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THE CENTER OF A GRADED CONNECTED LIE ALGEBRA IS A NICE IDEAL

by Yves FÉLIX

In this text graded vector spaces and graded Lie algebras are always defined over the field $\mathbb{Q}$; $\mathbb{L}(V)$ denotes the free graded Lie algebra on the graded connected vector space V. The notation $L \coprod L'$ means the free product of L and L' in the category of graded Lie algebras, UL denotes the enveloping algebra of the Lie algebra L and $(UL)_+$ denotes the canonical augmentation ideal of UL. The operator s is the usual suspension operator in the category of graded vector spaces, $(sV)_n = V_{n-1}$.

Let $(\mathbb{L}(V), d)$ be a graded connected differential Lie algebra and α be a cycle of degree n in $\mathbb{L}(V)$. An important problem in differential homological algebra consists to compute the homology of the differential graded Lie algebra $(\mathbb{L}(V \oplus \mathbb{Q}x), d)$, $d(x) = \alpha$, and in particular the kernel and the cokernel of the induced map

$$\varphi_\alpha : A = H(\mathbb{L}(V), d) \longrightarrow B = H(\mathbb{L}(V \oplus \mathbb{Q}x), d).$$

Clearly the homology of $(\mathbb{L}(V \oplus \mathbb{Q}x), d)$ and the map φ_α depend only on the class a of α, so that we can write φ_a instead of φ_α.

DEFINITIONS.

(i) *An element a in the Lie algebra A is nice if the kernel of the map φ_a is the ideal generated by a.*

(ii) *An ideal I in the Lie algebra A is nice if, for every element a into I, the kernel of the map φ_a is contained in I.*

Our first result reads

Key words: Differential graded Lie algebra – Inerty – Rational homotopy theory.
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THEOREM 1. — *Let* $(\mathbb{L}(V), d)$ *be a graded connected differential Lie algebra and* a *be an element in* A_n. *If* n *is even and* a *is in the center, then*

(1) *the element* a *is nice,*

(2) *There is a split short exact sequence of graded Lie algebras:*

$$0 \to \mathbb{L}(W) \to B \to A/(a) \to 0,$$

with W *a graded vector space isomorphic to* $s^{n+1}A/(a)_+$.

In the case n is odd, the Whitehead bracket $[a, a]$ is zero since a is in the center, and thus the triple Whitehead bracket $< a, a, a >$ is well defined. We first remark that $< a, a, a >$ belongs also to the center. More generally

PROPOSITION 1. — *The Whitehead triple bracket* $< \alpha, \beta, \gamma >$ *of three elements in the center belongs also to the center.*

THEOREM 2. — *Let* $(\mathbb{L}(V), d)$ *be a graded connected differential Lie algebra and* a *be an element in* A_n. *If* n *is odd,* a *belongs to the center and* $< a, a, a >= 0$, *then*

(1) *the element* a *is nice,*

(2) B *contains an ideal isomorphic to* $\mathbb{L}(W)$ *with* $W = s^{n+1}A/(a)_+$,

(3) *the Lie algebra* B *admits a filtration such that the graded associated Lie algebra* G *is an extension*

$$0 \to \mathbb{L}(W) \amalg K \to G \to A/(a) \to 0,$$

with $K = \mathbb{L}(\alpha, \beta)/([\alpha, \beta], [\alpha, \alpha])$, $|\alpha| = 2n + 1$, $|\beta| = 3n + 2$.

THEOREM 3. — *Let* $(\mathbb{L}(V), d)$ *be a graded connected differential Lie algebra and* a *be an element in* A_n. *If* n *is odd,* a *belongs to the center and* $< a, a, a >\neq 0$, *then*

(1) *the image of* φ_a *is* $A/(a, < a, a, a >)$,

(2) B *contains an ideal isomorphic to* $\mathbb{L}(W)$ *with*

$$W = s^{n+1}(A/(a, < a, a, a >))_+ ,$$

(3) *the Lie algebra* B *admits a filtration such that the graded associated Lie algebra* G *is an extension*

$$0 \to \mathbb{L}(W) \amalg (\gamma, \rho) \to G \to A/(a, < a, a, a >) \to 0 ,$$

with (γ, ρ) *an abelian Lie algebra on 2 generators* γ *and* ρ; $|\rho| = 2n+1$ *and* $|\gamma| = 4n+2$.

COROLLARY 1. — *When the element* a *is in the center, then the kernel of* φ_a *is always contained in the center and its dimension is at most two. In particular the center is a nice ideal.*

COROLLARY 2. — *If* dim A *is at least 3 and if* a *is in the center, then* B *contains a free Lie algebra on at least 2 generators.*

In all cases, B contains a free Lie algebra $\mathbb{L}(W)$ with W isomorphic to $(A/(a, < a, a, a >))_+$.

From those results on differential graded Lie algebras we deduce corresponding results for the rational homotopy Lie algebras of spaces.

