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ON THE AXIOMATIC OF HARMONIC FUNCTIONS I

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The aim of the present paper is to present some remarks on Brelot's axiomatic of harmonic functions [2] and to show that any space, which locally has a countable basis and on which there exists a positive superharmonic function, possesses a countable basis.

1. Let \( X \) be a locally compact connected space and \( \mathcal{H} \) a sheaf on \( X \), of real vector spaces of continuous functions \(^{(1)}\) called harmonic functions. An open set \( U \subset X \) is called regular if it is non-empty, relatively compact and, if for any continuous function \( f \) on the boundary \( \partial U \) of \( U \), there exists a unique function on \( \bar{U} \) equal to \( f \) on \( \partial U \) and harmonic on \( U \), non-negative if \( f \) is non-negative. The restriction of this function on \( U \) will be denoted by \( H_f^U \). For any \( x \in U \) the functional
\[
    f \to H_f^U(x)
\]
is linear and non-negative on the real vector space of continuous functions on \( \partial U \). There exists therefore a measure \( \omega^U_x = \omega_x \) on \( \partial U \), called harmonic measure, such that
\[
    H_f^U(x) = \int f \, d\omega^U_x
\]
for any continuous function \( f \) on \( \partial U \).

We assume that \( \mathcal{H} \) satisfies the following axioms.

\( A_1. \) The regular domains form a basis of \( X \).

\( A_2. \) The limit of any increasing sequence of harmonic functions on a domain is either harmonic or identically infinite.

\(^{(1)}\) The term « function » means, in this paper, « real finite function ».
If \( u \) is a non-negative harmonic function on a domain \( U \), then it follows from A2, considering the sequence \( \{nu\} \), that \( u \) is either positive or identically zero.

**Theorem 1.** — Let \( \mathcal{U} \) be an increasingly directed set of harmonic functions on a domain \( U \); the least upper bound of \( \mathcal{U} \) is either harmonic or identically infinite.

We shall prove the assertion using an idea from R. Nevanlinna ([Uniformisierung, Springer Verlag, 1953]).

Let us suppose the least upper bound of \( \mathcal{U} \) is not identically infinite and let \( x \) be a point of \( U \) at which it is finite. There exists an increasing sequence \( \{u_n\} \) of \( \mathcal{U} \) such that

\[
\lim_{n \to \infty} u_n(x) = \sup_{u \in \mathcal{U}} u(x).
\]

We denote

\[
u_0 = \lim_{n \to \infty} u_n.
\]

Then \( \nu_0 \) is harmonic by A2. Let \( y \) be a point of \( U \) different from \( x \) and \( \{\nu_n\} \) be an increasing sequence of \( \mathcal{U} \) such that \( u_n \leq \nu_n \) and

\[
\lim_{n \to \infty} \nu_n(y) = \sup_{u \in \mathcal{U}} u(y).
\]

We denote

\[
u_0 = \lim_{n \to \infty} \nu_n.
\]

Since \( \nu_0 \) is finite at \( x \) it is harmonic. Obviously \( u_0 \leq \nu_0 \). Since \( u_0 \) and \( \nu_0 \) are equal at \( x \) they coincide everywhere. It follows that

\[
u_0(y) = \sup_{u \in \mathcal{U}} u(y).
\]

\( y \) being arbitrary, \( u_0 \) is the least upper bound of \( \mathcal{U} \).

The theorem shows that axiom A2 is equivalent to axiom 3 [2].

2. A lower semi-continuous numerical function \(^{(2)}\) \( s \) on an open set \( V \), which does not take the value \(-\infty\), is called hyperharmonic if for any regular domain \( U, \overline{U} \subset V \), and \( x \in U \)

\[
s(x) \geq \int s \, d\omega_\nu^U.
\]

\(^{(2)}\) "Numerical function" will mean a function whose values are real numbers or \( \pm \infty \).
A hyperharmonic function on the open set V is called superharmonic if it is not identically infinite on any component of V. A function s is called hypoharmonic (resp. subharmonic) if — s is hyperharmonic (resp. superharmonic). A non-negative hyperharmonic function is called a potential if its greatest harmonic minorant is zero. A set A is called polar if for any \( x \in X \) there exists a positive superharmonic function on a neighbourhood U of \( x \) infinite on \( U \cap A \).

**Theorem 2.** — Let \( U \subset X \) be an open non-compact set on which there exists a positive superharmonic function. Then any superharmonic function on U non-negative outside a compact subset of U is non-negative.

