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SCATTERING THEORY FOR 3–PARTICLE SYSTEMS IN CONSTANT MAGNETIC FIELDS : DISPERSIVE CASE

by C. GÉRARD and I. ŁABA(*)

1. Introduction.

A system of N interacting particles of masses m_i and electric charges q_i in a constant magnetic field $\vec{B} = (0, 0, 2b)$ in $\mathbb{R}^3$ is described by the Hamiltonian

$$H = \sum_{1}^{N} \frac{1}{2m_i}(D_i - q_i J x_i)^2 + \sum_{i<j} V_{ij}(x_i - x_j), \tag{1.1}$$

where J is the vector potential associated with the field. We will use the transversal gauge in which J is the skew-symmetric matrix

$$J := \begin{pmatrix} 0 & -b & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.2}$$

We consider the case when $N = 3$ and assume that all of the particles are charged, i.e., $q_i \neq 0$ for $1 \leqslant i \leqslant 3$. No other conditions on the charges of particles will be needed. In particular, the system is allowed to have neutral proper subsystems (pairs).

The main task of scattering theory is to describe the large time asymptotic behaviour of the solutions of the Schrödinger equation

$$i\frac{\partial u}{\partial t} = Hu.$$

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For an N-particle system described by (1.1), with pair interactions V_{ij} vanishing at infinity, one expects that it either remains stable or breaks up into asymptotically independent stable subsystems called clusters. This statement cast into precise mathematical terms is called asymptotic completeness.

In [GL1] and [GL2], we proved asymptotic completeness for short-range and long-range systems of N charged particles in a constant magnetic field, under the assumption that they have no neutral proper subsystems. In this paper, the latter assumption is not required. However, our results are restricted to the 3−body case, and to Coulomb-type interactions.

We saw in [GL1] that the behaviour of noninteracting clusters depends on their electric charge. Bound states of charged clusters escape to infinity only along the field and their transversal to the field coordinates remain bounded. On the other hand, bound states of neutral clusters may travel across the field with a nonzero average velocity which depends on their internal structure. We emphasize that this has nothing to do with the free motion of the center of mass which occurs in the absence of external forces. *Noninteracting* charged particles, quantum or classical, can perform only bounded motion in the directions transversal to the magnetic field; whether the sum of their electric charges is zero or not is irrelevant. The motion of clusters of charged particles across the field becomes possible if the Lorentz forces are cancelled by the interactions between the particles. This can occur only for bound states of neutral clusters. We call this case *dispersive*, since the effective kinetic energy of such states is given by a certain dispersive Hamiltonian. For a more detailed discussion of the properties of bound states, we refer the reader to [GL1].

It is expected that if an N−particle system breaks up into clusters moving away from one another, the asymptotic behaviour of these clusters will be similar to that described above. Thus, if neutral clusters are present, there will be scattering channels corresponding to the unbounded motion of those clusters across the field. The mathematical analysis of these channels is significantly harder than that of channels involving only charged clusters. Before we state and prove our results rigorously, let us describe heuristically the general outline of the paper, the main difficulties we encountered in solving the problem and the methods we employed to overcome them.

Our main result – asymptotic completeness – is stated in Section 4 (Theorem 4.5). In order to formulate it rigorously, we first need to review our discussion of the separation of the center of mass ([GL1]), which will allow us to introduce a suitable notion of reduced Hamiltonians. We do this in Section 3. The bound states, scattering states, and channel identification operators are defined in Section 4. Our assumptions on the pair interactions V_{ij} are stated in Section 2, where we also review the notation used in this paper.

The main steps in our proof of asymptotic completeness are the following. We first prove the Mourre estimate (Sections 6 and 7), from which the minimal velocity estimate follows (Section 8). This shows that, for states orthogonal to the bound states of the system, the system breaks up into clusters moving away from one another with nonzero relative velocities. To decouple the different scattering channels, we apply standard arguments (*i.e.*, the construction of the *asymptotic velocity* – see Section 4) to the propagation in the direction of the field, and use the methods of [GL2] to treat the propagation in the transversal directions (Section 9). Finally, in Section 10 we combine these results to replace the exact evolution e^{-itH} of the system by the asymptotic evolutions used in the definition of the wave operators.

The proof of the Mourre estimate contains most of the new ideas in this article. Since this part of the paper is also quite technical, we explain it here in some detail.

Given two selfadjoint operators: H and B, we will say that $B \geqslant c$ at $H = \lambda$ if $E_\Delta(H)BE_\Delta(H) \geqslant cE_\Delta(H)^2$, provided that the interval $\Delta = (\lambda - \delta, \lambda + \delta)$ is sufficiently small (see Definition 5.1). We will also use the notation :

$$\inf_{H=\lambda} B = \sup\{c \in \mathbb{R} \mid B \geqslant c \text{ at } H = \lambda\}.$$

We say that H satisfies the *strict Mourre estimate* with the conjugate operator A at energy λ if

$$\inf_{H=\lambda} [H, iA] > 0.$$

The importance of the Mourre estimate in spectral and scattering theory for multiparticle systems is well-known (see e.g., [Mo], [PSS], [SS1–5], [CFKS], [DG], [HuSi]). If H satisfies the strict Mourre estimate at all points λ in an interval I, its spectrum in this interval is absolutely continuous. Moreover, one obtains the limiting absorption principle, local decay and minimal velocity estimates. In particular, the Mourre estimate is a key part of the existing proofs of asymptotic completeness ([SS1], [Gr], [De1]). For this purpose, one needs to show the Mourre estimate for all values of λ, except possibly for a closed and discrete subset of $\mathbb{R}$, consisting of *thresholds* and eigenvalues of H. (In fact, one first has to show that, up to a compact error, a similar estimate holds at all points, including possible eigenvalues of H, away from the threshold set, and then deduce from this the above statement. This step will not be discussed here; an abstract version of the argument involved is given in Lemma 5.4.)

In [GL1], where we proved the Mourre estimate for Hamiltonians of the form (1.1) under the assumption that all proper subsystems are charged, the main difficulty was that conventional separation of the center of mass motion was not possible. Instead, we had to exploit the fact that the magnetic Hamiltonian has a constant of motion called *pseudomomentum*, and define the reduced Hamiltonians using suitable unitary operators U_a. The rest of the proof of the Mourre estimate was similar to that for N-particle Hamiltonians without a magnetic field. For comparison purposes, let us sketch it briefly. Let $A = \frac{1}{2}(\langle D_z, z\rangle + \langle z, D_z\rangle)$ be the generator of dilations in the direction of the field. A standard argument involving a partition of unity shows that one can obtain the Mourre estimate for H, with A as the conjugate operator, from similar estimates for the Hamiltonians H_a of noninteracting clusters (given a *cluster decomposition* a, *i.e.*, a partition of the set $\{1, \ldots, N\}$ into disjoint nonempty sets, H_a is obtained by subtracting from H the interactions V_{ij} between particles belonging to different clusters). We assume that all clusters in any decomposition a consisting of at least two of them are charged. We then have

$$H_a = \frac{1}{2} D_{z_a}^2 + H^a,$$

where $\frac{1}{2} D_{z_a}^2$ is the kinetic energy of the motion of the centers of mass of the clusters in the direction of the magnetic field, and H^a – the energy of the motion within clusters together with the energy of the bounded motion of their centers of mass across the field.

To prove the Mourre estimate, we proceed by induction in a. If $a = a_{\min}$ is the decomposition into N clusters, each consisting of one particle, $H^{a_{\min}}$ has only pure point spectrum (the Landau levels of the noninteracting particles). Then

$$\inf_{H_{a_{\min}}=\lambda} [H_{a_{\min}}, iA] = \inf_{H_{a_{\min}}=\lambda} D_{z_{a_{\min}}}^2 > 0,$$

if $\lambda \notin \sigma_{\rm pp}(H^{a_{\rm min}})$. For $a \neq a_{\rm min}$, we obtain that

$$\begin{aligned}\inf_{H_a=\lambda}[H_a, iA] &= \inf_{H_a=\lambda}(D_{z_a}^2 + [H^a, iA]) \\ &= \inf_{\lambda_1+\lambda_2=\lambda}\Big(\inf_{\frac{1}{2}D_{z_a}^2=\lambda_1} D_{z_a}^2 + \inf_{H^a=\lambda_2}[H^a, iA]\Big) \\ &= \inf_{\lambda_1+\lambda_2=\lambda,\lambda_1\geqslant 0}\big(2\lambda_1 + \inf_{H^a=\lambda_2}[H^a, iA]\big).\end{aligned}$$

By the previous step of induction (*i.e.*, the Mourre estimate for H^a), $\inf_{H^a=\lambda_2}[H^a, iA] > 0$ if λ_2 is not a threshold or eigenvalue of H^a, Hence the right-hand side of the above equation is strictly positive unless $\lambda_1 = 0$ and $\lambda = \lambda_2$ is a threshold or eigenvalue of H^a.

If the total charge $Q = \sum q_i$ of the system is nonzero, one eventually obtains the Mourre estimate for H with the conjugate operator A at all energy values $\lambda \in \mathbb{R}\setminus\tau$, where the set τ (consisting of the eigenvalues of H_a for all a) is closed and discrete. If the system is neutral ($Q = 0$), the result is similar, except that the Mourre estimate is localized not only in energy, but also in pseudomomentum. Consequently, for all scattering states of the system (defined in Section 4), the clusters separate with nonzero relative velocities in the direction of the field.

This picture changes radically if we allow the system to have neutral subsystems. Let $N = 3$, and consider the Hamiltonian H_a of noninteracting clusters for a cluster decomposition $a = \{(1,2),(3)\}$ containing a neutral pair $(1,2)$. We have

$$H_a = \frac{1}{2}D_{z_a}^2 + H^{a,\rm c} + H^{a,\rm n},$$

where $\frac{1}{2}D_{z_a}^2$ is, as before, the kinetic energy of the motion of centers of mass of clusters along the field, $H^{a,\rm c}$ is the kinetic energy of the motion of the third particle across the field (the spectrum of $H^{a,\rm c}$ consists of its Landau levels Λ_j), and $H^{a,\rm n}$ is the Hamiltonian of the neutral pair $(1,2)$ with its center of mass motion along the field removed.

The separation of the center of mass motion across the field, and the corresponding notion of reduced Hamiltonians, for the neutral pair is rather different from either the charged case or the free case (with no magnetic field). We use the fact that the pseudomomentum k of the neutral pair commutes with $H^{a,\rm n}$, to write $H^{a,\rm n}$ as a direct integral

$$H^{a,\rm n} = \int_{\mathbb{R}^2}^{\oplus} H(k)dk,$$

where each $H(k)$ acts on $L^2(\mathbb{R}^3)$. If, as before, we let $A = \frac{1}{2}(\langle D_z, z\rangle + \langle z, D_z\rangle)$, we may have

$$\inf_{H_a=\lambda}[H_a, iA] = \inf_{H_a=\lambda} D^2_{z_a} = 0,$$

whenever $\lambda - \Lambda_j \in \sigma_{\text{pp}}(H(k))$ for some $j \in \mathbb{N}$ and $k \in \mathbb{R}$. The set of such values of λ cannot be expected to be discrete. In fact, suppose that there is an eigenvalue $E(k)$ of $H(k)$ with the corresponding eigenprojection $P(k)$, satisfying suitable regularity assumptions (stated in detail in Section 4). Let

$$P = \int_{\mathcal{U}}^{\oplus} P(k)dk.$$

Since A commutes with k, we have

$$P[H^{a,\mathrm{n}}, iA]P = \int_{\mathcal{U}}^{\oplus} P(k)[H(k), A]P(k)dk = 0,$$

by the virial theorem. (The same argument shows that the Mourre estimate cannot hold for H_a with any conjugate operator commuting with k.)

Let $B_{a,P} = \frac{1}{2}P(\langle \nabla_k E(k), D_k\rangle + \langle D_k, \nabla_k E(k)\rangle)P$. Then

$$P[H^{a,\mathrm{n}}, iB_{a,P}]P = \int_{\mathcal{U}}^{\oplus} |\nabla E(k)|^2 P(k)dk,$$

and its infimum at $H_a = \lambda$ is strictly positive, provided that $\lambda - \Lambda_j$ is not a critical point of $E(k)$ for any $j \in \mathbb{N}$. The set of such critical points can be excluded as a *secondary threshold set* (Definition 6.6).

This is the main idea behind the construction in Section 6.2. Namely, we define the conjugate operator for H_a as $A + cB_a$, where B_a is essentially the sum of operators $B_{a,P}$ of the above form for all possible eigenvalues and eigenprojections $E(k)$ and $P(k)$. One then has to use the local inequalities for commutators (for a fixed value of k, or for k in a neighbourhood of a point) and continuity of the operators involved to obtain inequalities uniform in k. This part of the proof is rather technical.

Note that the definition and properties of the operator D_k appearing above depend on the choice of the direct integral representation of $H^{a,\mathrm{n}}$. Throughout this paper, we will use the representation described in Section 3, and it is in this representation that D_k will always be defined.

The physical interpretation of the above argument is as follows. It was shown in [GL1, Proposition 6.4] that, for a bound state u of the neutral pair such that $Pu = u$, $\nabla_k E(k)$ is the average velocity of the center of mass of the pair. Thus B_a can be understood as $\langle y_a^{\mathrm{n}}, v_a^{\mathrm{n}} \rangle$, where y_a^{n} is the position of the center of mass of the pair, and v_a^{n} – its velocity. However, we must warn the reader not to take this too literally, since v_a^{n} is only the average velocity over long periods of time (the actual velocity is not constant), and neither is D_k simply identical to y_a^{n}.

The conjugate operator for H is obtained by gluing together the different conjugate operators for the channel Hamiltonians H_a : $A + cB_a$ for the cluster decompositions which contain a neutral pair, and A for those consisting of only charged clusters. The details of this construction are explained in Section 7. In particular, our argument showing that the "cross-terms" ($[H_b, B_a]$ for $a \neq b$) are small uses that for 3–particle systems the regions of configuration space corresponding to different two-cluster decompositions are disjoint.

Another point which we would like to mention here is the decoupling of the channels transversally to the magnetic field. To prove asymptotic completeness, we must show that the system cannot oscillate between various configurations containing neutral clusters. Once more, we use in an essential way the fact that we deal with a 3–particle system. Different 2–cluster configurations can then interact only through the free region where all three particles are well separated. However, since these particles are charged, an argument from [GL2] shows that the free region is inaccessible to scattering, which leads to the asymptotic decoupling of the 2–cluster channels.

The last step in the proof of asymptotic completeness is the analysis of the dynamics of the clusters moving away from one another with nonzero relative velocities. It is well-known that, while for short-range systems the motion of the centers of mass is asymptotically free, in the long-range case a modified asymptotic dynamics has to be used instead. More precisely, one shows that for large time the evolution of the system is approximated by

$$e^{iS_a(D_a,t)} e^{-itH^a},$$

where $S_a(\xi_a, t)$ is an approximate solution of the classical Hamilton-Jacobi equation, and e^{-itH^a} is the internal dynamics of the clusters. A key point is that S_a and H^a commute, so that the internal and external dynamics are decoupled.

For a neutral cluster in a magnetic field, its internal structure and the motion of its center of mass in directions transversal to the field are interdependent. If an additional long-range time-dependent force (the intercluster interaction) is present, we expect that, as for standard N-particle systems, the center of mass dynamics has to be modified. This, however, cannot be done without altering the internal structure of the cluster.

For the physically important Coulomb potentials (or, more generally, for *Coulomb-type* interactions – see Definition 2.1), the problem becomes much easier. Given a cluster decomposition consisting of a neutral pair $(1, 2)$ and a charged particle (3), we first replace the coordinates of the particles $1, 2$ by the coordinates of the center of mass of the pair. We then use that, for such potentials, the total interaction between the pair $(1, 2)$ and the particle (3) is short-range. Hence, although the neutral pair can escape to infinity across the field, no modification of the asymptotic evolution in transversal directions will be needed. We do not know how to solve the general long-range 3-body problem, which seems to be much more difficult.

The phase-space analysis of scattering and time-dependent approach to proving asymptotic completeness was initiated by Enss [E1]. This approach turned out to be particularly successful in the N-body scattering. The three-body long range problem was solved by Enss in [E2]. Asymptotic completeness for N-body systems with short-range potentials was proved by Sigal and Soffer [SS1], who also developed many of the techniques used in the long-range case [SS2], [SS3], [SS4]. A geometrical construction due to Graf [Gr] simplified considerably the original proof of [SS1]. Finally, asymptotic completeness in the long-range case was proved by Dereziński [De] and Sigal and Soffer [SS5].

One of the most important tools in scattering theory, which also turns out to be essential in our work, is the method of positive commutators. Although it can be traced back to earlier work of Kato, Putnam, and Lavine, its usefulness was limited until the discovery of the Mourre estimate [Mo]. For N-body systems the Mourre estimate was first proved in [PSS]; a simpler proof was given later in [FH]. This method was further developed in numerous papers, including [BG] and [SS1], [SS2], [SS3].

Prior to our work ([GL1], [GL2]), very little was known about multiparticle scattering in a magnetic field. In the case of one particle in a constant field, asymptotic completeness for short-range and Coulomb potentials was shown in [AHS1]; a different proof was given in [S]. The separation of the center of mass in the presence of a magnetic field was considered in [AHS2]. The general long-range one-body problem was solved in [L1] and [I1]. Long-range two-particle systems with total charge zero were studied in [L2] and [I1]. In [I2], Iwashita obtained the Mourre estimate for reduced 3−particle Hamiltonians in certain special cases. Asymptotic completeness for N-particle systems containing no neutral proper subsystems with short and long range interactions was proved in [GL1] and [GL2] respectively. Let us also mention the paper [VZ] on the spectral theory of N-particle Hamiltonians with a constant magnetic field, and several articles on one-particle scattering in a magnetic field vanishing at infinity ([BP], [E3], [LT1], [LT2], [Ni], [NR]). In this paper, we also draw upon certain ideas used by Dereziński in his study of dispersive Hamiltonians ([De2]).

2. Notation.

We will first review the notation of [GL1] and formulate the hypotheses we will impose on the interactions.

The coordinates in the configuration space $X = \mathbb{R}^{3N}$ will be denoted by

$$x = (x_1, \dots, x_N), \ x_i = (y_i, z_i),$$

where $(y_i, z_i) = (\hat{y}_i, \tilde{y}_i, z_i) \in \mathbb{R}^2 \times \mathbb{R}$ are the coordinates of the i-th particle in the plane transversal to $\vec{B}$ and along the direction of $\vec{B}$ respectively. We equip X with the metric

$$g(x, \bar{x}) = \sum_1^N m_i x_i \bar{x}_i.$$

Let us also introduce the N-particle vector potential A:

$$A(x_1, \dots, x_N) = (q_1 J x_1, \dots, q_N J x_N).$$

A is a antisymmetric mapping $A : X \to X'$, which we will also consider as an antisymmetric bilinear form on $X \times X$.

We consider the following subspaces of X:

$$Z := \{x \in X \mid Ax = 0\} = \{x \in X \mid y_i = 0, \ 1 \leqslant i \leqslant N\},$$
$$Y := Z^{\perp} = \{x \in X \mid z_i = 0, \ 1 \leqslant i \leqslant N\}.$$

Then

$$X = Y \oplus^{\perp} Z,$$

and the projections of a vector $x \in X$ onto these subspaces will be denoted by $y \in Y$ and $z \in Z$ respectively.

We can now rewrite H as

$$H = \frac{1}{2}(D - Ax)^2 + V(x) = \frac{1}{2}D_z^2 + \frac{1}{2}(D_y - Ay)^2 + V(x), \tag{2.1}$$

where

$$V(x) := \sum_{i<j} V_{ij}(x_i - x_j).$$

Let us recall some of the standard notation used in the N-body theory. $\mathcal{A}$ will stand for the set of all cluster decompositions, i.e., partitions $a = (C_1, \dots, C_k)$ of $\{1, \dots, N\}$ into disjoint non-empty sets C_i called clusters. The number of clusters of a will be denoted by $\#a$. We will say that a is a refinement of b and write $a \leqslant b$ if all clusters of a are subsets of clusters of b. The relation $\leqslant$ defines a natural lattice structure on $\mathcal{A}$ with the maximal and minimal elements

$$a_{\max} = (\{1, \dots, N\}),\ a_{\min} = (\{1\}, \dots, \{N\}).$$

The finest cluster decomposition c such that $a \leqslant c$ and $b \leqslant c$ will be denoted by $a \vee b$. We will also write $a < b$ if $a \leqslant b$ and $a \neq b$. For a pair $\{i, j\} \subset \{1, \dots, N\}$, (ij) will stand for the $N-1$-cluster partition

$$(ij) = (\{1\}, \dots, \{\hat{i}\}, \dots, \{\hat{j}\}, \dots, \{N\}, \{i, j\}).$$

Given a cluster decomposition a, X can be written as

$$X = X_a \oplus^{\perp} X^a,$$

where

$$X_a = \{x \in X \mid x_i = x_j \text{ for all } i, j \text{ such that } (ij) \leqslant a\},$$
$$X^a = \left\{x \in X \mid \sum_{i \in C_k} m_i x_i = 0 \text{ for all } C_k \in a\right\}.$$

It is easy to check that

$$a \leqslant b \text{ iff } X_b \subset X_a,$$
$$X_{a_{\max}} = \{x \in X \mid x_i = x_j \text{ for all } i, j\},\ X_{a_{\min}} = X.$$

For $a \in \mathcal{A}$, we will denote

$$Y_a := X_a \cap Y,\ Z_a := X_a \cap Z,$$
$$Y^a := X^a \cap Y,\ Z^a := X^a \cap Z.$$

Note that the projections on Z and X_a commute for all $a \in \mathcal{A}$, so that

$$X_a = Y_a \oplus Z_a,\ X^a = Y^a \oplus Z^a.$$

The symbols

$$x_a,\ y_a,\ z_a,$$
$$x^a,\ y^a,\ z^a,$$

will stand for the orthogonal projections of $x \in X$ on the above spaces. We will denote by $(Ax)_a$ the restriction of Ax to X_a, and by $A^a{}_a, A_a{}^a, A_{a\,a}, A^{a\,a}$ the restrictions of the bilinear form A to $X^a \times X_a, X_a \times X^a, X_a \times X_a, X^a \times X^a$ respectively.

For any $a \in \mathcal{A}$, we can write

$$H = H_a + I_a,$$

where

$$H_a := \frac{1}{2}(D - Ax)^2 + V^a(x^a),\ V^a(x^a) := \sum_{(ij)\leqslant a} V_{ij}(x_i - x_j),$$

is the cluster Hamiltonian, and

$$I_a := V(x) - V^a(x^a) = \sum_{(ij)\not\leqslant a} V_{ij}(x_i - x_j)$$

is the intercluster interaction.

As we mentioned in the introduction, we assume in this paper that all particles are charged, i.e.,

$$(Q) \qquad q_i \neq 0 \text{ for all } i.$$

In particular, our system is allowed to have neutral proper subsystems (pairs). We will say that $a \in \mathcal{A}^{\mathrm{n}}$ if a contains a neutral pair, and $a \in \mathcal{A}^{\mathrm{c}}$ otherwise.

Let us now state the hypotheses that we will impose on the interactions.

(V1) V_{ij} is a multiplication operator on $L^2(\mathbb{R}^3)$ such that V_{ij} is $-\Delta$ bounded with relative bound 0.

(V2) $z\nabla_z V_{ij}(x)$ is $-\Delta$ bounded. Moreover,

$$\text{i)}\quad |F\left(\frac{|x|}{R}\geqslant 1\right)V_{ij}(-\Delta+1)^{-1}\| = o(1),\ R\to\infty,$$

$$\text{ii)}\quad |F\left(\frac{|x|}{R}\geqslant 1\right)z\nabla_z V_{ij}(-\Delta+1)^{-1}\| = o(1),\ R\to\infty.$$

(V3)

$$\|(-\Delta+1)^{-1}|\partial_x^\alpha V_{ij}|(-\Delta+1)^{-1}\| < \infty,\ |\alpha|\leqslant 2.$$

(V')

$$\begin{aligned}
&V_{ij} = V_{ij}^s + V_{ij}^l,\ \text{where}\\
&\|V_{ij}^s(x)\langle x\rangle^2(-\Delta+1)^{-1}\| < \infty,\\
&|\partial_x^\alpha V_{ij}^l(x)| \leqslant C_\alpha\langle x\rangle^{-\epsilon-|\alpha|},\ \epsilon>0, |\alpha|\geqslant 0.
\end{aligned}$$

(LR)

$$\begin{aligned}
&V_{ij}(x) = V_{ij}^s + V_{ij}^l,\ \text{where}\\
&\|F\left(\frac{|x|}{R}\geqslant 1\right)V_{ij}^s(-\Delta+1)^{-1}\| \in L^1(dR),\\
&\|F\left(\frac{|x|}{R}\geqslant 1\right)\partial_x^\alpha V_{ij}^l\| \leqslant CR^{-|\alpha|-\mu},\ \mu>0,\ |\alpha|\leqslant 1.
\end{aligned}$$

Finally, we will assume that the interactions are of Coulomb type.

