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COMPLETE MINIMAL SURFACES OF ARBITRARY GENUS IN A SLAB OF $\mathbf{R}^3$

by C.J. COSTA(*) and P.A.Q. SIMÕES

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1. Introduction.

In [2] L.P. Jorge and F. Xavier have proved the existence of complete minimal surfaces between two parallel planes in $\mathbf{R}^3$ with the conformal structure of a disk. Afterwards H. Rosenberg and E. Toubiana [5], and F. Lopez [3], were able to extend the results of Jorge and Xavier by showing respectively the existence of the same type of surfaces with the conformal structure of an annulus and a projective plane minus a disk. Nevertheless the technique used to prove these results (Runge's approximation theorem) is not apparently sufficient to show the existence of such surfaces with higher genus.

Using lacunary power series F.F. Brito [1] has constructed explicit examples of minimal surfaces with the same properties of [2]. An important feature of Brito's technique is that it can be used to construct examples of higher genus. Indeed in this paper we construct for every $k = 1, 2, \ldots$ and $1 \leq N \leq 4$, examples of complete minimal surfaces of genus k and N ends in a slab of $\mathbf{R}^3$. More precisely we will prove:

THEOREM 1.1. — *For every $k = 1, 2, \ldots$ and $1 \leq N \leq 4$, there is a complete minimal immersion $X_{k,N} : M_{k,N} \to \mathbf{R}^3$, with infinite total curvature such that:*

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a) *$M_{k,1}$ and $M_{k,2}$ are respectively a compact Riemann surface of gen k minus one disk and two disks,*

b) *$M_{k,j+2}$, $j = 1,2$, are respectively $M_{k,j}$ punctured at two points,*

c) *$X_{k,N}$ $(M_{k,N})$ lies between two parallel planes of $\mathbf{R}^3$ and $X_{k,3}, X_k$ have two embedded planar ends.*

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2. Preliminaries.

A powerful method to obtain examples of minimal surfaces in I is the so-called Weierstrass-Enneper representation. We will conjugate tl method with a lemma of F.F. Brito [1] to prove Theorem 1.1. We summari this procedure in Theorem 2.1 and Lemma 2.1 below.

THEOREM 2.1. — *Let M be a non-compact Riemann surface. Su pose that g and η are respectively a meromorphic function and a holomc phic differential on M such that*

(C.1) *$p \in M$ is a pole of order m of g if and only if p is a zero of order 2 of η,*

(C.2) *for every closed curve γ in M,*

$$\operatorname{Re}\int_\gamma g\eta = 0, \quad \int_\gamma g^2\eta = \overline{\int_\gamma \eta} \quad \text{and}$$

(C.3) *for every divergent curve λ in M,*

$$\int_\lambda (1+ \mid g \mid^2) \mid \eta \mid = +\infty.$$

Then, $X : M \to \mathbf{R}^3$,

$$X(z) = \operatorname{Re}\int_{z_0}^{z} \left((1-g^2)\eta, i(1+g^2)\eta, 2g\eta\right)$$

is a complete minimal immersion.

Given a power series in C with radius of convergence 1,

$$h(z) = \sum_{j=1}^{\infty} a_j z^{n_j},$$

we say that it has Hadamard gaps if there is $\alpha > 1$ such that $\frac{n_{j+1}}{n_j} \geq \alpha > 1$, for all $j = 1, 2, \ldots$

LEMMA 2.1 (F. Brito [1]). — *Suppose that* $h(z) = \sum_{j=1}^{\infty} a_j z^{n_j}$ *has Hadamard gaps and satisfies*

(a) $\sum_{j=1}^{\infty} | a_j | < \infty$,

(b) $\lim_{j\to\infty} | a_j | \min \left\{ \frac{n_j}{n_{j-1}}, \frac{n_{j+1}}{n_j} \right\} = \infty$ *and*

(c) $\sum_{j=1}^{\infty} | a_j |^2 n_j = \infty$.

