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ON THE AXIOMATIC OF HARMONIC FUNCTIONS II

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In this paper we shall use constantly the notations and definitions from [2].

1. The fine topology of X is the least fine topology on X, which is finer than the given topology and with respect to which the superharmonic functions (non necessarily defined on the whole space X) are continuous. A set $V \ni x$ is a fine neighbourhood of x if and only if either x is an interior point of V for the initial topology or there exists a superharmonic function s, defined on a neighbourhood of x, such that

$$s(x) < \liminf_{X - V \ni y \to x} s(y).$$

THEOREM 1. — *Any point of* X *possesses a fundamental system of fine neighbourhoods which are compact and connected in the initial topology of* X.

Since the theorem has a local character we may suppose that there exists a harmonic positive function and a positive potential on X. Dividing the sheaf of harmonic functions by a positive harmonic function the fine topology does not change; we may suppose therefore that the constants are harmonic functions.

Let $x \in X$ and V be a fine neighbourhood of x. It is sufficient to suppose that x is not an interior point of V. There exists then a superharmonic function s on X such that

$$s(x) < \liminf_{X - V \ni y \to x} s(y).$$

Let α be a real number

$$s(x) < \alpha < \liminf_{X - V \ni y \to x} s(y)$$

and U be a regular domain containing x such that s is greater than α on $\overline{U} - V$. We denote by F the set

$$F = \{y \in \overline{U} | s(y) \leqslant \alpha\}$$

and by K the component of F containing x. It is sufficient to prove that K is a fine neighbourhood of x.

Let C be a component of F contained in U. There exists then an open set G, $C \subset G \subset \overline{G} \subset U$, $F \cap \partial G = \emptyset$. Since $s > \alpha$ on ∂G it follows $s > \alpha$ on G which contradicts the inequality $s \leqslant \alpha$ on C. Consequently any component of F has a non-empty intersection with ∂U.

Let β and ε be positive numbers such that $s + \beta > \alpha$ on ∂U and $s(x) + \varepsilon\beta < \alpha$. Let K′ be a compact set in $\partial U - K$ such that

$$\omega_x^U(\partial U - (K \cup K')) < \varepsilon.$$

We denote by u the function on $\overline{U}$ equal to

$$y \to \omega_y^U(\partial U - (K \cup K'))$$

on U equal to 1 on $\partial U - (K \cup K')$ and equal to 0 on $K \cup K'$. u is lower semicontinuous. We denote by F_ε the set

$$F_\varepsilon = \{y \in \overline{U} | s(y) + \beta u(y) \leqslant \alpha\}.$$

Obviously $F_\varepsilon \subset F$ and $x \in F_\varepsilon$. There exists an open set G, $K \subset G$, $K' \cap F \cap G = \emptyset$, $F \cap \partial G = \emptyset$. Let $y \in F_\varepsilon \cap G$ and C_y be the component of F_ε containing y. C_y is contained in G since $F_\varepsilon \cap \partial G = \emptyset$. $C_y \cap \partial U$ is not empty, as it was shown above; let $z \in C_y \cap \partial U$. If $z \notin K$ then

$$s(y) + \beta u(y) = s(y) + \beta > \alpha$$

which contradicts the relation $z \in F_\varepsilon$. Hence $z \in K$ and $C_y \subset K$, $y \in K$, $F_\varepsilon \cap G \subset K$. Since

$$\begin{aligned} \liminf_{U - K \ni y \to x} (s(y) + \beta u(y)) &= \liminf_{G - K \ni y \to x} (s(y) + \beta u(y)) \\ &\geqslant \liminf_{G - F_\varepsilon \ni y \to x} (s(y) + \beta u(y)) \geqslant \alpha > s(x) + \beta u(x). \end{aligned}$$

K is a fine neighbourhood of x.

2. We shall suppose in this paragraph that there exists a positive potential on X.

THEOREM 2. — *For any non-negative superharmonic function s on X and any set $E \subset X$, $\hat{R}_s^E$ is equal to s on the fine interior of* E.