DEFINITION 2.1 — *The interactions V_{ij} are said to be of Coulomb type if:*

$$V_{ij}^l(x) = q_i q_j V^l(x).$$

The key property of the Coulomb interactions which we will use in this paper is that if C_l is a neutral cluster, then $\sum_{i\in C_l} V_{ik}^l(x_i - x)$ vanishes if $x_i = x_j$ for $(ij)\in C_l$.

Hypothesis *(V1)* and a result of [GL1], which we state below as Proposition 2.2, ensure that H is self-adjoint.

PROPOSITION 2.2 — *Assume that (V1) holds. Then V is $\frac{1}{2}(D - Ax)^2$ bounded with relative bound 0. Consequently,*

$$H = \frac{1}{2}(D - Ax)^2 + V(x)$$

is selfadjoint with the magnetic Sobolev space

$$H^2_A(\mathbb{R}^{3N}) \ := \{u \in L^2(X) \mid (D - Ax)^2 u \in L^2(X)\}$$

as its domain.

Other consequences of hypotheses *(V)* will be given later in Lemma 3.1 in Section 3.

We use the following convention for cut-off functions. $F(\cdot \in \Omega)$ will stand for a smoothed out characteristic function of Ω, equal to 0 outside Ω and to 1 in a slightly smaller set, and 1_Ω will denote the sharp characteristic function of Ω. Finally, $\langle x \rangle$ is a smooth function greater than some $\epsilon > 0$ for all x and equal to $|x|$ for $|x| > 1$.

3. The pseudomomentum and the reduced Hamiltonians.

To formulate our main result, we will need some of the definitions of [GL1], in particular, the concepts of the bound and scattering states and the reduced Hamiltonians.

Let

$$K := D + Ax.$$

The observable K is the generator of the *magnetic translations* :

$$e^{i\langle x', K\rangle} u(x) = e^{i\langle x', Ax\rangle} u(x + x'), \ \forall x' \in X,$$

and satisfies

$$\left[K, \frac{1}{2}(D - Ax)^2\right] = 0. \tag{3.1}$$

One deduces from (3.1) and from the fact that

$$V(x + x') = V(x), \ x' \in X_{a_{\max}}$$

that the pseudomomentum

$$K_{a_{\max}} := D_{x_{a_{\max}}} + (Ax)_{a_{\max}}$$

satisfies $[H, iK_{a_{\max}}] = 0$. Similarly, for any cluster decomposition $a \in \mathcal{A}$ the external pseudomomentum

$$K_a := D_{x_a} + (Ax)_a$$

satisfies $[H_a, K_a] = 0$.

The commutation relations of the components of the external pseudomomenta can be summarized by

$$i[\langle x', K_a\rangle, \langle x'', K_a\rangle] = -2\langle x', Ax''\rangle,\ x', x'' \in X_a.$$

To understand these commutation relations it is necessary to analyse the restriction of A to X_a. Let us consider a cluster decomposition a equal to

$$a = (C_1, \cdots C_{\sharp a}).$$

We denote by

$$Q_{C_l} := \sum_{i \in C_l} q_i$$

the total charge of the cluster C_l, we call the cluster C_l *neutral* if $Q_{C_l} = 0$ and *charged* if $Q_{C_l} \neq 0$. We denote by $\sharp_{\mathrm{c}} a$ and $\sharp_{\mathrm{n}} a$ the numbers of charged and neutral clusters of a respectively. We define

$$\begin{aligned} X_{C_l} &:= \{x \in X \mid x_i = x_j \text{ if } (i,j) \in C_l,\ x_k = 0 \text{ if } k \notin C_l\}, \\ X^{C_l} &:= \{x \in \mid \sum_{i \in C_l} m_i x_i = 0,\ x_k = 0 \text{ if } k \notin C_l\}. \end{aligned}$$

We have

$$X_a = \bigoplus_1^{\sharp a} X_{C_l},\ X^a = \bigoplus_1^{\sharp a} X^{C_l}. \tag{3.2}$$

Clearly, the bilinear forms $A_{a\,a}$ and $A^{a\,a}$ can be decomposed into blocks with respect to the direct sum decomposition (3.2).

Let C_l be a cluster of a. If we identify X_{C_l} with $\mathbb{R}^3$, we see that the matrix of $\langle x', Ax\rangle$ restricted to X_{C_l} is equal to $Q_{C_l} J$. Consequently, the two orthogonal to the field components of $K_{C_l} = \sum_{i \in C_l} K_i$, where C_l is a cluster of a, commute if C_l is neutral and have the Heisenberg commutation relations if C_l is charged. One can use these facts to find unitary transformations reducing the channel Hamiltonians H_a to simpler models. Let us briefly recall the discussion of this reduction given in [GL1].

We define

$$X_a^{\mathrm{n}} := \bigoplus_{\text{neutral } C_l} X_{C_l}, X_a^{\mathrm{c}} := \bigoplus_{\text{charged } C_l} X_{C_l},$$
$$X^{a,\mathrm{n}} := \bigoplus_{\text{neutral } C_l} X^{C_l}, X^{a,\mathrm{c}} := \bigoplus_{\text{charged } C_l} X^{C_l},$$

so that

$$\begin{aligned} X_a &= X_a^{\mathrm{n}} \oplus X_a^{\mathrm{c}}, \\ X^a &= X^{a,\mathrm{n}} \oplus X^{a,\mathrm{c}}. \end{aligned}$$

We denote by $\pi_a^{\mathrm{n}}, \pi^{a\,\mathrm{n}}, \pi_a^{\mathrm{c}}, \pi^{a\,\mathrm{c}}$ the orthogonal projections on these spaces. The spaces $Y_a^{\mathrm{n}}, Y_a^{\mathrm{c}}, Y^{a\,\mathrm{n}}, Y^{a\,\mathrm{c}}$ are defined as the intersections with Y of the corresponding subspaces of X.

We saw in [GL1] that there exist bijective linear mappings

$$\begin{aligned} \tilde{M}_a^{\mathrm{c}} &: Y_a^{\mathrm{c}\prime} \to T^\star \mathbb{R}^{\sharp_{\mathrm{c}} a} \\ M_a^{\mathrm{c}} &: Y_a^{\mathrm{c}\prime} \to T^\star \mathbb{R}^{\sharp_{\mathrm{c}} a}, \end{aligned}$$

such that the following linear transformation is symplectic

$$\begin{aligned} &\chi_a : T^\star X \to T^\star \mathbb{R}^{\sharp_{\mathrm{c}} a} \times T^\star \mathbb{R}^{\sharp_{\mathrm{c}} a} \times T^\star Y_a^{\mathrm{n}} \times T^\star Z_a \times T^\star X^a \\ &(x, \xi) \mapsto (x', \xi'), \end{aligned}$$

where

$$\begin{cases} x' = (x_a', x^{a\prime}), x_a' = (\hat{y}_a^{\mathrm{c}\prime}, \tilde{y}_a^{\mathrm{c}\prime}, y_a^{\mathrm{n}\prime}, z_a'), \\ \xi' = (\xi_a', \xi^{a\prime}), \xi_a' = (\hat{\eta}_a^{\mathrm{c}\prime}, \tilde{\eta}_a^{\mathrm{c}\prime}, \eta_a^{\mathrm{n}\prime}, \zeta_a'), \end{cases}$$

and

$$\begin{cases} \tilde{M}_a^{\mathrm{c}}(\eta_a^{\mathrm{c}} + (Ax)_a^{\mathrm{c}}) =: (\tilde{y}_a^{\mathrm{c}\prime}, \tilde{\eta}_a^{\mathrm{c}\prime}), \\ M_a^{\mathrm{c}}(\eta_a^{\mathrm{c}} + A_a{}^{ac} x^a - A^{\mathrm{c}}_{a\,a} x_a) =: (\hat{y}_a^{\mathrm{c}\prime}, \hat{\eta}_a^{\mathrm{c}\prime}), \\ \pi_a^{\mathrm{n}}(\eta_a^{\mathrm{n}} + (Ax)_a^{\mathrm{n}}) =: \eta_a^{\mathrm{n}\prime}, \\ \pi_a^{\mathrm{n}}(y_a^{\mathrm{n}}) =: y_a^{\mathrm{n}\prime}, \\ z_a =: z_a', \\ \zeta_a =: \zeta_a' \\ \xi^a - A^a{}_a x_a =: \xi^{a\prime}, \\ x^a =: x^{a\prime}. \end{cases}$$

In the above equations, one copy of $\mathbb{R}^{\sharp_{\mathrm{c}} a}$ is denoted by $\hat{Y}_a^{\mathrm{c}}$ and the other copy by $\tilde{Y}_a^{\mathrm{c}}$. Note that if one identifies $(\tilde{y}_a^{\mathrm{c}}, \hat{y}_a^{\mathrm{c}})$ with the two orthogonal to the field components of y_a^{c} and $(\hat{\eta}_a^{\mathrm{c}}, \tilde{\eta}_a^{\mathrm{c}})$ with the two orthogonal to the field components of η_a^{c}, then one can identify the range of the symplectic transformation χ_a with T^*X, which leads to simpler formulas. However, we will not make this identification in the sequel.

As in [GL1], one can find a unitary transformation U_a implementing χ_a, such that

$$
\left\{\begin{array}{l} U_a : L^2(X) \to L^2(Z_a \times \hat{Y}_a^{\mathrm{c}} \times \tilde{Y}_a^{\mathrm{c}} \times Y_a^{\mathrm{n}} \times X^a), \\ U_a K_a U_a^\star = (D_{z_a}, D_{\tilde{y}_a^{\mathrm{c}}}, \tilde{y}_a^{\mathrm{c}}, D_{y_a^{\mathrm{n}}}). \end{array}\right. \tag{3.3}
$$

If we put

$$
U_a H_a U_a^\star =: \tilde{H}_a,
$$

we have

$$
\begin{aligned} \tilde{H}_a = \frac{1}{2} D_{z_a}^2 + R_a^{\mathrm{w}}(D_{\hat{y}_a^{\mathrm{c}}}, \hat{y}_a^{\mathrm{c}}, x^a) &+ \frac{1}{2}(D_{y_a^{\mathrm{n}}} - 2A_{a,\mathrm{n}}{}^a x^a)^2 \\ &+ \frac{1}{2}(D_{x^a} - A^{a\,a} x^a)^2 + V^a(x^a), \end{aligned} \tag{3.4}
$$

where $R_a(\hat{\eta}_a^{\mathrm{c}}, \hat{y}_a^{\mathrm{c}}, x^a)$ is a second order polynomial equal to

$$
R_a(\hat{\eta}_a^{\mathrm{c}}, \hat{y}_a^{\mathrm{c}}, x^a) = \frac{1}{2}\left((M_a^{\mathrm{c}})^{-1}(\hat{\eta}_a^{\mathrm{c}}, \hat{y}_a^{\mathrm{c}}) - 2A_{a,\mathrm{c}}{}^a x^a\right)^2,
$$

and the Weyl quantization is defined by

$$
P^{\mathrm{w}}(x, D)u(x) := (2\pi)^{-n} \int\int e^{i\langle x-y,\xi\rangle} P\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi. \tag{3.5}
$$

For $a \in \mathcal{A}$, we define

$$
\begin{aligned} \tilde{H}^a := R_a^{\mathrm{w}}(D_{y_a^{\mathrm{c}}}, y_a^{\mathrm{c}}, x^a) &+ \frac{1}{2}(D_{y_a^{\mathrm{n}}} - 2A_{a,\mathrm{n}}{}^a x^a)^2 \\ &+ \frac{1}{2}(D^a - A^{a\,a} x^a)^2 + V^a(x^a), \end{aligned} \tag{3.6}
$$

acting on $L^2(\tilde{Y}_a^{\mathrm{c}} \times \hat{Y}_a^{\mathrm{c}} \times Y_a^{\mathrm{n}} \times X^a)$. We also define

$$
H^a := H_a - \frac{1}{2} D_{z_a}^2 = U_a^\star \tilde{H}^a U_a.
$$

To end this section, we will now give some consequences of hypotheses *(V1)*, *(V2)* and *(V3)* which will be needed later.

LEMMA 3.1. — *Assume that V satisfies hypotheses (V1). Let $a \in \mathcal{A}$. Then*

$$
V_a \text{ is } \tilde{H}^{a_{\max}} \text{ bounded.} \tag{3.7}
$$

Assume that V satisfies hypotheses (V1) and (V2). Let $a \in \mathcal{A}$. Then

$$(3.8)\quad \text{i)}\ \left\|F\left(\frac{|x^a|}{R} \geqslant 1\right) V_a(\tilde{H}^{a_{\max}} + i)^{-1}\right\| = o(1),\ R \to \infty,$$

$$\text{ii)}\ \left\|F\left(\frac{|x^a|}{R} \geqslant 1\right) z^a \nabla_{z^a} V_a(\tilde{H}^{a_{\max}} + i)^{-1}\right\| = o(1),\ R \to \infty.$$

Let $q_a \in C_0^\infty(X^{a_{\max}})$ supported in $\{x^{a_{\max}} \,|\, |x^b| \geqslant \epsilon\langle x^{a_{\max}}\rangle,\ \forall b \not\leqslant a\}$. Then

$$(3.9)\quad \text{i)}\ q_a I_a(\tilde{H}^{a_{\max}} + i)^{-1} \text{ is compact on } L^2(X^{a_{\max}} \times Y^{\mathrm{c}}_{a_{\max}}),$$

$$\text{ii)}\ (\tilde{H}^{a_{\max}} + i)^{-1} q_a z \nabla_z I_a (\tilde{H}^{a_{\max}} + i)^{-1} \text{ is compact on } L^2(X^{a_{\max}} \times Y^{\mathrm{c}}_{a_{\max}}).$$

Assume that hypothesis (V3) holds. Then

$$(3.10)\qquad \|(H+i)^{-1} \partial_x^\alpha V (H+i)^{-1}\| < \infty,\ |\alpha| \leqslant 2.$$

Proof — To prove (3.7), (3.8) and (3.10) we use Kato's inequality (see for example [CFKS], Sect. 1.3) :

$$(3.11)\qquad |e^{-t(D-Ax)^2} u|(x) \leqslant e^{-t(D)^2} |u|(x).$$

We deduce from (3.11) that if $g(x)$ is a multiplication operator, we have

$$|g(x)(1+(D-Ax)^2)^{-1} u|(x) \leqslant |g(x)|(1+D^2)^{-1} |u|(x).$$

Applying this inequality we easily obtain (3.7), (3.8) and (3.10).

Let us now prove (3.9). Let us prove ii), the proof of i) being simpler. We write

$$\begin{aligned}
&(\tilde{H}^{a_{\max}} + i)^{-1} q_a z \nabla_z I_a (\tilde{H}^{a_{\max}} + i)^{-1} \\
&= (\tilde{H}^{a_{\max}} + i)^{-1} q_a z \nabla_z I_a F\left(\frac{|x^{a_{\max}}|}{R} \geqslant 1\right)(\tilde{H}^{a_{\max}} + i)^{-1} \\
&+ (\tilde{H}^{a_{\max}} + i)^{-1} q_a z \nabla_z I_a F\left(\frac{|x^{a_{\max}}|}{R} \leqslant 1\right)(\tilde{H}^{a_{\max}} + i)^{-1}.
\end{aligned}$$

The first term on the right hand side of (3.12) goes to 0 in norm when R goes to ∞ by (3.8) ii). The second one is seen to be compact using the fact that V_a is $\tilde{H}^{a_{\max}}$ bounded (see (3.7)) and writing

$$z\nabla_z V_a = [V_a, i\langle z, D_z\rangle].$$

This ends the proof of the lemma. □

4. Asymptotic completeness.

In this section we state the main result of this work — the asymptotic completeness of long-range wave operators. We start by recalling the discussion of the notions of bound states and scattering states given in [GL1].

The definition of a bound state depends on the total charge $Q = \sum_{i=1}^{n} q_i$ of the system. This reflects the fact that the properties of the pseudomomentum are different for charged and neutral systems.

DEFINITION 4.1 — *Assume that $Q \neq 0$. Let*

$$\mathcal{H}_{\text{scatt}} := U^{\star}_{a_{\max}}\big(\mathcal{H}_{\text{c}}(\tilde{H}^{a_{\max}}) \otimes L^2(Z_{a_{\max}})\big),$$
$$\mathcal{H}_{\text{bound}} := U^{\star}_{a_{\max}}\big(\mathcal{H}_{\text{pp}}(\tilde{H}^{a_{\max}}) \otimes L^2(Z_{a_{\max}})\big).$$

We have

$$L^2(X) = \mathcal{H}_{\text{scatt}} \oplus \mathcal{H}_{\text{bound}}.$$

If the system is neutral, $\tilde{H}^{a_{\max}}$ acting on $L^2(Y^{\text{n}}_{a_{\max}} \times X^{a_{\max}})$ can be written as a direct integral

$$\tilde{H}^{a_{\max}} = \int_{Y'_{a_{\max}}}^{\oplus} \tilde{H}^{a_{\max}}(\eta_{a_{\max}}) d\eta_{a_{\max}},$$

where the reduced Hamiltonians $\tilde{H}^{a_{\max}}(\eta_{a_{\max}})$ act on $L^2(X^{a_{\max}})$. We then define:

DEFINITION 4.2 — *Assume that $Q = 0$. Let*

$$\mathcal{H}_{\text{scatt}} := U^{\star}_{a_{\max}}\left(L^2(Z_{a_{\max}}) \otimes \int_{Y^{\text{n}\prime}_{\max}}^{\oplus} \mathcal{H}_{\text{c}}(\tilde{H}^{a_{\max}}(\eta^{\text{n}}_{a_{\max}}))d\eta^{\text{n}}_{a_{\max}}\right),$$
$$\mathcal{H}_{\text{bound}} := U^{\star}_{a_{\max}}\left(L^2(Z_{a_{\max}}) \otimes \int_{Y^{\text{n}\prime}_{\max}}^{\oplus} \mathcal{H}_{\text{pp}}(\tilde{H}^{a_{\max}}(\eta^{\text{n}}_{a_{\max}}))d\eta^{\text{n}}_{a_{\max}}\right).$$

We have

$$L^2(X) = \mathcal{H}_{\text{scatt}} \oplus \mathcal{H}_{\text{bound}}.$$

In both cases the states in $\mathcal{H}_{\text{bound}}$ describe a bounded cluster of particles. Their behaviour depends on the total charge of the system: a charged cluster can travel to infinity only in the direction of the field, whereas a neutral cluster can also travel across the field. The basic properties of the bound states, such as an expression for the asymptotic velocity of their center of mass, are given in[GL1], Propositions 6.3, 6.4.

The goal of scattering theory for Hamiltonians of the type (1.1) is the classification of the states in $\mathcal{H}_{\text{scatt}}$. A first classification, introduced in [GL1], can be expressed in terms of the observable of asymptotic velocity along the Z axis.

THEOREM 4.3 — *Assume that hypotheses (V1), (V2) hold. Let $J \in C_\infty(Z^{a_{\max}})$. Then there exists*

$$\text{s-}\lim_{t\to+\infty} e^{itH} J\left(\frac{z^{a_{\max}}}{t}\right) e^{-itH}. \tag{4.1}$$

Moreover, there exists a unique vector of commuting selfadjoint operators $\zeta^{a_{\max}+}$ with a dense domain which commute with H such that the limit (4.1) equals $J(\zeta^{a_{\max}+})$.

Theorem 4.3 gives a first classification of the states in $L^2(X)$. Indeed, if we put for $a \in \mathcal{A}$ $Z_a^{a_{\max}} = Z \cap X_a^{a_{\max}}$ and define

$$T_a := Z_a^{a_{\max}} \setminus \bigcup_{b \not\leq a} Z_b^{a_{\max}},$$

we have

$$\bigcup_{a\in\mathcal{A}} T_a = Z^{a_{\max}},\ T_a \cap T_b = \varnothing,\ a \neq b. \tag{4.2}$$

Thus

$$L^2(X) = \overset{\perp}{\bigoplus_{a\in\mathcal{A}}} E_{T_a}(\zeta^{a_{\max}+})(L^2(X)).$$

One of the results of [GL1] was that if the system has no neutral proper subsystems, then

$$E_{\{0\}}(\zeta^{a_{\max}+})(L^2(X)) = \mathcal{H}_{\text{bound}}.$$

This was a consequence of the Mourre estimate proved in [GL1] under this assumption with the generator of dilations in the z direction as the conjugate operator. If the system consists of a neutral interacting pair and a third (charged) particle, one cannot expect the Mourre estimate to hold with a conjugate operator acting only in the direction of the field (or, more generally, commuting with the pseudomomentum of the neutral pair), for reasons which were discussed in the introduction. We still have that $\mathcal{H}_{\text{bound}} \subset E_{\{0\}}(\zeta^{a_{\max}+})$, but $E_{\{0\}}(\zeta^{a_{\max}+})$ could also contain scattering states made of neutral clusters moving only transversally to the magnetic field. This is one of the main new difficulties encountered in the presence of neutral clusters. It will be adressed in Sections 6, 7 and 8.

Proving asymptotic completeness means giving a complete classification of the states in $\mathcal{H}_{\text{scatt}}$ in terms of the *wave operators*. In order to define the wave operators, we need to introduce the *channel identification operators* and the *modified free dynamics*. The latter, needed because of the long-range nature of the interactions, is not different from that used for standard $N-$body problems and will be discussed later. The channel identification operators, which we now define, deserve more attention.

The reduced Hamiltonian $\tilde{H}^a$ introduced in Section 3 can be written as

$$\tilde{H}^a = \int_{Y_a^{\mathrm{n}\prime}}^{\oplus} \tilde{H}^a(\eta_a^{\mathrm{n}})d\eta_a^{\mathrm{n}},$$

where

$$\tilde{H}^a(\eta_a^{\mathrm{n}}) = R_a^{\mathrm{w}}(D_{\hat{y}_a^{\mathrm{c}}}, \hat{y}_a^{\mathrm{c}}, x^a) + \frac{1}{2}(\eta_a^{\mathrm{n}} - 2A_{a,\mathrm{n}}{}^a x^a)^2 + \frac{1}{2}(D_{x^a} - A^{a\,a}x^a)^2 + V^a(x^a),$$

acting on $L^2(\hat{Y}_a^{\mathrm{c}} \times X^a)$. The Hamiltonian $\tilde{H}^a$ describes the dynamics of noninteracting clusters of particles, after the unitary transformation U_a has been applied. To introduce the channel identification operators, it is convenient to separate $\tilde{H}^a$ into parts corresponding to the charged and the neutral clusters. Using the decomposition $X^a = X^{a,\mathrm{n}} \oplus^{\perp} X^{a,\mathrm{c}}$, we can write the internal potential V^a as:

$$V^a(x^a) = V^{a,\mathrm{c}}(x^{a,\mathrm{c}}) + V^{a,\mathrm{n}}(x^{a,\mathrm{n}}),$$

where

$$V^{a,\mathrm{c}}(x^{a,\mathrm{c}}) := \sum_{(ij) \in \text{ charged } C_l} V_{ij},$$

$$V^{a,\mathrm{n}}(x^{a,\mathrm{n}}) := \sum_{(ij) \in \text{ neutral } C_l} V_{ij}.$$

Hence the reduced Hamiltonian $\tilde{H}^a(\eta_a^{\mathrm{n}})$ can be written as a sum:

$$\tilde{H}^a(\eta_a^{\mathrm{n}}) = \tilde{H}^{a,\mathrm{n}}(\eta_a^{\mathrm{n}}) + \tilde{H}_a^{\mathrm{c}},$$

where $\tilde{H}^{a,\mathrm{c}}$ and $\tilde{H}^{a,\mathrm{n}}(\eta_a^{\mathrm{n}})$ correspond to the charged and neutral clusters of a respectively. Precisely, we have

$$\text{(4.3)}\quad \tilde{H}^{a,\mathrm{n}}(\eta_a^{\mathrm{n}}) = \frac{1}{2}(\eta_a^{\mathrm{n}} - 2A_{a,\mathrm{n}}{}^a x^a)^2 + \frac{1}{2}(D_{x^{a,\mathrm{n}}} - A^{a\,a}x^{a,\mathrm{n}})^2 + V^{a,\mathrm{n}}(x^{a,\mathrm{n}}),$$

acting on $L^2(X^{a,\mathrm{n}})$, and

$$\text{(4.4)}\qquad \tilde{H}^{a,\mathrm{c}} = R_a^{\mathrm{w}}(D_{\hat{y}_a^{\mathrm{c}}}, \hat{y}_a^{\mathrm{c}}, x^a) + \frac{1}{2}(D_{x^{a,\mathrm{c}}} - A^{a\,a}x^{a,\mathrm{c}})^2 + V^{a,\mathrm{c}}(x^{a,\mathrm{c}}),$$

acting on $L^2(\hat{Y}_a^{\mathrm{c}} \times X^{a,\mathrm{c}})$. Thus, for example, if a is a cluster decomposition containing a neutral pair ($a = \{(1,2),(3)\}$, where $q_1 + q_2 = 0$), $H^{a,\mathrm{n}}$ and $H^{a,\mathrm{c}}$ are the reduced Hamiltonians of the neutral pair $(1,2)$ and the charged particle (3) respectively.