Then for every divergent curve γ *in* $D = \{z \in C; |z| < 1\}$

$$\int_\gamma |h'(z)|^2 |dz| = \infty.$$

3. Proof of Theorem 1.1.

The proof of Theorem 1.1 is founded on Proposition 3.1 and Lemma 3.1 which we will prove below. Nevertheless, first of all we will define the conformal structure of the Riemann surfaces $M_{k,q}$ of genus k and q ends, $k = 1, 2, \ldots, 1 \leq q \leq 4$ and homological basis for these surfaces.

For each integer $s \geq 3$ let $\bar{M}_s$ be the compact Riemann surfaces $\bar{M}_s = \{(z, w) \in (C \cup \{\infty\}); w^2 = z^s - 1\}$.

Observe that for every s, $z : \bar{M}_s \to C \cup \{\infty\}$ is a meromorphic function of degree two and $\bar{M}_s$ is a compact Riemann surface of genus k if $s = 2k + 1$ or $s = 2k + 2, k = 1, 2, \ldots$. For a fixed real number $\delta \geq 3$ let $V_s(\delta) = \{(z, w) \in \bar{M}_s; |z| \geq \delta\}$ and $p_1 = (0, i), p_2 = (0, -i) \in \bar{M}_s$. Then $V_s(\delta)$ is a closed disk if $s = 2k + 1$ or a disconnected union of two closed disks if $s = 2k + 2$. Now we define

(1) $M_{k,j} = \bar{M}_s \setminus V_s(\delta)$, $j = 1$ if $s = 2k + 1$ and $j = 2$ if $s = 2k + 2$

and

(2) $$M_{k,j+2} = M_{k,j} \setminus \{p_1, p_2\}, \quad j = 1, 2.$$

Now, we will construct a basis for the homology of $M_{k,q}$. We recall (see (1) and (2)) that with $k \geq 1$ fixed either $s = 2k+1$, if $q = 1,3$ or $s = 2k+2$ if $q = 2,4$. If $\varepsilon = e^{i\theta}$, where $\theta = \frac{2\pi}{s}$, let L_j be the arc $\left\{\varepsilon^j e^{it},\, 0 \leq t \leq \theta\right\}$, $j = 0, 1, \ldots, s-1$. Let $\tilde{\gamma_1}(t)$, $0 \leq t \leq 2\pi$, be a C^1 Jordan curve in $B_s = \left\{\rho e^{iu};\, \frac{1}{2} < \rho < 2,\, -\frac{\theta}{2} < u < \frac{3}{2}\theta\right\}$ that contains L_o in its interior and such that $\tilde{\gamma_1}(0) = \frac{3}{2}e^{i\frac{\theta}{2}}$. We observe that 1 and ε lie in the interior of $\tilde{\gamma_1}(t)$ and ε^j is contained in the exterior of $\tilde{\gamma_1}(t)$ for $2 \leq j \leq s-1$. So, in a neighbourhood of $\tilde{\gamma_1}(t)$ there exists a well-defined branch of $(z^s - 1)^{\frac{1}{2}}$. Let $\gamma_1(t) = (\tilde{\gamma_1}(t), w(t))$ a closed lift of $\tilde{\gamma_1}(t)$ to $M_{k,q} \subset \bar{M}_s$. Let Δ_n be the conformal diffeomorphism of $\bar{M}_s$, defined by $\Delta_n(z, w) = (\varepsilon^n z, w)$, $n = 0, 1, \ldots, s-1$. Observe that the restriction of Δ_n to $M_{k,q}$ is still a conformal diffeomorphism of $M_{k,q}$. Then

(3) $$\bar{\Lambda}_q = \{\gamma_n = \Delta_n \circ \gamma_1;\, n = 0, 1, \ldots, s-1\}$$

are closed curves of $M_{k,q}$ and $\bar{\Lambda}_q$ contains a basis for the homology of $\bar{M}_s$. Now we complete $\bar{\Lambda}_q$ to have a basis for the homology of $M_{k,q}$. If $q = 1,3$, where $s = 2k+1$, let $\alpha(t) = (\tilde{\alpha}(2t), w(t))$ to be a lift of $\tilde{\alpha}(2t)$ to $M_{k,q}$ where $\tilde{\alpha}(t) = \frac{1+\delta}{2}e^{it}$, $0 \leq t \leq 2\pi$ and denote by $\beta_j(t) = \left(\tilde{\beta}(t), w(t)\right)$, $j = 1, 2$ distinct lifts of $\tilde{\beta}(t) = \frac{1}{2}e^{it}, 0 \leq t \leq 2\pi$, to $M_{k,3}$. Then

(4) $$\Lambda_1 = \bar{\Lambda}_1 \cup \{\alpha\} \quad \text{and} \quad \Lambda_3 = \bar{\Lambda}_1 \cup \{\alpha, \beta_1, \beta_2\}$$

contain respectively a basis for the homology of $M_{k,1}$ and $M_{k,3}$.