Let x be a fine interior point of E. Then [3] (pag. 435)

$$\lim_{U, \mathfrak{U}} \overline{\int}_{(X-E) \cap \partial U} d\omega_x^U = 0,$$

where $\mathfrak{U}$ denotes the filter of sections of regular neighbourhoods of x. If s is bounded in a neighbourhood of x then

$$\begin{aligned} s(x) &\geqslant \hat{R}_s^E(x) = \lim_{U, \mathfrak{U}} \int R_s^E \, d\omega_x^U \\ &\geqslant \lim_{U, \mathfrak{U}} \sup \overline{\int}_{E \cap \partial U} R_s^E \, d\omega_x^U = \lim_{U, \mathfrak{U}} \sup \overline{\int}_{E \cap \partial U} s \, d\omega_x^U \\ &\geqslant \lim_{U, \mathfrak{U}} \int s \, d\omega_x^U - \lim_{U, \mathfrak{U}} \sup \overline{\int}_{(X-E) \cap \partial U} s \, d\omega_x^U = s(x). \end{aligned}$$

In the general case let $\mathcal{S}$ be the set of continuous finite positive superharmonic functions dominated by s. We have

$$s(x) \geqslant \hat{R}_s^E(x) \geqslant \sup_{s' \in \mathcal{S}} \hat{R}_{s'}^E(x) = \sup_{s' \in \mathcal{S}} s'(x) = s(x).$$

COROLLARY 1. — *A polar set has no fine interior points.*

THEOREM 3. — *Let G be a fine open set and s be a non-negative superharmonic function. Then*

a) $\hat{R}_s^G = R_s^G$;

b) $R_{R_s^G}^E = R_s^E$ *for any* $E \subset G$;

c) $R_s^G = \sup\limits_{s' \in \mathcal{S}} R_{s'}^G$, *where $\mathcal{S}$ is an increasingly directed set of superharmonic functions with* $s = \sup\limits_{s' \in \mathcal{S}} s'$ *on* G;

d) $\hat{R}_s^G = \sup\limits_{K \subset G} \hat{R}_s^K$ *where K is compact.*

COROLLARY 2. — *For any fine open set G and any measure μ with compact carrier we have*

$$\int s \, d\mu^G = \int \hat{R}_s^G \, d\mu \ (^1)$$

where s is an arbitrary non-negative superharmonic function.

(1) μ^G is the balayaged measure of μ on G [3] (p. 447).

This relation follows from Theorem 3 *c*) taking $\mathcal{S}$ as the set of all continuous finite positive superharmonic functions smaller than s.

LEMMA 1. — *Let s be a positive superharmonic function on* X *and* $F \subsetneq X$ *be a closed non empty set, non polar if* $X \notin \mathfrak{P}$. *s is resolutive for the normed Dirichlet problem on* $X - F$ *and we have*

$$R_s^F = H_s^{X-F} \text{ (2)}$$

on $X - F$.

Since F is non-polar if $X \notin \mathfrak{P}$, there exists a locally bounded positive potential on any component of $X - F$. Let s_0 be a positive continuous superharmonic function on X. We want to prove that $X - F$ is an MP_0-set [1]. Let $s' \in \overline{\mathcal{S}}_0^{X-F,X}$. Then $s' + \varepsilon s_0$ is non-negative outside a compact set contained in U for any $\varepsilon > 0$. From [2] (Theorem 2) it follows $s' + \varepsilon s_0 \geqslant 0$. ε being arbitrary we get $s' \geqslant 0$ and $X - F$ is an MP_0-set. By [1] (Corollary 3) the restrictions of the functions min (s, ns_0) on ∂U are resolutive. Since min $(s, ns_0) \uparrow s$ for $n \uparrow \infty$ it follows that the restriction of s on ∂U is resolutive.

Let $\bar{s} \in \overline{\mathcal{S}}_s^{X-F,X}$ and s' be the function on X equal to s on F and equal to min $(s, \bar{s})$ on $X - F$. s' is superharmonic and dominates s on F. Hence $\bar{s} \geqslant R_s^F$, $H_s^{X-F} \geqslant R_s^F$ on $X - F$. The converse inequality is trivial.