The channel identification operators can now be defined as follows:

$$\text{(4.5)}\qquad \Pi_a := U_a^* \left[E_{\mathrm{pp}}(\tilde{H}^{a,\mathrm{c}}) \otimes \int_{Y_a^{\mathrm{n}\prime}}^{\oplus} E_{\mathrm{pp}}(\tilde{H}^{a,\mathrm{n}}(\eta_a^{\mathrm{n}}))d\eta_a^{\mathrm{n}} \right] U_a.$$

Note that

$$E_{\mathrm{pp}}(\tilde{H}^{a,\mathrm{c}}) \otimes E_{\mathrm{pp}}(\tilde{H}^{a,\mathrm{n}}(\eta_a^{\mathrm{n}})) = E_{\mathrm{pp}}(\tilde{H}^a(\eta_a^{\mathrm{n}})),$$

and that

$$\mathcal{H}_{\mathrm{bound}} = \mathrm{Ran}\Pi_{a_{\max}}.$$

The following straightforward generalization of [GL1], Proposition 6.6 holds:

Proposition 4.4. — *Let $u \in \mathrm{Ran}\Pi_a$. Then:*

$$\lim_{R\to\infty} \sup_{t\in\mathbb{R}} \|F\left(\frac{|x^a|}{R} \geqslant 1\right) e^{-itH_a}u\| = 0,$$
$$\lim_{R\to\infty} \sup_{t\in\mathbb{R}} \|F\left(\frac{|y_a^{\mathrm{c}}|}{R} \geqslant 1\right) e^{-itH_a}u\| = 0.$$

Proposition 4.4 states that noninteracting charged clusters can escape to infinity only in the direction of the magnetic field. The behaviour of neutral clusters is rather different. To explain this, suppose that we have an eigenvalue

$$Y_a^{\mathrm{n}\prime} \ni \eta_a^{\mathrm{n}} \mapsto E(\eta_a^{\mathrm{n}}),$$

with the corresponding eigenprojection $P(\eta_a^{\mathrm{n}})$, satisfying suitable regularity assumptions (similar to those below), such that

$$\tilde{H}^{a,\mathrm{n}}(\eta_a^{\mathrm{n}})P(\eta_a^{\mathrm{n}}) = E(\eta_a^{\mathrm{n}})P(\eta_a^{\mathrm{n}}).$$

Then there exist states for which the kinetic energy of the center of mass of the neutral clusters moving under $e^{-itH_a}\Pi_a$ is given by an effective Hamiltonian

$$\frac{1}{2}D_{z_a}^2 + E(D_{y_a^{\mathrm{n}}} + (Ax)_a^{\mathrm{n}}).$$

To study the scattering channels associated with such effective kinetic energies, we need certain implicit assumptions on their behavior, which we will now describe.

We recall that $C^{n,1}(X)$ denotes the space of functions f in $C^n(X)$ such that $\partial_x^\alpha f$ is Lipschitz for $|\alpha| = n$. For a cluster decomposition a containing a neutral pair (i_1, i_2), we will denote by

$$\tau_a^{\mathrm{n}} = \left\{ \sum_{j=i_1,i_2} \frac{|q_j|}{m_j}|\vec{B}|(n_j + \frac{1}{2}),\ n_j \in \mathbb{N} \right\}$$

the set of the Landau levels of the neutral pair, and by Γ_a — the set

$$\Gamma_a = \{(\lambda, \eta_a^{\mathrm{n}}) \mid \lambda \in \sigma_{\mathrm{pp}}(\tilde{H}^{a,\mathrm{n}}(\eta_a^{\mathrm{n}})), \lambda \notin \tau_a^{\mathrm{n}}\} \subset \mathbb{R} \times Y_a^{\mathrm{n}\prime}.$$

Hypothesis (H) There is a locally finite covering of Γ_a by open sets $\mathcal{V}_j = I_j \times \Omega_j$, $I_j \subset \mathbb{R}$, $\Omega_j \subset Y_a^{\mathrm{n}\prime}$, $C^{1,1}$ projections $P_j : \Omega_j \to B(\mathcal{H})$, and $C^{2,1}$ functions $\omega_j : \Omega_j \to \mathbb{R}$, such that:

i) $\Gamma_a \cap \mathcal{V}_j = \{(\lambda, \eta_a^{\mathrm{n}}) \mid \eta_a^{\mathrm{n}} \in \Omega_j,\ \lambda = \omega_j(\eta_a^{\mathrm{n}})\}$;

ii) $E_{\{\omega_j(\eta_a^{\mathrm{n}})\}}(\tilde{H}^{a,\mathrm{n}}(\eta_a^{\mathrm{n}})) = P_j(\eta_a^{\mathrm{n}})$ for $\eta_a^{\mathrm{n}} \in \Omega_j$;

iii) The set τ_{a,Ω_j} of the critical values of ω_j on Ω_j is finite.

It follows from Theorem 6.1(iii) and (v) that these conditions are satisfied for all simple discrete eigenvalues of $\tilde{H}^{a,\mathrm{n}}(\eta_a^{\mathrm{n}})$. They seem very difficult to check for the possible eigenvalues of $\tilde{H}^{a,\mathrm{n}}(\eta_a^{\mathrm{n}})$ embedded in the essential spectrum. Nevertheless, we think that they allow to understand the behavior of a physical 3−body system with neutral clusters. Note also that similar conditions were imposed by Dereziński [De2] in his study of dispersive N-body Hamiltonians (*i.e.*, N−particle Hamiltonians with the quadratic kinetic energy term $\frac{1}{2}D^2$ replaced by a more general term $\omega(D)$). This fact illustrates a similarity between dispersive N−body Hamiltonians and N−body Hamiltonians in magnetic fields.

Let us now introduce the modified free dynamics that we will use to construct the wave operators. It is well-known that for long-range interactions the intercluster interaction cannot be entirely neglected as the clusters escape to infinity with non-zero relative velocities, so that one has to modify the free evolution appearing in the definition of the wave operators. The modification needed uses a solution of the classical Hamilton-Jacobi equation. For simplicity, we have chosen the Dollard modifier [Do], in which an approximate solution of the Hamilton-Jacobi equation is used. Let us also observe that since we assume that the potentials are of Coulomb type, the effective intercluster potential between a neutral pair and a third particle will be of short-range (see the remark after Definition 2.1). This is the reason why we only need to modify the dynamics in the direction of the magnetic field. Let us denote

$$|z|_a = \min_{(ij)\leqslant a} |z_i - z_j|,$$

and

$$I_a(t,x) := F\left(\frac{|z|_a \log t}{t} \geqslant 1\right) I_a^l(x). \tag{4.6}$$

Then $I_a(t,x)$ satisfies

$$|\partial_x^\alpha I_a(t,x)| \leqslant C_{\alpha,\epsilon}\langle t\rangle^{-|\alpha|-\mu+\epsilon},\ |\alpha| \leqslant 1 \tag{4.7}$$

for all $\epsilon > 0$. Define

$$S_a(t,\zeta_a) = \int_0^t \frac{1}{2}\zeta_a^2 + I_a(s,0,s\zeta_a)ds.$$

Note that $S_a(t,D_{z_a})$ and H^a commute and:

$$S_a(t,\zeta_a) = \frac{1}{2}t\zeta_a^2,\ \text{if } a \in \mathcal{A}^{\mathrm{n}}.$$

The main result of this paper is the following theorem:

THEOREM 4.5. — *Let H be given by (2.1), where V satisfies the hypotheses (V), (V′), and (LR) with $\mu > \sqrt{3}-1$. Moreover, assume that the condition (Q) holds and that the hypothesis (H) is satisfied for all cluster decompositions a containing neutral pairs. Then:*

i) $E_{\{0\}}(\zeta^{a_{\max}+})(L^2(X)) = \mathcal{H}_{\text{bound}}$.

ii) *the modified wave operators*

$$\Omega_a^+ := s - \lim_{t\to\infty} e^{itH} e^{-iS_a(t,D_{z_a})-itH^a}\Pi_a$$

exist and their ranges are mutually orthogonal.

iii) *the system is asymptotically complete :*

$$\bigoplus_{a\neq a_{\max}} \mathrm{Ran}\Omega_a^+ = \mathcal{H}_{\text{scatt}}.$$

Note that by i), the states with zero asymptotic velocity along the Z axis coincide with the bound states, as in the cases considered in [GL1], [GL2]. However, while in [GL1] and [GL2] this was an intermediate step in the proof of asymptotic completeness, here it is obtained as a consequence of it.

5. Local inequalities for operators.

In this section we present an abstract version of a few standard arguments of the Mourre theory for N−particle systems, which we will need in Section 6.

DEFINITION 5.1. — *Let A, B and H be self-adjoint operators on a Hilbert space $\mathcal{H}$, and let $c \in \mathbb{R}$. We will say that:*

(i) *$B \geqslant c$ at $H = E_0$ if there is a $\delta > 0$ such that for any function $f \in C_0^\infty(\mathbb{R})$ supported in $(E_0-\delta, E_0+\delta)$ we have*

$$f(H)Bf(H) \geqslant cf^2(H); \tag{5.1}$$

(ii) *B is almost positive (B a-$\geqslant 0$) at $H = E_0$ if $B \geqslant -\epsilon$ at $H = E_0$ for any $\epsilon > 0$;*

(iii) $B \geqslant A$ *(resp.,* B *a-*$\geqslant A$*) at* $H = E_0$ *if* $B - A \geqslant 0$ *(resp.,* $B - A$ *a-*$\geqslant 0$*) at* $H = E_0$.

DEFINITION 5.2. — *Let* $B(k,E)$ *and* $H(k)$ *be measurable families of self-adjoint operators on* $\mathcal{H}$ *for* $k \in \Omega_1 \subset \mathbb{R}^m$, $E \in I \subset \mathbb{R}$*. Let*

$$H = \int_{\Omega_1}^{\oplus} H(k)dk, \ B = B(E) = \int_{\Omega_1}^{\oplus} B(k,E)dk.$$

We will say that :

(i) $B \geqslant c$ *locally in* H *uniformly for* $(k,E) \in \Omega \subset \Omega_1 \times I$*, if there is a* $\delta > 0$ *such that for any* $f \in C_0^\infty(\mathbb{R})$ *supported in* $(-\delta,\delta)$ *we have*

$$f(H(k)-E)B(k,E)f(H(k)-E) \geqslant cf^2(H(k)-E) \text{ for all } (k,E) \in \Omega;$$

(ii) B *a-*$\geqslant 0$ *locally in* H *uniformly for* $(k,E) \in \Omega \subset \Omega_1 \times I$*, if for any* $\epsilon > 0$ $B \geqslant -\epsilon$ *locally in* H *uniformly for* $(k,E) \in \Omega$.

LEMMA 5.3. — *Let* $B(k,E)$ *and* $H(k)$ *be as in Definition 5.2. Assume that the mapping* $\Omega_1 \ni k \to H(k)$ *is continuous in the norm resolvent sense, and that the mapping* $\Omega_1 \times I \ni (k,E) \to f(H(k))B(k,E)f(H(k))$ *is norm continuous for any* $f \in C_0^\infty(\mathbb{R})$*. Let* Ω *be an open subset of* $\Omega_1 \times I$.

(i) *If* $B(k_0,E_0)$ *a-*$\geqslant 0$ *at* $H(k_0) = E_0$ *for some* $(k_0,E_0) \in \Omega$*, then for any* $\epsilon > 0$ *there is a neighbourhood* $\mathcal{U}$ *of* (k_0,E_0) *such that* $B \geqslant -\epsilon$ *locally in* H *uniformly for* $(k,E) \in \mathcal{U}$.

(ii) *If* B *a-*$\geqslant 0$ *locally in* H *pointwise on* Ω *(i.e.,* $B(k,E)$ *a-*$\geqslant 0$ *at* $H(k) = E$ *for any* $(k,E) \in \Omega$*), then* B *a-*$\geqslant 0$ *locally in* H *uniformly on any compact subset of* Ω.

Proof. — Since (ii) follows easily from (i) and a standard covering argument, we will only prove (i).

Pick $\epsilon > 0$. Then there is a $\delta_0 > 0$ such that for any C_0^∞ function f_0 supported in $(E_0 - \delta_0, E_0 + \delta_0)$ we have

$$f_0(H(k_0))B(k_0,E_0)f_0(H(k_0)) \geqslant -\frac{\epsilon}{2}f_0^2(H(k_0)).$$

Moreover, we can choose f_0 so that $f_0 = 1$ on $(E_0 - \delta_0/2, E_0 + \delta_0/2)$. By our assumptions on continuity, there is an open neighbourhood $\mathcal{U}_1$ of k_0 and $\delta_1 > 0$ such that for all $k \in \mathcal{U}_1$, $|E - E_0| < \delta_1$,

$$f_0(H(k))B(k,E)f_0(H(k)) \geqslant -\frac{\epsilon}{2}f_0^2(H(k)) - \frac{\epsilon}{2}. \tag{5.2}$$

Let $\delta_2 = \min(\delta_0, \delta_1)$, $\Delta = (E_0 - \delta_2/10, E_0 + \delta_2/10)$, and $\delta = \delta_2/10$. Then for any $E \in \Delta$ and for any function f supported in $(E - \delta, E + \delta)$ we have $f f_0 = f$. Multiplying (5.2) from both sides by f, we obtain that for all $k \in \mathcal{U}_1$,

$$f(H(k))B(k, E)f(H(k)) \geqslant -\epsilon f^2(H(k)).$$

Hence we can take $\mathcal{U} = \mathcal{U}_1 \times \Delta$. □

The main tool which we will use to establish local almost positivity of operators is the following lemma, which in fact is essentially known in the Mourre theory (see [FH]).

LEMMA 5.4. — *Let B and H be self-adjoint operators on $\mathcal{H}$ such that B is H-bounded, let c be a positive constant, and fix $E_0 \in \mathbb{R}$. Suppose that for any $\epsilon > 0$ there is a compact operator K_ϵ such that*

$$B \geqslant c - \epsilon + K_\epsilon \text{ at } H = E_0. \tag{5.3}$$

(i) *If E_0 is not an eigenvalue of H, we have B a-$\geqslant c$ at $H = E_0$.*

(ii) *Suppose that E_0 is an eigenvalue of H. Let $P = E_{\{E_0\}}(H)$ be the corresponding spectral projection, and let $c_0 = \|PBP\|$. Then*

$$B \text{ a-}\geqslant c(1 - P) - 2c_0 P \text{ at } H = E_0.$$

Proof. — Let us first note that, instead of arbitrary functions $f \in C_0^\infty(\mathbb{R})$ supported in $(E_0 - \delta, E_0 + \delta)$, it suffices to consider those for which in addition $0 \leqslant f \leqslant 1$ and $f = 1$ on $(E_0 - \delta/2, E_0 + \delta/2)$. Throughout the proof, f_δ will denote a function for which these conditions hold.

To prove (i), for a given $\epsilon > 0$ we first choose $\delta_0 > 0$ and a compact operator K_ϵ such that

$$f_{\delta_0}(H)Bf_{\delta_0}(H) \geqslant (c - \epsilon)f_{\delta_0}^2(H) + f_{\delta_0}(H)K_\epsilon f_{\delta_0}(H). \tag{5.4}$$

Multiplying then (5.4) from both sides by $f_\delta(H)$ for $\delta < \delta_0/2$ and using that $f_\delta(H) \to 0$ strongly if E_0 is not an eigenvalue of H, we obtain that for δ sufficiently small

$$f_\delta(H)Bf_\delta(H) \geqslant (c - \epsilon)f_\delta^2(H) + f_\delta(H)K_\epsilon f_\delta(H) \geqslant (c - \epsilon)f_\delta^2(H) - \epsilon f_\delta^2(H),$$

which shows that B a-$\geqslant c$ at $H = E_0$.

Let us now prove (ii). Let P^n be a sequence of finite-dimensional projections such that $P = \text{s-}\lim P^n$ and $P^n \leqslant P$. We write

$$B = C_1 + C_2 + C_2^* + C_3,$$

where

$$\begin{aligned} &C_1 = (1-P^n)B(1-P^n),\\ (5.5)\qquad &C_2 = (1-P)BP^n,\\ &C_3 = PBP - (P-P^n)B(P-P^n) \geqslant -2c_0P. \end{aligned}$$

Fix $\epsilon > 0$, and choose $\delta > 0$ and K_ϵ such that (5.3) holds for $f = f_\delta$. Then

$$\begin{aligned} (5.6)\quad f_\delta(H)C_1f_\delta(H) &\geqslant (c-\epsilon)f_\delta^2(H)(1-P^n) - f_\delta(H)(1-P^n)K_\epsilon(1-P^n)f_\delta(H)\\ &\geqslant (c-\epsilon)f_\delta^2(H)(1-P) - f_\delta(H)(1-P^n)K_\epsilon(1-P^n)f_\delta(H), \end{aligned}$$

where we have used that $P^n \leqslant P$. Since K_ϵ is compact and $P^n \to P$ strongly, for sufficiently large n we have

$$(5.7)\qquad \|(1-P^n)K_\epsilon(1-P^n) - (1-P)K_\epsilon(1-P)\| < \epsilon.$$

As in the proof of (i), we now use the fact that $f_\delta(H)(1-P) \to 0$ strongly as $\delta \to 0$ to obtain that for $\delta < \delta_0$,

$$(5.8)\qquad \|f_\delta(H)(1-P)K_\epsilon(1-P)f_\delta(H)\| < \epsilon.$$

Hence for large enough n and for $\delta < \delta_0$,

$$f_\delta(H)C_1f_\delta(H) \geqslant (c-\epsilon)f_\delta^2(H)(1-P) - 2\epsilon,$$

which implies that

$$(5.9)\qquad C_1 \ a\text{-}\!\geqslant c(1-P) \text{ at } H = E_0.$$

Finally, consider the term $C_2 + C_2^*$. Using the inequality

$$(\epsilon^{-1/2}C_2 + \epsilon^{1/2}P^n)(\epsilon^{-1/2}C_2^* + \epsilon^{1/2}P^n) \geqslant 0,$$

we obtain that

$$C_2 + C_2^* \geqslant -\epsilon P^n - \epsilon^{-1}C_2C_2^*.$$

Note that the operator

$$f_\delta(H)C_2C_2^*f_\delta(H) = f_\delta(H)(1-P)BP^nf_\delta^2(H)B(1-P)f_\delta(H)$$

is compact since P^n is a finite rank projection. Once more, we use that $f_\delta(H)(1-P) \to 0$ strongly as $\delta \to 0$ and argue as in (i) to show that

$$C_2 + C_2^* \ a\text{-}\geqslant 0 \text{ at } H = E_0. \tag{5.10}$$

This, together with (5.5) and (5.9), implies part (ii) of the lemma. □

6. The Mourre estimate for the channel Hamiltonians.

This section will be devoted to the proof of the Mourre estimate for the channel Hamiltonian $\tilde{H}_a^{a_{\max}}$, where $a \in \mathcal{A}$ is a cluster decomposition containing a neutral pair. Throughout this section, a will denote such a decomposition. Without loss of generality, we can assume that $a = (\{1,2\},\{3\})$, where $q_1 + q_2 = 0$.

Recall that in Section 3 we defined the reduced Hamiltonian $\tilde{H}^{a_{\max}}$, acting on $L^2(\tilde{Y}^{\mathrm{c}}_{a_{\max}} \times \hat{Y}^{\mathrm{c}}_{a_{\max}} \times X^{a_{\max}})$, and the Hamiltonian

$$H^{a_{\max}} := H - \frac{1}{2} D^2_{z_{a_{\max}}} = U^\star_{a_{\max}} \tilde{H}^{a_{\max}} U_{a_{\max}},$$

acting on $L^2(X^{a_{\max}} \times Y_{a_{\max}})$. This is the original 3−particle Hamiltonian after one has separated out the trivial part of the center of mass motion (along the z axis).

Let us first recall a few simple facts proved in Section 5 of [GL1]. For $a \in \mathcal{A}, a \neq a_{\max}$, we denote

$$\tilde{H}_a^{a_{\max}} := \tilde{H}^{a_{\max}} - I_a(x^{a_{\max}}).$$

We have seen in [GL1], Sect. 5 that $\tilde{H}_a^{a_{\max}}$ is unitarily equivalent to

$$\frac{1}{2} D^2_{z_a^{a_{\max}}} + \tilde{H}^a$$

acting on $L^2(Z_a^{a_{\max}} \times \tilde{Y}_a^{\mathrm{c}} \times \hat{Y}_a^{\mathrm{c}} \times Y_a^{\mathrm{n}} \times X^a)$. We will denote by $U_a^{a_{\max}}$ the unitary transformation implementing this equivalence. It has the following invariance property:

$$U_a^{a_{\max}} A^{a_{\max}} U_a^{a_{\max}\star} = A^{a_{\max}} \tag{6.1}$$

for

$$A^{a_{\max}} := \frac{1}{2}\left(\langle z^{a_{\max}}, D_{z^{a_{\max}}}\rangle + \langle D_{z^{a_{\max}}}, z^{a_{\max}}\rangle\right).$$

We have

$$\tilde{H}^a = R_a(D_{\hat{y}_a^c}, \hat{y}_a^c) + \frac{1}{2}(D_{y_a^n} - 2A_a{}^a x^a)^2 + \frac{1}{2}(D^a - A^{a\,a}x^a)^2 + V^a(x^a).$$

Note that $R_a(D_{\hat{y}_a^c}, \hat{y}_a^c)$ does not depend on x^a since the only charged cluster in a is a single particle and hence has no internal variables. Let

$$\tilde{H}^{a,\mathrm{n}}(\eta_a^\mathrm{n}) := \frac{1}{2}(\eta_a^\mathrm{n} - 2A_a{}^a x^a)^2 + \frac{1}{2}(D^a - A^{a\,a}x^a)^2 + V^a(x^a),$$

on $L^2(X^a)$, $\eta_a^\mathrm{n} \in Y_a^{\mathrm{n}\prime}$ (see (4.3)). This is the Hamiltonian of a neutral pair of particles after applying the separation of the center of mass as in [GL1], Sect. 3.1. Finally, we put :

$$A^a := \frac{1}{2}\left(\langle z^a, D_{z^a}\rangle + \langle D_{z^a}, z^a\rangle\right).$$

6.1. Spectral theory of $\tilde{H}^{a,\mathrm{n}}(\eta_a^\mathrm{n})$.

This subsection is devoted to a brief analysis of the spectral theory of $\tilde{H}^{a,\mathrm{n}}(\eta_a^\mathrm{n})$. We will prove the following theorem, which can also be found in [AHS2]. Recall from Section 4 that

$$\tau_a^\mathrm{n} := \left\{\sum_1^2 \frac{|q_i|}{m_i}|\vec{B}|\left(n_i + \frac{1}{2}\right), n_i \in \mathbb{N}\right\}, \tag{6.2}$$

denotes the set of Landau levels for the pair of particles $(1,2)$. Let

$$\Sigma_a^\mathrm{n} := \inf(\tau_a^\mathrm{n}) = \frac{1}{2}\sum_1^2 \frac{|q_i|}{m_i}|\vec{B}|.$$

THEOREM 6.1. — *Assume the hypotheses (V). Then:*

i) *$\tilde{H}^{a,\mathrm{n}}(\eta_a^\mathrm{n})$ is selfadjoint with domain*
$D := \{u \in L^2(X^a) | D_{x^a}^2 u, (y^a)^2 u \in L^2(X^a)\}$ *;*

ii) *the essential spectrum $\sigma_{\mathrm{ess}}(\tilde{H}^{a,\mathrm{n}}(\eta_a^\mathrm{n}))$ is equal to $[\Sigma_a^\mathrm{n}, +\infty[$;*

iii) *the eigenvalues of $\tilde{H}^{a,\mathrm{n}}(\eta_a^\mathrm{n})$ can accumulate only at τ_a^n ;*

iv) *if $E(\eta_a^\mathrm{n})$ is a discrete eigenvalue of $\tilde{H}^{a,\mathrm{n}}(\eta_a^\mathrm{n})$, then $\lim_{\eta_a^\mathrm{n} \to +\infty} E(\eta_a^\mathrm{n}) = \Sigma_a^\mathrm{n}$;*

v) *the map*

$$Y_a^{\mathrm{n}\prime} \ni \eta_a^\mathrm{n} \mapsto \tilde{H}^{a,\mathrm{n}}(\eta_a^\mathrm{n})$$

is analytic in norm resolvent sense.

To simplify the exposition, in all proofs in this section we will use the following notation. We will denote the parameter η_a^{n} by k, the Hamiltonian $\tilde{H}^{a,\mathrm{n}}(\eta_a^{\mathrm{n}})$ by $H(k)$, where

$$H(k) := \frac{1}{2}(k - A_e x)^2 + \frac{1}{2}(D - A_i x)^2 + V(x), \text{ acting on } L^2(X),$$

and X, x, V, A_e, A_i stand for $X^a, x^a, V^a, 2A_a{}^a, A^{a\,a}$ respectively. We will also write ζ, A and A_1 instead of ζ_a, A^a, and $A^{a_{\max}}$, and H instead of $\tilde{H}_a^{a_{\max}}$. To avoid any possible confusion, we will never use this simplified notation in the main text.