If $q = 2,4$, let $\mu_j(t) = (\tilde{\alpha}(t), w(t))$ be, $j = 1,2$, distinct lifts of $\tilde{\alpha}(t)$ to $M_{k,2}$ and let $\beta_{j+2}(t) = \left(\tilde{\beta}(t), w(t)\right)$ be, $j = 1,2$, two distinct lifts of $\tilde{\beta}(t)$ to $M_{k,4}$. Then

(5) $$\Lambda_2 = \bar{\Lambda}_2 \cup \{\mu_1, \mu_2\} \quad \text{and} \quad \Lambda_4 = \bar{\Lambda}_2 \cup \{\mu_1, \mu_2, \beta_3, \beta_4\}$$

contain respectively a basis for the homology of $M_{k,2}$ and $M_{k,4}$.

In order to construct minimal immersions $X_{k,q} : M_{k,q} \to \mathbf{R}^3$, as required in Theorem 1.1 we will define Weierstrass data $(g_{k,q}, \eta_{k,q})$ on $M_{k,q}$ satisfying the hyphotesis of Theorem 2.1. To make it possible we will require that the group of symmetries of $M_{k,q}$ generated by the conformal diffeomorphisms Δ_n, will be carried via $(g_{k,q}, \eta_{k,q})$ on a group of symmetries of $X_{k,q}(M_{k,q}) \subset \mathbf{R}^3$. With this purpose with $k \geq 1$ fixed let us define in $D = \{z \in C; |z| < 1\}$ the Hadamard's gaps series

(6) $$h_q(z) = \frac{c_q}{s}z^s + \frac{1}{s}\sum_{j=1}^{\infty} a_{jq} z^{sn_j}, \quad q = 1, \ldots, 4$$

where

$$n_j = 2^j! + 1, \quad a_{jq} = 2^{-j} e^{it_{jq}}, \quad j = 1, 2, \dots, 1 \leq q \leq 4 \tag{7}$$

and c_q, t_{jq} are real numbers, to be chosen later.

Finaly we define the Weierstrass data $(g_q, \eta_q) = (g_{k,q}, \eta_{k,q})$ by the expressions

$$g_p = \frac{\lambda_p}{w} h_p' \left(\frac{z}{\delta}\right), \quad g_{p+2} = \lambda_{p+2} w^{-1} \left(\frac{z}{\delta}\right)^2 h_{p+2}' \left(\frac{z}{\delta}\right), \tag{8}$$

$$\eta_p = w dz \quad \text{and} \quad \eta_{p+2} = \left(\frac{\delta}{z}\right)^2 w dz, \quad p = 1, 2 \tag{9}$$

where λ_q are complex numbers to be prescribed later.

Observe that if $(z, w) \in M_{k,q}$ then $\left|\frac{z}{\delta}\right| < 1$. So for each $q = 1, 2, 3, 4$ g_q is a well-defined meromorphic function on $M_{k,q}$ and η_q is a holomorphic differential on $M_{k,q}$. Since, $g_q \eta_q$ is an exact differential we obtain

$$\int_\gamma g_q \eta_q = 0 \tag{10}$$

for every closed curve $\gamma \subset M_{k,q}$.

