Yves Félix

The center of a graded connected Lie algebra is a nice ideal


<http://www.numdam.org/item?id=AIF_1996__46_1_263_0>
THE CENTER OF A GRADED CONNECTED
LIE ALGEBRA IS A NICE IDEAL

by Yves FÉLIX

In this text graded vector spaces and graded Lie algebras are always defined over the field \( \mathbb{Q} \); \( L(V) \) denotes the free graded Lie algebra on the graded connected vector space \( V \). The notation \( L \coprod L' \) means the free product of \( L \) and \( L' \) in the category of graded Lie algebras, \( UL \) denotes the enveloping algebra of the Lie algebra \( L \) and \( (UL)_+ \) denotes the canonical augmentation ideal of \( UL \). The operator \( s \) is the usual suspension operator in the category of graded vector spaces, \( (sV)_n = V_{n-1} \).

Let \( (L(V), d) \) be a graded connected differential Lie algebra and \( \alpha \) be a cycle of degree \( n \) in \( L(V) \). An important problem in differential homological algebra consists to compute the homology of the differential graded Lie algebra \( (L(V \oplus \mathbb{Q}x), d), d(x) = \alpha, \) and in particular the kernel and the cokernel of the induced map

\[
\varphi_\alpha : A = H(L(V), d) \rightarrow B = H(L(V \oplus \mathbb{Q}x), d).
\]

Clearly the homology of \( (L(V \oplus \mathbb{Q}x), d) \) and the map \( \varphi_\alpha \) depend only on the class \( a \) of \( \alpha \), so that we can write \( \varphi_a \) instead of \( \varphi_\alpha \).

**Definitions.**

(i) An element \( a \) in the Lie algebra \( A \) is nice if the kernel of the map \( \varphi_a \) is the ideal generated by \( a \).

(ii) An ideal \( I \) in the Lie algebra \( A \) is nice if, for every element \( a \) into \( I \), the kernel of the map \( \varphi_a \) is contained in \( I \).

Our first result reads

*Key words:* Differential graded Lie algebra – Inertia – Rational homotopy theory.

Theorem 1. — Let \((L(V), d)\) be a graded connected differential Lie algebra and \(a\) be an element in \(A_n\). If \(n\) is even and \(a\) is in the center, then
(1) the element \(a\) is nice,
(2) There is a split short exact sequence of graded Lie algebras:
\[
0 \to L(W) \to B \to A/(a) \to 0,
\]
with \(W\) a graded vector space isomorphic to \(s^{n+1}A/(a)_+\).

In the case \(n\) is odd, the Whitehead bracket \([a, a]\) is zero since \(a\) is in the center, and thus the triple Whitehead bracket \(<a, a, a>\) is well defined. We first remark that \(<a, a, a>\) belongs also to the center. More generally

Proposition 1. — The Whitehead triple bracket \(<\alpha, \beta, \gamma>\) of three elements in the center belongs also to the center.

Theorem 2. — Let \((L(V), d)\) be a graded connected differential Lie algebra and \(a\) be an element in \(A_n\). If \(n\) is odd, \(a\) belongs to the center and \(<a, a, a>\neq 0\), then
(1) the element \(a\) is nice,
(2) \(B\) contains an ideal isomorphic to \(L(W)\) with \(W = s^{n+1}A/(a)_+\),
(3) the Lie algebra \(B\) admits a filtration such that the graded associated Lie algebra \(G\) is an extension
\[
0 \to L(W) \amalg K \to G \to A/(a) \to 0,
\]
with \(K = L(\alpha, \beta)/([\alpha, \beta], [\alpha, \alpha]), |\alpha| = 2n + 1, |\beta| = 3n + 2\).

Theorem 3. — Let \((L(V), d)\) be a graded connected differential Lie algebra and \(a\) be an element in \(A_n\). If \(n\) is odd, \(a\) belongs to the center and \(<a, a, a>\neq 0\), then
(1) the image of \(\varphi_a\) is \(A/(a, <a, a, a>)\),
(2) \(B\) contains an ideal isomorphic to \(L(W)\) with
\[
W = s^{n+1}(A/(a, <a, a, a>))_+,\n\]
(3) the Lie algebra \(B\) admits a filtration such that the graded associated Lie algebra \(G\) is an extension
\[
0 \to L(W) \amalg (\gamma, \rho) \to G \to A/(a, <a, a, a>) \to 0,
\]
with $(\gamma, \rho)$ an abelian Lie algebra on 2 generators $\gamma$ and $\rho$; $|\rho| = 2n + 1$ and $|\gamma| = 4n + 2$.

**Corollary 1.** — When the element $a$ is in the center, then the kernel of $\varphi_a$ is always contained in the center and its dimension is at most two. In particular the center is a nice ideal.

**Corollary 2.** — If $\dim A$ is at least 3 and if $a$ is in the center, then $B$ contains a free Lie algebra on at least 2 generators.

In all cases, $B$ contains a free Lie algebra $L(W)$ with $W$ isomorphic to $(A/(a, <a, a, a>))^+$. 