Proof of Theorem 6.1. — Let us first prove i). Using Kato's inequality as in Lemma 3.1, we deduce from hypothesis *(V1)* that V is relatively bounded with relative bound 0 with respect to

$$\frac{1}{2}(k - A_e x)^2 + \frac{1}{2}(D - A_i x)^2.$$

The above operator is a second order elliptic operator (*i.e.*, its symbol $h(k; y, \xi)$ satisfies $\frac{1}{c}(\xi^2 + y^2) \leqslant h(k; y, \xi) \leqslant c(\xi^2 + y^2)$), from which one easily deduces that it is selfadjoint with domain D.

Let us now prove ii). We denote

$$\begin{aligned} H_0(k) &:= \frac{1}{2}(k - A_e x)^2 + \frac{1}{2}(D_x - A_i x)^2 \\ &= \frac{1}{2}D_z^2 + \frac{1}{2}(k - A_e y)^2 + \frac{1}{2}(D_y - A_i y)^2 \\ &=: \frac{1}{2}D_z^2 + \hat{H}_0(k). \end{aligned} \tag{6.3}$$

Let us start by studying the spectrum of $\hat{H}_0(k)$. We claim that $\sigma(\hat{H}_0(k))$ is independent of k. Indeed, if we denote by $T(k)$ the unitary transformation implementing the symplectic transformation

$$\begin{aligned} y &\to y + y_0 \\ \eta &\to \eta - A_i y_0, \end{aligned}$$

for a vector $y_0 \in Y$ such that $A_e y_0 = k$, we have

$$T(k)\hat{H}_0(k)T(k)^\star = \hat{H}_0(0).$$

Moreover, by the arguments of Section 3, we see that

$$\int^\oplus \hat{H}_0(k)dk$$

is unitarily equivalent to

$$\sum_1^2 \frac{1}{2m_i}(D_{y_i} - q_i J y_i)^2,$$

whose spectrum is the set τ_a^{n} of the Landau levels of the neutral pair. Hence

$$\sigma(\hat{H}_0(k)) = \tau_a^{\mathrm{n}}. \tag{6.4}$$

We deduce immediately from (6.4) and (6.3) that

$$\sigma(H_0(k)) = [\Sigma_a^{\mathrm{n}}, +\infty[.$$

Thus, to prove ii), it suffices to show that

$$(H(k)+i)^{-1}V(H_0(k)+i)^{-1} \text{ is compact.} \tag{6.5}$$

This follows from i) and Lemma 6.4.

iii) follows from Lemma 3.1 below and the abstract Mourre theory [Mo].

To prove iv), we compute

$$T(k)H(k)T(k)^{\star} = \frac{1}{2}(A_e y)^2 + \frac{1}{2}(D - A_i x)^2 + V(x+y_0) =: H_T(k), \tag{6.6}$$

where we recall that $A_e y_0 = k$. We will prove that as $k \to \infty$, $H_T(k)$ converges in norm resolvent sense to

$$H_{0,T} := \frac{1}{2}(A_e y)^2 + \frac{1}{2}(D - A_i x)^2. \tag{6.7}$$

This fact clearly implies iv). To prove our claim it suffices to show that

$$\lim_{k\to\infty} V(x+y_0)(y^2 + D_x^2 + i)^{-1} = 0. \tag{6.8}$$

This is an easy consequence of hypotheses *(V1)* and *(V2) i)*. Finally, v) follows by differentiating $(H(k)-z)^{-1}$ with respect to k and using the characterization of the domain of $H(k)$ given in i). □

LEMMA 6.2. — *Let $f_\delta \in C_0^\infty(\mathbb{R})$ be a cutoff function supported in $[E-\delta, E+\delta]$. Then:*

$$f_\delta(\tilde{H}^{a,\mathrm{n}}(\eta_a^{\mathrm{n}}))[\tilde{H}^{a,\mathrm{n}}(\eta_a^{\mathrm{n}}), iA^a] f_\delta(\tilde{H}^{a,\mathrm{n}}(\eta_a^{\mathrm{n}})) = f_\delta(\tilde{H}^{a,\mathrm{n}}(\eta_a^{\mathrm{n}}))D_{z^a}^2 f_\delta(\tilde{H}^{a,\mathrm{n}}(\eta_a^{\mathrm{n}})) + K_1(\eta_a^{\mathrm{n}}),$$

where

$$\text{(6.9)}\qquad \begin{aligned} &\text{i) } K_1(\eta_a^{\mathrm{n}}) \textit{ is compact on } L^2(X^a),\\ &\text{ii) } \eta_a^{\mathrm{n}} \mapsto K_1(\eta_a^{\mathrm{n}}) \textit{ is norm continuous,}\\ &\text{iii) } \|K_1(\eta_a^{\mathrm{n}})\| \to 0 \textit{ as } \eta_a^{\mathrm{n}} \to \infty. \end{aligned}$$

An operator

$$K^1 = \int_{Y_a^{\mathrm{n}\prime}}^{\oplus} K_1(\eta_a^{\mathrm{n}}) d\eta_a^{\mathrm{n}},$$

where $K_1(\eta_a^{\mathrm{n}})$ satisfies (6.9), is called *a−compact* (see [PSS].)

Recall that in all proofs in this section we are using the simplified notation defined at the beginning of the proof of Theorem 6.1.

Proof of Lemma 6.2. — Since

$$[H(k), iA] = D_z^2 - \langle z, \nabla_z V\rangle,$$

we have

$$K_1(k) = -f_\delta(H(k))\langle z, \nabla_z V\rangle f_\delta(H(k)).$$

This operator is compact on $L^2(X)$ by Lemma 3.1 and Theorem 6.1 i). To handle the limit $k \to \infty$, we write as in the proof of Theorem 6.1 iv):

$$T(k)K_1(k)T(k)^\star = -f_\delta(H(k))z\nabla_z V(x + y_0)f_\delta(H(k))$$

for $y_0 = A_e^{-1}k$. To show that this operator goes to 0 in norm when k goes to ∞, we use hypothesis *(V2) ii)* and argue as in the proof of (6.8). □

We will denote by

$$\text{(6.10)}\qquad d^{\mathrm{n}}(E) := \mathrm{dist}(E, \tau_a^{\mathrm{n}}),$$

the distance from E to the threshold set for the neutral pair τ_a^{n}. Note that we will not follow the usual convention, according to which $d^{\mathrm{n}}(E)$ would stand for the distance only to the thresholds to the left of E. Later, this will allow us to avoid certain technical complications due to the lack of continuity of the latter function.

LEMMA 6.3. — *Let $E \in \mathbb{R}$. For any $\epsilon > 0$ there is a $\delta > 0$ such that for any function $f_\delta \in C_0^\infty((E - \delta, E + \delta))$ we have*

(6.11)

$$f_\delta(\tilde H^{a,\mathrm{n}}(\eta_a^{\mathrm{n}}))D_{z^a}^2 f_\delta(\tilde H^{a,\mathrm{n}}(\eta_a^{\mathrm{n}})) \geqslant (2d^{\mathrm{n}}(E) - \epsilon) f_\delta^2(\tilde H^{a,\mathrm{n}}(\eta_a^{\mathrm{n}})) + K_2(\eta_a^{\mathrm{n}}),$$

where $K_2(\eta_a^{\mathrm{n}})$ *satisfy :*

$$(6.12)\qquad \begin{aligned} &\text{i) } K_2(\eta_a^{\mathrm{n}}) \textit{ is compact on } L^2(X^a),\\ &\text{ii) } \eta_a^{\mathrm{n}} \mapsto K_2(\eta_a^{\mathrm{n}}) \textit{ is norm continuous,}\\ &\text{iii) } \|K_2(\eta_a^{\mathrm{n}})\| \to 0 \textit{ as } \eta_a^{\mathrm{n}} \to \infty. \end{aligned}$$

Proof. — Since $D_z^2 = 2(H_0(k) - \hat{H}_0(k))$, D_z commutes with $\hat{H}_0(k)$, and $\sigma(\hat{H}_0(k)) = \tau_a$, for sufficiently small $\delta > 0$ and all k we have

$$(6.13)\qquad f_\delta(H_0(k))D_z^2 f_\delta(H_0(k)) \geqslant 2d^{\mathrm{n}}(E)f_\delta(H_0(k))^2.$$

Using that $f(H_0(k)) - f(H(k))$ is compact on $L^2(X)$ for any function $f \in C_0^\infty(\mathbb{R})$ and for any k, we obtain (6.11). The properties i)–iii) of K_2 are shown as in Lemma 6.2. □

The following Mourre estimate for $\tilde{H}^{a,\mathrm{n}}(\eta_a^{\mathrm{n}})$ is an immediate consequence of Lemmas 6.2 and 6.3.

LEMMA 6.4. — *For any* $\eta_a^{\mathrm{n}} \in Y_a^{\mathrm{n}\prime}$, $E \in \mathbb{R}$, *and* $\epsilon > 0$, *there is an operator* $K_\epsilon(\eta_a^{\mathrm{n}})$ *compact on* $L^2(X^a)$ *such that*

$$[\tilde{H}^{a,\mathrm{n}}(\eta_a^{\mathrm{n}}), iA^a] > 2d^{\mathrm{n}}(E) - \epsilon + K_\epsilon(\eta_a^{\mathrm{n}}) \text{ at } \tilde{H}^{a,\mathrm{n}}(\eta_a^{\mathrm{n}}) = E.$$

In the next subsection we will also need the following lemma :

LEMMA 6.5. — i) *Let* $E \in \mathbb{R}$ *and* $\epsilon > 0$. *Then there is a constant* $C > 0$ *such that*

$$(6.14)\qquad [\tilde{H}^{a,\mathrm{n}}(\eta_a^{\mathrm{n}}), iA^a] > 2d^{\mathrm{n}}(E) - \epsilon \textit{ at } \tilde{H}^{a,\mathrm{n}}(\eta_a^{\mathrm{n}}) = E$$

uniformly for $|\eta_a^{\mathrm{n}}| > C$.

ii) *If* $I \subset \mathbb{R}$ *is a compact interval, for any* $\epsilon > 0$ *there is a* $C > 0$ *such that (6.14) holds uniformly for* $|\eta_a^{\mathrm{n}}| > C$ *and* $E \in I$.

Proof. — i) Choose a $\delta > 0$ such that (6.11) holds for a function $f_\delta \in C_0^\infty([E-\delta, E+\delta])$ such that $f_\delta \equiv 1$ on $[E-\delta/2, E+\delta/2]$. We can apply Lemmas 6.2 and 6.3 to this function, and choose a constant $C > 0$ such that $\|K_1(k)\| < \epsilon$ and $\|K_2(k)\| < \epsilon$ for $|k| > C$. Then

$$(6.15)\qquad \begin{aligned} f_\delta(H(k))[H(k), iA]f_\delta(H(k)) &\geqslant f_\delta(H(k))D_z^2 f_\delta(H(k)) - \epsilon\\ &\geqslant (2d^{\mathrm{n}}(E) - \epsilon)f_\delta^2(H(k)) - 2\epsilon \end{aligned}$$

for all $|k| > C$, which implies i) (with 3ϵ instead of ϵ).

ii) follows from i) by a standard covering argument. □

6.2. The Mourre estimate for $\tilde{H}_a^{a_{\max}}$.

In Section 4, we introduced the set *(H)* of implicit conditions on the point spectrum of $\tilde{H}^{a,\mathrm{n}}(\eta_a^{\mathrm{n}})$. In the rest of this section, we will assume that these conditions are satisfied.

We will first define the threshold set. Let

$$\tau_a := \left\{ \sum_1^3 \frac{|q_i|}{m_i} |\vec{B}| \Big(n_i + \frac{1}{2} \Big), n_i \in \mathbb{N} \right\} \tag{6.16}$$

be the set of Landau levels of all particles. As we will see later, the Mourre estimate may fail at these energy levels. We will also have to exclude other energy levels, which are defined below.

Definition 6.6. — *The secondary threshold set $\tilde{\tau}_a$ is the set*

$$\tilde{\tau}_a := \{ \Lambda + \kappa \mid \Lambda \in \tau_a^c, \ \kappa \in \tilde{\tau}_a^{\mathrm{n}} \},$$

where $\tau_a^c = \left\{ \frac{|q_3|}{m_3} |\vec{B}| \Big(j + \frac{1}{2} \Big) \mid j \in \mathbb{N} \right\}$ is the set of the Landau levels of the third particle, and

$$\tilde{\tau}_a^{\mathrm{n}} := \bigcup_j \tau_{a,\Omega_j},$$

where τ_{a,Ω_j} are defined in hypothesis (H).

The physical meaning of this definition can be understood as follows. We have seen in [GL1, Sect. 6] that the motion of the center of mass of a bound state of a neutral cluster of particles is given by a dispersive Hamiltonian $\tilde{H}^{a,\mathrm{n}}(\eta_a^{\mathrm{n}})$, so that its effective kinetic energy is given by an eigenvalue $\omega(\eta_a^{\mathrm{n}})$. The set $\tilde{\tau}_a^{\mathrm{n}}$ is the set of critical values of this kinetic energy.

Lemme 6.7. — *The set $\tau_a \cup \tilde{\tau}_a$ is discrete and closed.*

Proof. — Recall that we are using the same simplified notation (defined at the beginning of the proof of Theorem 6.1) as in the previous subsection. Clearly, it suffices to show that $\tilde{\tau}_a$ can accumulate only at

τ_a, which in turn will follow if we prove that the only possible points of accumulation of $\tilde{\tau}_a^{\mathrm{n}}$ are in τ_a^{n} (the set τ_a^{n} of the Landau levels of the neutral pair is defined in (6.2)).

Suppose that $\lambda \notin \tau_a^{\mathrm{n}}$. By Lemma 6.5 i) and virial theorem, there are $C > 0$ and $\epsilon > 0$ such that $(\lambda - \epsilon, \lambda + \epsilon)$ is disjoint from τ_a^{n} and

$$\Gamma_a \cap ((\lambda - \epsilon, \lambda + \epsilon) \times Y_a^{\mathrm{n}\prime})) \subset (\lambda - \epsilon, \lambda + \epsilon) \times \{|k| \leqslant C\}$$

(recall that Γ_a was defined in Section 4, before the statement of the Hypothesis *(H)*). Consequently, it is enough to show that λ is not a point of accumulation of critical values of ω_j for $|k| \leqslant C$. By Lemma 6.2 of [GL1], the set $\tilde{\Gamma}_a = \Gamma_a \cup (\tau_a^{\mathrm{n}} \times Y_a^{\mathrm{n}\prime})$ is closed. Hence the set

$$\begin{aligned} &\Gamma_a \cap \big([\lambda - \epsilon, \lambda + \epsilon] \times \{|k| \leqslant C\}\big) \\ &\qquad = \tilde{\Gamma}_a \cap \big([\lambda - \epsilon, \lambda + \epsilon] \times \{|k| \leqslant C\}\big) \end{aligned}$$

is compact. We can therefore cover it by a finite number of open sets $\mathcal{V}_j$ as in Hypothesis *(H)*. Hence the set $\tilde{\tau}_a^{\mathrm{n}} \cap (\lambda - \epsilon, \lambda + \epsilon)$ is finite, and, in particular, λ is not an accumulation point of $\tilde{\tau}_a^{\mathrm{n}}$. □

We can now state and prove the main result of this section — the Mourre estimate for $\tilde{H}_a^{a_{\max}}$.

THEOREM 6.8. — *Let $\lambda \in \mathbb{R} \backslash \tau_a \cup \tilde{\tau}_a$. Then there is an operator*

$$\tilde{B}_a(\lambda) = \frac{1}{2}\left(\langle m(\lambda; D_{z_a^{a_{\max}}}, D_{y_a^{\mathrm{n}}}), y_a^{\mathrm{n}}\rangle + \langle y_a^{\mathrm{n}}, m(\lambda; D_{z_a^{a_{\max}}}, D_{y_a^{\mathrm{n}}})\rangle\right)$$

for some $C^{1,1}$ function $m(\zeta_a^{a_{\max}}, \eta_a^{\mathrm{n}})$, and a function $\alpha_0(\lambda) > 0$, such that

$$[\tilde{H}_a^{a_{\max}}, iA^{a_{\max}} + c\tilde{B}_a] \geqslant \min(1, c)\alpha_0(\lambda) \text{ at } \tilde{H}_a^{a_{\max}} = \lambda. \tag{6.17}$$

Proof. — We will fix the value of λ as in the statement of the theorem; although the constants, operators, etc., appearing in the proof will usually depend on λ, we will not display that dependence explicitly unless it is necessary to do so.

Let f_δ denote a function in $C_0^\infty([\lambda - \delta, \lambda + \delta])$ such that $f_\delta \equiv 1$ on $[\lambda - \delta/2, \lambda + \delta/2]$. We have

$$\begin{aligned} &f_\delta(H)[H, iA_1]f_\delta(H) \\ &= \sum_{j=0}^{\infty} \int^{\oplus} f_\delta\Big(\frac{1}{2}\zeta^2 + \Lambda_j + H(k)\Big)\left(\zeta^2 + [H(k), iA]\right) f_\delta\Big(\frac{1}{2}\zeta^2 + \Lambda_j + H(k)\Big) d\zeta dk, \end{aligned}$$

where $\Lambda_j = \frac{|q_3|}{m_3}|\vec{B}|\left(j+\frac{1}{2}\right)$, $j \in \mathbb{N}$, are the Landau levels of the third particle. Since $H(k)$ are bounded from below uniformly in k, there are only finitely many values of j ($j = 1, 2, \ldots, N_0$) for which $f_\delta\left(\frac{1}{2}\zeta^2+\Lambda_j+H(k)\right) \neq 0$, and the sum on the right-hand side is in fact finite. For $1 \leqslant j \leqslant N_0$, let $\lambda_j = \lambda-\Lambda_j$ and $f_j(s) = f_\delta(s+\Lambda_j)$ (so that f_j is supported in $(\lambda_j-\delta, \lambda_j+\delta)$). If δ is small enough, the supports of f_j's are disjoint for different values of j.

Let

$$d(E) := \operatorname{dist}(E, \tau_a)$$

be the distance from E to the threshold set τ_a. Note that $d(E)$ is a continuous function of E. Moreover,

$$d(E+s) \leqslant d(E)+s \text{ for all } E \in \mathbb{R},\ s > 0, \tag{6.18}$$

and

$$d^{\mathrm{n}}(E-\Lambda_j) \geqslant d(E) \tag{6.19}$$

(recall that d^{n} is defined in (6.10)).

Since $H(k)$ is bounded from below uniformly in k, for any function $f \in C_0^\infty([\lambda_j-1, \lambda_j+1])$ we have $f\left(H(k)+\frac{1}{2}\zeta^2\right) = 0$ if $|\zeta|$ is large enough ($|\zeta| \geqslant C_1$). By Lemma 6.5ii), for any $\epsilon > 0$ there is a constant $C_2 = C_2(\epsilon)$ such that for $1 \leqslant j \leqslant N_0$,

$$[H(k), iA] \geqslant 2d^{\mathrm{n}}\left(\lambda_j - \frac{1}{2}\zeta^2\right) - \epsilon \geqslant 2d\left(\lambda - \frac{1}{2}\zeta^2\right) - \epsilon \text{ at } H(k) = \lambda_j - \frac{1}{2}\zeta^2$$

uniformly for $|k| > C_2(\epsilon)$ and $|\zeta| < C_1$ (the second inequality follows from (6.19)). Let $C(\epsilon) = \max(C_1, C_2(\epsilon))$. Then, if δ is sufficiently small, we have

$$\begin{aligned}
&\int_{|(\zeta,k)|\geqslant C(\epsilon)}^{\oplus} f_j\left(\frac{1}{2}\zeta^2+H(k)\right)\left(\zeta^2+[H(k), iA]\right) f_j\left(\frac{1}{2}\zeta^2+H(k)\right) d\zeta dk \\
&\geqslant \int_{|(\zeta,k)|\geqslant C(\epsilon)}^{\oplus} f_j\left(\frac{1}{2}\zeta^2+H(k)\right)\left(\zeta^2+2d(\lambda-\frac{1}{2}\zeta^2)-\epsilon\right) f_j\left(\frac{1}{2}\zeta^2+H(k)\right) d\zeta dk \\
&\geqslant \int_{|(\zeta,k)|\geqslant C(\epsilon)}^{\oplus} (2d(\lambda)-\epsilon)\, f_j^2\left(\frac{1}{2}\zeta^2+H(k)\right) d\zeta dk,
\end{aligned}$$

where we have used that by (6.18),

$$d\left(\lambda-\frac{1}{2}\zeta^2\right)+\frac{1}{2}\zeta^2 \geqslant d(\lambda). \tag{6.20}$$

Since $\lambda \notin \tau_a$, we can pick $\epsilon = d(\lambda) > 0$, and for $C_0 = C(d(\lambda))$ we obtain :

$$(6.21)\quad \int_{|(\zeta,k)|\geqslant C_0}^{\oplus} f_j\Big(\frac{1}{2}\zeta^2+H(k)\Big)\,(\zeta^2+[H(k),iA])\,f_j\Big(\frac{1}{2}\zeta^2+H(k)\Big)d\zeta dk \geqslant \int_{|(\zeta,k)|\geqslant C_0}^{\oplus} d(\lambda) f_j^2\Big(\frac{1}{2}\zeta^2+H(k)\Big)d\zeta dk.$$

Let

$$\Sigma_j := \left\{(\zeta,k)|\lambda_j \in \sigma_{\mathrm{pp}}\Big(\frac{1}{2}\zeta^2+H(k)\Big)\right\},$$

and let $M_0 = \{|(\zeta,k)| \leqslant C_0\}$.

By Lemmas 6.4 and 5.4, and by (6.19), for each $(\zeta,k) \in \Sigma_j^{\mathrm{c}}$ we have

$$(6.22)\quad [H(k),iA]\ a\text{-}\!\geqslant 2d^{\mathrm{n}}\Big(\lambda_j-\frac{1}{2}\zeta^2\Big) \geqslant 2d\Big(\lambda-\frac{1}{2}\zeta^2\Big) \text{ at } H(k)=\lambda_j-\frac{1}{2}\zeta^2.$$

Note that (6.22), Lemma 5.3 (i), and the virial theorem (see e.g., [CFKS], Thm 4.6) imply that Σ_j is a closed set, hence $\Sigma_j \cap M_0$ is compact.

Let $(\zeta_0,k_0) \in \Sigma_j$. By hypothesis *(H)*, there exists a neighborhood $\mathcal{V}^j$ of (ζ_0,k_0), a $C^{1,1}$ projection $P^j(k)$, and a $C^{2,1}$ function $\omega^j(k)$, such that

$$(6.23)\quad \begin{aligned} &\omega^j(k)+\frac{1}{2}\zeta^2=\lambda_j,\ (\zeta,k)\in\mathcal{V}^j\cap\Sigma_j,\\ &P^j(k)=E_{\{\omega^j(k)\}}(H(k)),\ (\zeta,k)\in\mathcal{V}^j. \end{aligned}$$

Moreover, since $\lambda_j \notin \tilde{\tau}_a^{\mathrm{n}}$, we can choose $\mathcal{V}^j$ such that on $\mathcal{V}^j \cap \{\zeta=0\}$ we have

$$(6.24)\quad |\nabla\omega^j(k)| > 0.$$

In particular, (6.24) implies that $\zeta^2+|\nabla\omega^j(k)|^2>0$ for all $(\zeta,k)\in\Sigma_j$. Since $\Sigma_j \cap M_0$ is compact, there is a constant $\theta_0>0$ such that for $1 \leqslant j \leqslant N_0$,

$$(6.25)\quad \inf\{\zeta^2+|\nabla\omega^j(k)|^2 \mid (\zeta,k)\in\Sigma_j\cap M_0\} > \theta_0.$$

Let $\alpha_0 = \min(d(\lambda),\theta_0/2)$, and let $\epsilon=\alpha_0$.

By Lemmas 5.4, 6.4, and the virial theorem, for each $(\zeta,k)\in\Sigma_j$ we have

$$(6.26)\quad [H(k),iA]\ a\text{-}\!\geqslant 2d\Big(\lambda-\frac{1}{2}\zeta^2\Big)(1-P^j(k)) \text{ at } H(k)=\lambda_j-\frac{1}{2}\zeta^2,$$

where, as before, we have used (6.19) to replace d^{n} by d.

Let us now fix j, $1 \leqslant j \leqslant N_0$, and consider the points (ζ, k) in the compact set $M_0 = \{|(\zeta, k)| \leqslant C_0\}$. For each $(\zeta, k) \in \Sigma_j \cap M_0$, let $\mathcal{V}^j$ be its open neighbourhood chosen as above (*i.e.*, so that (6.23) and (6.24) are satisfied). For $(\zeta, k) \in M_0 \setminus \Sigma_j$, choose an open neighbourhood $\mathcal{U}^j$ such that $\mathcal{U}^j \cap \Sigma_j = \varnothing$.