On the other hand

$$\left[h_q'(z)\right]^2 = \sum_{j=1}^{\infty} d_{jq} z^{m_{jq}}, \quad 1 \leq q \leq 4 \tag{11}$$

where

$$d_{jq} \in C \quad \text{and} \quad m_{jq} \equiv -2 (\operatorname{mod} s). \tag{12}$$

Then, as $\Delta_n^* w = w$ and $\Delta_n^* dz = \varepsilon^n dz$, we find from (8), (9), (11) and (12) that $\Delta_n^* \left(g_q^2 \eta_q\right) = \varepsilon^{-n} g_q^2 \eta_q, \Delta_n^* \eta_q = \varepsilon^n \eta_q$ if $q = 1, 2$ and $\Delta_n^* \left(g_q^2 \eta_q\right) = \varepsilon^n g_q^2 \eta_q, \Delta_n^* \eta_q = \varepsilon^{-n} \eta_q$, if $q = 3, 4$. So

$$\int_{\gamma_n} g_q^2 \eta_q = \int_{\gamma_1} \Delta_n^* \left(g_q^2 \eta_q\right) = \varepsilon^{-n} \int_{\gamma_1} g_q^2 \eta_q, \quad q = 1, 2$$

and

$$\int_{\gamma_n} \eta_q = \int_{\gamma_1} \Delta_n^* (\eta_q) = \varepsilon^n \int_{\gamma_1} \eta_q, \quad q = 1, 2.$$

So we conclude that

$$\int_{\gamma_n} g_q^2 \eta_q = \overline{\int_{\gamma_n} \eta_q} \Longleftrightarrow \int_{\gamma_1} g_q^2 \eta_q = \overline{\int_{\gamma_1} \eta_q}, \quad n = 1, 2, \dots s. \tag{13}$$

Suppose now that (g_q, η_q) define minimal surfaces $X_{k,q} : M_{k,q} \to \mathbf{R}^3$. Then the results below together with Theorem 2.1 allow us to conclude that

$$2(x_1 + ix_2)(p) = \overline{\int_{p_0}^{p} \eta} - \int_{p_0}^{p} g^2\eta = \delta_{q,n}(x_1 + ix_2)\left(\Delta_n(p)\right) + A_n$$

and

$$x_3(p) = \mathrm{Re}\int_{p_0}^{p} g\eta = x_3\left(\Delta_n(p)\right)$$

where $A_n = -\delta_{q,n}(x_1 + ix_2)\left(\Delta_n(p_0)\right)$ is a constant, $\delta_{q,n} = \varepsilon^n$ if $q = 1,2$, $\delta_{q,n} = \varepsilon^{-n}$ if $q = 3,4$ and $X_{k,q} = (x_1 + ix_2, x_3)$.

Then if L_n are the rigid motion of $\mathbf{R}^3$ give by

$$L_n\left(x_1 + ix_2, x_3\right) = \left(\delta_{q,n}(x_1 + ix_2) + A_n, x_3\right)$$

where we identify $\mathbf{R}^3 \equiv C \times R$ we conclude that L_n generates the desired group of symmetries of $\mathbf{R}^3$ for the surfaces $X_{k,q}(M_{k,q})$.

Notice that from (8) and (9), $g_q\eta_q$ is a exact differential on $M_{k,q}$ and

$$x_3(p) = \mathrm{Re}\int g_q\eta_q = \mathrm{Re}\left[\delta\lambda_q h_q\left(\frac{z}{\delta}\right)\right]. \tag{14}$$

As h_q is bounded on $M_{k,q}$ we conclude that third coordinate of the immersion is bounded.

The symmetries just exploited and a proposition and a lemma mentioned at the beginning of this section are exactly what we need to prove Theorem 1.1.

So, let us state and prove the proposition and the lemma.

PROPOSITION 3.1. — *For each integer $s \geq 3$ let $\bar{M}_s$ be the compact Riemann surface $\bar{M}_s = \{(z,w) \in (C \cup \infty)^2; w^2 = z^s - 1\}$, $\theta = \dfrac{2\pi}{s}$ and let $\tilde{\gamma}(t)$, $0 \leq t \leq 2\pi$, be a C^1 Jordan curve in $B_s = \left\{\rho e^{iu}, \frac{1}{2} < \rho < 2, -\theta/2 < u < \frac{3}{2}\theta\right\}$ that contains $L_0 = \left\{e^{iu}; 0 \leq u \leq \theta\right\}$ in its interior. If $\gamma(t) = (\tilde{\gamma}(t), w(t))$ is a lift of $\tilde{\gamma}(t)$ to $\bar{M}_s$ and $r = 2s - 2$ or $r = 2s$, we have that*