From those results on differential graded Lie algebras we deduce corresponding results for the rational homotopy Lie algebras of spaces.

Let $X$ denote a finite type simply connected CW complex and $Y$ the space obtained by attaching a cell to $X$ along an element $u$ in $\pi_{n+1}(X)$.

$$Y = X \bigcup_ue^{n+2}.$$ 

The graded vector space $L_X = \pi_*(\Omega X) \otimes \mathbb{Q}$ together with the Whitehead product is then a graded Lie algebra. Moreover by the Milnor-Moore theorem the Hurewicz map induces an isomorphism of Hopf algebras $U\pi_*(\Omega X) \otimes \mathbb{Q} \cong H_*(\Omega X; \mathbb{Q})$.

Now some notations: we denote by $\Sigma$ the isomorphism $\pi_n(\Omega X) \rightarrow \pi_{n+1}(X)$ and we put $a = \Sigma^{-1}(u)$. For sake of simplicity, an element in $\pi_*(\Omega X) \otimes \mathbb{Q}$ and its image in $H_*(\Omega X; \mathbb{Q})$ will be denoted by the same letter.

Recall that the Quillen minimal model of the space $X$ ([5], [9], [1]) is a differential graded Lie algebra $(\mathbb{L}(V), d)$, unique up to isomorphism, and equiped with natural isomorphisms

1. $V \cong s^{-1}H_*(X; \mathbb{Q})$,
2. $\theta_X : H(\mathbb{L}(V), d) \cong L_X$.

The differential $d$ is an algebrization of the attaching map. More precisely, denote by $\alpha$ a cycle in $(\mathbb{L}(V), d)$ with $\theta_X([\alpha]) = a$, then the differential graded Lie algebra $(\mathbb{L}(V \oplus \mathbb{Q}x), d)$, $d(x) = \alpha$ is a Quillen model of $Y$ and the injection

$$(\mathbb{L}(V), d) \longrightarrow (\mathbb{L}(V \oplus \mathbb{Q}x), d),$$
is a Quillen model for the topological injection \( i \) of \( X \) into \( Y \). In particular \( \varphi_a \) is the induced map \( L_X \to L_Y \).

The injection \( i : X \to Y \) induces a sequence of Hopf algebra morphisms

\[
H_*(\Omega X; \mathbb{Q}) \xrightarrow{f} H_*(\Omega X; \mathbb{Q})/(a) \xrightarrow{g} H_*(\Omega Y; \mathbb{Q}).
\]

The attachment is called inert if \( g \circ f \) is surjective (this is equivalent to the surjectivity of \( g \)), and is called nice if \( g \) is injective ([7]). Clearly the attachment is nice if the element \( a \) is a nice element in \( L_X \).

The structure of the Lie algebra \( L_X = \pi_*(\Omega X) \otimes \mathbb{Q} \), for \( X \) a space with finite Lusternik-Schnirelmann, has been at the origin of a lot of recent works (cf. [3], [2], [4]). In particular the radical of \( L_X \) (union of all solvable ideals) is finite dimensional and each ideal of the form \( I_1 \times \cdots I_r \) satisfies \( r \leq \text{LS cat} \, X \).

We deduce from Theorems 1, 2 and 3 above the following results concerning attachment of a cell along an element in the center.

**Theorem 4.** — If \( n \) is even and \( a \) is in the center, then

1. the attachment is nice,
2. there is a split short exact sequence of graded Lie algebras:

\[
0 \to L(W) \to L_Y \to L_X/(a) \to 0,
\]

with \( W \) a graded vector space isomorphic to \( s^{n+1}(H_*(\Omega X; \mathbb{Q})/(a))_+ \). The splitting is given by the natural map \( L_X \to L_Y \).

**Theorem 5.** — If \( n \) is odd, \( a \) belongs to the center and \( \langle a, a, a \rangle \geq 0 \), then

1. the attachment is nice,
2. \( L_Y \) contains an ideal isomorphic to \( L(W) \) with

\[
W = s^{n+1}(H_*(\Omega X; \mathbb{Q})/(a))_+,
\]

3. the Lie algebra \( L_Y \) admits a filtration such that the graded associated Lie algebra \( G \) is an extension

\[
0 \to L(W) \amalg K \to G \to L_X/(a) \to 0,
\]

with \( K = L(\alpha, \beta)/([\alpha, \beta], [\alpha, \alpha]), \ |\alpha| = 2n + 1, |\beta| = 3n + 2 \).
Example 1. — Let $X = P^\infty(C)$ and $u : S^2 \to P^\infty(C)$ be the canonical injection. Then $Y = X/S^2$ and its rational cohomology is $\mathbb{Q}[x_4, y_6]/(x_4^3 - y_6)$. In this case the Lie algebra $L_Y$ is isomorphic to $K$ and the graded vector space $W$ is zero.

Example 2. — Let $X = P^3(C)$ and $u : S^2 \to P^3(C)$ be the canonical injection. Then $Y = X/S^2 \cong S^4 \vee S^6$. In this case $W$ is the graded vector space generated by the brackets $\text{ad}^n(i_4)(i_6)$, $n \geq 1$, where $i_4$ and $i_6$ denote the canonical injections of the spheres $S^4$ and $S^6$ into $Y$.