Let \( s_0 \) be a positive superharmonic function on U and \( s \) be a superharmonic function on U non-negative outside a compact set. For any positive real number \( \alpha \) we denote

\[
K_\alpha = \{ x \in U \mid s(x) + \alpha s_0(x) \leq 0 \}.
\]

\( K_\alpha \) is a compact set and we have \( K_\alpha \subset K_\beta \) for \( \alpha \geq \beta \) and

\[
K_\alpha = \bigcap_{\alpha > \beta} K_\beta.
\]

Suppose \( s \) negative at a point. Then since \( \inf s > -\infty \) there exists a real number \( \alpha > 0 \) such that \( K_\alpha \neq \emptyset \) and \( K_\beta = \emptyset \) for \( \beta > \alpha \). The function \( s + \alpha s_0 \) is superharmonic and non-negative. Let \( x \) be a point of \( K_\alpha \) and V the component of U which contains \( x \). Since \( s + \alpha s_0 \) vanishes at \( x \) it vanishes on the whole V ([2] Theorem 3 (i)). It follows \( V \subset K_\alpha \) which is a contradiction since V is non-compact. Hence \( s \) is non-negative.

**Corollary 1.** — Let \( U \subset X \) be an open non-compact set and \( s_0 \) a superharmonic function on U such that

\[
\inf s_0 > 0.
\]

Then any superharmonic function \( s \) on U for which

\[
\lim_{x \to a_U} \inf s(x) \geq 0,
\]

where \( a_U \) is the Alexandroff point of U, is non-negative.

(*) This means either X non-compact or \( U \neq X \).
Indeed for any $\varepsilon > 0$ the function $s + \varepsilon s_0$ is non-negative outside a compact set and therefore non-negative on $U$.

**Theorem 3.** — Let $f$ be a lower semi-continuous numerical function on $X$, which does not take the value $-\infty$. The greatest lower bound of the set of hyperharmonic functions which dominate $f$ is hyperharmonic, continuous at any point $x$ where $f$ is continuous; if, moreover, it is superharmonic and different at $x$ from $f(x)$ then it is harmonic on a neighbourhood of $x$.

Let $\mathcal{G}$ denote the set of hyperharmonic functions which dominate $f$, $s_0$ be its greatest lower bound, $U$ a regular domain and $y \in U$. We have

$$s_0(y) = \inf_{s \in \mathcal{G}} s(y) \geq \inf_{s \in \mathcal{G}} \int s \, d\omega^y \geq \int s_0 \, d\omega^y.$$  

From this relation it follows that the regularised function $\hat{s}_0$ is hyperharmonic ([2] Theorem 7). Since $f$ is lower semi-continuous $s_0 \in \mathcal{G}$, $\hat{s}_0 = s_0$.

Let $x$ be a point at which $f$ is continuous and $s \in \mathcal{G}$ such that $s(x) > f(x)$.

There exists a harmonic functions $u$, defined on a neighbourhood of $x$, for which

$$f(x) < u(x) < s(x).$$

Let $U$ be a regular neighbourhood of $x$, where these inequalities still hold. For any $y \in U$ we have

$$\int s \, d\omega^y \geq \int u \, d\omega^y = u(y) > f(y)$$

and therefore the balayaged function of $s$ relative to $X - U$, $\mathcal{R}_x^{x-U}$, belongs to $\mathcal{G}$. Herefrom it follows that if $s_0$ is superharmonic and $s_0(x) \neq f(x)$, then $s_0$ is harmonic on a neighbourhood of $x$. Further we get

$$\limsup_{y \to x} s_0(y) \leq \limsup_{y \to x} \mathcal{R}_x^{x-U}(y) = \mathcal{R}_x^{x-U}(y) \leq s(x).$$

(4) This theorem was proved in the classical case by M. BRELOT, *Journ. de Math. Pures et Appl.*, 24, 1945, 1-32.
Let $U'$ be a domain, $K$ a compact set in $U'$ and $f'$ the function defined on $U'$ equal to $f$ on $K$ and equal to $s_0$ on $U' - K$. We denote by $g'$ the set of hyperharmonic functions on $U'$ which dominate $f'$ and by $s_0'$ its greatest lower bound. Obviously $s_0' \leq s_0$ on $U'$ and $s_0' = s_0$ on $U' - K$. The function on $X$ equal to $s_0$ on $X - K$ and equal to $s_0'$ on $U'$ is hyperharmonic ([2] Theorem 4) and dominates $f$. Hence $s_0 = s_0'$ on $U'$.