$$\int_{\gamma} z^r \frac{dz}{w} \neq 0.$$

Proof. — We observe that $z^r\dfrac{dz}{w}$ and $z^{r+1-s}wdz$ are holomorphic differentials in a neighbourhood of $\gamma(t)$. Also as over $\tilde{\gamma}(t)$ there exists a well-defined branch of $\sqrt{z^s - 1}$ then γ is a closed curve. Furthermore, as

$$d\left(z^{r+1-s}w\right) = (r+1-s)z^{r-s}wdz + \frac{s}{2}z^r\frac{dz}{w}$$

it is enough to prove that

$$I = \int_\gamma z^{r-s}wdz \neq 0. \tag{15}$$

Now we choose an open disk $U \subset C$, such that $\tilde{\gamma}(t)$ lies in U, $\{e^{it}; 0 \leq t \leq \theta\} \subset U$ and $e^{ip\theta} \notin U$, $p = 2, 3, \ldots, s-1$. In U let ω be the branch of $(z^s - 1)^{1/2}$ such that $\gamma(t) = (\tilde{\gamma}(t), \omega(\tilde{\gamma}(t)))$. So, if we collapse $\tilde{\gamma}(t)$ to the arc $\left\{e^{it} \in C; 0 \leq t \leq \theta\right\}$ we find from (15)

$$I = 2i\int_0^\theta e^{i(r-s)t}e^{it}\left(e^{ist} - 1\right)^{\frac{1}{2}}dt.$$

Then, $I \neq 0$, if and only if,

$$\int_0^\theta e^{i\left(r+1-\frac{3s}{4}\right)t}\left(\sin\frac{st}{2}\right)^{\frac{1}{2}}dt \neq 0.$$

Now setting $v = \dfrac{st}{2} - \dfrac{\pi}{2}$ we conclude that the last expression is equivalent to

$$J_0 = \int_0^{\frac{\pi}{2}} \cos\beta v(\cos v)^{\frac{1}{2}}dv \neq 0 \tag{16}$$

where

$$\beta = \frac{2r+2}{s} - \frac{3}{2}. \tag{17}$$

On the other hand if $m \geq 0$ is an integer and

$$J_m = \int_0^{\frac{\pi}{2}} \cos[(\beta - m)v](\cos v)^{m+\frac{1}{2}}dv, \tag{18}$$

then, by using integration by parts we obtain:

$$\begin{aligned} J_m &= \int_0^{\frac{\pi}{2}} \cos[(\beta - m - 1)v](\cos v)^{m+\frac{3}{2}}dv \\ &\quad - \int_0^{\frac{\pi}{2}} \sin[(\beta - m - 1)v]\sin v(\cos v)^{m+\frac{1}{2}}dv \\ &= \left[1 - \frac{2}{2m+3}(\beta - m - 1)\right]J_{m+1}. \end{aligned}$$

This result implies that for every integer $m \geq 0$,

$$J_0 = J_{m+1}\prod_{j=1}^{m+1}\left[1 - \frac{2}{2j+1}(\beta - j)\right]. \tag{19}$$

We observe that the hypothesis of Proposition 3.1 and (17) imply that

$$1 - \frac{2}{2j+1}(\beta - j) \neq 0, \quad \forall j = 0, 1, \ldots \tag{20}$$

Also, the same hypothesis show that $|\beta - 2| < 1$ if $r = 2s - 2$ and $|\beta - 3| < 1$ if $r = 2s$. So, in each case we can find an integer $m \geq 2$ such that $|\beta - m| < 1$. These results together with (18), (19) and (20) show that (16) is true. This finishes the proof of Proposition 3.1.

LEMMA 3.1. — *There exist real numbers c_q, t_{jq}, complex numbers $\lambda_q \neq 0$, where $j = 1, 2, \ldots$, and a real number $L \geq 3$ such that for every closed curve $\gamma = \gamma_q$ in $M_{k,q}$ and for every $\delta \geq L$*

$$\int_\gamma g_q^2 \eta_q = \overline{\int_\gamma \eta_q}\,, \quad q = 1, 2, 3, 4$$

where g_q and η_q are defined in (8) and (9) and $M_{k,q}$ are given by (1) and (2).