Theorem 6. — If $n$ is odd, $a$ belongs to the center and $< a, a, a > \neq 0$, then

1. the image of $L_X$ in $L_Y$ is $L_X/(a, < a, a, a >)$,
2. $L_Y$ contains an ideal isomorphic to $L(W)$ with
   \[ W = s^{n+1}(H^*(\Omega X; \mathbb{Q})/(a, < a, a, a >))_. \]
3. the Lie algebra $L_Y$ admits a filtration such that the graded associated Lie algebra $G$ is an extension
   \[ 0 \to L(W) \Join (\gamma, \rho) \to G \to L_X/(a, < a, a, a >) \to 0, \]
   with $(\gamma, \rho)$ an abelian Lie algebra on 2 generators $\gamma$ and $\rho$; $|\rho| = 2n + 1$ and $|\gamma| = 4n + 2$.

Corollary 1'. — When the element $a$ is in the center, then the kernel of the map $L_X \to L_Y$ is always contained in the center and its dimension is at most two.

Corollary 2'. — If $\text{dim } L_X$ is at least 3 and if $a$ is in the center, then $L_Y$ contains a free Lie algebra on at least 2 generators.

Example 3. — Let $X$ be either $S^{2n+1} \times K(Z, 2n)$ or else $P^n(C)$, then the attachment of a cell of dimension $2n + 2$ along a nonzero element generates only one new rational homotopy class of degree $2n + 3$. The space $Y$ has in fact the rational homotopy type either of $S^{2n+3} \times K(Z, 2n)$ or else of $P^{n+1}(C)$. These are rationally the only situations where $L_X$ has dimension two and the attachment does not generate a free Lie algebra.

Example 4. — Let $Z$ be a simply connected finite CW complex not rationally contractible. Then for $n \geq 1$ the rational homotopy Lie algebra
$L_Y$ of $Y = S^{2n+1} \times Z/(S^{2n+1} \times \{\star\})$ contains a free Lie algebra on at least two generators. It is enough to see that $Y$ is obtained by attaching a cell along the sphere $S^{2n+1}$ in $X = S^{2n+1} \times Z$.

In all cases, $L_Y$ contains a free Lie algebra $\mathbb{L}(W)$ with $W$ isomorphic to $(H_*(\Omega X) \otimes \mathbb{Q}/(a, <a, a, a>))$. The elements of $W$ have the following topological description.

Let $\beta$ be an element of degree $r-1$ in $L_X/(a, <a, a, a>)$ and $b = \Sigma \beta$. The Whitehead bracket

$$S^{n+r}[i_{n+1}, i_r] S^{n+1} \vee S^r u \rightarrow X,$$

extends to $D^{n+r+1}$ because the element $a$ is in the center. On the other hand in $Y$ the map $u$ extends to $D^{n+2}$, this gives the commutative diagram

$$\begin{array}{ccc}
D^{n+r+1} & \longrightarrow & D^{n+2} \vee S^r \\
\uparrow & & \uparrow \\
S^{n+r} & \longrightarrow & S^{n+1} \vee S^r \\
\downarrow & & \downarrow \\
Y & \longrightarrow & X
\end{array}$$

These two extensions of the Whitehead product to $D^{r+n+1}$ define an element $\varphi(\beta)$ in $\pi_{r+n+1}(Y) \otimes \mathbb{Q}$. Now for every element $\alpha = \alpha_1 \ldots \alpha_n$ in $H_+((\Omega X; \mathbb{Q})/(a, <a, a, a>)$, with $\alpha_i$ in $L_X$, we define

$$\varphi(\alpha) = [\alpha_n, [\alpha_{n-1}, \ldots, [\alpha_2, \varphi(\alpha_1)] \ldots]].$$

This follows directly from the construction of $W$ given in section 1.

**Proposition 2.** — When $\{\beta_i\}$ runs along a basis of $H_+((\Omega X; \mathbb{Q})/(a, <a, a, a>)$, the elements $\{\Sigma^{-1}\varphi(\beta_i)\}$ form a basis of a sub free Lie algebra of $L_Y$.

**Corollary 1** means that the center of $L_X$ is a nice ideal. We conjecture:

**Conjecture.** — Let $X$ be a simply connected finite type CW complex with finite Lusternik-Schnirelmann category, then the radical of $L_X$ is a nice ideal.

We now prove this conjecture in a very particular case.

**Theorem 7.** — If the ideal $I$ generated by $a$ has dimension two and is contained in $(L_X)_{\text{even}}$, then the element $a$ is nice and $L_Y$ is an extension of a free Lie algebra $\mathbb{L}(W)$ by $L_X$ with $W \supset s^n(U(L_X/I)_+)$.  