We take $U'$ as being a regular neighbourhood of $x$ and $K$ a compact neighbourhood of $x$. For any $\varepsilon > 0$ we have

$$s_0' + \varepsilon H_1^{U'} \equiv g'$$

and

$$s_0'(x) + \varepsilon H_1^{U'}(x) \neq f'(x).$$

From the preceding considerations we have, since $f'$ is continuous at $x$,

$$\limsup_{y \to x} s_0(y) = \limsup_{y \to x} s_0'(y) \leq s_0'(x) + \varepsilon H_1^{U'}(x) = s_0(x) + \varepsilon H_1^{U'}(x).$$

$\varepsilon$ being arbitrary $s$ is continuous at $x$.

**Corollary 2.** — A superharmonic function which dominates a continuous function is equal to the least upper bound of the set of its continuous finite superharmonic minorants.

**Corollary 3**([2] Proposition 12). — Let $F$ be a closed set with a non-empty interior. If there exists a potential on $X$ then there exists a continuous positive potential on $X$ harmonic on $X - F$.

It is sufficient to take $f$ as being a continuous non-negative function, $f \neq 0$, whose carrier lies in $F$.

3. We shall denote by $\mathcal{B}$ (resp. $\mathcal{H}$) the class of spaces $(X, \mathcal{H})$ for which there exists at least a positive potential (resp. a positive harmonic function) on $X$. The type $\mathcal{B}$ (resp. $\mathcal{H}$) of $X$ is not altered by the multiplication of all the functions of $\mathcal{H}$ by a positive continuous function. An open set $U \subset X$ is said to be of type $\mathcal{B}$ (resp. $\mathcal{H}$) if any component of $U$ belongs to $\mathcal{B}$ (resp. $\mathcal{H}$). The spaces of type $\mathcal{B} \cup \mathcal{H}$ (resp. $\mathcal{B}$) are exactly those on which there exists a positive superharmonic (resp. positive superharmonic non-harmonic) function. On a space of type $\mathcal{H} - \mathcal{B}$ any two positive superharmonic functions are
proportional. If \( X \in \mathcal{P} \) (resp. \( \mathcal{H} \)) and \( U \) is a domain in \( X \), then \( U \in \mathcal{P} \) (resp. \( \mathcal{H} \)). This is trivial for \( \mathcal{H} \) and for \( \mathcal{P} \) it results from the fact that there exists a positive superharmonic function on \( X \) which is not harmonic on \( U \). If \( U \subset X \), \( U \in \mathcal{P} \) and \( X - U \) is polar, then \( X \) also belongs to \( \mathcal{P} \) since any potential can be extended to a superharmonic function on \( X \), ([2] page 125) which is obviously a potential. This result does not hold if we take \( \mathcal{H} \) instead of \( \mathcal{P} \).

Let \( X_1 \) denote the real axis and \( X_2 \) the unit circumference in the complex plane and let \( \mathcal{H}_1 \) (resp. \( \mathcal{H}_2 \)) be the sheaf of solutions of the equation \( u'' + \alpha u = 0 \) on \( X_1 \) (resp. \( X_2 \)), where \( \alpha \) is a real number. If \( \alpha \) is positive, no positive superharmonic function exists on the spaces \( (X_1, \mathcal{H}_1) \), \( (X_2, \mathcal{H}_2) \). If \( \alpha \) is positive and irrational, the only harmonic function on \( X_2 \) is identically zero. For \( \alpha = 0 \) we obtain examples of spaces of the type \( \mathcal{H} - \mathcal{P} \). For \( \alpha < 0 \), \( X_1 \) belongs to \( \mathcal{P} - \mathcal{H} \) and \( X_2 \) belongs to \( \mathcal{P} - \mathcal{H} \). We do not know if there exists non-compact spaces of the type \( \mathcal{P} - \mathcal{H} \).

**Lemma 1** (5). — If \( \mathcal{H} \) satisfies axiom \( A_1 \) the axiom \( A_2 \) is equivalent to the following assertion. Let \( U \) be a domain, \( V \subset U \) an open set, \( K \subset V \) a compact set and \( x \in U \). There exists a positive number \( \alpha = \alpha (U, V, K, x) \) such that for any non-negative superharmonic function \( s \) on \( U \) harmonic on \( V \)

\[
\sup_{y \in K} s(y) \leq \alpha s(x).
\]

Suppose \( A_2 \) fulfilled. If \( \alpha \) does not exist there exists for any natural number \( n \) a non-negative superharmonic function \( s_n \) on \( U \), harmonic on \( V \) and such that

\[
\sup_{y \in K} s_n(y) > n, \quad s_n(x) < \frac{1}{n^2}.
\]

This leads to a contradiction since the function

\[
\sum_{n=1}^{\infty} s_n
\]

is superharmonic on \( U \) and infinite on a component of \( V \).