Let X denote a finite type simply connected CW complex and Y the space obtained by attaching a cell to X along an element u in $\pi_{n+1}(X)$.

$$Y = X \bigcup_u e^{n+2}.$$

The graded vector space $L_X = \pi_*(\Omega X) \otimes \mathbb{Q}$ together with the Whitehead product is then a graded Lie algebra. Moreover by the Milnor-Moore theorem the Hurewicz map induces an isomorphism of Hopf algebras $U\pi_*(\Omega X) \otimes \mathbb{Q} \xrightarrow{\cong} H_*(\Omega X; \mathbb{Q})$.

Now some notations : we denote by Σ the isomorphism $\pi_n(\Omega X) \to \pi_{n+1}(X)$ and we put $a = \Sigma^{-1}(u)$. For sake of simplicity, an element in $\pi_*(\Omega X) \otimes \mathbb{Q}$ and its image in $H_*(\Omega X; \mathbb{Q})$ will be denoted by the same letter.

Recall that the Quillen minimal model of the space X ([5], [9], [1]) is a differential graded Lie algebra $(\mathbb{L}(V), d)$, unique up to isomorphism, and equiped with natural isomorphisms

(i) $V \cong s^{-1}H_*(X; \mathbb{Q})$,

(ii) $\theta_X : H(\mathbb{L}(V), d) \cong L_X$.

The differential d is an algebrization of the attaching map. More precisely, denote by α a cycle in $(\mathbb{L}(V), d)$ with $\theta_X([\alpha]) = a$, then the differential graded Lie algebra $(\mathbb{L}(V \oplus \mathbb{Q}x), d)$, $d(x) = \alpha$ is a Quillen model of Y and the injection

$$(\mathbb{L}(V), d) \longrightarrow (\mathbb{L}(V \oplus \mathbb{Q}x), d),$$

is a Quillen model for the topological injection i of X into Y. In particular φ_a is the induced map $L_X \to L_Y$.

The injection $i : X \to Y$ induces a sequence of Hopf algebra morphisms

$$H_*(\Omega X;\mathbb{Q}) \xrightarrow{f} H_*(\Omega X;\mathbb{Q})/(a) \xrightarrow{g} H_*(\Omega Y;\mathbb{Q}).$$

The attachment is called *inert* if $g \circ f$ is surjective (this is equivalent to the surjectivity of g), and is called *nice* if g is injective ([7]). Clearly the attachment is nice if the element a is a nice element in L_X.

The structure of the Lie algebra $L_X = \pi_*(\Omega X) \otimes \mathbb{Q}$, for X a space with finite Lusternik-Schnirelmann, has been at the origin of a lot of recent works (cf. [3], [2], [4]). In particular the radical of L_X (union of all solvable ideals) is finite dimensional and each ideal of the form $I_1 \times \cdots I_r$ satisfies $r \leqslant \text{LS cat } X$.

We deduce from Theorems 1, 2 and 3 above the following results concerning attachment of a cell along an element in the center.

THEOREM 4. — *If n is even and a is in the center, then*

(1) *the attachment is nice,*

(2) *there is a split short exact sequence of graded Lie algebras :*

$$0 \to \mathbb{L}(W) \to L_Y \to L_X/(a) \to 0,$$

with W a graded vector space isomorphic to $s^{n+1}(H_*(\Omega X;\mathbb{Q})/(a))_+$. The splitting is given by the natural map $L_X \to L_Y$.

THEOREM 5. — *If n is odd, a belongs to the center and $< a,a,a >= 0$, then*

(1) *the attachment is nice,*

(2) *L_Y contains an ideal isomorphic to $\mathbb{L}(W)$ with*

$$W = s^{n+1}(H_*(\Omega X;\mathbb{Q})/(a))_+,$$

(3) *the Lie algebra L_Y admits a filtration such that the graded associated Lie algebra G is an extension*

$$0 \to \mathbb{L}(W) \amalg K \to G \to L_X/(a) \to 0,$$

with $K = \mathbb{L}(\alpha,\beta)/([\alpha,\beta],[\alpha,\alpha])$, $|\alpha| = 2n+1$, $|\beta| = 3n+2$.

Example 1. — Let $X = P^\infty(\mathbb{C})$ and $u : S^2 \to P^\infty(\mathbb{C})$ be the canonical injection. Then $Y = X/S^2$ and its rational cohomology is $\mathbb{Q}[x_4, y_6]/(x_4^3 - y_6^2)$. In this case the Lie algebra L_Y is isomorphic to K and the graded vector space W is zero.

Example 2. — Let $X = P^3(\mathbb{C})$ and $u : S^2 \to P^3(\mathbb{C})$ be the canonical injection. Then $Y = X/S^2 \cong S^4 \vee S^6$. In this case W is the graded vector space generated by the brackets $\mathrm{ad}^n(i_6)(i_4)$, $n \geqslant 1$, where i_4 and i_6 denote the canonical injections of the spheres S^4 and S^6 into Y.