Proof. — First of all we will prove that there exist $c_q, t_{jq} \in \mathbf{R}$, $\lambda_q \in C$ and $L \geq 3$ such that

$$\int_{\gamma_n} g_q^2 \eta_q = \overline{\int_{\gamma_n} \eta_q}\,, \quad n = 0, 1, \ldots, s-1 \tag{21}$$

for every $\delta \geq L$, where γ_n are the closed paths that appear in (4) if $q = 1, 3$ or in (5) if $q = 2, 4$. For this, it suffices to verify the second equality in (13). First we specify γ_1 to be the lift of $\tilde{\gamma}_1$ with $\gamma_1(0) = (\tilde{\gamma}_1(0), w(0))$, where $-iw(0)$ is a positive real number. We define respectively on $C \setminus \bigcup_{j=0}^{k-1} L_{2j+1} \cup [1, \infty)$ and on $C \setminus \bigcup_{j=0}^{k} L_{2j}$ branches $\omega_p, p = 1, 2$ of $\left(z^{2k+p} - 1\right)^{\frac{1}{2}}$ wich satisfy $w \circ \gamma_1 = \omega_p \circ \tilde{\gamma}_1$. Then

$$\int_{\gamma_1} \eta_p = \int_{\tilde{\gamma}_1} \omega_p dz \quad \text{and} \quad \int_{\gamma_1} \eta_{p+2} = \delta^2 \int_{\tilde{\gamma}_1} \omega_p \frac{dz}{z^2}, \quad p = 1, 2.$$

In order to evaluate the integrals on the right side, we collapse $\tilde{\gamma}_1$ to the arc $L_0 = \left\{e^{it} \in C; 0 \leq t \leq \theta\right\}$. Then

$$\int_{\gamma_1} \eta_p = \int_0^\theta \left(e^{ist} - 1\right)^{\frac{1}{2}} ie^{it} dt - \int_\theta^0 \left(e^{ist} - 1\right)^{\frac{1}{2}} ie^{it} dt$$

$$= (2i)^{\frac{3}{2}} \int_0^\theta e^{\left(1 + \frac{s}{4}\right)it} \sqrt{\sin\left(\frac{st}{2}\right)}\, dt \neq 0, \quad p = 1, 2 \tag{22}$$

and

$$\int_{\gamma_1} \eta_{p+2} = \delta^2 (2i)^{\frac{3}{2}} \int_0^\theta e^{\left(-1+\frac{s}{4}\right)it} \sqrt{\sin\left(\frac{st}{2}\right)} dt \neq 0, \quad p = 1, 2. \tag{23}$$

On the other hand, we obtain from (6) and (7)

$$\left[h_q'(z)\right]^2 = c_q^2 z^{2s-2} + 2c_q \sum_{j=1}^{\infty} n_j a_{jq} z^{(n_j+1)s-2} + f_q(z), \quad 1 \leq q \leq 4$$

where $f_q(z)$, is a Hadamard's gap series in the disk $D \subset C$. As t_{jq}, $j = 1, 2, \ldots$ are real numbers to be fixed later (see (7)) we can write

$$f_q(z) = \sum_{r=1}^{\infty} b_{rq} z^{n_r}$$

where b_{rq} are complex numbers depending on the variables t_{jq}. So, we obtain from (8) and (9)

$$c_q^{-1} \lambda_q^{-2} \delta^{r_q} \int_{\gamma_1} g_q^2 \eta_q = c_q I_{0q} + 2 \sum_{j=1}^{\infty} e^{it_{jq}} I_{jq} + c_q^{-1} J_q \tag{24}$$

where

$$I_{0q} = \int_{\gamma_1} z^{r_q} \frac{dz}{w}, \quad \delta^{(m_{jq}-r_q)} I_{jq} = 2^{-j} n_j \int_{\gamma_1} z^{m_{jq}} \frac{dz}{w},$$

$r_1 = r_2 = 2s - 2$, $r_3 = r_4 = 2s$, $m_{j1} = m_{j2} = (n_j + 1)s - 2$, $m_{j3} = m_{j4} = (n_j + 1)s$ and

$$\delta^{r_q} J_q = \int_{\gamma_1} f_q\left(\frac{z}{\delta}\right) \frac{dz}{w}, \quad 1 \leq q \leq 4.$$