(5) This lemma was inspired by a similar result of R.-M. Hervé ([3] n° 2; propriété 7).
Suppose now the existence of an α asserted in the lemma, and let \( \{u_n\} \) be an increasing sequence of harmonic functions on \( U \). If there exists a point \( x \in U \) for which \( \{u_n(x)\} \) is convergent, then \( \{u_n(x) - u_{n-1}(x)\} \) converges to zero. Hence

\[
\{u_n - u_{n-1}\}
\]

converges to zero uniformly on any compact set of \( U \). It follows immediately that \( \lim_{n \to \infty} u_n \) is harmonic.

The axiom \( A_2 \) can be strengthened by requiring

\[
\lim_{k \to x} \alpha(U, U, K, x) = 1.
\]

This assertion was called axiom 3' ([2], page 147). From this axiom it follows that the positive harmonic functions on \( U \) equal to 1 at a point of \( U \) form an equicontinuous set of functions.

**Lemma 2.** — Let \( x \in X \), \( I \) be an increasingly directed ordered set and for any \( i \in I \) let \( U_i \) be a domain on \( X \) containing \( x \), \( V_i \) be an open subset of \( U_i \) and \( s_i \) a non-negative superharmonic function on \( U_i \), harmonic on \( V_i \) equal to 1 at \( x \). We suppose \( U_i \subset U_{x_i} \), \( V_i \subset V_x \) for any \( i \leq x \). Let \( \mathcal{U} \) be an ultrafilter on \( I \) finer than the filter of sections of \( I \). If \( \mathcal{H} \) satisfies the axiom 3' then \( s_i \) converges uniformly along \( \mathcal{U} \) on any compact subset of \( \bigcup_{i \in I} V_i \) to a harmonic function.

Let \( K \) be a compact set in \( \bigcup_{i \in I} V_i \) and \( x \in I \) such that \( K \subset V_x \).

Since \( \{s_i| i \geq x\} \) is an equicontinuous family of functions on \( K \), \( s_i \) converges uniformly along \( \mathcal{U} \) on \( K \). Its limit is therefore harmonic on \( \bigcup_{i \in I} V_i \).

**Theorem 4.** — Let \( X \) be non-compact. If \( \mathcal{H} \) satisfies the axiom 3' and any relatively compact domain of \( X \) belongs to \( \mathcal{H} \) then \( X \) belongs to \( \mathcal{H} \).

Let \( x \in X \), \( I \) be the set of relatively compact domains containing \( x \) ordered by the inclusion relation and for any \( i \in I \) denote \( U_i = V_i = : \) and let \( s_i \) be a positive harmonic

\[(\ast) \text{ This is filter generated by the family of sets } \{i \in I | i \geq x\}. \]
function on $U$, equal to $1$ at $x$. By means of lemma 2 one can construct a positive harmonic function on $X$.

**Corollary 4** [3] (7). — If $\mathcal{H}$ satisfies the axiom 3' and $X$ belongs to $\mathcal{B}$ and is non-compact, then $X$ belongs to $\mathcal{H}$.

**Theorem 5** [3] (7). — If $\mathcal{H}$ satisfies the axiom 3' and $X$ belongs to $\mathcal{B}$ then there exists for any point $x \in X$ a positive potential on $X$ harmonic on $X - \{x\}$.

Let $p$ be a positive potential on $X$ and $I$ be the set of compact neighbourhoods of $x$ ordered by the inverse inclusion relation. For any $i \in I$ we denote

$$U_i = X, \quad V_i = X - i,$$

$$s_i = \frac{\hat{R}_p}{\hat{R}_p(x_0)},$$

where $x_0$ is a fixed point different from $x$. Let $U$ be an ultrafilter on $I$ finer than the filter of sections of $I$ and for any $y \in X$

$$s(y) = \lim_{i \in U} s_i(y).$$

By lemma 2 $s$ is harmonic on $X - \{x\}$.

Let $U$ be a regular neighbourhood of $x$ and $y \in U$. Since by lemma 2 $s_i$ converges uniformly along $U$ to $u$ on $\partial U$ we have

$$s(y) = \lim_{i \in U} s_i(y) \supseteq \lim_{i \in U} \int s_i \, d\omega_y = \int s \, d\omega_y.$$

The regularised function $\hat{s}$ of $s$ is therefore superharmonic. From the above uniform convergence we deduce the existence of a positive number $\alpha$ and a $x \in I$ such that

$$s_i \leq \alpha p$$

on $\partial U$ for any $i \geq x$. It follows ([3] Lemma 3.1)

$$s_i \leq \alpha p, \quad s \leq \alpha p$$

on $X - U$. Hence $\hat{s}$ is a potential. It cannot be harmonic on a neighbourhood of $x$ since then it would be harmonic on $X$ and therefore zero.