THEOREM 6. — *If n is odd, a belongs to the center and $< a, a, a > \neq 0$, then*

(1) *the image of L_X in L_Y is $L_X/(a, < a, a, a >)$,*

(2) *L_Y contains an ideal isomorphic to $\mathbb{L}(W)$ with*

$$W = s^{n+1}(H_*(\Omega X; \mathbb{Q})/(a, < a, a, a >))_+ .$$

(3) *the Lie algebra L_Y admits a filtration such that the graded associated Lie algebra G is an extension*

$$0 \to \mathbb{L}(W) \amalg (\gamma, \rho) \to G \to L_X/(a, < a, a, a >) \to 0,$$

with (γ, ρ) an abelian Lie algebra on 2 generators γ and ρ; $|\rho| = 2n + 1$ and $|\gamma| = 4n + 2$.

COROLLARY 1′. — *When the element a is in the center, then the kernel of the map $L_X \to L_Y$ is always contained in the center and its dimension is at most two.*

COROLLARY 2′. — *If* dim L_X *is at least 3 and if a is in the center, then L_Y contains a free Lie algebra on at least 2 generators.*

Example 3. — Let X be either $S^{2n+1} \times K(\mathbb{Z}, 2n)$ or else $P^n(\mathbb{C})$, then the attachment of a cell of dimension $2n + 2$ along a nonzero element generates only one new rational homotopy class of degree $2n+3$. The space Y has in fact the rational homotopy type either of $S^{2n+3} \times K(\mathbb{Z}, 2n)$ or else of $P^{n+1}(\mathbb{C})$. These are rationally the only situations where L_X has dimension two and the attachment does not generate a free Lie algebra.

Example 4. — Let Z be a simply connected finite CW complex not rationally contractible. Then for $n \geqslant 1$ the rational homotopy Lie algebra

L_Y of $Y = S^{2n+1} \times Z/(S^{2n+1} \times \{*\})$ contains a free Lie algebra on at least two generators. It is enough to see that Y is obtained by attaching a cell along the sphere S^{2n+1} in $X = S^{2n+1} \times Z$.

In all cases, L_Y contains a free Lie algebra $\mathbb{L}(W)$ with W isomorphic to $(H_*(\Omega X) \otimes \mathbb{Q}/(a, < a, a, a >))_+$. The elements of W have the following topological description.

Let β be an element of degree $r-1$ in $L_X/(a, < a, a, a >)$ and $b = \Sigma\beta$. The Whitehead bracket

$$S^{n+r} \xrightarrow{[i_{n+1}, i_r]} S^{n+1} \vee S^r \xrightarrow{u \vee b} X,$$

extends to D^{n+r+1} because the element a is in the center. On the other hand in Y the map u extends to D^{n+2}, this gives the commutative diagram

$$\begin{array}{ccccc} D^{n+r+1} & \longrightarrow & D^{n+2} \vee S^r & \longrightarrow & Y \\ \uparrow & & \uparrow & & \uparrow \\ S^{n+r} & \xrightarrow{[i_{n+1}, i_r]} & S^{n+1} \vee S^r & \xrightarrow{u \vee b} & X \end{array}$$

These two extensions of the Whitehead product to D^{r+n+1} define an element $\varphi(\beta)$ in $\pi_{r+n+1}(Y) \otimes \mathbb{Q}$. Now for every element $\alpha = \alpha_1 \ldots \alpha_n$ in $H_+(\Omega X; \mathbb{Q})/(a, < a, a, a >)$, with α_i in L_X, we define

$$\varphi(\alpha) = [\alpha_n, [\alpha_{n-1}, \ldots, [\alpha_2, \varphi(\alpha_1)] \ldots] .$$

This follows directly from the construction of W given in section 1.

PROPOSITION 2. — *When $\{\beta_i\}$ runs along a basis of $H_+(\Omega X; \mathbb{Q})/(a, < a, a, a >)$, the elements $\{\Sigma^{-1}\varphi(\beta_i)\}$ form a basis of a sub free Lie algebra of L_Y.*

Corollary 1 means that the center of L_X is a nice ideal. We conjecture:

CONJECTURE. — Let X be a simply connected finite type CW complex with finite Lusternik-Schnirelmann category, then the radical of L_X is a nice ideal.

We now prove this conjecture in a very particular case.

THEOREM 7. — *If the ideal I generated by a has dimension two and is contained in $(L_X)_{\text{even}}$, then the element a is nice and L_Y is an extension of a free Lie algebra $\mathbb{L}(W)$ by L_X with $W \supset s^n(U(L_X/I)_+)$.*

Example 5. — Let X be the geometric realization of the commutative differential graded algebra $(\wedge(x, c, y, z, t), d)$ with $d(x) = d(c) = 0$, $d(y) =$

xc, $d(z) = yc$, $d(t) = xyz$, $|x| = |c| = 3, |y| = 5, |z| = 7$, and $|t| = 14$. We denote by u the element of $\pi_3(X)$ satisfying $\langle u, x\rangle = 1$ and $\langle u, c\rangle = 0$. The ideal generated by $a = \Sigma^{-1}u$ has dimension three and is concentrated in even degrees, but the element a is not nice.