**Example 5.** — Let $X$ be the geometric realization of the commutative differential graded algebra $(\wedge(x, c, y, z, t), d)$ with $d(x) = d(c) = 0, d(y) = \ldots$
The center of a graded connected Lie algebra 269

\[ xc, d(z) = yc, d(t) = xyz, |x| = |c| = 3, |y| = 5, |z| = 7, \text{ and } |t| = 14. \]

We denote by \( u \) the element of \( \pi_3(X) \) satisfying \( \langle u, x \rangle = 1 \) and \( \langle u, c \rangle = 0 \). The ideal generated by \( a = \Sigma^{-1} u \) has dimension three and is concentrated in even degrees, but the element \( a \) is not nice.

The rest of the paper is concerned with the proof of Theorems 1, 2, 3 and 7.

1. Proof of Theorem 1.

Denote by \((L(V), d)\) and \((L(V \oplus \mathbb{Q}x), d)\) free differential graded connected Lie algebras, \( d(x) = \alpha, [\alpha] = a \).

By putting \( V \) in gradation 0 and \( x \) in gradation 1, we make \((L(V \oplus \mathbb{Q}x), d)\) into a filtered differential graded algebra. The term \((E^1, d^1)\) of the associated spectral sequence has the form

\[ (E^1, d^1) = (A \coprod L(x), d), \quad A = H_*(L(V), d) = L_X, \quad d(x) = a. \]

The ideal \( I \) generated by \( x \) is the free Lie algebra on the elements \([x, \beta_i]\), with \( \{\beta_i\} \) a graded basis of \( UA \) and where by definition, we have

\[ [x, 1] = x \]

\[ [x, \alpha_1 \alpha_2 \ldots \alpha_n] = \ldots [x, \alpha_1], \alpha_2, \ldots \alpha_n, \quad \alpha_i \in L_X. \]

Since \( a \) is in the center, if \( \beta_i \in (UA)_+ \), then \( d([x, \beta_i]) = 0 \). Therefore the ideal \( J \) generated by \([x, x]\) and the \([x, \beta_i], \beta_i \in (UA)_+ \) is a subdifferential graded Lie algebra. The ideal \( J \) is in fact the free Lie algebra on \([x, x]\) and the elements \([x, \beta_i]\) and \([x, [x, \beta_i]]\), with \( \{\beta_i\} \) a basis of \((UA)_+\).

A simple computation shows that

\[ d[x, x] = 2[a, x] \]

\[ d[x, \beta_i] = 0 \]

\[ d[x, [x, \beta_i]] = -[x, \beta_i a]. \]

This shows that \( H(J, d) \) is isomorphic to the free graded Lie algebra \( \mathbb{L}(W) \) where \( W \) is the vector space formed by the elements \([x, \beta_i]\) with \( \beta_i \in (UA/(a))_+ \).

The short exact sequence \( 0 \rightarrow J \rightarrow A \coprod L(x) \rightarrow A \oplus (x) \rightarrow 0 \) yields thus a short exact sequence in homology

\[ 0 \rightarrow \mathbb{L}(W) \rightarrow H(A \coprod L(x)) \rightarrow A/(a) \rightarrow 0. \]
This implies that the term $E^2$ is generated in degrees 0 and 1. The spectral sequence degenerates thus at the $E^2$ level: $E^2 = E^\infty$.

Denote now by $u_{\beta_i}$ a class in $H(\mathbb{L}(V \oplus \mathbb{Q}x), d)$ whose representative in $E^\infty_{1,*}$ is $[x, \beta_i]$. The Lie algebra generated by the $u_{\beta_i}$ admits a filtration such that the graded associated Lie algebra is free. This Lie algebra is therefore free, and its quotient is $A/(a)$. This proves the theorem. \qed

2. Proof of Proposition 1.

The elements $\alpha$, $\beta$ and $\gamma$ are represented by cycles $x$, $y$ and $z$ in $(\mathbb{L}(V), d)$. Since the elements $\alpha$, $\beta$ and $\gamma$ are in the center, there exist elements $a$, $b$ and $c$ in $\mathbb{L}(V)$ such that

$$d(a) = [x, y], \quad d(b) = [y, z], \quad d(c) = [z, x].$$

The triple Whitehead product is then represented by the element

$$\omega = (-1)^{|x|} [c, y] + (-1)^{|x|} [b, x] + (-1)^{|x|} [a, z].$$

We will show that the class of $\omega$ is central, i.e. for every cycle $t$ the bracket $[\omega, t]$ is a boundary. First of all, since $\alpha$, $\beta$ and $\gamma$ are in the center there exists elements $x_1$, $y_1$ and $z_1$ such that

$$d(x_1) = [x, t], \quad d(y_1) = [y, t], \quad d(z_1) = [z, t].$$

We now easily check that the three following elements $\alpha_1$, $\alpha_2$ and $\alpha_3$ are cycles:

$$\alpha_1 = (-1)^{|x|+|t|} [t, c] + (-1)^{|x|+|x|+|t|+1} [x, x_1] + (-1)^{|x|+|x|} [z, x_1],$$

$$\alpha_2 = (-1)^{|x|+|x|} [t, b] + (-1)^{|x|+|x|+|t|} [z, y_1] + (-1)^{|y|+|t|+1} [y, y_1],$$

$$\alpha_3 = (-1)^{|x|+|x|} [t, a] + (-1)^{|y|+|t|+|x|} [y, x_1] + (-1)^{|z|+|x|} [x, y_1].$$