(7) Théorème 16-1.
Theorem 6. — On a compact space of the type $\mathcal{B}$ any superharmonic function is a potential. Particularly any superharmonic function is non-negative.

If $s$ is a superharmonic function then $-\min(s, 0)$ is a subharmonic function. Since the space is compact and of type $\mathcal{B}$ it is dominated by a potential. Hence it vanishes and $s$ is non-negative. The greatest harmonic minorant of $s$ vanishes being dominated by a potential. $s$ is therefore a potential.

Remark. — A space of the type $\mathcal{B} \cap \mathcal{H}$ is non-compact.

Theorem 7. — Let $X \in \mathcal{B} \cup \mathcal{H}$ and $U$ be a domain on $X$. If $X - U$ is non-polar then $U \in \mathcal{B} \cap \mathcal{H}$.

Let $s$ be a positive superharmonic function on $X$. Suppose its restriction on $U$ is a potential. There exists then a positive superharmonic function $s'$ on $U$ such that

$$\lim_{U \ni x \rightarrow \partial U} s'(x) = \infty$$

([1] Lemma 1). If we extend $s'$ to a function on $X$ equal to $+\infty$ on $X - U$ we obtain a superharmonic function. This is a contradiction since $X - U$ is non-polar. Hence the restriction of $s$ on $U$ is not a potential and $U \in \mathcal{H}$.

$\partial U$ is non-polar. This is obvious if $\partial U = X - U$. If

$$\partial U \neq X - U,$$

$\partial U$ is non-polar since $X - \partial U$ is non-connected. There exists therefore a point $x \in \partial U$ such that the intersection of any neighbourhood $V$ of $x$ with $\partial U$ is non-polar in $V$. Let $V$ be a regular domain which contains $x$ and $K$ be a compact non-polar set, $K \subset V \cap \partial U$. The reduced function $(R^x_K)_V$ of $s$ relative to $K$, where the operation is made on $V$, does not vanish, it converges to zero at the boundary of $V$ and is harmonic on $V - K$. The function $s'$ on $U$ equal to $s$ on $U - V$ and equal to $s - (R^x_K)_V$ on $V \cap U$ is superharmonic and non-proportional to $s$. Hence $U \in \mathcal{B}$.

Corollary 5. — If $X \in \mathcal{B} \cup \mathcal{H}$ and $U$ is an open non-connected set, then $U$ is of the type $\mathcal{B} \cap \mathcal{H}$.

Let $V$ be a component of $U$. Since $X - V$ has interior points it is non-polar. $V$ is therefore of the type $\mathcal{B} \cap \mathcal{H}$. 
4. Lemma 3. — The sum of a sequence of potentials convergent at a point is a potential.

Let \( \{p_n\} \) be a sequence of potentials such that

\[
s = \sum_{n=1}^{\infty} p_n
\]

be finite at a point. Let \( u \) be a harmonic minorant of \( s \). We shall prove inductively that

\[
u \leq \sum_{n=m}^{\infty} p_n.
\]

Suppose

\[
u \leq \sum_{n=m}^{\infty} p_n.
\]

Then \( u - \sum_{n=m+1}^{\infty} p_n \) is a subharmonic minorant of \( p_n \) and therefore non-positive.

A sequence \( \{U_n\} \) of relatively compact domains on \( X \) is called a pseudo-exhaustion of \( X \) if

\[
\overline{U}_n \subset U_{n+1}
\]

for any \( n \) and

\[
X = \bigcup_{n=1}^{\infty} U_n
\]

is polar.

Theorem 8. — Any space of the type \( \mathcal{B} \cup \mathfrak{S} \) possesses a pseudo-exhaustion.

Let \( K \subset X \) be a compact non-polar set such that \( X - K \) contains only a finite number of components, let \( p \) be a positive potential on \( X - K \) and \( \mathcal{B} \) be the set of functions \( (\hat{R}_p^{x-U})_{x-K} \), where \( U \) is a relatively compact domain which contains \( K \). The greatest lower bound of \( \mathcal{B} \), being a non-negative harmonic minorant of \( p \), is equal to zero. There exists therefore a sequence \( \{U_n\} \) of relatively compact domains containing \( K \), such that for any \( n \) \( U_n \subset U_{n+1} \) and

\[
\sum_{n=1}^{\infty} (\hat{R}_p^{x-U_n})_{x-K}
\]

is a superharmonic function on \( X - K \), infinite on \( X - \bigcup_{n=1}^{\infty} U_n \).
**Theorem 9.** — If $X \in \mathcal{B}$ and $\{U_n\}$ is a pseudo-exhaustion of $X$ there exists a continuous potential on $X$, which is infinite exactly on $X - \bigcup_{n=1}^{\infty} U_n$.