The rest of the paper is concerned with the proof of Theorems 1,2,3 and 7.

1. Proof of Theorem 1.

Denote by $(\mathbb{L}(V), d)$ and $(\mathbb{L}(V \oplus \mathbb{Q}x), d)$ free differential graded connected Lie algebras, $d(x) = \alpha$, $[\alpha] = a$.

By putting V in gradation 0 and x in gradation 1, we make $(\mathbb{L}(V \oplus \mathbb{Q}x), d)$ into a filtered differential graded algebra. The term (E^1, d^1) of the associated spectral sequence has the form

$$(E^1, d^1) = (A \coprod \mathbb{L}(x), d)\,, \qquad A = H_*(\mathbb{L}(V), d) = L_X\,, \qquad d(x) = a\,.$$

The ideal I generated by x is the free Lie algebra on the elements $[x, \beta_i]$, with $\{\beta_i\}$ a graded basis of UA and where by definition, we have

$$\begin{aligned}[x, 1] &= x \\ [x, \alpha_1\alpha_2 \dots \alpha_n] &= [\dots[x, \alpha_1], \alpha_2], \dots \alpha_n]\,, \qquad \alpha_i \in L_X.\end{aligned}$$

Since a is in the center, if $\beta_i \in (UA)_+$, then $d([x, \beta_i]) = 0$. Therefore the ideal J generated by $[x, x]$ and the $[x, \beta_i]$, $\beta_i \in (UA)_+$ is a subdifferential graded Lie algebra. The ideal J is in fact the free Lie algebra on $[x, x]$ and the elements $[x, \beta_i]$ and $[x, [x, \beta_i]]$, with $\{\beta_i\}$ a basis of $(UA)_+$. A simple computation shows that

$$\begin{aligned}d[x, x] &= 2[a, x] \\ d[x, \beta_i] &= 0 \\ d[x, [x, \beta_i]] &= -[x, \beta_i a].\end{aligned}$$

This shows that $H(J, d)$ is isomorphic to the free graded Lie algebra $\mathbb{L}(W)$ where W is the vector space formed by the elements $[x, \beta_i]$ with $\beta_i \in (UA/(a))_+$.

The short exact sequence $0 \to J \to A \coprod \mathbb{L}(x) \to A \oplus (x) \to 0$ yields thus a short exact sequence in homology

$$0 \to \mathbb{L}(W) \to H(A \coprod \mathbb{L}(x)) \to A/(a) \to 0.$$

This implies that the term E^2 is generated in degrees 0 and 1. The spectral sequence degenerates thus at the E^2 level : $E^2 = E^\infty$.

Denote now by u_{β_i} a class in $H(\mathbb{L}(V \oplus \mathbb{Q}x), d)$ whose representative in $E^\infty_{1,*}$ is $[x, \beta_i]$. The Lie algebra generated by the u_{β_i} admits a filtration such that the graded associated Lie algebra is free. This Lie algebra is therefore free, and its quotient is $A/(a)$. This proves the theorem. □

2. Proof of Proposition 1.

The elements α, β and γ are represented by cycles x, y and z in $(\mathbb{L}(V), d)$. Since the elements α, β and γ are in the center, there exist elements a, b and c in $\mathbb{L}(V)$ such that

$$d(a) = [x, y]\,, \qquad d(b) = [y, z]\,, \qquad d(c) = [z, x].$$

The triple Whitehead product is then represented by the element

$$\omega = (-1)^{|zy|}[c, y] + (-1)^{|xy|}[b, x] + (-1)^{|xz|}[a, z].$$

We will show that the class of ω is central, i.e. for every cycle t the bracket $[\omega, t]$ is a boundary. First of all, since α, β and γ are in the center there exists elements x_1, y_1 and z_1 such that

$$d(x_1) = [x, t]\,, \qquad d(y_1) = [y, t]\,, \qquad d(z_1) = [z, t].$$

We now easily check that the three following elements α_1, α_2 and α_3 are cycles:

$$\begin{aligned}
\alpha_1 &= (-1)^{|tx|+|t|}[t, c] + (-1)^{|xz|+|x|+1+|zt|}[x, z_1] + (-1)^{|zt|+|z|}[z, x_1] \\
\alpha_2 &= (-1)^{|t|+|tz|}[t, b] + (-1)^{|z|+|zy|+1+|yt|}[z, y_1] + (-1)^{|y|+|yt|}[y, z_1] \\
\alpha_3 &= (-1)^{|t|+|ty|}[t, a] + (-1)^{|y|+|ty|+1+|xt|}[y, x_1] + (-1)^{|x|+|xt|}[x, y_1].
\end{aligned}$$