From Proposition 3.1 we conclude that $I_{0q} \neq 0$, $1 \leq q \leq 4$. Then, we can choose sequences of real numbers t_{jq}, $j = 1, 2, \ldots$ such that

$$\operatorname{Arg}\left(e^{it_{jq}} I_{jq}\right) = \operatorname{Arg} I_{0q}, \quad \text{if} \quad I_{jq} \neq 0 \quad \text{and} \quad t_{jq} = 0 \quad \text{if} \quad I_{jq} = 0 \tag{25}$$

for every $c_q > 0$ and $\delta \geq 3$.

Suppose that t_{jq}, are fixed such that (25) is satisfied. Since $|J_q| = |J_q(\delta)|$ is a bounded real function of the real variable δ we can find $L \geq 3$ and c_q large sufficiently such that in (24)

$$\frac{1}{\lambda_q^2} \int_{\gamma_1} g_q^2 \eta_q \neq 0, \quad q = 1, 2, 3, 4, \quad \text{for every} \quad \delta \geq L. \tag{26}$$

Finally, from (22), (23) and (26) we conclude that for every $\delta \geq L$ there exists $\lambda_q = \lambda_q(\delta) \in C$ such that the second equality of (13) is satisfied. This concludes the proof of (21).

In order to finish the proof of Lemma 3.1 we need to show that

$$(27)\qquad \int_\alpha g_p^2\eta_p = \overline{\int_\alpha \eta_p}\,,\quad \int_{\beta_j} g_3^2\eta_3 = \overline{\int_{\beta_j}\eta_3}\,,\quad p=1,3\,,\quad j=1,2$$

and

$$(28)\qquad \int_{\mu_j} g_p^2\eta_p = \overline{\int_{\mu_j} \eta_p}\,,\quad \int_{\beta_{j+2}} g_4^2\eta_4 = \overline{\int_{\beta_{j+2}}\eta_4},\quad p=2,4\,,\quad j=1,2$$

where α, β_j and μ_j are given in (4) and (5). First we observe that w appears in (8) and (9) with an odd exponent and α covers twice the closed curve $\tilde{\alpha}(t)$, $0 \le t \le 2\pi$, each time with a distinct determination of $\left(z^{2k+1}-1\right)^{1/2}$. This implies that the first integrals in (27) are null.

On the other hand, let ω be the branch of $\left(z^{2k+2}-1\right)^{\frac{1}{2}}$ on $\{z \in C; 1 < |z| < \delta\}$ such that $\omega\left(\frac{1+\delta}{2}\right) = v > 0$. Suppose that $\mu_1\left(\frac{1+\delta}{2}\right) = \left(\tilde{\alpha}\left(\frac{1+\delta}{2}\right), -v\right)$, $\mu_2\left(\frac{1+\delta}{2}\right) = \left(\tilde{\alpha}\left(\frac{1+\delta}{2}\right), v\right)$. Then

$$\int_{\mu_j}\eta_q = -(-1)^j \int_{\tilde{\alpha}} \omega dz = -(-1)^j 2\pi i \operatorname*{Res}_{z=\infty} \omega dz, \qquad q = 2,4.$$

To calculate $\operatorname*{Res}_{z=\infty} \omega dz$ we write $z = u^{-1}$ and

$$(29)\qquad \omega(u) = \sum_{n=0}^{\infty} a_n u^{(2n-1)(k+1)}, \quad dz = -u^{-2}du, \quad a_n \in \mathbf{R}.$$