Let $p$ be a continuous finite potential on $X$ and, for any $n$, let $f_n$ denote a continuous non-negative function on $X$ equal to 0 on $U_n$, equal at most to $p$ on $U_{n+1} - U_n$ and equal to $p$ on $X - U_{n+1}$. The function

$$p_n = R_{f_n}^X$$

is a continuous finite potential, harmonic on $U_n$ (Theorem 3) and $p_n \geq p_{n+1}$. Let $u$ denote the limit of the sequence $\{p_n\}$. The function $u$ is locally bounded and harmonic on $\bigcup_{n=1}^{\infty} U_n$.

Since $X - \bigcup_{n=1}^{\infty} U_n$ is polar there exists a harmonic function on $X$ equal to $u$ on $\bigcup_{n=1}^{\infty} U_n$. Being a harmonic minorant of $p$ it vanishes. Hence $u$ is equal to zero on $\bigcup_{n=1}^{\infty} U_n$. We may therefore assume that the function

$$p_0 = \sum_{n=1}^{\infty} p_n$$

is finite at a certain point. Since $p_n$ is harmonic on $U_n$, $p_0$ is continuous and finite on $\bigcup_{n=1}^{\infty} U_n$. According to lemma 3 $p$ is a potential and it is equal to infinite on $X - \bigcup_{n=1}^{\infty} U_n$.

**Corollary 6.** — Let $f$ be a finite non-negative upper semicontinuous function on $X \in \mathcal{B}$. $R_f^X$ is the greatest lower bound of the set of continuous hyperharmonic majorants of $f$. If $R_f^X$ is a potential, $R_f^X$ is the greatest lower bound of the set of continuous potentials which dominate $f$.

Let $x \in X$ and $s$ be a superharmonic majorant of $f$. Let further $\{U_n\}$ be a pseudo-exhaustion of $X$, $U_1 \ni x$, $U = \bigcup_{n=1}^{\infty} U_n$ and let $p$ be a continuous potential on $X$ finite at $x$ and equal to $\infty$ on $X - U$. Since $U$ is a normal space there exists a
continuous finite function \( g \) on \( U \), \( f \leq g \leq s \). Let \( g_0 \) be the lower semi-continuous function on \( X \) equal to \( g \) on \( U \) and equal to 0 on \( X - U \). The function \( R_{g_0}^x \) is superharmonic and continuous on \( U \) according to theorem 3. Hence the function \( s_0 = R_{g_0}^x + \varepsilon p \) is a continuous superharmonic majorant of \( f \) for any \( \varepsilon > 0 \) and we have
\[
s_0(x) \leq s(x) + \varepsilon p(x)
\]

In order to prove the last assertion it is sufficient to show that there exists a potential which dominates \( f \). The function
\[
\hat{u} = \lim_{n \to \infty} \hat{R}_f^x(u_n)
\]
is a harmonic function on \( U \). Since \( \hat{R}_f^x \) is locally bounded \( u \) is locally bounded. There exists therefore a harmonic function on \( X \) equal to \( u \) on \( U \). This function is a minorant of \( \hat{R}_f^x \). Hence \( u \) vanishes on \( U \). We may therefore assume that
\[
\sum_{n=1}^{\infty} R_f^x(u_n(x))
\]
is convergent. Let us denote
\[
U_0 = \emptyset, \quad G_n = U_{n+1} - \overline{U}_{n-1}.
\]
We have
\[
\sum_{n=1}^{\infty} \hat{R}_f^g(x) < \infty.
\]
The function
\[
p + \sum_{n=1}^{\infty} \hat{R}_f^g
\]
is a potential which dominates \( f \).

5. **Lemma 4.** — Let \( X \) be a locally compact locally connected space, \( F \) a closed nowhere disconnecting set in \( X \), and \( \aleph \) a cardinal number. If \( X - F \) possesses a basis whose cardinal is at most equal to \( \aleph \) and if there exists a set of continuous functions on \( X \) whose cardinal is at most equal to \( \aleph \) and which separates the points of \( F \), then \( X \) possesses a basis whose cardinal is at most equal to \( \aleph \).

Let \( U \) be an open set on \( X \) and \( \{ U_i \}_{i \in I} \) the family of components of \( U \). Since \( F \) is nowhere disconnecting, \( \{ U_i - F \}_{i \in I} \).
are exactly the family of components of \( U - F \). Since \( X - F \) possesses a basis whose cardinal is most equal to \( \aleph \), the cardinal of I is at most equal to \( \aleph \).