By (6.22), (6.26), and Lemma 5.3 (i), we can assume that

$$[H(k), iA] \geqslant 2\left(d\Big(\lambda - \frac{1}{2}\zeta^2\Big) - \epsilon\right),$$

and

$$[H(k), iA] \geqslant 2d\Big(\lambda - \frac{1}{2}\zeta^2\Big)(1 - P^j(k)) - \epsilon,$$

at $H(k) = \lambda_j - \dfrac{1}{2}\zeta^2$ uniformly on each of the neighbourhoods $\mathcal{U}^j$ and $\mathcal{V}^j$ respectively. (Note that the continuity assumptions of Lemma 5.3 are satisfied by Theorem 6.1(v), Lemma 6.2, and Hypothesis *(H)*.)

Since the set M_0 is compact, we can choose a finite subcovering of it: $\{\mathcal{V}_i^j\}_{i=1,\ldots,l}, \{\mathcal{U}_i^j\}_{i=1,\ldots,m}$ $(l = l(j),\ m = m(j))$. Take a partition of unity such that $\operatorname{supp}\chi_{i,j} \subset \mathcal{V}_i^j$, $\operatorname{supp}\phi_{i,j} \subset \mathcal{U}_i^j$,

$$\sum_1^l \chi_{i,j}^2(\zeta, k) + \sum_1^m \phi_{i,j}^2(\zeta, k) = 1 \text{ on } M_0.$$

Then, if $\delta > 0$ is sufficiently small,

$$\begin{aligned}
&\int_{M_0}^{\oplus} f_j\Big(\frac{1}{2}\zeta^2 + H(k)\Big)\left(\zeta^2 + [H(k), iA]\right) f_j\Big(\frac{1}{2}\zeta^2 + H(k)\Big) d\zeta dk \\
&\geqslant \sum_{i=1}^{l} \int_{M_0}^{\oplus} \phi_{i,j}^2(\zeta, k)\left(\zeta^2 + 2d\Big(\lambda - \frac{1}{2}\zeta^2\Big) - \epsilon\right) f_j^2\Big(\frac{1}{2}\zeta^2 + H(k)\Big) d\zeta dk \\
&+ \sum_{i=1}^{m} \int_{M_0}^{\oplus} \chi_{i,j}^2(\zeta, k)\left(\zeta^2 + \left(2d\Big(\lambda - \frac{1}{2}\zeta^2\Big) - \epsilon\right)(1 - P_i^j(k))\right) \\
&\quad f_j^2\Big(\frac{1}{2}\zeta^2 + H(k)\Big) d\zeta dk \\
&\geqslant \sum_{i=1}^{l} \int_{M_0}^{\oplus} \phi_{i,j}^2(\zeta, k)(2d(\lambda) - \epsilon) f_j^2\Big(\frac{1}{2}\zeta^2 + H(k)\Big) d\zeta dk \\
&+ \sum_{i=1}^{m} \int_{M_0}^{\oplus} \chi_{i,j}^2(\zeta, k)(2d(\lambda) - \epsilon) f_j^2\Big(\frac{1}{2}\zeta^2 + H(k)\Big)(1 - P_i^j(k)) d\zeta dk
\end{aligned} \tag{6.27}$$

$$+\sum_{i=1}^{m}\int_{M_0}^{\oplus}\chi_{i,j}^2(\zeta,k)f_j^2\Big(\frac{1}{2}\zeta^2+H(k)\Big)\zeta^2P_i^j(k)d\zeta dk$$
$$\geqslant\sum_{i=1}^{l}d(\lambda)\int_{M_0}^{\oplus}\phi_{i,j}^2(\zeta,k)f_j^2\Big(\frac{1}{2}\zeta^2+H(k)\Big)d\zeta dk$$
$$+\sum_{i=1}^{m}d(\lambda)\int_{M_0}^{\oplus}\chi_{i,j}^2(\zeta,k)f_j^2\Big(\frac{1}{2}\zeta^2+H(k)\Big)(1-P_i^j(k))d\zeta dk$$
$$+\sum_{i=1}^{m}\int_{M_0}^{\oplus}\chi_{i,j}^2(\zeta,k)f_j^2\Big(\frac{1}{2}\zeta^2+H(k)\Big)\zeta^2P_i^j(k)d\zeta dk.$$

We define

$$\tilde{B}_a^j=\frac{1}{2}\sum_{i=1}^{m}\chi_{i,j}(\zeta,k)P_i^j(k)\left(\langle\nabla\omega_i^j(k),D_k\rangle+\langle D_k,\nabla\omega_i^j(k)\rangle\right)P_i^j(k)\chi_i^j(\zeta,k),$$

and

$$\tilde{B}_a=\sum_{j=1}^{N_0}\tilde{B}_a^j.$$

Using the identity

$$\begin{aligned}(6.28)\qquad &[H(k),iP^j(k)D_kP^j(k)]\\ &=[\omega^j(k),iP^j(k)D_kP^j(k)]=P^j(k)\nabla\omega^j(k)P^j(k),\end{aligned}$$

we obtain

$$\begin{aligned}(6.29)\quad &f_j(H)[H,ic\tilde{B}_a^j]f_j(H)\\ &\geqslant\sum_{i=1}^{m}\int_{M_0}^{\oplus}\chi_{i,j}^2(\zeta,k)f_j^2\Big(\frac{1}{2}\zeta^2+H(k)\Big)P_i^j(k)c|\nabla\omega_i^j(k)|^2d\zeta dk.\end{aligned}$$

By (6.25),

(6.30)

$$f_j^2\Big(\frac{1}{2}\zeta^2+H(k)\Big)(\zeta^2+c|\nabla\omega_i^j(k)|^2)P_i^j(k)\geqslant\min(1,c)\alpha_0f_j^2\Big(\frac{1}{2}\zeta^2+H(k)\Big)P_i^j(k).$$

Moreover, if $j\neq l$, and if the neighbourhoods $\mathcal{V}_i^j$ and $\delta>0$ are sufficiently small, we have $f_j\Big(\frac{1}{2}\zeta^2+H(k)\Big)\chi_{i,l}(\zeta,k)=0$, since $f_j\Big(\frac{1}{2}\zeta^2+H(k)\Big)$ is, by definition, supported in $\left|\frac{1}{2}\zeta^2+H(k)-\lambda_j\right|\leqslant\delta$, and, by continuity of $H(k)$ (Theorem 6.1(v)), $\chi_{i,l}$ is supported in $\left|\frac{1}{2}\zeta^2+H(k)-\lambda_l\right|\leqslant\epsilon_0$ for some small $\epsilon_0 > 0$. Hence

$$f_j(H)[H, ic\tilde{B}_a^l]f_j(H) = 0, \text{ if } j \neq l.$$

Collecting (6.21), (6.27), (6.29), and using (6.30), we obtain:

$$f_\delta(H)[H, i(A_1 + c\tilde{B}_a)]f_\delta(H) \geqslant \min(1, c)\alpha_0 f_\delta^2(H),$$

which completes the proof of the theorem. □

7. The Mourre estimate for the 3-particle Hamiltonian.

We will now prove the Mourre estimate for $\tilde{H}^{a_{\max}}$ under the assumption that the total charge of the system Q is nonzero. The main idea here is to glue together the estimates obtained for the channel Hamiltonians $\tilde{H}_a^{a_{\max}}$ for all 2−cluster partitions a. If a contains a neutral pair, we will use the conjugate operator constructed in Theorem 6.17 in Section 6. Otherwise, we will use the Mourre estimate proved in [GL1], Section 5 with the conjugate operator $A^{a_{\max}}$.

Note that one cannot usually obtain a global Mourre estimate from local estimates for different conjugate operators, because of the cutoff errors. However, for 3−particle systems these errors are localized in the free region and therefore can be estimated relatively easily.

Another difficulty that we will have to face comes from the separation of the center of mass motion for the full Hamiltonian $H^{a_{\max}}$. If a contains a neutral pair, the unitary transformations $U_{a_{\max}}$ and U_a are incompatible in the sense that one cannot identify their ranges in a natural way, so that one cannot make sense out of for example $U_a U_{a_{\max}}^{-1}$. A similar difficulty was encountered in [GL1]. One way to deal with this problem is (as in [GL1]) simply not to separate the center of mass motion. We will, however, have to use the fact that the pseudomomentum of the center of mass $K_{a_{\max}}$ is conserved. This will allow us to localize the Mourre estimate in the (noncommuting) pseudomomentum variables. Such localization will be needed to control certain delicate error terms.

In this section, we will impose the conditions *(V)* and *(V′)* on the potentials, and the condition *(H)*.

Let us first define the threshold set. We will write:

$$\begin{aligned}\tau_{a_{\min}} &:= \sigma(\tilde{H}^{a_{\min}}),\ \ \sigma_{a_{\min}} := \varnothing,\\ \tau_a &:= \bigcup_{b<a} \tau_b \cup \sigma_b,\end{aligned}$$

where $\sigma_a := \sigma_{pp}(\tilde{H}^a)$ if $a \in \mathcal{A}^{\mathrm{c}}$, and $\sigma_a = \tilde{\tau}_a$ if $a \in \mathcal{A}^{\mathrm{n}}$ (recall that $\tilde{\tau}_a$ was defined in Definition 6.6). Then:

$$\tau = \tau_{a_{\max}} \cup \bigcup_{\sharp a=2} \tilde{\tau}_a. \tag{7.1}$$

It follows from Lemma 6.7 that τ is closed and discrete.

Next, we will glue together the operators constructed in the previous section for different cluster decompositions into an auxiliary conjugate operator. We fix an $N-$body partition of unity

$$1 = \sum_{a\in\mathcal{A}} q_a^2(x^{a_{\max}}),$$

with the following properties:

$$\begin{aligned}&\operatorname{supp} q_a \subset \{x^{a_{\max}} \in X^{a_{\max}} \mid |x^{a_{\max}}| \geqslant 1, |x^a| \leqslant \delta^a |x^{a_{\max}}|,\\ &\qquad\qquad |x|_a \geqslant \delta_a |x^{a_{\max}}|\},\ \text{ for } a \neq a_{\max}\\ &|\partial_x^\alpha q_a| = O(\langle x\rangle^{-|\alpha|}),\ \text{ for } a \neq a_{\max},\\ &q_{a_{\max}} \in C_0^\infty(X^{a_{\max}}).\end{aligned} \tag{7.2}$$

Since $N = 3$, we can also assume that

$$q_a q_b = 0,\ \text{ if } \sharp a = \sharp b = 2,\ a \neq b. \tag{7.3}$$

If $\sharp a = 2$ and $a \in \mathcal{A}^{\mathrm{n}}$, *i.e.*, a contains a neutral pair, we define

$$\begin{aligned}B_a &:= U_a^\star \tilde{B}_a U_a,\\ M_a &:= q_a F_a B_a q_a F_a,\end{aligned}$$

where $\tilde{B}_a$ is the operator constructed in Theorem 6.17, and F_a is a cutoff function of the form:

$$F_a = F\Big(\frac{|y_a^{\mathrm{c}}|}{\langle x^{a_{\max}}\rangle} \leqslant 1\Big).$$

If $\sharp a = 2$ and a does not contain a neutral pair, we put $M_a = 0$. Then:

$$M := \sum_{\sharp a=2} M_a. \tag{7.4}$$

In Theorem 7.1 below, we will use the conjugate operator $A^{a_{\max}} + cM$, for c small enough. Using Nelson's commutator theorem (see e.g., [RSII]), we see that $A^{a_{\max}} + cM$ is essentially selfadjoint on $C_0^\infty(X)$.

Recall that the total pseudomomentum $K_{a_{\max}} = (D_{a_{\max}} + (Ax)_{a_{\max}})$ is a constant of motion for the Hamiltonian H. For $\chi \in C_0^\infty(\mathbb{R}^2)$, we denote by $\chi(K_{a_{\max}})$ the operator

$$\chi(K_{a_{\max}}) = U^\star_{a_{\max}} \chi^{\mathrm{w}}(\tilde{y}^{\mathrm{c}}_{a_{\max}}, D_{\tilde{y}^{\mathrm{c}}_{a_{\max}}}) U_{a_{\max}}. \tag{7.5}$$

The main result of this section is the following theorem, which will be a key step in the study of the scattering states of H.

THEOREM 7.1. — *Assume hypotheses (Q), (V), (V′) and (H). Let $\lambda \notin \tau$. Then there exist $\alpha(\lambda) > 0$ and $c > 0$ such that :*

$$[H^{a_{\max}}, i(A^{a_{\max}} + cM)] \geqslant \alpha(\lambda) + R_{a_{\max}} \text{ at } H^{a_{\max}} = \lambda,$$

where $R_{a_{\max}}$ is an operator such that for any $\chi \in C_0^\infty(\mathbb{R}^2)$ $\chi(K_{a_{\max}})R_{a_{\max}}$ is compact on $X^{a_{\max}} \times Y^{\mathrm{c}}_{a_{\max}}$.

Since in this section we will never use the original Hamiltonian H, or the operator $A = \frac{1}{2}(\langle z, D_z\rangle + \langle D_z, z\rangle)$, we will simplify the notation and write H and A instead of $H^{a_{\max}}$ and $A^{a_{\max}}$. Consequently, the superscript $a_{\max}$ will also be omitted from symbols such as, for instance, $\tilde{H}_a^{a_{\max}}$. Note, however, that we will have to distinguish between x and $x^{a_{\max}}$, and therefore we will continue to write e.g., $x^{a_{\max}}$ and $y^{a_{\max}}$ (*i.e.*, symbols such as x and y will retain their usual meaning).

We will obtain the Mourre estimate by gluing together the estimates proved in the previous section for the cluster decompositions containing a neutral pair, and estimates similar to those of [GL1] for cluster decompositions in which all clusters are charged. To this end, we will use the channel expansion in Proposition 7.7, which essentially says that the error terms due to M are localized in the free region. We will then choose the constant c sufficiently small so that in the free region the commutator is dominated by the positive term $[H_{a_{\min}}, iA_{a_{\min}}]$, and use Theorem 6.17 to demonstrate that the terms corresponding to $a \in \mathcal{A}$, $\sharp a = 2$, are positive.

The first two lemmas deal with the cutoffs F_a and the errors related to them. The reason why these cutoffs are necessary is as follows. The Mourre estimate for $\tilde{H}_a$, proved in Section 6, implies that, for the bound states of a neutral pair, $|x_a^{\mathrm{n}}|$ grows linearly in t. This, however, does not automatically mean that so does $|x^{a_{\max}}|$. To ensure that, we need the cutoffs F_a. In particular, note that the property i) in Lemma 7.5 below, which will be used later to prove the minimal velocity estimate, would be false without these cutoffs.

On the other hand, one can show that the errors due to F_a are small. The intuitive reason for this is that a charged particle cannot escape to infinity across the field on its own, so that no propagation should take place in $\operatorname{supp} q_a \setminus \operatorname{supp} F_a$.

LEMMA. 7.2. — *Let $a \in \mathcal{A}$ be a 2–cluster decomposition with a neutral pair. Then on* $\operatorname{supp} q_a F_a$ *one has:*

$$|y_a^{\mathrm{n}}| \leqslant C\langle x^{a_{\max}}\rangle.$$

Proof. — We have

$$y^{a_{\max}} = \pi^{a_{\max}} y_a^{\mathrm{n}} + \pi^{a_{\max}} y_a^{\mathrm{c}} + \pi^{a_{\max}} y^a.$$

On $\operatorname{supp} q_a$ one has $|y^a| \leqslant \delta^a \langle x^{a_{\max}}\rangle$, and on $\operatorname{supp} F_a$, $|y_a^{\mathrm{c}}| \leqslant \langle x^{a_{\max}}\rangle$. Hence on $\operatorname{supp} q_a F_a$, one has

$$|\pi^{a_{\max}} y_a^{\mathrm{n}}| \leqslant C\langle x^{a_{\max}}\rangle.$$

Since $\pi^{a_{\max}}$ is invertible on Y_a^{n}, this proves the lemma. □

The next lemma will be needed to estimate errors related to the cutoffs F_a.

LEMMA 7.3. — *Let $f \in C_0^\infty(\mathbb{R})$, $\chi \in C_0^\infty(\mathbb{R}^2)$, and let $a \in \mathcal{A}$ be a cluster decomposition with $\sharp a = 2$ containing a neutral pair. We have :*

$$\|\chi(K_{a_{\max}}) f(H)(1 - F_a) q_a \langle x^{a_{\max}}\rangle\| < \infty.$$

Proof. — Clearly, we have

$$\|\chi(K_{a_{\max}})(D + Ax)_{a_{\max}}\| < \infty,$$

and

$$\|f(H)(D - Ax)_{a_{\max}}\| < \infty.$$

Taking the difference, we obtain:

$$\|\chi(K_{a_{\max}})f(H)(Ax)_{a_{\max}}\| < \infty, \tag{7.6}$$

To fix the notation, let us put $a = (\{1,2\},\{3\})$ so that

$$y_a^{\mathrm{c}} = (0,0,y_3),\ y_a^{\mathrm{n}} = \left(\frac{m_1y_1+m_2y_2}{m_1+m_2}, \frac{m_1y_1+m_2y_2}{m_1+m_2}, 0\right).$$

Using the fact that $q_1 + q_2 = 0, q_3 \neq 0$, we see that $(Ax)_{a_{\max}} = \left(\frac{m_1}{M}v, \frac{m_2}{M}v, \frac{m_3}{M}v\right)$, where $v = J(q_1(y_1-y_2)+q_3y_3)$, $M = m_1+m_2+m_3$, so that on $\operatorname{supp} q_a$ we have

$$\begin{aligned} |y_3| &\leqslant c_1\langle (Ax)_{a_{\max}}\rangle + c_2(y_1-y_2) \leqslant c_1\langle (Ax)_{a_{\max}}\rangle + c_3(y^a) \\ &\leqslant c_1\langle (Ax)_{a_{\max}}\rangle + c_4\delta^a\langle x^{a_{\max}}\rangle, \end{aligned} \tag{7.7}$$

where c_i denote constants which depend only on the masses and charges of the particles. On the other hand, on $\operatorname{supp}(1-F_a)$ we have

$$\langle x^{a_{\max}}\rangle \leqslant |y_a^{\mathrm{c}}| = |y_3|. \tag{7.8}$$

Combining (7.7) and (7.8), we obtain that on $\operatorname{supp} q_a(1-F_a)$,

$$\langle x^{a_{\max}}\rangle \leqslant C\langle (Ax)_{a_{\max}}\rangle,$$

provided that $c_4\delta^a < 1$. This, together with (7.6), completes the proof of the lemma. □

The following properties of the operator M and of the energy cutoffs $f(\tilde{H})$ for $f \in C_0^\infty(\mathbb{R})$ will be used in the proof of Theorem 7.1.

LEMMA 7.4. — i) *Let $q \in C^\infty(X^{a_{\max}})$ with*

$$|\partial^\alpha_{x^{a_{\max}}} q| \leqslant C_\alpha \langle x^{a_{\max}}\rangle^{-|\alpha|},$$

and let $f \in C_0^\infty(\mathbb{R})$. Then

$$[f(\tilde{H}), iq] = K(\tilde{H}+i)^{-1},$$

where K is a compact operator on $L^2(X^{a_{\max}} \times Y^{\mathrm{c}}_{a_{\max}})$.

ii) *Let $q_a \in C^\infty(X^{a_{\max}})$ such that*

$$\begin{aligned} &|\partial^\alpha_{x^{a_{\max}}} q_a| \leqslant C_\alpha\langle x^{a_{\max}}\rangle^{-|\alpha|}, \\ &\operatorname{supp} q_a \subset \{x^{a_{\max}} \mid |x^b| \geqslant \epsilon\langle x^{a_{\max}}\rangle,\ \forall b \not\leqslant a\}, \end{aligned}$$

and let $f \in C_0^\infty(\mathbb{R})$. *Then*

$$q_a f(\tilde{H}) - q_a f(\tilde{H}_a) = K(\tilde{H}+i)^{-1},$$

where K *is a compact operator on* $L^2(X^{a_{\max}} \times Y^{\mathrm{c}}_{a_{\max}})$.

Proof. — To prove that various operators are compact, we will use the following property, which follows from Proposition 2.2 and formula (3.6):

(7.9) $\langle x^{a_{\max}}\rangle^{-\epsilon}(\tilde{H}+i)^{-1}$ is compact on $L^2(X^{a_{\max}} \times Y^{\mathrm{c}}_{a_{\max}})$ for $\epsilon > 0$.

We will use the following functional calculus formula for $f \in C_0^\infty(\mathbb{R})$ (see [HeSj]) :

$$f(A) = \frac{i}{2\pi}\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{f}(z)(z-A)^{-1}dz \wedge d\overline{z}. \tag{7.10}$$

Here $\tilde{f} \in C_0^\infty(\mathbb{C})$ is an almost analytic extension of f satisfying:

$$\begin{aligned} \tilde{f}_{|\mathbb{R}} &= f, \\ |\partial_{\overline{z}}\tilde{f}(z)| &\leqslant C_k|\mathrm{Im}z|^k, \ \forall k \in \mathbb{N}. \end{aligned} \tag{7.11}$$

Let us first prove i). Using (7.10), we get

$$\begin{aligned} K &= [f(\tilde{H}), iq](\tilde{H}+i) \\ &= \frac{i}{2\pi}\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{f}(z)(z-\tilde{H})^{-1}[\tilde{H}, iq](z-\tilde{H})^{-1}(\tilde{H}+i)dz \wedge d\overline{z}. \end{aligned}$$

We write

$$\begin{aligned} &(z-\tilde{H})^{-1}[\tilde{H}, iq](z-\tilde{H})^{-1}(\tilde{H}+i) \\ &\qquad = (z-\tilde{H})^{-1}(\tilde{H}+i)(\tilde{H}+i)^{-1}[\tilde{H}, iq](z-\tilde{H})^{-1}(\tilde{H}+i). \end{aligned}$$

The term $(\tilde{H}+i)^{-1}[\tilde{H}, iq]$ is compact on $L^2(X^{a_{\max}} \times Y^{\mathrm{c}}_{a_{\max}})$, and we have

$$\|(z-\tilde{H})^{-1}(\tilde{H}+i)\| \leqslant C|\mathrm{Im}z|^{-1},$$

for $z \in \mathrm{supp}\tilde{f}$. Using the estimate (7.11), we see that K is compact as a norm convergent integral of compact operators.