Since $(2n-1)(k+1) - 2 \neq -1$ for every n we obtain

$$\operatorname*{Res}_{u=0} \omega(u)u^{-2}du = 0.$$

This show that $\int_{\mu_j}\eta_q = 0$. Also, from (8), (11) and (12) we have that

$$(30)\qquad \int_{\mu_j} g_q^2\eta_q = (-1)^j \lambda_q^2 \sum_{j=1}^{\infty} e_{jq} \int_{\tilde{\alpha}} z^{r_{jq}}\frac{dz}{\omega}, \quad q = 2,4$$

where $e_{jq} \in \mathbf{R}$, $r_{j2} \equiv -2(\mathrm{mod}(2k+2))$ and $r_{j4} \equiv 0(\mathrm{mod}(2k+2))$. But $\int_{\tilde{\alpha}} z^{r_{jq}}\frac{dz}{\omega} = -2\pi i \operatorname*{Res}_{z=\infty} z^{r_{jq}}\frac{dz}{\omega}$. Also, $(2n-1)(k+1) - r_{jq} - 2 \neq -1$, $q = 2,4$ and for every $n = 1,2\ldots$, $j = 1,2,\ldots$. These results together with (29) imply that

$$\operatorname*{Res}_{z=\infty} z^{r_{jq}}\frac{dz}{w} = 0, \quad j = 1,2\ldots$$

So, the integrals in (30) are null and the first equality in (28) are satisfied. Also, an easy computation shows that

$$\operatorname*{Res}_{z=0} \quad z^m\frac{dz}{w} = \operatorname*{Res}_{z=0} \quad \frac{wdz}{z^2} = 0, \quad m = 1,2,\ldots$$

This complete the proof of (27) and (28) and finishes the proof of Lemma 3.1.

Proof of Theorem 1.1. — Let $M_{k,q}$ $q = 1,2,3,4$, be the Riemann surfaces given in (2) and (1). We will prove that (g_q, η_q) as defined in Lemma 3.1 satisfies the hypothesis of Theorem 2.1. First, observe that w and dz are respectively a holomorphic function and a holomorphic differential on $M_{k,q}$ with simple zeros at the points $q_n = \left(\varepsilon^n e^{\frac{2\pi i}{s}}, 0\right)$. We remember that $s = 2k+1$ if $q = 1,3$ and $s = 2k+2$ if $q = 2,4$. Also as $h_q(z)$ is a holomorphic function on $D = \{z \in C, |z| < 1\}$ we can choose $\delta > L$ such that $h'_q\left(\frac{1}{\delta}e^{it}\right) \neq 0, 0 \leq t \leq 2\pi$, $1 \leq q \leq 4$, where L is given in Lemma 3.1. With this choice of δ, g_q and η_q have respectively simple poles and double zeros at the points q_n. So, (g_q, η_p) satisfies (C.1) of Theorem 2.1.

On the other hand, $g_q.\eta_q$ is an exact differential on $M_{k,q}$. These results together with Lemma 3.1, implies that (g_q, η_q) satisfy (C.2) of Theorem 2.1.

Finally, let $l(t) = \left(\tilde{l}(t), w(t)\right), 0 \leq t < 1$, be a divergent curve on $M_{k,q}$. Suppose that $q = 3,4$ and $\lim_{t\to 1} \tilde{l}(t) = 0$. In this situation since η_q has double poles at $(0, \pm i)$ we conclude that $l(t)$ has infinite length. So, we can suppose that for each q that $\lim_{t\to 1} |\tilde{l}(t)| = \delta$ and that there exists $0 < \varepsilon^* < 1$ such that $\frac{\delta+1}{2} \leq |\tilde{l}(t)| \leq \delta$ for every $\varepsilon^* < t < 1$. Then, as h_q satisfy Lemma 2.1

$$\int_l (1 + |g_q^2|)\,|\eta_q| \geq \int_l |g_q^2 \eta_q| \geq \frac{\lambda_q^2}{4(1+\delta^s}\int_{\tilde{l}} |h'_q\left(\frac{z}{\delta}\right)|^2 |dz| = \infty.$$

So, (g_q, η_q) satisfies (C.3) of Theorem 2.1. This proves that (g_q, η_q) defines a complete minimal immersion $X_q : M_{k,q} \to \mathbf{R}^3$. Furthermore, as in (14) the third coordinate of the immersion verifies

$$|X_q^3(z,w)| = |\operatorname{Re} \int_{(1,0)}^{(z,w)} g_q \eta_q| = |\operatorname{Re} \frac{\lambda}{\delta} h\left(\frac{z}{\delta}\right)| < \infty.$$

This completes the proof of Theorem 1.1.

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