There exists a set \( \mathcal{F} \) whose cardinal is at most equal to \( \aleph \) of continuous functions on \( X \) which separates the points of \( X \). For any \( f \in \mathcal{F} \) and any two rational numbers \( \alpha, \beta \) we denote

\[
U(f; \alpha, \beta) = \{ x \in X | \alpha < f(x) < \beta \}.
\]

Let \( \mathcal{B}' \) denote the family

\[
\mathcal{B}' = \{ U(f; \alpha, \beta) | f \in \mathcal{F}, \alpha, \beta \text{ rational numbers} \}.
\]

The cardinal number of \( \mathcal{B}' \) is at most \( \aleph \). Let us denote by \( \mathcal{B} \) the system of components of the sets of the form \( \bigcap_{i=1}^{n} U_i \), \( U_i \in \mathcal{B}' \). According to the above remark the cardinal number of \( \mathcal{B} \) is at most equal to \( \aleph \).

We want to prove that \( \mathcal{B} \) is a basis of \( X \). Let \( x \) be a point of \( X \) and \( U \) be a relatively compact neighbourhood of \( x \). For any \( y \in \partial U \) let \( f_y \) be a function of \( \mathcal{F} \) such that

\[
f_y(x) \neq f_y(y).
\]

There exist two rational numbers \( \alpha_y, \beta_y \) such that

\[
x \in U(f_y; \alpha_y, \beta_y), \quad y \notin U(f_y; \alpha_y, \beta_y).
\]

Since \( \partial U \) is compact we may find a finite number of points \( \{ y_i | i = 1, \ldots, n \} \) on \( \partial U \) such that

\[
x \in \bigcap_{i=1}^{n} U(f_{y_i}; \alpha_{y_i}, \beta_{y_i}), \quad \left( \bigcap_{i=1}^{n} U(f_{y_i}; \alpha_{y_i}, \beta_{y_i}) \right) \cap \partial U = \emptyset.
\]

Let \( V \) denote the component of \( \bigcap_{i=1}^{n} U(f_{y_i}; \alpha_{y_i}, \beta_{y_i}) \) which contains \( x \). Since \( V \cap \partial U = \emptyset \) we have \( V \subset U \). \( \mathcal{B} \) is hence a basis, since \( V \in \mathcal{B} \).

**Theorem 10.** — Let \( X \in \mathbb{P} \cup \mathbb{Q} \), \( F \) be a closed polar set on \( X \) and let \( \aleph \) be a cardinal number. If for any point of \( X - F \) there exists a neighbourhood which possesses a basis whose cardinal is at most equal to \( \aleph \), then \( X \) possesses a basis whose cardinal is at most equal to \( \aleph \).
Since any space of the type $\mathcal{B} \cup \mathcal{S}$ can be covered with a finite system of domains of the type $\mathcal{B}$, it is sufficient to prove the theorem for the case $X \in \mathcal{B}$.

Let $\{U_n\}$ be a pseudo-exhaustion on $X - F$. Since $F$ is polar $\{U_n\}$ is a pseudo-exhaustion of $X$. We denote $U = \bigcup_{n=1}^{\infty} U_n$ and assume $F = X - \bigcup_{n=1}^{\infty} U$. Let $p$ be a continuous potential on $X$ such that $p$ is infinite exactly on $F$. Since any $U_n$ possesses a basis whose cardinal is at most equal to $\aleph_1$, $U$ possesses a basis $\mathcal{B}$ whose cardinal is at most equal to $\aleph_1$. For any two relatively compact sets $V, W \in \mathcal{B}$, $V \subseteq W$, let $f_{v,w}$ denote a continuous function on $X$, $0 \leq f_{v,w} \leq 1$, equal to $1$ on $V$ and equal to $0$ on $X - W$. We denote by $\mathcal{F}$ the set of functions of the form

$$\max_{1 \leq i \leq n} f_{v_i,w_i}.$$ 

The cardinal number of $\mathcal{F}$ is at most equal to $\aleph_1$. We denote further for any $f \in \mathcal{F}$

$$s_f = R^X_p$$

and

$$\mathcal{I} = \{s_f | f \in \mathcal{F}\}.$$ 

The cardinal number of $\mathcal{I}$ is at most equal to $\aleph_1$. Hence, according to the preceding lemma, it remains only to prove that $\mathcal{I}$ separates the points of $F$.