We deduce elements β_1, β_2 and β_3 satisfying

$$d(\beta_1) = [\alpha_1, y]\,, \qquad d(\beta_2) = [\alpha_2, x]\,, \qquad d(\beta_3) = [\alpha_3, z].$$

Now we verify that $[\omega, t]$ is the boundary of

$$\begin{aligned}
&(-1)^{|zy|+|ct|+|yc|+1}[y_1, c] + (-1)^{|yc|+|bt|+|bx|+1}[x_1, b] \\
&+ (-1)^{|xz|+|az|+|at|+1}[z_1, a] + (-1)^{1+|tx|+|ty|+|xy|}\beta_2 \\
&+ (-1)^{1+|yz|+|ty|+|tz|}\beta_1 + (-1)^{1+|tx|+|tz|+|xz|}\beta_3.
\end{aligned}$$ □

3. Proof of Theorems 2 and 3.

We filter the Lie algebra $(\mathbb{L}(V \oplus \mathbb{Q}x), d)$ by putting V in gradation 0 and x in gradation 1. We obtain a spectral sequence $E^r_{*,*}$. We will first compute the term $(E^1_{*,*}, d^1)$ and its homology E^2. We will see that E^2 is generated only in filtration degrees 0, 1 and 2. The differential d^2 is thus defined by its value on $E^2_{2,*}$. Under the hypothesis of Theorem 2, we show that $d^2 = 0$ and that the spectral sequence collapses at the E^2-level. Under the hypothesis of Theorem 3, we show that the image of $d^2 : E^2_{2,*} \to E^2_{0,*}$ has dimension 1 and that a basis is given by the class of the Whitehead bracket $\langle a, a, a\rangle$. We will compute explicitely the term $E^3_{*,*}$. This term is generated only in filtration degrees 0 and 1, so that the spectral sequence collapses at the E^3-level in that case.

3.1. Description of (E^1, d^1).

The term (E^1, d^1) has the form

$$(E^1, d^1) = (A \coprod \mathbb{L}(x), d), \qquad A = H_*(\mathbb{L}(V), d) = L_X, \qquad d(x) = a.$$

We denote by I the ideal generated by x,

$$I = \mathbb{L}([x, \beta_i], \text{ with } \{\beta_i\} \text{ a basis of } UA),$$

and by J the ideal of I generated by the $[x, \beta_i]$ with β_i in $(UA)_+$.

$$J = \mathbb{L}([x, [x, \cdots [x, \beta_i] \cdots], \text{ with } \{\beta_i\} \text{ a basis of } (UA)_+).$$

For sake of simplicity we introduce the notation

$$\varphi_1(\beta) = [x, \beta], \quad \varphi_n(\beta) = [x, \varphi_{n-1}(\beta)], n \geqslant 2.$$

LEMMA 1.

(1) $d(\varphi_1(\beta)) = 0$ *for* $\beta \in (UA)_+$.

(2) $[a, \varphi_n(\beta)] =_{(J^2)} \quad -\varphi_n(a\beta)$, $n \geqslant 1$.

(3) $d(\varphi_n(\beta)) =_{(J^2)} \quad -(n-1)\varphi_{n-1}(a\beta)$, $n \geqslant 2$.

In the above formulas $=_{(J^2)}$ *means equality modulo decomposable elements in the Lie algebra* J.

Proof.

(1) $d\varphi_1(\beta) = d[x,\beta] = [a,\beta] = 0$.

(2) The Jacobi identity shows that $[a, \varphi_1(\beta)] = [a,[x,\beta]] = -[[x,a],\beta] = -[x, a\beta]$. By induction we deduce $[a,\varphi_n(\beta)] =_{(J^2)} [x,[a,\varphi_{n-1}(\beta)]] = -\varphi_n(a\beta)$.

(3) $d\varphi_n(\beta)) = [a, \varphi_{n-1}(\beta)] + [x, d\varphi_{n-1}(\beta)] =_{(J^2)} -(n-1)\varphi_{n-1}(a\beta)$.

□

LEMMA 2.

(1) $d\varphi_1(a) = 0, d\varphi_2(a) = 0$

(2) $d\varphi_n(a) \in \mathbb{L}(\varphi_1(a), \cdots \varphi_{n-2}(a))$

(3) $d\varphi_n(a) + \alpha_n[\varphi_1(a), \varphi_{n-2}(a)]$ *is a decomposable element in* $\mathbb{L}(\varphi_2(a), \ldots, \varphi_{n-3}(a))$, *for* $n \geqslant 3$, *with* $\alpha_3 = 1$ *and for* $n \geqslant 4$, $\alpha_n = 5 + \dfrac{(n+3)(n-4)}{2}$.