Let us now prove ii). Using again (7.10), we write:

$$\begin{aligned} K &= q_a\left(f(\tilde{H}) - f(\tilde{H}_a)\right)(\tilde{H}+i) \\ &= \frac{i}{2\pi}\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{f}(z)q_a(z-\tilde{H})^{-1}I_a(z-\tilde{H}_a)^{-1}(\tilde{H}+i)dz \wedge d\overline{z}. \end{aligned}$$

We write

$$q_a(z-\tilde{H})^{-1}I_a(z-\tilde{H}_a)^{-1} = (z-\tilde{H})^{-1}q_aI_a(z-\tilde{H}_a)^{-1} + (z-\tilde{H})^{-1}[\tilde{H}, iq_a](z-\tilde{H})^{-1}I_a(z-\tilde{H}_a)^{-1}.$$

We conclude as in the proof of i), using (3.8) and (3.9) of Lemma 3.1. The details are left to the reader. □

The next lemma describes some of the properties of M. For $m \in \mathbb{R}$, we denote by $S^m(X^{a_{\max}})$ the space of functions f such that

$$|\partial^\alpha_{x^{a_{\max}}} f| \leqslant C_\alpha \langle x^{a_{\max}}\rangle^{m-|\alpha|}.$$

LEMMA 7.5. — *Assume the hypotheses (H). Then the following estimates hold:*

$$\text{i)} \quad \|\langle x^{a_{\max}}\rangle^{-1} M\| < \infty.$$

Let $q_\epsilon \in S^\epsilon(X^{a_{\max}})$ *for* $\epsilon \in \mathbb{R}$. *Then*

$$\text{ii)} \quad \|[M, iq_\epsilon]\langle x^{a_{\max}}\rangle^{-\epsilon}\| < \infty.$$

Let $q_i \in S^0(X^{a_{\max}})$, $i = 1,2$ *and* $q_1 \equiv 0$ *on* $\operatorname{supp} q_2$. *Then:*

$$\text{iii)} \quad \|q_1 M q_2 \langle x^{a_{\max}}\rangle\| < \infty.$$

Proof. — It suffices to prove the lemma with M replaced by one of the M_a. Using the invariance of the Weyl calculus under linear symplectic transformations, we have

$$B_a = \frac{1}{2}\langle m_a^{\mathrm{w}}(D_{z_a^{a_{\max}}}, D_{y_a^{\mathrm{n}}} + (Ax)_a^{\mathrm{n}}), y_a^{\mathrm{n}}\rangle + \mathrm{hc}$$

and

$$\tilde{B}_a = \frac{1}{2}\langle m_a^{\mathrm{w}}(D_{z_a^{a_{\max}}}, D_{y_a^{\mathrm{n}}}), y_a^{\mathrm{n}}\rangle + \mathrm{hc}.$$

Since $\langle y, Ay'\rangle = 0$ for $y, y' \in Y_a^{\mathrm{n}}$, we can as in [GL1], Sect. 3.1 find a unitary transformation U wich conserves the position operator $x^{a_{\max}}$ and sends B_a onto $\tilde{B}_a$. (Note that the transformation U_a introduced in Section 3 does not have this property). We can therefore replace M_a by $q_aF_a\langle m_a(D), y_a^{\mathrm{n}}\rangle q_aF_a$, for some $C^{1,1}$ function m_a. Property i) follows then from Lemma 7.2. Applying Lemma A.3 in the Appendix for $m=1$, we obtain ii) and iii). □

PROPOSITION 7.6. — *We have*

$$\|(H+i)^{-1}[H, iM](H+i)^{-1}\| < \infty.$$

Proof. — We can replace M by one of the terms M_a. We have

$$\begin{aligned}(H+i)^{-1}[H, iM_a](H+i)^{-1} &= (H+i)^{-1}[H, iq_aF_a]B_aq_aF_a(H+i)^{-1} + \mathrm{hc}\\ &\quad + (H+i)^{-1}q_aF_a[H_a, iB_a]q_aF_a(H+i)^{-1}\\ &\quad + (H+i)^{-1}q_aF_a[I_a, iB_a]q_aF_a(H+i)^{-1}\\ &=: I_1 + I_2 + I_3.\end{aligned}$$

Let us first consider I_1. We note that $(H+i)^{-1}[H, iq_aF_a]\langle x^{a_{\max}}\rangle$ is bounded, which by Lemma 7.5 i) implies that I_1 is bounded. The term I_2 is bounded by the arguments of Section 6. Let us now consider the term I_3. Using the cutoff q_a, we have

$$q_aF_a[I_a, iB_a]q_aF_a = q_aF_a[\tilde{I}_a, iB_a]q_aF_a,$$

where due to hypotheses *(V′)* and Lemma 3.1 we have

$$\begin{aligned}&\tilde{I}_a = \tilde{I}_a^s + \tilde{I}_a^l,\\ (7.12)\qquad &\tilde{I}_a^s(x^{a_{\max}})\langle x^{a_{\max}}\rangle^{1+\epsilon}(H+i)^{-1} \text{ is bounded,}\\ &|\partial^\alpha_{x^{a_{\max}}}\tilde{I}_a^l(x^{a_{\max}})| \leqslant C_\alpha\langle x^{a_{\max}}\rangle^{-\epsilon-|\alpha|},\ \forall\alpha.\end{aligned}$$

We estimate the term containing $[\tilde{I}_a^s, iB_a]$ by 'undoing' the commutator and writing

$$\begin{aligned}M_aI_a^s(H+i)^{-1} &= q_aF_aB_a\langle x^{a_{\max}}\rangle^{-1}\\ &\times\langle x^{a_{\max}}\rangle\tilde{I}_a^s(H+i)^{-1}\times(H+i)q_aF_a(H+i)^{-1},\end{aligned}$$

which shows that

$$(H+i)^{-1}q_aF_a[\tilde{I}_a^s, iM_a]q_aF_a(H+i)^{-1}$$

is bounded. To estimate the term containing $\tilde{I}_{a,l}$, we use Lemma 7.5 ii).

PROPOSITION 7.7. — *Let j_a be an N-body partition of unity such that, in addition to properties similar to (7.2), one has*

$$\begin{aligned}(7.13)\qquad &q_a \equiv 1 \text{ on } \operatorname{supp} j_a \text{ for } \sharp a = 2,\\ &q_b \equiv 0 \text{ on } \operatorname{supp} j_a \text{ for } \sharp a = 2,\ b \neq a.\end{aligned}$$

Then

$$\begin{aligned} f_\delta(H)[H,iA+icM]f_\delta(H) &= \sum_{\sharp a=2} j_a f_\delta(H_a)[H_a,iA+icB_a]f_\delta(H_a)j_a \\ &+ j_{a_{\min}} f_\delta(H_{a_{\min}})[H_{a_{\min}},iA]f_\delta(H_{a_{\min}})j_{a_{\min}} \\ &+ \sum_{\sharp b=2} j_{a_{\min}} f_\delta(H_{a_{\min}})[H_{a_{\min}},icM_b]f_\delta(H_{a_{\min}})j_{a_{\min}} + R, \end{aligned} \tag{7.14}$$

where R denotes an operator such that $\chi(K_{a_{\max}})R$ is compact for $\chi \in C_0^\infty(\mathbb{R})$.

Proof. — The criterion that we will use to ensure compactness of operators is (7.9). Let us first consider the commutator

$$f_\delta(H)[H,iA]f_\delta(H).$$

Arguing as in [GL1], Sect. 5, and using hypotheses *(V)*, we obtain the following channel expansion:

$$\begin{aligned} f_\delta(H)[H,iA]f_\delta(H) &= \sum_{a\in\mathcal{A}} j_a^2 f_\delta(H)[H,iA]f_\delta(H) \\ &= \sum_{a\neq a_{\max}} j_a f_\delta(H_a)[H_a,iA]f_\delta(H_a)q_a + R. \end{aligned} \tag{7.14}$$

Let us now consider the channel expansion of the commutator $f_\delta(H)[H,iM]f_\delta(H)$. We have :

$$\begin{aligned} f_\delta(H)[H,iM]f_\delta(H) &= \sum_{a\in\mathcal{A}} j_a^2 f_\delta(H)[H,iM]f_\delta(H) \\ &= \sum_{a\neq a_{\max}} j_a^2 f_\delta(H)[H,iM]f_\delta(H) + R, \end{aligned}$$

using Proposition 7.6. Using Lemma 7.4, we have then

$$\begin{aligned} j_a^2 f_\delta(H)[H,iM]f_\delta(H) &= j_a f_\delta(H_a)j_a[H,iM]f_\delta(H) + R \\ &= j_a f_\delta(H_a)j_a[H_a,iM]f_\delta(H) + j_a f_\delta(H_a)j_a[I_a,iM]f_\delta(H) + R. \end{aligned}$$

Let us first consider the term $j_a f_\delta(H_a)j_a[I_a,iM]$. Let $j_{a,1}$ be a cutoff function with the same properties as j_a equal to 1 on suppj_a. By Lemma 7.5 iii), we have

$$f_\delta(H_a)j_a M I_a f_\delta(H_a) = f_\delta(H_a)j_a M j_{a,1} I_a f_\delta(H_a) + R,$$

which shows that

$$j_a f_\delta(H_a)j_a[I_a,iM]f_\delta(H) = j_a f_\delta(H_a)j_a[\tilde{I}_a,iM]f_\delta(H) + R,$$

where $\tilde{I}_a = j_{a,1} I_a = \tilde{I}_a^s + \tilde{I}_a^l$ satisfies the properties (7.12). We prove that the term containing $\tilde{I}_{a,s}$ is compact by the same arguments as in the proof of Proposition 7.6. To handle the term containing $\tilde{I}_{a,l}$ we use Lemma 7.5 ii). We obtain that

$$j_a f_\delta(H_a) j_a [I_a, iM] f_\delta(H) = R.$$

Let us now consider the term $j_a f_\delta(H_a) j_a [H_a, iM] f_\delta(H)$. We have

$$j_a f_\delta(H_a) j_a [H_a, iM] f_\delta(H) = \sum_{b \ni \text{neutral pair}} j_a f_\delta(H_a) j_a [H_a, iM_b] f_\delta(H).$$

Let us consider different cases.

Case 1 : $\sharp a = 2$. Then by property (7.13) of the cutoffs q_b, we have

$$j_a [H_a, iM_b] = 0, \ b \neq a.$$

It remains to consider the term

$$j_a f_\delta(H_a) j_a [H_a, iM_a] f_\delta(H). \tag{7.16}$$

We have

$$\begin{aligned} [H_a, iM_a] &= [H_a, iq_a] F_a B_a F_a j_a + \text{hc} \\ &\quad + q_a [H_a, iF_a] B_a F_a q_a + \text{hc} + q_a F_a [H_a, iB_a] F_a q_a \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

Using Lemma 7.5 iii) and the property (7.13) of j_a, we see that

$$f_\delta(H_a) j_a I_1 f_\delta(H) = R.$$

Using Lemma 7.3, we get similarly that

$$f_\delta(H_a) j_a I_2 f_\delta(H) = R.$$

It remains to consider I_3. We have

$$U_a [H_a, iB_a] U_a^\star = [\tilde{H}_a, i\tilde{B}_a] = n(D_{z_a^{a_{\max}}}, D_{y_a^{\mathrm{n}}}),$$

for some $C^{1,1}$ function n (see (6.28)). Using Lemma 7.5 ii), we get :

$$\begin{aligned} & j_a f_\delta(H_a) j_a q_a F_a [H_a, iB_a] F_a q_a f_\delta(H) \\ &\quad = j_a f_\delta(H_a) q_a F_a [H_a, iB_a] F_a q_a j_a f_\delta(H) + R \\ &\quad = j_a f_\delta(H_a) q_a F_a [H_a, iB_a] F_a q_a f_\delta(H_a) j_a + R. \end{aligned}$$

Hence

(7.17)
$$j_a f_\delta(H_a) j_a [H_a, iM_a] f_\delta(H) = j_a f_\delta(H_a) q_a F_a [H_a, iB_a] F_a q_a f_\delta(H_a) j_a + R.$$

Using then Lemma 7.3 and Lemma 7.4, we can get rid of the cutoffs q_a and F_a modulo a term R. We finally obtain

$$(7.18)\qquad j_a f_\delta(H_a) q_a [H_a, iM_a] f_\delta(H) = j_a f_\delta(H_a)[H_a, iB_a] f_\delta(H_a) j_a + R.$$

Case 2 : $a = a_{\min}$. We have

$$j_{a_{\min}}[H_{a_{\min}}, iM_b] = j_{a_{\min}}[H_b, iM_b] - j_{a_{\min}}[V^b, iM_b].$$

Using Lemma 7.5 iii), we see that

$$\begin{aligned} &f_\delta(H_{a_{\min}}) j_{a_{\min}} [V^b, iM_b] j_{a_{\min}} f_\delta(Ha_{\min}) \\ &\qquad = f_\delta(H_{a_{\min}}) j_{a_{\min}} [\tilde{V}^b, iM_b] j_{a_{\min}} f_\delta(Ha_{\min}) + R, \end{aligned}$$

where $\tilde{V}^b = j_{a_{\min},1} V^b = \tilde{V}^{b,s} + \tilde{V}^{b,l}$ satisfies the properties (7.12). By the same argument as in the proof of Proposition 7.6, we see that

$$(7.19)\qquad f_\delta(H_{a_{\min}}) j_{a_{\min}} [\tilde{V}^b, iM_b] j_{a_{\min}} f_\delta(Ha_{\min}) = R.$$

Next, writing $[H_b, iM_b]$ as above, we get that

$$(7.20)\qquad \begin{aligned} &j_{a_{\min}} f_\delta(H_{a_{\min}}) j_{a_{\min}} [H_b, iM_b] f_\delta(H) \\ &= j_{a_{\min}} f_\delta(H_{a_{\min}}) [H_b, iM_b] j_{a_{\min}} f_\delta(H) + R \\ &= j_{a_{\min}} f_\delta(H_{a_{\min}}) [H_b, iM_b] f_\delta(H_{a_{\min}}) j_{a_{\min}} + R. \end{aligned}$$

Putting together (7.15), (7.18) and (7.20), we obtain the lemma. □

Proof of Theorem 7.1. — Let j_a and R be as in Proposition 7.7. We will use the expansion (7.14) for $f_\delta \in C_0^\infty([\lambda - \delta, \lambda + \delta])$ with $\delta > 0$ sufficiently small. The terms in (7.14) with $\sharp a = 2$ will be treated using Theorem 6.17. To handle the term with $a = a_{\min}$, we note that since

$$[\tilde{H}_{a_{\min}}, iA] = D^2_{z^{a_{\max}}},$$

and $\lambda \notin \tau_{a_{\min}}$, we have

$$(7.21)\qquad f_\delta(H_{a_{\min}})[H_{a_{\min}}, iA] f_\delta(H_{a_{\min}}) \geqslant \alpha_0 f_\delta^2(H_{a_{\min}}).$$

On the other hand, using Proposition 7.6 i), we have

$$\sum_{\sharp b=2} f_\delta(H_{a_{\min}})[H_{a_{\min}}, icM_b] f_\delta(H_{a_{\min}}) \geqslant -\alpha_1 c f_\delta^2(H_{a_{\min}}).$$

We pick then c such that

$$\alpha_1 c \leqslant \frac{1}{2}\alpha_0,$$

and we have

$$\begin{aligned}(7.22)\qquad & j_{a_{\min}} f_\delta(H_{a_{\min}})[H_{a_{\min}}, iA] f_\delta(H_{a_{\min}}) j_{a_{\min}} \\ & + \sum_{\sharp b=2} j_{a_{\min}} f_\delta(H_{a_{\min}})[H_{a_{\min}}, icM_b] f_\delta(H a_{\min}) j_{a_{\min}} \\ & \geqslant \frac{1}{2}\alpha_0 j_{a_{\min}} f_\delta^2(H_{a_{\min}}) j_{a_{\min}}.\end{aligned}$$

Applying then Theorem 6.17 to the terms with $\sharp a = 2$ in (7.14), we get:

$$f_\delta(H)[H, iA + icM] f_\delta(H) \geqslant c_0 \sum_{a \neq a_{\max}} j_a f_\delta^2(H_a) j_{a_{\min}} + R = c_0 f_\delta^2(H) + R.$$

This completes the proof of the theorem. □

8. Minimal velocity estimate.

This section will be devoted to the proof of a minimal velocity estimate for the evolution of a state in $\mathcal{H}_{\text{scatt}}$. In this section, we will assume the hypotheses *(V')* on the potentials and the hypothesis *(H)*. We will also assume that the total charge Q of the system is nonzero. (If $Q = 0$ and all particles are charged, all pairs are also charged. This case was treated in [GL1] and [GL2].)

Recall that we defined in Section 7 the operator $\chi(K_{a_{\max}})$ for $\chi \in C_0^\infty(Y'_{a_{\max}})$. As a first step, we need the following abstract propagation estimate which is due to Sigal-Soffer [SS3]. Its proof will be given in the Appendix.

PROPOSITION 8.1. — *Let $\mathcal{H}$ be a Hilbert space, H – a self-adjoint operator on $\mathcal{H}$ with domain $D(H)$, and A – a self-adjoint operator such that $\mathrm{ad}_A^\alpha (H + i)^{-1}$ is bounded for $\alpha = 1, 2$. Assume that for $\Delta \subset \mathbb{R}$ we have*

$$(8.1)\qquad E_\Delta(H)[H, iA] E_\Delta(H) \geqslant c_0 E_\Delta(H).$$

Then for all $g \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp} g \subset]-\infty, c_0[$ and $f \in C_0^\infty(\Delta)$, we have

$$(8.2)\qquad \int_1^{+\infty} \left\|g\left(\frac{A}{t}\right) f(H) e^{-itH} u\right\|^2 \frac{dt}{t} \leqslant C\|u\|^2.$$

Moreover,

$$\text{s-}\lim_{t\to\infty} g\left(\frac{A}{t}\right) f(H)e^{-itH} = 0. \tag{8.3}$$

We will use a version of this propagation estimate which is localized in pseudomomentum. We first state a consequence of Theorem 7.1.

PROPOSITION 8.2. — *Let $f \in C_0^\infty(\mathbb{R})$ be a cutoff function supported in $(\lambda-\delta, \lambda+\delta)$, where $\lambda \notin \tau \cup \sigma_{\mathrm{pp}}(H^{a_{\max}})$, and let $\chi \in C_0^\infty(Y'_{a_{\max}})$. Then, if $\delta > 0$ is sufficiently small, there exists $c_0 > 0$ such that*

$$\begin{aligned}\chi(K_{a_{\max}})f(H^{a_{\max}})[H^{a_{\max}}, iA^{a_{\max}} + iM]f(H^{a_{\max}})\chi(K_{a_{\max}})\\ \geqslant c_0\chi(K_{a_{\max}})f^2(H^{a_{\max}})\chi(K_{a_{\max}}).\end{aligned}$$

Proof. — This follows from Theorem 7.1 and Lemma 5.4 i). □

The following lemma will also be proved in the Appendix.

LEMMA 8.3. — *Assume that hypotheses (H) and (V′) are satisfied. Then*

$$\|\mathrm{ad}^\alpha_{A+cM}(H+i)^{-1}\| < \infty, \ \alpha = 1, 2.$$

Let us denote by A_1 the operator $A + cM$ (recall that this operator depends on the energy level λ). Using Lemma 8.3 and Proposition 8.2, we obtain the following propagation estimate.

PROPOSITION 8.4. — *Let $f \in C_0^\infty((\lambda-\delta, \lambda+\delta))$, where $\lambda \notin \tau \cup \sigma_{\mathrm{pp}}(H^{a_{\max}})$ and $\delta > 0$ is sufficiently small, and let $\chi \in C_0^\infty(Y'_{a_{\max}})$. Then for $g \in C_0^\infty(\mathbb{R})$, $\operatorname{supp} g \subset]-\infty, c_0[$, we have*

$$\text{i)}\quad \int_1^{+\infty} \left\|g\Big(\frac{A_1}{t}\Big)\chi(K_{a_{\max}})f(H^{a_{\max}})u_t\right\|^2 \frac{dt}{t} \leqslant C\|u\|^2,$$

$$\text{ii)}\quad \text{s-}\lim_{t\to\infty} g\Big(\frac{A_1}{t}\Big)\chi(K_{a_{\max}})f(H^{a_{\max}})e^{-itH^{a_{\max}}} = 0.$$

To deduce a minimal velocity estimate from Proposition 8.4, we will need the following lemma.

LEMMA 8.5. — *Let f be as in Proposition 8.4. For ϵ_0 small enough, we have*

$$\left\| f(H^{a_{\max}})F\Big(\frac{|x^{a_{\max}}|}{t} \leqslant \epsilon_0\Big)F\Big(\frac{|A_1|}{t} \geqslant c_0\Big)\right\| = O(t^{-1}).$$

Combining Lemma 8.5 and Proposition 8.4, we obtain:

PROPOSITION 8.6. — *Let $f \in C_0^\infty(\mathbb{R})$ be supported away from $\tau \cup \sigma_{\mathrm{pp}}(H^{a_{\max}})$ and let $\chi \in C_0^\infty(Y'_{a_{\max}})$. Then we have for ϵ_0 small enough:*

$$\text{i)} \quad \int_1^{+\infty} \|F\Big(\frac{|x|}{t} \leqslant \epsilon_0\Big)\chi(K_{a_{\max}})f(H^{a_{\max}})u_t\|^2 \frac{dt}{t} \leqslant C\|u\|^2,$$

$$\text{ii)} \quad s\text{-}\lim_{t\to\infty} F\Big(\frac{|x^{a_{\max}}|}{t} \leqslant \epsilon_0\Big)\chi(K_{a_{\max}})f(H^{a_{\max}})e^{-itH^{a_{\max}}} = 0.$$

Proof. — Since the conjugate operator A_1 depends on the energy level λ, we cannot apply Lemma 8.5 and Proposition 8.4 directly to a function f as in the statement of the proposition. Instead, we have to use a covering argument. For each $\lambda \in \operatorname{supp} f$, we choose a $\delta > 0$ such that Lemma 8.5 and Proposition 8.4 hold for functions supported in $(\lambda - \delta, \lambda + \delta)$, and deduce that i) and ii) hold for such functions. We then choose a finite subcovering of $\operatorname{supp} f$ by intervals $(\lambda - \delta, \lambda + \delta)$, and write f as a sum of a finite number of functions supported in these intervals. □

Proof of Lemma 8.5. — Denote by R the operator $f(H)F\Big(\frac{|x^{a_{\max}}|}{t} \leqslant \epsilon_0\Big)$, and by R_1 the operator $f_1(H)F_1\Big(\frac{|x^{a_{\max}}|}{t} \leqslant \epsilon_0\Big)$ for C_0^∞ functions F_1 and f_1 of slightly larger supports such that $ff_1 = f$, $FF_1 = F$, and f_1 is supported away from $\tau \cup \sigma_{\mathrm{pp}}(H^{a_{\max}})$. It is easy to check the following estimates:

$$\begin{aligned} &\|R(1 - R_1)\| = O(t^{-1}), \\ &\|R(1 - R_1)\langle x^{a_{\max}}\rangle\| = O(1), \\ &\|[R, A_1]\| = O(1). \end{aligned} \tag{8.4}$$

Let

$$\tilde{A}_1(t) := R_1 A_1 R_1^\star.$$

Picking $\epsilon_0 \ll 1$, we have

$$\|\tilde{A}_1(t)\| \leqslant \frac{1}{2}c_0 t,$$

so that

$$F\Big(\frac{|\tilde{A}_1(t)|}{t} \geqslant c_0\Big) = 0. \tag{8.5}$$

It therefore suffices to prove that

$$R\left(F\Big(\frac{A_1}{t}\Big) - F\Big(\frac{\tilde{A}_1(t)}{t}\Big)\right) = O(t^{-1}). \tag{8.6}$$

We write $F(s) = (s+i)F_{-1}(s)$, where:

$$|\partial_s^\alpha F_{-1}(s)| \leqslant C_\alpha \langle s\rangle^{-1-\alpha},\ |\alpha| \geqslant 0.$$

We have

$$\begin{aligned}
&R\left(F\Big(\frac{A_1}{t}\Big) - F\Big(\frac{\tilde{A}_1(t)}{t}\Big)\right) \qquad (8.7)\\
&= R\left(\frac{A_1}{t} - \frac{\tilde{A}_1(t)}{t}\right) F\Big(\frac{\tilde{A}_1(t)}{t}\Big) + R\Big(\frac{A_1}{t} + i\Big)\left(F_{-1}\Big(\frac{A_1}{t}\Big) - F_{-1}\Big(\frac{\tilde{A}_1(t)}{t}\Big)\right)\\
&= R\left(\frac{A_1}{t} - \frac{\tilde{A}_1(t)}{t}\right) F\Big(\frac{\tilde{A}_1(t)}{t}\Big) + \frac{A_1}{t} + i)R\left(F_{-1}\Big(\frac{A_1}{t}\Big) - F_{-1}\Big(\frac{\tilde{A}_1(t)}{t}\Big)\right)\\
&= O(t^{-1}),
\end{aligned}$$

by (8.4). Moreover, by (8.4),

$$R(A_1 - \tilde{A}_1(t)) = RA_1 - RR_1A_1R_1 = O(1).$$

Hence the first term on the r.h.s of (8.7) is $O(t^{-1})$ as claimed. To handle the second term, we use the following functional calculus formula (see [G]):

$$F_{-1}(B) = \frac{i}{2\pi}\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{f}_{-1}(z)(z-B)^{-1}dz \wedge d\overline{z}. \tag{8.8}$$

Here $\tilde{f}_{-1}$ is an almost analytic extension of f satisfying:

$$\begin{aligned}
&\tilde{f}_{-1|\mathbb{R}} = F_{-1}, \qquad (8.9)\\
&|\partial_{\overline{z},z}^\alpha \partial_{\overline{z}}\tilde{f}(z)| \leqslant C_{\alpha,N}\langle z\rangle^{-2-|\alpha|-N}|\mathrm{Im} z|^N,\ \forall N \in \mathbb{N},\ \alpha \in \mathbb{N}^2.
\end{aligned}$$

Hence:

$$\begin{aligned}
&R\left(F_{-1}\Big(\frac{A_1}{t}\Big) - F_{-1}\Big(\frac{\tilde{A}_1(t)}{t}\Big)\right)\\
&\frac{i}{2\pi}\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{f}_{-1}(z)R\Big(z - \frac{A_1}{t}\Big)^{-1}\Big(\frac{A_1}{t} - \frac{\tilde{A}_1(t)}{t}\Big)\Big(z - \frac{\tilde{A}_1(t)}{t}\Big)dz \wedge d\overline{z}.
\end{aligned}$$

Now

$$\begin{aligned}
&\Big(\frac{A_1}{t}+i\Big)R\Big(z-\frac{A_1}{t}\Big)^{-1}\Big(\frac{A_1}{t}-\frac{\tilde{A}_1(t)}{t}\Big)\Big(z-\frac{\tilde{A}_1(t)}{t}\Big)^{-1}\\
&=\Big(\frac{A_1}{t}+i\Big)\Big(z-\frac{A_1}{t}\Big)^{-1}R\Big(\frac{A_1}{t}-\frac{\tilde{A}_1(t)}{t}\Big)\Big(z-\frac{\tilde{A}_1(t)}{t}\Big)^{-1}\\
&+\Big(\frac{A_1}{t}+i\Big)\Big(z-\frac{A_1}{t}\Big)^{-1}\Big[R,\frac{A_1}{t}\Big]\Big(z-\frac{A_1}{t}\Big)^{-1}\Big(\frac{A_1}{t}-\frac{\tilde{A}_1(t)}{t}\Big)\Big(z-\frac{\tilde{A}_1(t)}{t}\Big)^{-1}.
\end{aligned}$$

Recall that for any selfadjoint operator B, one has

$$\|(B+i)(z-B)^{-1}\| \leqslant C\langle z\rangle|\mathrm{Im}z|^{-1}. \tag{8.10}$$

Using (8.10) and the estimates (8.9), we see that the second term on the r.h.s. of (8.7) is $O(t^{-1})$. This completes the proof of the lemma. □

9. Propagation estimates for the free region.

In this section we prove estimates on the propagation for e^{-itH} in the free region where all particles are separated. To do this, we will need the *center of orbit* observable, which is defined as follows:

DEFINITION 9.1. — *The center of orbit observable is*

$$C := \frac{1}{2}(y + A^{-1}D_y),$$

where we consider A as an invertible mapping $Y \to Y'$.