Let $x, y \in F$, $x \neq y$ and $V$ be a neighbourhood of $x$, $y \notin V$. We denote by $\mathcal{I}_y$ the family of functions $s_f \in \mathcal{I}$ for which the carrier of $f$ is contained in $V$. $\mathcal{I}_y$ is an upper directed family of superharmonic functions. Its least upper bound $s$ is therefore superharmonic. We have

$$s \leq R^X_p, \quad s(y) \leq R^X_p(y) < \infty$$

and $s = p$ on $V - F$. Since $F$ is polar we have $s = p$ on $V$ and therefore $s(x) = \infty$. There exists therefore an $s_f \in \mathcal{I}_y$ such that

$s_f(x) > s_f(y)$.

**Corollary 7.** — *If* $X \in \mathcal{B} \cup \mathcal{S}$ *and any point of* $X$ *possesses a neighbourhood with a countable basis,* $X$ *possesses a countable basis. Particularly if* $X$ *is a manifold, and* $X \in \mathcal{B} \cup \mathcal{S}$, *$X$ possesses a countable basis.*
There exist for any cardinal number $\aleph$ examples of spaces on which the constants are harmonic and which possess points for which the cardinal number of any fundamental system of neighbourhoods is at least equal to $\aleph$. Let $M$ be a set whose cardinal is $\aleph$ and $I$ the set of points of the complex plane $\{e^{\theta} \mid \theta \text{ real number}\}$. For any finite set $I \subseteq M$ we denote by $X_I$ the topological space obtained from the topological space $I' \times I$, where $I$ is considered with the discrete topology, identifying the points $(1, i)$ with $i \in I$. We denote by $a_i$ this point of $X_I$. The harmonic functions on $X_I - \{a_i\}$ will be the functions which are linear in $\theta_i$. A continuous function $u$ defined on a neighbourhood of $a_i$ is harmonic if it is harmonic outside $\{a_i\}$ and for sufficiently small $\varepsilon > 0$

$$u(a_i) = \frac{1}{2n} \sum_{i \in I} [u(e^{i\theta}, \varepsilon) + u(e^{-i\theta}, \varepsilon)],$$

where $n$ is the cardinal number of $I$. It is easy to verify that the harmonic functions satisfy the axioms $A_1, A_2$.

For any $I \subseteq J$ we denote by $\varphi_{IJ}$ the map $X_J \to X_I$ defined by

$$\varphi_{IJ}(z, i) = \begin{cases} (z, i) & \text{if } i \in I; \\ a_i & \text{if } i \notin I; \end{cases} \quad \varphi_{IJ}(a_j) = a_i.$$

The system $\{X_I, \varphi_{IJ}\}$ is a projective system of topological spaces. Let $\{X, \varphi_i\}$, be its projective limit and $a$ the point of $X$ corresponding to the points $a_i$. $X$ is compact and the cardinal number of any fundamental system of neighbourhoods of $a$ is at least equal to $\aleph$. The harmonic functions on $X$ will be the functions of the form $u \circ \varphi_i$, where $u$ is a harmonic function on $X_I$. It can be verified that the sheaf of harmonic functions on $X$ satisfies the required axioms (and even the axiom $3'$).

**Theorem 11. —** The set of non-relatively compact components of an open set on $X \in \mathcal{B} \cup \mathcal{G}$ is at most countable.

Let $\{G_i\}_{i \in I}$ be a family of pairwise disjoint domains on $X$ and $U$ be a relatively compact domain on $X$. We denote by $I_U$ the set of $i \in I$ for which

$$G_i \cap U \neq \emptyset, \quad G_i - \overline{U} \neq \emptyset.$$
For any $i \in I_U$ we denote by $f_i$ the function on $\partial U$ equal to $1$ on $G_i \cap \partial U$ and equal to $0$ on $\partial U - G_i$. This function is resolutive with respect to $U$ [1] and let $H^U_{f_i}$ denote its solution. This function doesn't vanish since in the contrary case there would exist a non-negative superharmonic function $s$ on $U$ converging to infinite at any point of $G_i \cap \partial U$. The function on $U \cup G_i$ equal to $s$ on $U$ and equal to infinite on $G_i - U$ would be a superharmonic function infinite on an open set. This is a contradiction. From

$$\sum_{i \in I_U} H^U_{f_i} \leq H^U_1$$

is follows that $I_U$ is at most countable.

Let $G$ be an open set $\{G_i\}_{i \in I}$ be the family of its non-relatively compact components and $\{U_n\}$ be a pseudo-exhaustion of $X$. From the above proof its follows that $I = \bigcup_{n=1}^{\infty} I_{U_n}$ is at most countable.

BIBLIOGRAPHY