We deduce elements $\beta_1$, $\beta_2$ and $\beta_3$ satisfying

$$d(\beta_1) = [\alpha_1, y], \quad d(\beta_2) = [\alpha_2, x], \quad d(\beta_3) = [\alpha_3, z].$$

Now we verify that $[\omega, t]$ is the boundary of

$$(-1)^{|x|+|t|+|t|}[y_1, c] + (-1)^{|x|+|t|+|t|} [x_1, b]$$

$$+ (-1)^{|x|+|t|+|t|+1} [z_1, a] + (-1)^{|y|+|t|+|x|} \beta_2$$

$$+ (-1)^{|x|+|t|+|t|+1} [z_1, a] + (-1)^{|x|+|t|+|t|+1} [z_1, a].$$
3. Proof of Theorems 2 and 3.

We filter the Lie algebra \((L(V \oplus \mathbb{Q}x), d)\) by putting \(V\) in gradation 0 and \(x\) in gradation 1. We obtain a spectral sequence \(E^r_{*,*}\). We will first compute the term \((E^1_{*,*}, d^1)\) and its homology \(E^2\). We will see that \(E^2\) is generated only in filtration degrees 0, 1 and 2. The differential \(d^2\) is thus defined by its value on \(E^2_{2,*}\). Under the hypothesis of Theorem 2, we show that \(d^2 = 0\) and that the spectral sequence collapses at the \(E^2\)-level. Under the hypothesis of Theorem 3, we show that the image of \(d^2 : E^2_{2,*} \to E^3_{0,*}\) has dimension 1 and that a basis is given by the class of the Whitehead bracket \(\langle a, a, a \rangle\). We will compute explicitly the term \(E^3_{*,*}\). This term is generated only in filtration degrees 0 and 1, so that the spectral sequence collapses at the \(E^3\)-level in that case.

3.1. Description of \((E^1, d^1)\).

The term \((E^1, d^1)\) has the form

\[
(E^1, d^1) = (A \coprod L(x), d), \quad A = H_*(L(V), d) = L_X, \quad d(x) = a.
\]

We denote by \(I\) the ideal generated by \(x\),

\[
I = L([x, \beta_1], \text{ with } \{\beta_1\} \text{ a basis of } UA),
\]

and by \(J\) the ideal of \(I\) generated by the \([x, \beta_i]\) with \(\beta_i \in (UA)_+\).

\[
J = L([x, [x, \cdots [x, \beta_i] \cdots]], \text{ with } \{\beta_i\} \text{ a basis of } (UA)_+).
\]

For sake of simplicity we introduce the notation

\[
\varphi_1(\beta) = [x, \beta], \quad \varphi_n(\beta) = [x, \varphi_{n-1}(\beta)], n \geq 2.
\]

**Lemma 1.**

1. \(d(\varphi_1(\beta)) = 0\) for \(\beta \in (UA)_+\).
2. \([a, \varphi_n(\beta)] = (J^2) - \varphi_n(a\beta), n \geq 1.\)
3. \(d(\varphi_n(\beta)) = (J^2) - (n - 1)\varphi_{n-1}(a\beta), n \geq 2.\)

In the above formulas \((J^2)\) means equality modulo decomposable elements in the Lie algebra \(J\).
Proof.

(1) \( d\varphi_1(\beta) = d[x, \beta] = [a, \beta] = 0 \).

(2) The Jacobi identity shows that \( [a, \varphi_1(\beta)] = [a, [x, \beta]] = -[[x, a], \beta] = -[x, a\beta] \). By induction we deduce \( [a, \varphi_n(\beta)] = (J^2) [x, [a, \varphi_n-1(\beta)]] = -\varphi_n(a\beta) \).

(3) \( d\varphi_n(\beta)) = [a, \varphi_{n-1}(\beta)] + [x, d\varphi_{n-1}(\beta)] = (J^2) -(n-1)\varphi_{n-1}(a\beta). \)

\(\square\)

Lemma 2.

(1) \( d\varphi_1(a) = 0, d\varphi_2(a) = 0 \)

(2) \( d\varphi_n(a) \in \mathbb{L}(\varphi_1(a), \cdots, \varphi_{n-2}(a)) \)

(3) \( d\varphi_n(a) + \alpha_n[\varphi_1(a), \varphi_{n-2}(a)] \) is a decomposable element in \( \mathbb{L}(\varphi_2(a), \cdots, \varphi_{n-3}(a)) \), for \( n \geq 3 \), with \( \alpha_3 = 1 \) and for \( n \geq 4 \), \( \alpha_n = 5 + \frac{(n+3)(n-4)}{2} \).