Proof. — We first check that $[a, \varphi_1(a)] = 0$ and that

$$[a, \varphi_2(a)] = -[\varphi_1(a), \varphi_1(a)] .$$

Now, using Jacobi identity we find that

$$[a, \varphi_n(a)] = -\sum_{p=1}^{n-1} \binom{n-1}{p} [\varphi_p(a), \varphi_{n-p}(a)] .$$

Using once again Jacobi identity we find

$$[x, [\varphi_n(a), \varphi_m(a)]] = [\varphi_{n+1}(a), \varphi_m(a)] + [\varphi_n(a), \varphi_{m+1}(a)] .$$

The derivation formula valid for $n \geqslant 2$

$$d\varphi_{n+1}(a) = [a, \varphi_n(a)] + [x, d\varphi_n(a)] ,$$

gives now point (3) of the lemma. □

3.2. Computation of $E^2 = H(E^1, d^1)$.

LEMMA 3. — *The homology* $H(\mathbb{L}(\varphi_n(a), n \geqslant 1), d)$ *is the quotient of the free Lie algebra* $\mathbb{L}(\varphi_1(a), \varphi_2(a))$ *by the ideal generated by the relations* $[\varphi_1(a), \varphi_1(a)]$ *and* $[\varphi_1(a), \varphi_2(a)]$.

Proof. — Since the differential d is purely quadratic, the graded Lie algebra $(\mathbb{L}(\varphi_n(a), n \geqslant 1), d)$ represents a formal space Z with rational cohomology $H^*(Z;\mathbb{Q})$ isomorphic to the dual of the suspension of the graded vector space generated by the $\varphi_n(a)$, $n \geqslant 1$.

The rational cup product in $H^*(Z;\mathbb{Q})$ is given by the dual of the differential. This means that $H^*(Z;\mathbb{Q})$ is generated by elements u_1 and u_2 defined by $\langle u_i, s\varphi_j(a)\rangle = 1$ if $i = j$ and 0 otherwise. The description of $d(\varphi_5(a))$ yields the relation $u_1^3 = \frac{9}{8}u_2^2$. Now since the Poincaré series of $H^*(Z;\mathbb{Q})$ and $\mathbb{Q}[u_1,u_2]/\left(u_1^3 - \frac{9}{8}u_2^2\right)$ are both equal to $\frac{1}{1-t^{n+1}} - t^{n+1}$, there is no other relation. Therefore

$$H^*(Z;\mathbb{Q}) \cong \mathbb{Q}[u_1,u_2]/\left(u_1^3 - \frac{9}{8}u_2^2\right).$$

The Lie algebra $H(\mathbb{L}(\varphi_n(a), n \geqslant 1), d)$ is thus isomorphic to the rational homotopy Lie algebra L_Z; its dimension is three and a basis is given by the elements $\varphi_1(a), \varphi_2(a)$ and $[\varphi_2(a), \varphi_2(a)]$. This implies the result. □

The differential ideal J is thus the free product of two differential ideals

$$J = \mathbb{L}((\varphi_n(a)), n \geqslant 1) \coprod \mathbb{L}((\varphi_n(\beta_i), \varphi_n(a\beta_i), n \geqslant 1, \quad \text{with}\ \{\beta_i\}\ \text{a basis of}\ (UA/a)_+).$$

Each factor is stable for the differential. Therefore

$$H(J) = \frac{\mathbb{L}(\varphi_1(a), \varphi_2(a))}{([\varphi_1(a), \varphi_1(a)], [\varphi_1(a), \varphi_2(a)])} \coprod \mathbb{L}((\varphi_1(\beta_i)), \quad \text{with}\ \{\beta_i\}\ \text{a basis of}\ (UA/a)_+).$$

The short exact sequence of Lie algebras

$$0 \to J \to A \coprod \mathbb{L}(x) \to A \oplus \mathbb{Q}x \to 0,$$

closes the description of the term E^2 of the spectral sequence

Corollary. — *The term E^2 satisfies $E^2_{0,*} = A/(a)$, and $E^2_{+,*} = H(J)$. In particular E^2 is generated in filtration degrees 0, 1 and 2.*

3.3. Description of the differential d^2.

Recall that a is in the center. The element $[\alpha, \alpha]$ is thus a boundary : there exists some element b with $d(b) = [\alpha, \alpha]$. Then the element $[b, \alpha]$ is also a cycle and its homology class is the triple Whitehead bracket $< a, a, a >$.

