, which cover $S_i$, and have the following property: If we denote $Y_{ij} := \Phi_i^{-1}(S_{ij})$ and $\Phi_{ij} := \Phi_i|_{Y_{ij}}$, we get that for the families $\Phi_{ij} : Y_{ij} \to S_{ij}$ the number $C_{ij} := h^2(\Phi_{ij}^{-1}(s), \mathcal{O}_{\Phi_{ij}^{-1}(s)})$ is constant for $s \in S_{ij}$. Hence, $C_0 := \max_{j=1, \ldots, l_i} C_{ij}$ has the desired property.

Remark. — Proposition 4.3 can also be proved as follows. By a result of Milnor ([Mil]) the Betti numbers of the variety $X' := \Phi_{9rK_X}(X)$ have estimates depending on its degree $d_X \leq 9^3 r^3 k$ only. From the standard exact cohomology sequences and dualities it easily follows that $h^{2,0}(X)$ may be estimated by Betti numbers of $X'$.

Using Proposition 4.3 and Theorem 2.2 we can choose an integer $p = p(r, k)$, such that $p$ is divisible by $r$, $p \geq 9r$ and

$$\Phi_{pK_Y} : Y \to \mathbb{P}^M$$

is birational onto its image, where, by ([MatMum])

$$M = h^0(Y, pK_Y) - 1 \leq p^3 k + 3.$$  

(1) Remark that by an easy argument like in the proof of Proposition 3.2 any two such desingularizations have the same $h^2(\pi^{-1}(s), \mathcal{O}_{X(s)})$. 

LEMMA 4.4.

1) The degree of $Y' := \Phi_{pK_Y}(Y) \subset \mathbb{P}^M$ is smaller than $\deg X' < p^3 k$.

2) For any map $f \in R(X',Y')$, the degree $d_f$ of its graph $\Gamma_f \subset \mathbb{P}^N \times \mathbb{P}^M$ is not greater than $8p^3 k$.

Lemma 4.4 is a particular case of part 2 and 3 of Proposition 2 of [BanMar] for $n = 3$, applied to the threefolds $X_c, Y_c$ and linear systems $|pK_{X_c}|, |pK_{Y_c}|$. We have to note only that Proposition 2 and Lemma 1 in [BanMar] is stated for Cartier divisors. But only the fact that they are $\mathbb{Q}$-Cartier is used in their proofs.

By Proposition 1 of the same paper ([BanMar]), there exist algebraic families $(X, p_X, T)$, $(Z, p_Z, V)$, $(\mathcal{Y}, p_Y, U)$ with constructive bases and projections $\pi_U : V \to U, \pi_T : V \to T$, with the following properties:

1) For any $X_c \in X(r,k)$, there is a point $t \in T$, such that $X_c$ is birational to $X = p_X^{-1}(t)$, and all points $t \in T$ have this property.

2) For any $Y$ with $R(X,Y) \neq \emptyset$ for some $X_c \in X(r,k)$, there is a point $u \in U$, such that $Y$ is birational to $Y = p_Y^{-1}(u)$, and all points $u \in U$ have this property.

3) For any dominant rational map $f : p_X^{-1}(t) = X \to Y = p_Y^{-1}(u)$, there is a point $v \in V$, such that $\pi_U(v) = u, \pi_T(v) = t, p_Z^{-1}(v)$ is a graph of the map $f$, and all points $v \in V$ have this property.

Let $\tilde{V} = \{(t,v) \in T \times V | \pi_T(v) = t\}$, and denote by $p_T$ resp. $p_V$ the projections to the first resp. to the second factor. Through the composed map $\pi_U \circ p_V : \tilde{V} \to U$ the variety $\tilde{V}$ is also a variety over $U$. Let $\tilde{\mathcal{Y}} = \tilde{V} \times_U \mathcal{Y}$ be obtained by base change, and denote the projection to the first factor by $p_{\tilde{\mathcal{Y}}} : \tilde{\mathcal{Y}} \to \tilde{V}$. Then we have

$$\tilde{\mathcal{Y}} \xrightarrow{p_{\tilde{V}}} \tilde{V} \xrightarrow{p_T} T.$$ 

In this diagram, for every $t \in T$, the set $p_T^{-1}(t)$ can be considered as the set of graphs of dominant rational maps $f : X \to Y$, where $X = p_X^{-1}(t)$, and $p_{\tilde{\mathcal{Y}}} : \tilde{\mathcal{Y}} \to \tilde{V}$ is the universal family of threefolds $Y$ over the graphs of $f : X \to Y$.

By applying the process of local uniform desingularization, described in Proposition 4.3, to the family $p_{\tilde{\mathcal{Y}}} : \tilde{\mathcal{Y}} \to \tilde{V}$, we obtain a finite number of smooth families $(p_{\tilde{\mathcal{Y}}})_i : (\tilde{\mathcal{Y}})_i \to (\tilde{V})_i, i = 1, \ldots, l$, the bases $(\tilde{V})_i$ of which are connected and cover $\tilde{V}$, and the fibers of which are desingularizations of the fibers of $p_{\tilde{\mathcal{Y}}} : \tilde{\mathcal{Y}} \to \tilde{V}$. For any $i$ the map $(p_{\tilde{\mathcal{Y}}})_i : (\tilde{\mathcal{Y}})_i \to (\tilde{V})_i$ is a smooth
family of projective threefolds of general type over a connected base $(\tilde{V})_i$. By a theorem of J. Kollár and Sh. Mori ([KolMor], Theorem 12.7.6.2) there is an algebraic map $\phi_i$ from $(\tilde{V})_i$ to the birational equivalence classes of the fibers of $(p_{\tilde{V}})_i : (\tilde{Y})_i \rightarrow (\tilde{V})_i$. Moreover, all these fibers have the same Hilbert function.