THEOREM 4. — *Let s be a non-negative superharmonic function. For any* $E \subset X$ *such that s is finite on* E

$$R_s^E = \inf_{G \supset E} R_s^G,$$

where G *is fine open.*

Obviously

$$R_s^E \leqslant \inf_{G \supset E} R_s^G.$$

Let s' be a non-negative superharmonic function on X, $s' \geqslant s$ on E and θ be a real number, $0 < \theta < 1$. The set

$$G = \{x \in X | s'(x) > \theta s(x)\}$$

(2) The normed Dirichlet problem and the associated notions were introduced in [1].

is fine open and contains E. We have $\frac{s'}{\theta} \geqslant s$ on G and therefore

$$\frac{s'}{\theta} \geqslant \mathrm{R}_s^{\mathrm{G}} \geqslant \inf_{\mathrm{G} \supset \mathrm{E}} \mathrm{R}_s^{\mathrm{G}}.$$

s', θ being arbitrary we get

$$\mathrm{R}_s^{\mathrm{E}} \geqslant \inf_{\mathrm{G} \supset \mathrm{E}} \mathrm{R}_s^{\mathrm{G}}.$$

THEOREM 5 [3]. — *Let s_1, s_2 be non-negative superharmonic functions and* E *be an arbitrary set. Then*

$$\mathrm{R}_{s_1+s_2}^{\mathrm{E}} = \mathrm{R}_{s_1}^{\mathrm{E}} + \mathrm{R}_{s_2}^{\mathrm{E}}, \qquad \hat{\mathrm{R}}_{s_1+s_2}^{\mathrm{E}} = \hat{\mathrm{R}}_{s_1}^{\mathrm{E}} + \hat{\mathrm{R}}_{s_2}^{\mathrm{E}}.$$

If E is compact the relation follows from lemma 1. By theorem 3 *d)* it can be extended to fine open sets and by theorem 4 to arbitrary E subjected to the condition that $s_1 + s_2$ is finite on E. In the general case we have

$$\mathrm{R}_{s_1+s_2}^{\mathrm{E}} \leqslant \mathrm{R}_{s_1}^{\mathrm{E}} + \mathrm{R}_{s_2}^{\mathrm{E}}$$

and

$$\mathrm{R}_{s_1+s_2}^{\mathrm{E}} = \mathrm{R}_{s_1}^{\mathrm{E}} + \mathrm{R}_{s_2}^{\mathrm{E}}$$

on E. We denote by E′ the set

$$\mathrm{E}' = \{y \in \mathrm{E} \mid s_1(y) + s_2(y) < \infty\}.$$

Let $x \in \mathrm{X} - \mathrm{E}$. If $\mathrm{R}_{s_1+s_2}^{\mathrm{E}}(x) = \infty$ the required equality holds at x. On the contrary case there exists a non-negative superharmonic function s_0 on X finite at x and $s_0 \geqslant s_1 + s_2$ on E. For any non-negative superharmonic function s on X and any $\varepsilon > 0$ we have

$$\mathrm{R}_s^{\mathrm{E}'} \leqslant \mathrm{R}_s^{\mathrm{E}} \leqslant \mathrm{R}_s^{\mathrm{E}'} + \varepsilon s_0.$$

Hence

$$\mathrm{R}_s^{\mathrm{E}'}(x) = \mathrm{R}_s^{\mathrm{E}}(x).$$

We have therefore

$$\mathrm{R}_{s_1+s_2}^{\mathrm{E}}(x) = \mathrm{R}_{s_1+s_2}^{\mathrm{E}'}(x) = \mathrm{R}_{s_1}^{\mathrm{E}'}(x) + \mathrm{R}_{s_2}^{\mathrm{E}'}(x) = \mathrm{R}_{s_1}^{\mathrm{E}}(x) + \mathrm{R}_{s_2}^{\mathrm{E}}(x).$$

The second equality follows immediately from the first one.

[3] This theorem was proved by R.-M. Hervé under the supplementary hypothesis that X has a countable basis, and either s_1, s_2 are continuous or E is closed or E is open or the axiom D is fulfilled [3].

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