We will denote by c the corresponding classical observable

$$c := \frac{1}{2}(y + A^{-1}\eta). \tag{9.1}$$

There are at least two reasons why the observable C is useful in the study of $N-$particle systems in a constant magnetic field. Since C is a constant of motion for a Hamiltonian of non-interacting particles, it is convenient to use C instead of y to prove estimates on propagation across the field. On the other hand, estimates on C and y are essentially equivalent, since the difference between C and y is bounded by the total energy of the system (see Lemma 9.4 ii) below). In [GL2], the observable C was used to obtain a bound on the size of charged clusters of particles. We will first recall some of the properties of C. We refer the reader to [GL2] for the proofs.

The center of orbit is an observable with values in Y which satisfies the following commutation relations:

$$[\langle C,\eta'\rangle, i\langle C,\eta''\rangle] = -\frac{1}{2}\langle A^{-1}\eta'',\eta'\rangle. \tag{9.2}$$

We will need a functional calculus for the (noncommuting) observable C which is defined as follows.

DEFINITION 9.2. — *For $F \in C_0^\infty(Y)$, we define*

$$F(C) := (2\pi)^{-2N}\int \hat{F}(\eta)e^{i\langle C,\eta\rangle}d\eta. \tag{9.3}$$

Let us discuss briefly some of the properties of the functional calculus defined by (9.3). First of all, $F(C)$ is selfadjoint if F is a real-valued function. More importantly, this functional calculus is equivalent to the Weyl calculus defined in (3.5).

PROPOSITION 9.3. — *For $F \in C_0^\infty(Y)$, we have*

$$F(C) = F(c)^{\mathrm{w}}(x,D). \tag{9.4}$$

Proof. — See [GL2], Prop. 7.3.

The following lemma, most of which was proved in [GL2], summarizes the basic properties of C.

LEMMA 9.4. — i) *Let $F \in L^\infty(Y)$ and $\alpha > 0$. Then*

$$\left\|(H+i)F\Big(\frac{C}{t^\alpha}\Big)(H+i)^{-1}\right\| = O(\|F\|_\infty).$$

ii) *For any $\chi \in C_0^\infty(\mathbb{R})$ and $F \in C_0^\infty(Y)$, we have*

$$\chi(H)\left(F\Big(\frac{C}{t^\alpha}\Big) - F\Big(\frac{y}{t^\alpha}\Big)\right) = O(t^{-\alpha}).$$

iii) *For any $\chi \in C_0^\infty(\mathbb{R})$ and $F \in C_0^\infty(Y)$, $F_1 \in C^\infty(Y)\cap L^\infty(Y)$, such that $F_1 \equiv 1$ on $\operatorname{supp} F$, we have*

$$\chi(H)\left(F\Big(\frac{C}{t^\alpha}\Big) - F\Big(\frac{C}{t^\alpha}\Big)F_1\Big(\frac{y}{t^\alpha}\Big)\right) = O(t^{-2\alpha}).$$

iv) *Assume that (V3) holds. Then for $F \in C_0^\infty(Y)$, we have*

$$\left[(H+i)^{-1}, iF\left(\frac{C}{t^\alpha}\right)\right] = \frac{1}{t^\alpha}[(H+i)^{-1}, iC]\nabla_c F\left(\frac{C}{t^\alpha}\right) + O(t^{-2\alpha}).$$

Proof. — Statements i), iii) and iv) are proven in [GL2], Lemma, 7.4. To prove ii), we apply the identity

$$e^{iA} - e^{iB} = \int_0^1 e^{isA}(A-B)e^{i(1-s)B}ds$$

to $A = \langle \frac{C}{t^\alpha}, \eta\rangle$, $B = \langle \frac{y}{t^\alpha}, \eta\rangle$. We obtain:

$$\chi(H)\left(F\Big(\frac{C}{t^\alpha}\Big) - F\Big(\frac{y}{\alpha}\Big)\right)$$
$$= t^{-\alpha}\int \hat{F}(\eta)d\eta \int_0^1 \chi(H)e^{is\langle \frac{C}{t^\alpha}, \eta\rangle}\left(\langle A-B, \eta\rangle\right)e^{i(1-s)\langle \frac{y}{t^\alpha}, \eta\rangle}ds.$$

Since

$$C - y = \frac{1}{2}A^{-1}(D_y - Ay),$$

we see that $(H+i)^{-1}(C-y)$ is bounded, which using i) gives that

$$\chi(H)\left(F\Big(\frac{C}{t^\alpha}\Big) - F\Big(\frac{y}{\alpha}\Big)\right) = O(t^{-\alpha}).$$

This completes the proof of ii). □

Let H be an $N-$particle Hamiltonian in a constant magnetic field satisfying the hypotheses *(V)*. We will denote by $W(t,x)$ a time-dependent potential satisfying:

$$|\partial_x W(t,x)| \leqslant C\langle t\rangle^{-1-\mu}, \ \mu > 0,$$

$U(t)$ will stand for the unitary evolution generated by $H(t) = H + W(t,x)$, and D_t will denote the Heisenberg derivative associated with the evolution $U(t)$:

$$\mathrm{D}_t := \partial_t + [H(t), i\,\cdot\,].$$

Let us first establish existence of the asymptotic energy (for proof, see [GL2], Lemma 7.5).

LEMMA 9.5. — *Let $\chi \in C_0^\infty(\mathbb{R})$. Then the norm limit*

$$\lim_{t\to+\infty} U(t)^\star \chi(H) U(t) =: \gamma(\chi) \tag{9.5}$$

exists. Moreover, there is an unbounded self-adjoint operator H^+ with a dense domain such that the limit in (9.5) equals $\chi(H^+)$.

Our next result says that the center of orbit (and hence the transversal to the field coordinates of the particles) cannot grow faster than $O(t)$ along the evolution.

LEMMA 9.6. — *Let $F \in C_0^\infty(Y)$ satisfy $F(y) = F(|y|)$ with $0 \leqslant F \leqslant 1$, $F \equiv 1$ near 0 and $F' = -f^2$. Let*

$$F_R(t) : F\left(\frac{|y|}{tR}\right).$$

Then for $R \geqslant R_0$ the limits

$$F_R^+ = s - \lim_{t\to\infty} U(t)^* F_R(t) U(t) \tag{9.6}$$

exist. Moreover,

$$\text{s-}\lim_{R\to\infty} F_R^+ = 1, \tag{9.7}$$

and

$$[H^+, F_R^+] = 0. \tag{9.8}$$

Using Lemma 9.4 ii), we deduce from Lemma 9.6 the following corollary.

COROLLAIRE 9.7. — *Let $F \in C_0^\infty(Y)$ such that $F(y) = F(|y|)$ with $0 \leqslant F \leqslant 1$, $F \equiv 1$ near 0 and $F' = -f^2$. Then for $R \geqslant R_0$ the limit*

$$F_{R,C}^+ = s - \lim_{t\to\infty} U(t)^* F\left(\frac{|C|}{tR}\right) U(t) \tag{9.9}$$

exist and equal F_R^+.

Proof of Lemma 9.6. — We compute for $\chi \in C_0^\infty(\mathbb{R})$:

(9.10)

$$\mathrm{D}_t\left(\chi(H)F\left(\frac{|y|}{Rt}\right)\chi(H)\right) = \chi(H)\left(-\frac{|y|}{Rt^2}f^2\left(\frac{|y|}{Rt}\right) + f\left(\frac{|y|}{Rt}\right)M(t)f\left(\frac{|y|}{Rt}\right)\right)$$
$$+[W(t,x), i\chi(H)]F\left(\frac{|y|}{Rt}\right)\chi(H) + \text{hc},$$

for

$$M(t) = \frac{1}{2}\left(\langle \frac{y}{|y|Rt} D - Ax\rangle + \mathrm{hc}\right).$$

Using the functional calculus formula (7.10), we obtain:

$$[\chi(H), W(t,x)] = O(t^{-1-\mu}). \tag{9.11}$$

If $\chi_1 \in C_0^\infty(\mathbb{R})$ is another cutoff function with $\chi_1\chi = \chi$, we have

$$\begin{aligned}&\chi(H)f\Big(\frac{|y|}{Rt}\Big)M(t)f\Big(\frac{|y|}{Rt}\Big)\chi(H)\\ &\chi(H)f\Big(\frac{|y|}{Rt}\Big)\chi_1(H)M(t)\chi_1(H)f\Big(\frac{|y|}{Rt}\Big)\chi(H) + O(t^{-1}R^{-1}),\end{aligned}$$

On the other hand, we have

$$\|\chi_1(H)M(t)\chi_1(H)\| \leqslant \frac{C}{Rt},$$

which , together with (9.10), (9.12), gives:

(9.13)

$$\mathrm{D}_t\left(\chi(H)F\Big(\frac{|y|}{Rt}\Big)\chi(H)\right) \geqslant \frac{1}{2t}\chi(H)f^2\Big(\frac{|y|}{Rt}\Big)\chi(H) + O(t^{-2}R^{-2}) + O(t^{-1-\mu}),$$

for $R \geqslant R_0$. Next, we note that by a density argument, it suffices to prove the existence of the limit (9.9) for a state u such that $\chi^2(H^+)u = u$ for some $\chi \in C_0^\infty(\mathbb{R})$. For such u, we have

$$\begin{aligned}U(t)^*F\Big(\frac{|y|}{t}\Big)U(t)u &= U(t)^*F\Big(\frac{|y|}{t}\Big)\chi^2(H)U(t)u + o(1)\\ &= U(t)^*\chi(H)F\Big(\frac{|y|}{t}\Big)\chi(H)U(t)u + o(1).\end{aligned}$$

Following a standard procedure (see [SS1]), we first deduce from (9.13) that

$$\mathrm{s-}\lim_{t\to\infty} U(t)^*\chi(H)F\Big(\frac{|y|}{Rt}\Big)\chi(H)U(t)$$

exists. By a density argument,

$$\mathrm{s-}\lim_{t\to\infty} U(t)^*F\Big(\frac{|y|}{Rt}\Big)U(t) =: F_R^+$$

exists and satisfies $0 \leqslant F_R^+ \leqslant 1$. Using again (9.13), we have for $u \in L^2(X)$ such that $u = \chi^2(H)u$:

$$(F_R^+u|u) \geqslant (U(t_0)^*\chi(H)F\Big(\frac{|y|}{Rt_0}\Big)\chi(H)U(t_0)u|u) + O(t_0^{-1}R^{-2}) + o(t_0^0).$$

Since for fixed t_0 $F\Big(\dfrac{|y|}{Rt_0}\Big)$ tends strongly to 1 when R goes to ∞, we obtain that

$$\lim_{R\to\infty}(F_R^+u|u) = \|u\|^2,$$

which shows that s$-\lim_{R\to\infty} F_R^+ = 1$. This completes the proof of the lemma. □

We define the "free region" $\mathcal{Y}_{a_{\min}}$ as

$$\mathcal{Y}_{a_{\min}} = \{y \in Y \;\| y^a| > 0,\ \forall a \neq a_{\min}\}. \tag{9.14}$$

This is the region of the configuration space in which all three particles are separated. The following proposition shows that no propagation takes place in $\mathcal{Y}_{a_{\min}}$.

Proposition 9.8. — *Let $g \in C_0^\infty(\mathcal{Y}_{a_{\min}})$, $g \geqslant 0$, and let $\chi \in C_0^\infty(\mathbb{R})$. Then for any $\alpha > 1/2$,*

$$\begin{aligned}&\int_1^{+\infty} \Big\|g\Big(\frac{C}{t^\alpha}\Big)\chi(H)U(t)u\Big\|^2\frac{dt}{t} < \infty,\\ &\lim_{t\to\infty}\Big\|g\Big(\frac{C}{t^\alpha}\Big)u_t\Big\| = 0.\end{aligned} \tag{9.15}$$

Using Lemma 9.4 ii), we deduce from Proposition 9.8 the following corollary.

Corollary 9.9. — *Let $g \in C_0^\infty(\mathcal{Y}_{a_{\min}})$, $g \geqslant 0$, and let $\chi \in C_0^\infty(\mathbb{R})$. Then for any $\alpha > 1/2$*

$$\begin{aligned}&\int_1^{+\infty} \Big\|g\Big(\frac{y}{t^\alpha}\Big)\chi(H)U(t)u\Big\|^2\frac{dt}{t} < \infty,\\ &\lim_{t\to\infty}\Big\|g\Big(\frac{y}{t^\alpha}\Big)u_t\Big\| = 0,\end{aligned} \tag{9.16}$$

Proof of Proposition 9.8. — Let g be as in the statement of the proposition. Pick a smooth function F with $\operatorname{supp} F \subset \mathcal{Y}_{a_{\min}}$, homogeneous of degree 0 for large enough y, such that $\langle y, \nabla F(y)\rangle \geqslant g^2(y)$. Consider the propagation observable

$$\Phi(t) = \chi(H)F\Big(\frac{C}{t^\alpha}\Big)\chi(H).$$

The Heisenberg derivative of $\Phi(t)$ is equal to

$$\mathrm{D}_t\Phi(t) = \chi(H)\partial_t F\Big(\frac{C}{t^\alpha}\Big)\chi(H) + \chi(H)\left[H, iF\Big(\frac{C}{t^\alpha}\Big)\right]\chi(H) + O(t^{-1-\mu}).$$

Let us first compute $\frac{\partial}{\partial t}\Phi(t)$. By Proposition 9.3, we have

$$\partial_t F\Big(\frac{C}{t^\alpha}\Big) = \partial_t G^{\mathrm{w}}(t, y, D_y),$$

for $G(t,y,\eta) = F\Big(\frac{c(y,\eta)}{t^\alpha}\Big)$. Since $\langle y, \nabla_y F(y)\rangle \geqslant g^2(y)$, we obtain

$$\partial_t G(t,y,\eta) \leqslant \frac{-\alpha}{t^{1+\alpha}} g^2\left(\frac{c(y,\eta)}{t^\alpha}\right).$$

Using sharp Gårding's inequality (see [Hö], Thm. 18.6.7) and symbolic calculus, we get

$$\partial_t F\Big(\frac{C}{t^\alpha}\Big) \leqslant -\frac{\alpha}{t}\left(g\Big(\frac{C}{t^\alpha}\Big)\right)^2 + O(t^{-1-\alpha}). \tag{9.17}$$

Let us now estimate the term $\chi(H)\left[H, iF\Big(\frac{C}{t^\alpha}\Big)\right]\chi(H)$. Using Lemma 9.4 iii), we have

$$\begin{aligned}\chi(H)\left[H, iF\Big(\frac{C}{t^\alpha}\Big)\right]\chi(H) &= \chi(H)\left[H_{a_{\min}}, iF\Big(\frac{C}{t^\alpha}\Big)\right]\chi(H) \\ &+\chi(H)\left[I(t,x), F\Big(\frac{C}{t^\alpha}\Big)\right]\chi(H) + O(t^{-2\alpha}),\end{aligned} \tag{9.18}$$

where

$$I(t,x) = F_1\left(\frac{y}{t^\alpha}\right) I_{a_{\min}}(x),$$

and $F_1 \in C_0^\infty(\mathcal{Y}_{a_{\min}})$ is a cutoff function satisfying $F_1 \equiv 1$ on supp$F$. The first term on the right-hand side of (9.18) is 0 since $[H_{a_{\min}}, C] = 0$. To estimate the second one, we note that hypotheses *(V')* imply that

$$\begin{aligned}&I(t,x) = I^s(t,x) + I^l(t,x),\\ &\|(H+i)^{-1}I^s(t,x)\| \leqslant Ct^{-(1+\epsilon)\alpha},\\ &|\nabla I^l(t,x)| \leqslant Ct^{-1-\mu}.\end{aligned} \tag{9.19}$$

Using formula (9.3), we write:

$$\left[I(t,x), iF\Big(\frac{C}{t^\alpha}\Big)\right] = \int \hat{F}(\eta)\int_0^1 e^{is\langle\frac{C}{t^\alpha},\eta\rangle}\left[I(t,x), i\frac{C}{t^\alpha}\right]e^{i(1-s)\langle\frac{C}{t^\alpha},\eta\rangle}ds\,d\eta.$$

Using (9.19) and Lemma 9.4 i), we finally obtain

$$\chi(H)\left[H, iF\left(\frac{C}{t^{\alpha}}\right)\right]\chi(H) = O(t^{-2\alpha}). \tag{9.20}$$

(9.17) and (9.20) imply that

$$-\mathrm{D}_t\Phi(t) \geqslant \frac{\alpha}{t}\chi(H)\left(g(\frac{C}{t^{\alpha}})\right)^2\chi(H) + O(t^{-2\alpha}),$$

from which it follows by a standard argument that

$$\int_1^{\infty}\|g\left(\frac{C}{t^{\alpha}}\right)\chi(H)U(t)u\|^2\frac{dt}{t} < \infty, \tag{9.21}$$

and that the limit

$$\lim_{t\to\infty}\langle U(t)u, \Phi(t)U(t)u\rangle$$

exists for u such that $\chi(H)u = u$. Repeating the above argument with F replaced by g^2, we obtain that

$$\lim_{t\to\infty}\|g\left(\frac{C}{t^{\alpha}}\right)\chi(H)U(t)u\| \text{ exists.} \tag{9.22}$$

By (9.21), this limit is zero, which completes the proof of the proposition. □

10. Proof of asymptotic completeness.

In this section we prove the results of Section 4 for short-range and Coulomb-type interactions.

Recall that in Section 4 we introduced the asymptotic velocity observable $\zeta^{a_{\max}+}$ for the evolution e^{-itH}. To prove the asymptotic completeness of the wave operators, we will consider separately the states in $\mathrm{Ran}E_{T_a}(\zeta^{a_{\max}+})$, for all $a \in \mathcal{A}$. As explained in Section 4, the space $\mathrm{Ran}E_{\{0\}}(\zeta^{a_{\max}+})$ could also contain scattering states such that neutral clusters are moving only transversally to the magnetic field. We will see later, as a consequence of asymptotic completeness, that such states do not exist. However, we cannot exclude them at this stage of the analysis.

Let us now introduce some notation which will be used later. For $a \in \mathcal{A}, a \neq a_{\max}$, we denote by $U_{a,i}(t), i = 0, 1$, the unitary evolutions generated by the time-dependent Hamiltonians:

$$H_{a,0}(t) = H_a + \tilde{I}_a(t,x),$$
$$H_{a,1}(t) = H_a + I_a(t,x),$$

and

$$H_{a,2}(t) = \begin{cases} H_a + \tilde{I}_a(t,x_a), & \text{if } a \in \mathcal{A}^{\mathrm{n}}, \\ H_a + \tilde{I}_a(t,z_a), & \text{if } a \in \mathcal{A}^{\mathrm{c}}, \end{cases}$$

$$H_{a,3}(t) = \begin{cases} H_a + I_a(t,x_a), & \text{if } a \in \mathcal{A}^{\mathrm{n}}, \\ H_a + I_a(t,z_a), & \text{if } a \in \mathcal{A}^{\mathrm{c}}, \end{cases}$$

where the effective time-dependent potential $I_{a,t}(x)$ was defined in (4.6), and

$$\tilde{I}_{a,t}(x) := F\left(\frac{|y|_a \log t}{t} \geqslant 1\right) I_a^l(x).$$

Note that as $I_a(t,x)$, the effective potential $\tilde{I}_a(t,x)$ satisfies the estimates (4.7).

We will denote by $\zeta_i^{a_{\max}+}$, $i = 0, \cdots, 3$ the observables of asymptotic velocity along Z for the evolutions $U_{a,i}(t)$. Before proceeding to the proof of asymptotic completeness, we have to analyze in detail the states in $E_{\{0\}}(\zeta^{a_{\max}+}) \cap \mathcal{H}_{\mathrm{scatt}}$. For such states, we will use the results of Sections 9 and 8 to replace the evolution e^{-itH} by $U_{a,0}(t)$.

Proposition 10.1 — *Assume that hypotheses (Q), (V′) and (H) hold. Let $u \in E_{\{0\}}(\zeta^{a_{\max}+}) \cap \mathcal{H}_{\mathrm{scatt}}$. Then*

$$u = \sum_{\sharp a = 2, a \in \mathcal{A}^{\mathrm{n}}} u_a,$$

where

$$\lim_{t \to \infty} U_{a,0}(t)^* e^{-itH} u_a \text{ exists.}$$

Proof. — Let $u \in E_{\{0\}}(\zeta^{a_{\max}+}) \cap \mathcal{H}_{\mathrm{scatt}}$. Note that for $\chi \in C_0^\infty(Y'_{a_{\max}})$, we have

$$\chi(K_{a_{\max}})(\mathcal{H}_{\mathrm{scatt}}) \subset \mathcal{H}_{\mathrm{scatt}}$$

and that $\chi(K_{a_{\max}})$ tends strongly to 1 when χ tends to 1. Therefore, by a density argument, we may assume that

$$u = \chi(K_{a_{\max}})u, \ \chi \in C_0^\infty(Y'_{a_{\max}}),$$
$$u = f(H^{a_{\max}})u, \ \mathrm{supp} f \cap \tau \cup \sigma_{\mathrm{pp}}(H^{a_{\max}}) = \varnothing.$$

It follows then from Proposition 8.6 that

$$u_t = F\Big(\frac{|x^{a_{\max}}|}{t} \geqslant \epsilon_0\Big) u_t + o(1).$$

On the other hand, since $u \in E_{\{0\}}(\zeta^{a_{\max}+})$, we have for any $\epsilon > 0$:

$$u_t = F\Big(\frac{|z^{a_{\max}}|}{t} \leqslant \epsilon\Big) u_t + o(1).$$

Combining these two estimates, we get:

$$u_t = F\Big(\frac{|y^{a_{\max}}|}{t} \geqslant \epsilon_0\Big) u_t + o(1).$$

Using also Lemma 9.6, we can assume that

$$u_t = F\Big(\epsilon_0 \leqslant \frac{|y^{a_{\max}}|}{t} \leqslant C_1\Big) u_t + o(1). \tag{10.1}$$

Let now

$$1 = \sum_{a \in \mathcal{A}} q_a(y^{a_{\max}})$$

be a standard $N-$body partition of unity on $Y^{a_{\max}}$ with

$$\begin{aligned} &\mathrm{supp} q_a \subset \{y^{a_{\max}} | |y^a| \leqslant \epsilon_1, \ |y^b| \geqslant \epsilon_2, \ \forall b \not\leqslant a\}, \\ &q_a \equiv 1 \text{ on } \{y^{a_{\max}} | |y^a| \leqslant \frac{1}{2}\epsilon_1, \ |y^b| \geqslant 2\epsilon_2, \ \forall b \not\leqslant a\} \end{aligned}$$

for some ϵ_1, ϵ_2 with $\epsilon_1 < \epsilon_0$. Using (10.1), the fact that $N = 3$, and Corollary 9.9 for $\alpha = 1$, we obtain:

$$u_t = \sum_{\sharp a = 2} q_a\Big(\frac{y^{a_{\max}}}{t}\Big) u_t + o(1).$$

We claim now that the limits:

$$\begin{aligned} &\lim_{t \to \infty} e^{itH} q_a\Big(\frac{y^{a_{\max}}}{t}\Big) e^{-itH} u = u_a, \\ &\lim_{t \to \infty} U_{a,0}(t)^* q_a\Big(\frac{y^{a_{\max}}}{t}\Big) e^{-itH} u_a, \end{aligned} \tag{10.2}$$

exist. Let us indicate how to prove the existence of the first limit in (10.2), the proof for the second one being similar. We compute the Heisenberg derivative of $M_a(t) = \chi(H) q_a\Big(\frac{y^{a_{\max}}}{t}\Big)\chi(H)$ and obtain

$$\mathbf{D}_t M_a(t) = \frac{1}{t}\chi(H)\left(-\frac{y^{a_{\max}}}{t} + \Big\langle D - Ax, \nabla q_a\Big(\frac{y^{a_{\max}}}{t}\Big)\Big\rangle\right)\chi(H).$$

Since $N = 3$, the function ∇q_a is supported in the free region $\mathcal{Y}_{a_{\min}}$ defined in (9.14) and in the region $\left\{|y^{a_{\max}}| \leqslant \frac{1}{2}\epsilon_0\right\}$. We can then use the propagation estimates in Corollary 9.9 and Proposition 8.6 for the two evolutions e^{-itH} and $U_{a,0}(t)$ and Lemma A.1 in the Appendix to obtain the existence of the above limit. It remains to check that $u_a = 0$ for $a \in \mathcal{A}^{\mathrm{c}}$, or equivalently that

$$\text{s-}\lim_{t\to\infty} q_a\Big(\frac{y^{a_{\max}}}{t}\Big)U_{a,0}(t) = 0,\ a \in \mathcal{A}^{\mathrm{c}}. \tag{10.3}$$

Since a does not contain neutral clusters, it follows from [GL2], Cor. 7.7 and e.g., Lemma 9.4 ii) that for $\alpha > \frac{1}{2}$:

$$\text{s-}\lim_{t\to\infty} F\Big(\frac{|y^{a_{\max}}|}{t^\alpha} \geqslant 1\Big)U_{a,0}(t) = 0,$$

which implies (10.3). This completes the proof of the proposition. □

The following proposition follows from the arguments in [GL1], Thm. 6.6.