From this fact two conclusions can be derived:

1. Since the index of a canonical threefold can be bounded in terms of the Hilbert function ([KolMor], p. 666), the indices of the canonical models of the fibers of the family $(p_{\tilde{V}})_i : (\tilde{Y})_i \rightarrow (\tilde{V})_i$ vary in a finite set of natural numbers, only.

2. Let $(p_T)_i := p_T|_{(V)_i}$, and $n_i(t)$ be the number of irreducible components of $(p_T)_i^{-1}(t)$ (it may be zero). Define, for $X = p_X^{-1}(t)$, $G(X) = \{Y|(Y,f) \in F(X)\}$, and let $\sim$ denote birational equivalence on $G(X)$. Since $\#(G(X)/\sim) < \infty$, it follows that the restriction $\phi_i$ to $(p_T)_i^{-1}(t)$ has to be constant on the connected components of $(p_T)_i^{-1}(t)$. Then

$$\#(G(X)/\sim) \leq \sum_{i=1}^{I} n_i(t).$$

Since from the beginning the constructions of all the families were algebraic and defined only by the constants $r$ and $k$, we have proved Theorem 4.2, and also the following

**Lemma 4.5.** — There exists a universal constant $C_2(r,k)$, depending only on $r$ and $k$, such that we have

$$\#(G(X)/\sim) \leq C_2(r,k).$$

Next, we look at the map

$$(\pi_T, \pi_U) : V \rightarrow T \times U.$$ 

It is algebraic and any point of the fiber over $(t,u) \in T \times U$ defines a map from $R(\pi_T^{-1}(t), \pi_U^{-1}(u))$. The last set is finite for all $(t,u)$, so we get:

**Lemma 4.6.** — There exists a universal constant $C_3(r,k)$, depending only on $r$ and $k$, such that

$$\#R(X,Y) \leq C_3(r,k).$$
From Lemma 4.5 and Lemma 4.6 the statement of Theorem 4.1 for 3-dimensional targets is immediate. The desired bound may be chosen as $C_2(r, k)C_3(r, k)$. □

b) Now we consider targets $((Y, f)/\sim) \in (\mathcal{F}(X)/\sim)$ with $\dim_{\mathbb{C}} Y \leq 2$. For these targets we know that the indices of the $Y_c$ are 1 or 2. So we can repeat the same argument as above, omitting however Proposition 4.3. The only change which has to be done is replacing the moduli spaces due to [KolMor] by the respective moduli spaces for surfaces or curves. So we get Theorem 4.1, and in particular Lemma 4.5 and Lemma 4.6, also for these kinds of targets. □

Remark. — According to [BanMar], there exists a universal function $\sigma$ in two variables, such that $\#R(X,Y) \leq \sigma(r \cdot r_{Y_c}, k)$. This fact, together with Theorem 4.2, yields an alternative proof of Lemma 4.6.

5. A conjecture of Kobayashi for threefolds of general type.

In this section we prove

THEOREM 5.1. — For any complex variety $X$ there is a number $c(X)$ such that

$$\#R(X,Y) \leq c(X)$$

for any complex variety $Y$ of general type with $\dim_{\mathbb{C}} Y \leq 3$. If $X$ is a threefold of general type, then $c(X)$ can be expressed only in terms of $r_{X_c}$ and $K_{X_c}^3$.

Proof of Theorem 5.1. — Like in Section 3 we may assume that $X$ and $Y$ are smooth projective varieties. By [KobOch], $\#R(X,Y)$ is finite for every fixed $Y$. By Theorem 3.1, we know that for given $X$ there exist only finitely many such $Y$, up to birational equivalence. Since birational equivalence does not affect the number $\#R(X,Y)$, the first statement follows.

Let $X$ now be a projective threefold of general type. Then the second statement is just Lemma 4.6. □

Remark 5.2. — The estimate which is given in Theorem 5.1 is not effective.
BIBLIOGRAPHY


ESTIMATES OF THE NUMBER OF RATIONAL MAPPINGS


M. Reid, Canonical threefolds, A. Beauville (Ed.): Algebraic Geometry, Angers 1979, 273-310, Sijthoff and Noordhoff, 1980.


P. Samuel, Complements a un article de Hans Grauert sur la conjecture de Mordell, Publ. Math. IHES, 29 (1966), 311-318.


I.H. Tsai, Dominant maps and dominated surfaces of general type, Preprint (1996), 44 pages.

I.H. Tsai, Chow varieties and finiteness theorems for dominant maps, Preprint (1996), 33 pages.

K. Ueno, Classification theory of algebraic varieties and compact complex spaces, LNM 439 (1975), Springer Verlag.