LEMMA 4. — *Denote by* $[\varphi_2(\alpha)]$ *the class of* $\varphi_2(\alpha)$ *in the* E^2*-term of the spectral sequence. We then have*

$$d^2([\varphi_2(a)]) = -\frac{3}{2}\langle[\alpha, b]\rangle,$$

where $\langle - - \rangle$ *means the class of a cycle in the* E^2 *term.*

Proof. — We easily verify that in the differential Lie algebra $(\mathbb{L}(V \oplus \mathbb{Q}x), d)$, we have

$$d\left(\varphi_2(\alpha) - \frac{3}{2}[x, b]\right) = -\frac{3}{2}[\alpha, b].$$

Since $[x, b]$ is in filtration degree 1, and $[\alpha, b]$ in filtration degree 0, this gives the result by definition of the differential d^2. □

3.4. End of the proof of Theorem 2.

If $\langle[b, \alpha]\rangle = 0$, then $d^2 = 0$, the spectral sequence degenerates at the E^2 level and Theorem 2 is proved. □

3.5. Computation of the term $E^3_{*,*}$.

Henceforth, we suppose $\langle[b, \alpha]\rangle \neq 0$. A simple computation using Jacobi identity gives the following identity.

LEMMA 5. — $d^2([\varphi_2(a), \varphi_2(a)]) = 3[\varphi_1(a), \varphi_1(\langle[\alpha, b]\rangle)]$.

Let $\{\beta_i\}$, $i \in I$, denote a basis of $(UA/a)_+$ such that $\langle[\alpha, b]\rangle = \beta_{i_0}$ for some index i_0. The elements $\varphi_1(a)$ and $[\varphi_2(a), \varphi_2(a)]$ together with the elements $\varphi_1(\beta_i)$ generate an ideal M in the graded Lie algebra $E^2_{+,*}$. The Lie algebra M is generated by the elements $\varphi_1(\beta_i)$, $[\varphi_2(a), \varphi_1(\beta_i)]$, $[\varphi_2(a), \varphi_2(a)]$ and $\varphi_1(a)$ and satisfies the two relations $[\varphi_1(a), \varphi_1(a)] = 0$ and $[[\varphi_2(a), \varphi_2(a)], \varphi_1(a)] = 0$. Denote by N the graded

Lie algebra

$$N = \mathbb{L}(\varphi_1(\beta_i), [\varphi_2(a), \varphi_1(\beta_i)]) \coprod \frac{\mathbb{L}([\varphi_2(a), \varphi_2(a)], \varphi_1(a))}{([\varphi_1(a), \varphi_1(a)], [[\varphi_2(a), \varphi_2(a)], \varphi_1(a)])}.$$

Since M and N have the same Poincaré series they coincide. Therefore as a Lie algebra, M can be written

$$M = \mathbb{L}(\varphi_1(\beta_i), [\varphi_2(a), \varphi_1(\beta_i)]) \coprod \frac{\mathbb{L}([\varphi_2(a), \varphi_2(a)], \varphi_1(a))}{([\varphi_1(a), \varphi_1(a)], [[\varphi_2(a), \varphi_2(a)], \varphi_1(a)])}.$$

The equation

$$d^2[\varphi_2(a), \varphi_1(\beta_i)] = \frac{3}{2}\varphi_1(\beta_i[a, b]).$$

shows that the Lie algebra M decomposes into the free product of three differential graded Lie algebras, the first one being acyclic :

$$M = \mathbb{L}(\varphi_1(\beta_i\cdot\langle[\alpha, b]\rangle), [\varphi_2(a), \varphi_1(\beta_i)], i \in I) \coprod \mathbb{L}(\varphi_1(\beta_i), i \in I\backslash\{i_0\}) \coprod K,$$

$$K = \mathbb{L}(\varphi_1(\langle[\alpha, b]\rangle) \coprod \frac{\mathbb{L}([\varphi_2(a), \varphi_2(a)], \varphi_1(a))}{([[\varphi_2(a), \varphi_2(a)], \varphi_1(a)], [\varphi_1(a), \varphi_1(a)])}.$$

We thus have

$$H(M) = \mathbb{L}(\varphi_1(\beta_i), i \in I\backslash\{i_0\}) \coprod H(K).$$

To compute the homology of K we put $t = \varphi_1\langle[\alpha, b]\rangle$, $y = \varphi_1(a)$ and $z = [\varphi_2(a), \varphi_2(a)]$.

LEMMA 6. — *Let* $(\mathcal{L}, d) = (\mathbb{L}(y, z, t)/([y, z], [y, y]), d)$ *be a differential graded Lie algebra with* t *and* z *in* $\mathcal{L}_{\text{even}}$, y *in* $\mathcal{L}_{\text{odd}}$, *and where the differential* d *is defined by* $d(t) = d(y) = 0$ *and* $d(z) = [t, y]$. *Then* $H(\mathcal{L}, d)$ *is a* $\mathbb{Q}$*-vector space of dimension two generated by the classes of* t *and* y.

Proof. — Denote by R the ideal generated by t. As a Lie algebra R is the free Lie algebra generated by t, $w = [t, y]$, the elements $u_n = ad^n(z)(t)$, for $n \geqslant 1$ and the elements $w_n = ad^n(z)[t, y]$, for $n \geqslant 1$.