Proposition 10.2. — *Assume that the hypotheses (V) hold. Let $a \neq a_{\max}$. Then the limits*

$$\text{s-}\lim_{t\to\infty} U_{a,1}(t)^* e^{-itH} E_{T_a}(\zeta^{a_{\max}+}),$$
$$\text{s-}\lim_{t\to\infty} e^{itH} U_{a,1}(t) E_{T_a}(\zeta_1^{a_{\max}+}),$$

exist.

Combining Propositions 10.1 and 10.2, we see that we can replace the evolution $e^{-itH}u$ of any state $u \in \mathcal{H}_{\mathrm{scatt}}$ by a superposition of the effective evolutions $U_{a,0}(t)$, $U_{a,1}(t)$. It remains to replace these evolutions by the simpler evolutions $U_{a,2}(t)$, $U_{a,3}(t)$. This is done in a slightly more abstract setting in the next proposition.

Proposition 10.3. — *Let $W(t,x)$ be a time-dependent potential such that*

$$|\nabla_x W(t,x)| \leqslant C\langle t\rangle^{-1-\mu},\ \mu > 0.$$

For $a \neq a_{\max}$, let $U_a(t)$ be the unitary evolution generated by $H_a + W(t,x)$. Let $U_{a,\mathrm{eff}}(t)$ be the unitary evolution generated by

$$H_a + W(t, z_a) \text{ if } a \in \mathcal{A}^{\mathrm{c}},$$
$$H_a + W(t, x_a) \text{ if } a \in \mathcal{A}^{\mathrm{n}}.$$

Let us denote by $\zeta^{a_{\max}+}, \zeta_{\mathrm{eff}}^{a_{\max}+}$ the observables of the asymptotic velocity along $Z^{a_{\max}}$ for the evolutions $U_a(t)$ and $U_{a,\mathrm{eff}}(t)$. Assume that $\mu > \sqrt{3}-1$. Then the limits

$$
\begin{aligned}
&\mathrm{s-}\lim_{t\to\infty} U_{a,\mathrm{eff}}(t)^* U_a(t) E_{T_a}(\zeta^{a_{\max}+}),\\
&\mathrm{s-}\lim_{t\to\infty} U_a(t)^* U_{a,\mathrm{eff}}(t) E_{T_a}(\zeta_{\mathrm{eff}}^{a_{\max}+}),
\end{aligned}
$$

exist.

Proof. — If $a \in \mathcal{A}^{\mathrm{c}}$, then the proposition follows from [GL2], Prop. 8.1, Equ. 8.14. Let us therefore assume that $a \in \mathcal{A}^{\mathrm{n}}$. Since $\mu > \sqrt{3}-1$, we can first apply the arguments of Dereziński [De1] in the z variable as in [GL2] to obtain the existence of the limits

$$
\begin{aligned}
&\mathrm{s-}\lim_{t\to\infty} U_{a'}(t)^* U_a(t) E_{T_a}(\zeta^{a_{\max}+}),\\
&\mathrm{s-}\lim_{t\to\infty} U_a(t)^* U_{a'}(t) E_{T_a}(\zeta_{\prime}^{a_{\max}+}),
\end{aligned}
$$

where $U_{a'}(t)$ is the evolution generated by $H_a + W(t, y^a, x_a)$. We then apply Corollary 9.9. to the Hamiltonian H^a acting on $L^2(X^a \times Y_a^{\mathrm{n}})$ with a time-dependent potential $W(t, y^a)$ equal to $\tilde{I}_{a,t}(y^a, x_a)$ or $I_{a,t}(y^a, x_a)$, where the variable x_a plays the role of a parameter. Since $\sharp a = 2$, we obtain the estimates:

$$
\begin{aligned}
&\mathrm{s-}\lim_{t\to\infty} F\Big(\frac{|y^a|}{t^\alpha} \geqslant 1\Big) U_{a'}(t) = 0,\\
&\int_1^{+\infty} \Big\|F\Big(\frac{|y^a|}{t^\alpha} = 1\Big)\chi(H_a) U_{a'}(t) u\Big\|^2 \frac{dt}{t} < \infty,
\end{aligned}
\tag{10.4}
$$

for $\alpha > \dfrac{1}{2}$. For $\mu > \dfrac{1}{2}$, we can apply Lemma A.2 in the Appendix with the operator $\Phi(t)$ equal to $F\Big(\dfrac{|y^a|}{t^\alpha} \leqslant 1\Big)$ to obtain the existence of the limits

$$
\mathrm{s-}\lim_{t\to\infty} U_{a,\mathrm{eff}}(t)^* U_{a'}(t),\ i = 0, 1.
$$

To check that the hypotheses of Lemma A.2 are satisfied, we use (10.4) and the fact that

$$
F\left(\frac{|y^a|}{t^\alpha} \leqslant 1\right)(W(t, y^a, x_a) - W(t, x_a)) = O(t^{-1-\mu+\alpha}) \in L^1(dt).
$$

This completes the proof of the proposition. □

Proof of Theorem 4.5. — Let us first prove iii), *i.e.*, the completeness of the modified wave operators. Property i) will easily follow from iii). We first consider the vectors in $\mathrm{Ran}E_{T_a}(\zeta^{a_{\max}+})$, for $a \neq a_{\max}$. Combining Propositions 10.2 and 10.3, we obtain the existence of the limit

$$\lim_{t\to\infty} U_{a,3}(t)^* e^{-itH} E_{T_a}(\zeta^{a_{\max}+}). \tag{10.5}$$

If $a \in \mathcal{A}^{\mathrm{c}}$, *i.e.*, a does not contain neutral clusters, then by [GL2], Prop. 8.2, we deduce from (10.5) the existence of

$$\mathrm{s}-\lim_{t\to\infty} e^{iS_a(t,D_{z_a})+itH^a} e^{-itH} E_{T_a}(\zeta^{a_{\max}+}) = \Xi_a^+. \tag{10.6}$$

Moreover, we have (see [GL2], Equ. 8.5)

$$\Xi_a^+ = \Pi_a \Xi_a^+, \tag{10.7}$$

which proves that

$$\mathrm{Ran}E_{T_a}(\zeta^{a_{\max}+}) \subset \mathrm{Ran}\Omega_a^+, \text{ for } a \in \mathcal{A}^{\mathrm{c}}. \tag{10.8}$$

Let us now consider the case of $a \in \mathcal{A}^{\mathrm{n}}$ (*i.e.*, a contains a neutral cluster). Since $a \in \mathcal{A}^{\mathrm{n}}$ and the long-range part of the interaction is of Coulomb type, we have $I_a(t, x_a) = 0$, *i.e.*, $U_{a,3}(t) = e^{-itH_a}$, and $S_a(t, D_{z_a^2}) = \frac{1}{2}tD_{z_a}^2$, hence the limit (10.6) exists. Let us denote by ζ^{a+} the observable of asymptotic velocity along Z^a for the Hamiltonian H^a acting on $L^2(X^a \times Y_a^{\mathrm{n}})$. We have

$$E_{\{0\}}(\zeta^{a+})\Xi_a^+ = \Xi_a^+.$$

On the other hand, applying [GL1], Thm. 6.7 to H^a, we obtain:

$$E_{\{0\}}(\zeta^{a+}) = \Pi_a, \tag{10.9}$$

so that

$$\mathrm{Ran}E_{T_a}(\zeta^{a_{\max}+}) \subset \mathrm{Ran}\Omega_a^+, \text{ for } a \in \mathcal{A}^{\mathrm{n}}. \tag{10.10}$$

Let u be a vector in $E_{\{0\}}(\zeta^{a_{\max}+}) \cap \mathcal{H}_{\mathrm{scatt}}$. Combining Propositions 10.1 and 10.3, we can write

$$u = \sum_{\sharp a = 2, a \in \mathcal{A}^{\mathrm{n}}} u_a$$

where the limit

$$\lim_{t\to\infty} U_{a,2}(t)^* e^{-itH} u_a = u_{a,2}$$

exists. Note that since $a \in \mathcal{A}^{\mathrm{n}}$, we have $U_{a,2}(t) = e^{-itH_a}$. Moreover, since $u_a \in E_{\{0\}}(\zeta^{a_{\max}+})$, we have

$$\lim_{t\to\infty} F\left(\frac{|z_a|}{t} \geqslant \epsilon\right) e^{-itH_a} u_{a,2} = 0,\ \forall \epsilon > 0.$$

By standard arguments in the z variable, this implies that $u_{a,2} = 0$ and hence $u_a = 0$. This proves that $E_{\{0\}}(\zeta^{a_{\max}+}) = \Pi_{a_{\max}}$, which completes the proof of i) and iii).

Let us now prove ii) *i.e.*, the existence of the wave operators. If $a \in \mathcal{A}^{\mathrm{c}}$, the existence of the limit

$$\mathrm{s-}\lim_{t\to\infty} e^{itH} e^{-iS_a(t,D_{z_a})-itH^a}\Pi_a$$

was proved in [GL2], Thm. 4.7. Let us consider now the case $a \in \mathcal{A}^{\mathrm{n}}$. Applying standard arguments in the z variable, (see e.g., [GL2], Prop. 8.2), we see that there exists a dense subset $\mathcal{D}$ in $L^2(X)$ such that for $u \in \mathcal{D}$ there exists a cutoff function $q_a \in C_0^\infty(Z_a)$ supported in

$$\{z_a \in Z_a \| z_a^b| \geqslant \epsilon_0,\ \forall b \not\leqslant a\},$$

such that

$$e^{-itH_a}\Pi_a u = q_a(\frac{z_a}{t})e^{-itH_a}u + o(1).$$

Moreover, by (10.9), we have

$$\lim_{t\to\infty} F\left(\frac{|z^a|}{t} \geqslant \epsilon\right) e^{-itH_a}\Pi_a u = 0,\ \forall \epsilon > 0.$$

Hence, applying Proposition 10.3 to the time-dependent potential $W(t,x) = I_a(t,x)$, we obtain the existence of the limit

$$\lim_{t\to\infty} U_{a,1}(t)^* e^{-itH_a}\Pi_a u.$$

Using then Proposition 10.2, we show the existence of

$$\lim_{t\to\infty} e^{itH} e^{-itH_a}\Pi_a u = \Omega^+ u,$$

which completes the proof of the theorem. $\square$

A. Appendix

We collect here some technical results needed in the main text.

We start by recalling a few abstract arguments used in scattering theory. The first one is a version of the Putnam-Kato theorem which was developed by Sigal-Soffer [SS1].

Let $H(t)$ denote a time-dependent self-adjoint operator on a Hilbert space $\mathcal{H}$ and let $U(t)$ be the evolution generated by $H(t)$. We will denote by $\mathbf{D}\Phi(t)$ the Heisenberg derivative associated with $U(t)$:

$$\mathbf{D}\Phi(t) \ := \ \frac{d}{dt}\Phi(t) + i[H(t), \Phi(t)].$$

LEMMA A.1. — *Suppose that* $\Phi(t) \in C^1(\mathbb{R}^+, B(\mathcal{H}))$ *is a uniformly bounded function whose values are self-adjoint operators. Assume that there exist* $C_0 > 0$ *and operator valued functions* $B(t)$ *and* $B_i(t)$, $i = 1, \dots, n$, *such that*

$$\mathbf{D}\Phi(t) \geqslant C_0 B^*(t)B(t) - \sum_{i=1}^{n} B_i^*(t)B_i(t). \tag{A.11}$$

Suppose that for $i = 1, \dots, n$,

$$\int_1^\infty \|B_i(t)U(t)\phi\|^2 dt \leqslant C\|\phi\|^2.$$

Then there exists C_1 *such that*

$$\int_1^\infty \|B(t)U(t)\phi\|^2 dt \leqslant C_1\|\phi\|^2. \tag{A.12}$$

The next argument is a version of Cook's method due to Kato. Let $H_1(t)$ and $H_2(t)$ be two time-dependent self-adjoint operators with a fixed domain $\mathcal{D}(H)$. Let $U_i(t)$ be the unitary evolutions generated by $H_i(t)$. We put

$$\mathbf{D}_{1,2}\Phi(t) := \frac{d}{dt}\Phi(t) + iH_2(t)\Phi(t) - i\Phi(t)H_1(t).$$

LEMMA A.2. — *Suppose that* $\Phi(t) \in C^1(\mathbb{R}^+, B(\mathcal{H}))$ *is a uniformly bounded operator-valued function. Assume that*

$$|(\psi_2|\mathbf{D}_{1,2}\Phi(t)\psi_1)| \leqslant \sum_{i=1}^{n} \|B_{2i}(t)\psi_2\|\|B_{1i}(t)\psi_1\|,$$

$$\int_1^\infty \|B_{2i}(t)U_2(t)\phi\|^2 dt \leqslant c\|\phi\|^2,$$

and for ϕ in a dense set $\mathcal{D}_1$,

$$\int_1^\infty \|B_{1i}(t)U_1(t)\phi\|^2 dt \leqslant c\|\phi\|^2.$$

Then the limit

$$\mathrm{s-}\lim_{t\to\infty} U_2^*(t)\Phi(t)U_1(t)$$

exists.

We can now prove Proposition 8.1.

Proof of Proposition 8.1. — Let $g \in C_0^\infty(]-\infty, c_1])$ with $c_1 < c_0$. Let

$$F = \int_s^{+\infty} g^2(s_1)ds_1.$$

We consider the propagation observable:

$$\Phi(t) = f(H)F\Big(\frac{A}{t}\Big)f(H),$$

and compute

$$\mathbf{D}\Phi(t) = \partial_t\Phi(t) + [H, i\Phi(t)].$$

We have

$$\mathbf{D}\Phi(t) = f(H)\left(\Big[H, iF\Big(\frac{A}{t}\Big)\Big] - \frac{A}{t^2}g^2\Big(\frac{A}{t}\Big)\right)f(H). \tag{A.13}$$

Let us compute the term $\Big[H, iF\Big(\frac{A}{t}\Big)\Big]$. We have

$$\begin{aligned}
&\Big[(H+i)^{-1}, iF\Big(\frac{A}{t}\Big)\Big]\\
&= (2\pi)^{-1}\int \hat{F}(\sigma)[(H+i)^{-1}, ie^{i\sigma\frac{a}{t}}]d\sigma\\
&= (2\pi)^{-1}\frac{1}{t}\int \hat{F}(\sigma)\sigma e^{i\sigma\frac{a}{t}}\int_0^1 e^{-i\sigma\theta\frac{a}{t}}[(H+i)^{-1}, iA]e^{i\sigma\theta\frac{a}{t}}d\theta d\sigma\\
&= (2\pi)^{-1}t^{-1}\iiint_0^1 \hat{g}(\sigma-\sigma')\hat{g}(\sigma')e^{i\sigma(1-\theta)\frac{a}{t}}[(H+i)^{-1}, iA]e^{i(\sigma)\theta\frac{a}{t}}d\theta d\sigma d\sigma'\\
&= (2\pi)^{-1}t^{-1}\iiint_0^1 \hat{g}(\sigma)\hat{g}(\sigma')e^{i(\sigma+\sigma')(1-\theta)\frac{a}{t}}[(H+i)^{-1}, iA]e^{i(\sigma+\sigma')\theta\frac{a}{t}}d\theta d\sigma d\sigma'.
\end{aligned}$$

Let

$$B := [(H+i)^{-1}, A], \ \ B(s) := e^{-isA}Be^{isA}.$$

Note that since $ad_A^2(H+i)^{-1}$ is bounded, we have

$$\|B(s) - B(s')\| \leqslant C|s - s'|. \tag{A.14}$$

We have

$$\begin{aligned} e^{i(\sigma+\sigma')(1-\theta)\frac{a}{t}}Be^{i(\sigma+\sigma')\theta\frac{a}{t}} &= e^{i\sigma\frac{a}{t}}B\Big(\frac{\sigma\theta+(1-\theta)\sigma'}{t}\Big)e^{i\sigma'\frac{a}{t}} \\ &= e^{i\sigma\frac{a}{t}}Be^{i\sigma'\frac{a}{t}} + O(t^{-1}). \end{aligned} \tag{A.15}$$

Using (A.15), and the fact that $\hat{g} \in L^1(\mathbb{R})$, we obtain:

$$\begin{aligned} \Big[(H+i)^{-1},\ iF\Big(\frac{A}{t}\Big)\Big] &= (H+i)^{-1}\left[H, iF\Big(\frac{A}{t}\Big)\right](H+i)^{-1} \\ &= \frac{1}{t}g\Big(\frac{A}{t}\Big)(H+i)^{-1}[H,iA](H+i)^{-1}g\Big(\frac{A}{t}\Big) + O(t^{-2}). \end{aligned} \tag{A.16}$$

Note that a similar argument shows that for $g, h \in C_0^\infty(\mathbb{R})$:

$$\left[g\Big(\frac{A}{t}\Big), h(H)\right] = O(t^{-1}). \tag{A.17}$$

Multiplying this identity from both sides by $f(H)(H+i)$, we obtain:

$$\begin{aligned} &f(H)\left[H, iF\Big(\frac{A}{t}\Big)\right]f(H) \\ &= \frac{1}{t}f(H)(H+i)g\Big(\frac{A}{t}\Big)(H+i)^{-1}[H,iA](H+i)^{-1}g\Big(\frac{A}{t}\Big)(H+i)f(H) \\ &= \frac{1}{t}f(H)g\Big(\frac{A}{t}\Big)[H,iA]g\Big(\frac{A}{t}\Big)f(H) + O(t^{-2}). \end{aligned}$$

Let $f_1 \in C_0^\infty(\mathbb{R})$ with $f_1 f = f$. Using then (8.1) and (A.17), we get:

$$\begin{aligned} f(H)\left[H, iF\Big(\frac{A}{t}\Big)\right]f(H) &= \frac{1}{t}f(H)g\Big(\frac{A}{t}\Big)[H,iA]g\Big(\frac{A}{t}\Big)f(H) + O(t^{-2}) \\ &= \frac{1}{t}f(H)g\Big(\frac{A}{t}\Big)f_1(H)[H,iA]f_1(H)g\Big(\frac{A}{t}\Big)f(H) + O(t^{-2}) \\ &\geqslant c_0\frac{1}{t}f(H)g^2\Big(\frac{A}{t}\Big)f(H) + O(t^{-2}). \end{aligned}$$

Since $\mathrm{supp} g \subset]-\infty, c_1]$, for $c_1 < c_0$, we have finally

$$\mathrm{D}\Phi(t) \geqslant (c_0 - c_1)\frac{1}{t}f(H)g^2\left(\frac{A}{t}\right)f(H) + O(t^{-2}),$$

from which (8.2) follows by the standard method of propagation estimates. Computing then the Heisenberg derivative of $f(H)g\left(\frac{A}{t}\right)f(H)$ and using (8.2), we obtain the existence of the limit in (8.3). By (8.2), this limit has to be zero. This completes the proof of the proposition. □

Let us now prove Lemma 8.3.

Proof of Lemma 8.3. — Note that the case of first order commutators has already been settled in Proposition 7.6. Since here we will need their exact expression, let us recall it. We will replace M by one of the M_a. We have

$$\begin{aligned}
&\mathrm{ad}_A(H+i)^{-1} = (H+i)^{-1}(D^2_{z^{a_{\max}}} - z^{a_{\max}}\nabla_{z^{a_{\max}}}V^{a_{\max}})(H+i)^{-1},\\
&\mathrm{ad}_{M_a}(H+i)^{-1} = (H+i)^{-1}[H, M_a](H+i)^{-1},
\end{aligned}$$

and using the computations in the proof of Proposition 7.6

$$\begin{aligned}
&[H, M_a] = [H, q_aF_a]B_aq_aF_a + \mathrm{hc}\\
&q_aF_an_a^{\mathrm{w}}(D_{z_a^{a_{\max}}}, D_{y_a^{\mathrm{n}}} + (Ax)_a^{\mathrm{n}})q_aF_a + q_aF_a[I_a, B_a]q_aF_a\\
&=: I_1 + I_2 + I_3
\end{aligned}$$

for $n_a \in C^{0,1}(Z_a^{a_{\max}} \times Y_a^{\mathrm{n}})$. Moreover (see the proof of Proposition 7.6):

$$q_aF_a[I_a, B_a]q_aF_a = q_aF_a[\tilde{I}_a^s, B_a]q_aF_a + q_aF_a[I_a^l, B_a]q_aF_a.$$

To estimate the second order commutators, we have to estimate the four terms $\mathrm{ad}^2_A(H+i)^{-1}$, $\mathrm{ad}^2_{M_a,A}(H+i)^{-1}$, $\mathrm{ad}^2_{A,M_a}(H+i)^{-1}$, $\mathrm{ad}^2_{M_a,M_b}(H+i)^{-1}$.

1) To see that the term $\mathrm{ad}^2_A(H+i)^{-1}$ is bounded, we use *(V′)* : we write $V^a = V^{a,s} + V^{a,l}$ and 'undo' the various commutators involving $V^{a,s}$.

2) The only term in $\mathrm{ad}^2_{A,M_a}(H+i)^{-1}$ which deserves some attention is $(H+i)^{-1}\mathrm{ad}^2_{A,M_a}H(H+i)^{-1}$. The terms containing $[A, I_1]$ and $[A, I_2]$ are easily seen to be bounded. The term containing $[A, I_3]$ is bounded by undoing the commutators containing $\tilde{I}_a^s$.

3) Similarly, to estimate $\mathrm{ad}^2_{M_a,A}(H+i)^{-1}$, it suffices to examine the term $(H+i)^{-1}\mathrm{ad}^2_{M_a,A}H(H+i)^{-1}$. It is easy to see that $(H+i)^{-1}[D^2_{z_a^{a_{\max}}}, M_a](H+i)^{-1}$ is bounded. It remains to consider

$$\begin{aligned}
q_aF_a[z^{a_{\max}}\nabla V^{a_{\max}}, B_a]q_aF_a &= q_aF_a[z^{a_{\max}}\nabla I_a, B_a]q_aF_a\\
&= q_aF_a[z^{a_{\max}}\nabla \tilde{I}_a, B_a]q_aF_a.
\end{aligned}$$

By the usual argument, this term is bounded.

4) Finally, let us consider the terms $\mathrm{ad}^2_{M_a,M_b}(H+i)^{-1}$. Again, it suffices to examine

$$(H+i)^{-1}\mathrm{ad}^2_{M_a,M_b}H(H+i)^{-1}.$$

Note that if $a \neq b$, since $q_a q_b = 0$ (see (7.3)), we have $\mathrm{ad}^2_{M_a,M_b}H = 0$, so that we may assume that $a = b$. The term $\mathrm{ad}_{M_a} I_1$ is seen to be bounded using the argument in the proof of Lemma 7.5. The term $\mathrm{ad}_{M_a} I_2$ is also bounded : in fact $[B_a, q_a F_a]$ and $[B_a, n_a^{\mathrm{w}}(D_{z_a^{a_{\max}}}, D_{y_a^{\mathrm{n}}} + (Ax)_a^{\mathrm{n}})]$ are bounded using Lemma A.3 below. The term $\mathrm{ad}_{B_a} I_3$ is also bounded by undoing all the commutators containing $\tilde{I}_a^s$. □

The following lemma is an extension of Leibniz rule to pseudodifferential calculus due to Calderon (see [M], Thm. 6 p. 306). Recall that $S^\epsilon(X)$ denotes the space of functions f such that

$$|\partial_x^\alpha f(x)| \leqslant C_\alpha \langle x\rangle^{\epsilon-|\alpha|}.$$

Lemma A.3. — *Let $f(x) \in S^\epsilon(X)$ and $g \in C^{m-1,1}(X')$ such that*

$$|\partial_\xi^\alpha g(\xi)| \leqslant C_\alpha, \quad |\alpha| \leqslant m.$$

Then we have

$$f(x)g(D) = \sum_{|\alpha|\leqslant m-1} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha g(D) \partial_x^\alpha f(x) + R_m,$$

where $R_m\langle x\rangle^{m-\epsilon}$ is bounded on $L^2(X)$.

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