Proof. — We first check that \( [a, \varphi_1(a)] = 0 \) and that

\[
[a, \varphi_2(\beta)] = -[\varphi_1(\beta), \varphi_1(\beta)].
\]

Now, using Jacobi identity we find that

\[
[a, \varphi_n(a)] = -\sum_{p=1}^{n-1} \binom{n-1}{p} [\varphi_p(a), \varphi_{n-p}(a)].
\]

Using once again Jacobi identity we find

\[
[x, [\varphi_n(a), \varphi_m(a)]] = [\varphi_{n+1}(a), \varphi_m(a)] + [\varphi_n(a), \varphi_{m+1}(a)].
\]

The derivation formula valid for \( n \geq 2 \)

\[
d\varphi_{n+1}(a) = [a, \varphi_n(a)] + [x, d\varphi_n(a)],
\]

gives now point (3) of the lemma. \(\square\)

3.2. Computation of \( E^2 = H(E^1, d^1) \).

Lemma 3. — The homology \( H(\mathbb{L}(\varphi_n(a), n \geq 1), d) \) is the quotient of the free Lie algebra \( \mathbb{L}(\varphi_1(a), \varphi_2(a)) \) by the ideal generated by the relations \([\varphi_1(a), \varphi_1(a)]\) and \([\varphi_1(a), \varphi_2(a)]\).
Proof. — Since the differential $d$ is purely quadratic, the graded Lie algebra $(L(\varphi_n(a), n \geq 1), d)$ represents a formal space $Z$ with rational cohomology $H^*(Z; \mathbb{Q})$ isomorphic to the dual of the suspension of the graded vector space generated by the $\varphi_n(a)$, $n \geq 1$.

The rational cup product in $H^*(Z; \mathbb{Q})$ is given by the dual of the differential. This means that $H^*(Z; \mathbb{Q})$ is generated by elements $u_1$ and $u_2$ defined by $\langle u_i, s\varphi_j(a) \rangle = 1$ if $i = j$ and 0 otherwise. The description of $d(\varphi_5(a))$ yields the relation $u_1^3 = \frac{9}{8} u_2^2$. Now since the Poincaré series of $H^*(Z; \mathbb{Q})$ and $\mathbb{Q}[u_1, u_2]/\left(u_1^3 - \frac{9}{8} u_2^2\right)$ are both equal to $\frac{1}{1 - t^{n+1}} - t^{n+1}$, there is no other relation. Therefore

$$H^*(Z; \mathbb{Q}) \cong \mathbb{Q}[u_1, u_2]/\left(u_1^3 - \frac{9}{8} u_2^2\right).$$

The Lie algebra $H(L(\varphi_n(a), n \geq 1), d)$ is thus isomorphic to the rational homotopy Lie algebra $L_Z$; its dimension is three and a basis is given by the elements $\varphi_1(a), \varphi_2(a)$ and $[\varphi_2(a), \varphi_2(a)]$. This implies the result. \qed

The differential ideal $J$ is thus the free product of two differential ideals

$$J = L((\varphi_n(a)), n \geq 1) \coprod L((\varphi_n(\beta_i), \varphi_n(a\beta_i), n \geq 1,$$

with $\{\beta_i\}$ a basis of $(UA/a)_+)$.

Each factor is stable for the differential. Therefore

$$H(J) = \frac{L(\varphi_1(a), \varphi_2(a))}{([\varphi_1(a), \varphi_1(a)], [\varphi_1(a), \varphi_2(a)])} \coprod L((\varphi_1(\beta_i)),$$

with $\{\beta_i\}$ a basis of $(UA/a)_+)$.

The short exact sequence of Lie algebras

$$0 \to J \to A \coprod L(x) \to A \oplus \mathbb{Q}x \to 0,$$

closes the description of the term $E^2$ of the spectral sequence.

**Corollary.** The term $E^2$ satisfies $E^2_{0,*} = A/(a)$, and $E^2_{+,*} = H(J)$. In particular $E^2$ is generated in filtration degrees 0, 1 and 2.
### 3.3. Description of the differential $d^2$.

Recall that $a$ is in the center. The element $[a, a]$ is thus a boundary: there exists some element $b$ with $d(b) = [a, a]$. Then the element $[b, a]$ is also a cycle and its homology class is the triple Whitehead bracket $<a, a, a>$.

**Lemma 4.** Denote by $[\varphi_2(\alpha)]$ the class of $\varphi_2(\alpha)$ in the $E^2$-term of the spectral sequence. We then have

$$d^2([\varphi_2(a)]) = -\frac{3}{2}([\alpha, b]),$$

where $\langle \cdot \rangle$ means the class of a cycle in the $E^2$ term.

**Proof.** We easily verify that in the differential Lie algebra $(L(V \oplus \mathbb{Q}x), d)$, we have

$$d \left( \varphi_2(\alpha) - \frac{3}{2} [x, b] \right) = -\frac{3}{2} [\alpha, b].$$

Since $[x, b]$ is in filtration degree 1, and $[\alpha, b]$ in filtration degree 0, this gives the result by definition of the differential $d^2$. $\square$

### 3.4. End of the proof of Theorem 2.

If $\langle [b, \alpha] \rangle = 0$, then $d^2 = 0$, the spectral sequence degenerates at the $E^2$ level and Theorem 2 is proved. $\square$

### 3.5. Computation of the term $E^3_{*,*}$.

Henceforth, we suppose $\langle [b, \alpha] \rangle \neq 0$. A simple computation using Jacobi identity gives the following identity.