Using the Jacobi identity, we get the following sequence of identities:

$$\begin{cases} d(t) = 0 \\ d(w) = 0 \\ d(u_1) = [t, w] \\ d(u_2) = 2[u_1, w] - [w_1, t] \\ \dots \\ d(u_n) = n[u_{n-1}, w], \quad \text{modulo } \mathbb{L}(u_1, \dots u_{n-2}, t, w_i) \\ d(w_1) = -[w, w] \\ d(w_2) = -3[w_1, w] \\ \dots \\ d(w_n) = -(n+1)[w_{n-1}, w], \quad \text{modulo } \mathbb{L}(w_1, \dots w_{n-2}). \end{cases}$$

This shows that the cohomology of the cochain algebra on R is $\mathbb{Q}[w^v] \otimes \wedge(t^v)$, with w^v and t^v 1-cochains satisfying $\langle w^v, w\rangle = 1$ and $\langle t^v, t\rangle = 1$. The interpretation of $H(R, d)$ as the dual of the vector space of indecomposable elements of the Sullivan minimal model of $C^*(R)$ shows that $H(R, d) \cong \mathbb{Q}w \oplus \mathbb{Q}t$.

The examination of the short exact sequence of differential complexes

$$0 \to (R, d) \to (\mathcal{L}, d) \to (\mathbb{Q}y \oplus \mathbb{Q}z, 0) \to 0,$$

shows that $H(\mathcal{L}, d) \cong \mathbb{Q}t \oplus \mathbb{Q}y$. □

This shows that $H(M)$ is isomorphic to the free product of $\mathbb{L}(\varphi_1(\beta_i), i \in I\backslash\{i_0\})$ with the abelian Lie algebra on the two elements $\varphi_1(a)$ and $\varphi_1(\langle[\alpha, b]\rangle)$.

3.6. End of the proof of Theorem 3.

From the short exact sequence of chain complexes

$$0 \to M \to E^2 \to E^2_{0,*} \oplus \varphi_2(a)\mathbb{Q} \to 0,$$

we deduce the isomorphism of graded vector spaces

$$E^3 = H(E^2, d^2) \cong H(M) \oplus A/(a, \langle[\alpha, b]\rangle).$$

Since E^3 is generated by elements in gradation 0 and 1, the spectral sequence degenerates at the term E^3, $E^3 = E^\infty$. This closes the proof of Theorem 3. □

4. Proof of Theorem 7.

We suppose that the ideal generated by a is composed of a and $b = [a, c]$. We choose an ordered basis $\{u_i\}$, $i = 1, \dots$ of L_X with $u_1 = c$, $u_2 = a$ and $u_3 = b$. We consider the set of monomials of UL_X of the form $\beta_i = u_{i_1} u_{i_2} \dots u_{i_n}$ with $i_n \leqslant i_{n-1} \leqslant \dots \leqslant i_2 \leqslant i_1$ and $i_j \neq i_{j+1}$ when the degree of u_j is odd. This set of monomials forms a basis of UL_X.

The ideal generated by x in $L_X \coprod \mathbb{L}(x)$ is then the free Lie algebra on the elements $[x, \beta_i]$. For sake of simplicity, we denote $x' = [x, c]$.

In particular the ideal J generated by the elements $[x, x]$, $[x', x']$ and the $[x, \beta_i]$ for $\beta_i \notin \{1, c\}$ is a differential sub Lie algebra that is a free Lie algebra on two types of elements:

First type: $[x, x], [x, x'], [x', x'], [x', [x, x']], [x', [x, x]]$
Second type: $[x, \beta_i], [x, [x, \beta_i]], [x', [x, \beta_i]], [x', [x, [x, \beta_i]]]$, $\beta_i \neq 1, c$.

We have

$d([x, x]) = -2[x, a]$

$d([x', x']) = -2[x, cb]$

$d([x, x']) = -[x, ca] - [x, b]$

$d([x', [x, x]]) = -2[x, [x, b]] + 2[x', [x, a]]$

$d([x, \beta_i]) = 0$

$d([x, [x, \beta_i]]) = -[x, \beta_i a]$

$d([x', [x, \beta_i]]) = -[x, \beta_i b]$

$d([x', [x, [x, \beta_i]]]) = -[x, [x, \beta_i b]] + [x', [x, \beta_i a]]$ modulo decomposable elements.

Looking at the linear part of the differential we see directly that $H(J)$ is isomorphic to the free Lie algebra on the element $[x, ca]$ and the elements $[x, \beta_i]$ with β_i a non empty word in the variables u_j different of a and b. □

Example 6. — Let X be the total space of the fibration with fibre S^7 and base $S^3 \times S^5$ whose Sullivan minimal model is $(\wedge(x, y, z), d)$, $d(x) = d(y) = 0$, $d(z) = xy$, $|x| = 3$, $|y| = 5$ and $|z| = 7$. If we attach a cell along the sphere S^3 we obtain the space $Y = (S^5 \times S^{10}) \vee S^{12}$.