**Lemma 5.** $d^2([\varphi_2(a), \varphi_2(a)]) = 3[\varphi_1(a), \varphi_1([\alpha, b])].$

Let $\{\beta_i\}$, $i \in I$, denote a basis of $(UA/a)_{+}$ such that $\langle [\alpha, b] \rangle = \beta_{i_0}$ for some index $i_0$. The elements $\varphi_1(a)$ and $[\varphi_2(a), \varphi_2(a)]$ together with the elements $\varphi_1(\beta_i)$ generate an ideal $M$ in the graded Lie algebra $E^3_{*,*}$. The Lie algebra $M$ is generated by the elements $\varphi_1(\beta_i)$, $[\varphi_2(a), \varphi_1(\beta_i)]$, $[\varphi_2(a), \varphi_2(a)]$ and $\varphi_1(a)$ and satisfies the two relations $[\varphi_1(a), \varphi_1(a)] = 0$ and $[[\varphi_2(a), \varphi_2(a)], \varphi_1(a)] = 0$. Denote by $N$ the graded
Lie algebra

\[ N = \mathbb{L}(\varphi_1(\beta_i), [\varphi_2(a), \varphi_1(\beta_i)]) \prod \mathbb{L}([\varphi_2(a), \varphi_2(a)], \varphi_1(a)) \frac{\mathbb{L}([\varphi_2(a), \varphi_2(a)], \varphi_1(a))}{([\varphi_1(a), \varphi_1(a)], [[\varphi_2(a), \varphi_2(a)], \varphi_1(a)])}. \]

Since \( M \) and \( N \) have the same Poincaré series they coincide. Therefore as a Lie algebra, \( M \) can be written

\[ M = \mathbb{L}(\varphi_1(\beta_i), [\varphi_2(a), \varphi_1(\beta_i)]) \prod \mathbb{L}([\varphi_2(a), \varphi_2(a)], \varphi_1(a)) \frac{\mathbb{L}([\varphi_2(a), \varphi_2(a)], \varphi_1(a))}{([\varphi_1(a), \varphi_1(a)], [[\varphi_2(a), \varphi_2(a)], \varphi_1(a)])}. \]

The equation

\[ d^2[\varphi_2(a), \varphi_1(\beta_i)] = \frac{3}{2} \varphi_1(\beta_i[a, b]). \]

shows that the Lie algebra \( M \) decomposes into the free product of three differential graded Lie algebras, the first one being acyclic:

\[ M = \mathbb{L}(\varphi_1(\beta_i), ([a, b]), [\varphi_2(a), \varphi_1(\beta_i)], i \in I) \prod \mathbb{L}(\varphi_1(\beta_i), i \in I \setminus \{i_0\}) \prod K, \]

\[ K = \mathbb{L}(\varphi_1([a, b])) \prod \mathbb{L}([\varphi_2(a), \varphi_2(a)], \varphi_1(a)) \frac{\mathbb{L}([\varphi_2(a), \varphi_2(a)], \varphi_1(a))}{([\varphi_2(a), \varphi_2(a)], \varphi_1(a), [\varphi_1(a), \varphi_1(a)])}. \]

We thus have

\[ H(M) = \mathbb{L}(\varphi_1(\beta_i), i \in I \setminus \{i_0\}) \prod H(K). \]

To compute the homology of \( K \) we put \( t = \varphi_1([a, b]), y = \varphi_1(a) \) and \( z = [\varphi_2(a), \varphi_2(a)] \).

**Lemma 6.** — Let \((\mathcal{L}, d) = (\mathbb{L}(y, z, t)/(y, z, [y, z]), d)\) be a differential graded Lie algebra with \( t \) and \( z \) in \( \mathcal{L}_{\text{even}}, y \) in \( \mathcal{L}_{\text{odd}}, \) and where the differential \( d \) is defined by \( d(t) = d(y) = 0 \) and \( d(z) = [t, y] \). Then \( H(\mathcal{L}, d) \) is a \( \mathbb{Q} \)-vector space of dimension two generated by the classes of \( t \) and \( y \).

**Proof.** — Denote by \( R \) the ideal generated by \( t \). As a Lie algebra \( R \) is the free Lie algebra generated by \( t, w = [t, y], \) the elements \( u_n = ad^n(z)(t), \) for \( n \geq 1 \) and the elements \( w_n = ad^n(z)[t, y], \) for \( n \geq 1 \).
Using the Jacobi identity, we get the following sequence of identities:

\[
\begin{align*}
  d(t) &= 0 \\
  d(w) &= 0 \\
  d(u_1) &= [t, w] \\
  d(u_2) &= 2[u_1, w] - [w, t] \\
  \vdots \\
  d(u_n) &= n[u_{n-1}, w], \mod L(u_1, \ldots, u_{n-2}, t, w_i) \\
  d(w_1) &= -[w, w] \\
  d(w_2) &= -3[w_1, w] \\
  \vdots \\
  d(w_n) &= -(n + 1)[w_{n-1}, w], \mod L(w_1, \ldots, w_{n-2}).
\end{align*}
\]

This shows that the cohomology of the cochain algebra on \( R \) is \( \mathbb{Q}[w^v] \otimes \wedge(t^v) \), with \( w^v \) and \( t^v \) 1-cochains satisfying \( \langle w^v, w \rangle = 1 \) and \( \langle t^v, t \rangle = 1 \). The interpretation of \( H(R, d) \) as the dual of the vector space of indecomposable elements of the Sullivan minimal model of \( C^*(R) \) shows that \( H(R, d) \cong \mathbb{Q}w \oplus \mathbb{Q}t \).

The examination of the short exact sequence of differential complexes

\[
0 \to (R, d) \to (L, d) \to (\mathbb{Qy} \oplus \mathbb{Qz}, 0) \to 0,
\]

shows that \( H(L, d) \cong \mathbb{Q}t \oplus \mathbb{Q}y. \)

This shows that \( H(M) \) is isomorphic to the free product of \( \mathbb{L}(\varphi_1(\beta_i), i \in I \setminus \{i_0\}) \) with the abelian Lie algebra on the two elements \( \varphi_1(a) \) and \( \varphi_1(\langle[a, b]\rangle) \).

### 3.6. End of the proof of Theorem 3.

From the short exact sequence of chain complexes

\[
0 \to M \to E^2 \to E^2_0, \oplus \varphi_2(a)\mathbb{Q} \to 0,
\]

we deduce the isomorphism of graded vector spaces

\[
E^3 = H(E^2, d^2) \cong H(M) \oplus A/(a, \langle[a, b]\rangle).
\]

Since \( E^3 \) is generated by elements in gradation 0 and 1, the spectral sequence degenerates at the term \( E^3, E^3 = E^\infty \). This closes the proof of Theorem 3. \( \square \)

We suppose that the ideal generated by $a$ is composed of $a$ and $b = [a, c]$. We choose an ordered basis $\{u_i\}$, $i = 1, \ldots$ of $L_X$ with $u_1 = c$, $u_2 = a$ and $u_3 = b$. We consider the set of monomials of $UL_X$ of the form $\beta_i = u_{i_1} u_{i_2} \cdots u_{i_n}$ with $i_n \leq i_{n-1} \leq \cdots \leq i_2 \leq i_1$ and $i_j \neq i_{j+1}$ when the degree of $u_j$ is odd. This set of monomials forms a basis of $UL_X$.

The ideal generated by $x$ in $L_X \bigoplus \mathbb{L}(x)$ is then the free Lie algebra on the elements $[x, \beta_i]$. For sake of simplicity, we denote $x' = [x, c]$.

In particular the ideal $J$ generated by the elements $[x, x], [x', x']$ and the $[x, \beta_i]$ for $\beta_i \notin \{1, c\}$ is a differential sub Lie algebra that is a free Lie algebra on two types of elements:

First type: $[x, x], [x, x'], [x', x'][x', [x, x']], [x', [x, x]]$

Second type: $[x, \beta_i], [x, [x, \beta_i]], [x', [x, \beta_i]], [x', [x, [x, \beta_i]]], \beta_i \neq 1, c.$

We have

\[
\begin{align*}
    d([x, x]) &= -2[x, a] \\
    d([x', x']) &= -2[x, cb] \\
    d([x, x']) &= -[x, ca] - [x, b] \\
    d([x', [x, x]]) &= -2[x, [x, b]] + 2[x', [x, a]] \\
    d([x, \beta_i]) &= 0 \\
    d([x, [x, \beta_i]]) &= -[x, \beta_i a] \\
    d([x', [x, \beta_i]]) &= -[x, \beta_i b] \\
    d([x', [x, [x, \beta_i]]]) &= -[x, [x, \beta_i b]] + [x', [x, \beta_i a]] \text{ modulo decomposable elements.}
\end{align*}
\]

Looking at the linear part of the differential we see directly that $H(J)$ is isomorphic to the free Lie algebra on the element $[x, ca]$ and the elements $[x, \beta_i]$ with $\beta_i$ a non empty word in the variables $u_j$ different of $a$ and $b$. \hfill \Box

**Example 6.** Let $X$ be the total space of the fibration with fibre $S^7$ and base $S^3 \times S^5$ whose Sullivan minimal model is $(\wedge(x, y, z), d)$, $d(x) = d(y) = 0$, $d(z) = xy$, $|x| = 3$, $|y| = 5$ and $|z| = 7$. If we attach a cell along the sphere $S^3$ we obtain the space $Y = (S^5 \times S^{10}) \vee S^{12}$. 
BIBLIOGRAPHY


Manuscrit reçu le 3 avril 1995,
accepté le 6 septembre 1995.

Yves FÉLIX,
Institut de Mathématique
2, Chemin du Cyclotron
1348 Louvain-La-Neuve (Belgique).
felix@agel.ucl.ac.be