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DUALITY THEOREMS FOR HARDY AND BERGMAN SPACES ON CONVEX DOMAINS OF FINITE TYPE IN $\mathbb{C}^n$

by S.G. KRANTZ[(*)] and S.-Y. LI

1. Introduction.

Let Ω be a smoothly bounded, convex domain of finite type in $\mathbb{C}^n$ (see [K1] for a discussion of finite type). We define $\mathcal{H}^p(\Omega)$ to be the usual Hardy space of holomorphic functions on Ω (see [K1]). We may identity it as a closed subspace of $L^p(\partial\Omega)$ by passing to the (almost everywhere) radial limit function $\tilde{f}$ on $\partial\Omega$. Let d be a quasimetric on $\partial\Omega$ (see [KL1] or [CHR] for a discussion of quasimetrics). Then $BMO(\partial\Omega)$ can be defined in the usual way, in terms of the quasimetric d and the area measure on $\partial\Omega$: the semi-norm on BMO is

$$\|g\|_{BMO} = \sup_{x,r} \frac{1}{|B(x,r)|} \int_{B(x,r)} |g(t) - g_{B(x,r)}|\, d\sigma(t).$$

Here the balls $B(x,r)$ are defined using the quasimetric, $g_{B(x,r)}$ is the average of g over the ball, $d\sigma$ is $(2n-1)$-dimensional area measure on the boundary of Ω, and $|B(x,r)| = \sigma(B(x,r))$. Of course in practice it is important to select a quasimetric that is compatible with the complex structure.

Now $BMOA(\Omega)$ denotes the space of holomorphic functions in $\mathcal{H}^1(\Omega)$ whose boundary function is in $BMO(\partial\Omega)$ with norm $\|f\|_* = \|\tilde{f}\|_1 +$

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$\|\tilde{f}\|_{BMO}$. It is easy to prove that $BMOA(\Omega)$ is a closed proper subspace of $BMO(\partial\Omega)$.

Let $A^p(\Omega)$ be the holomorphic subspace of the Lebesgue space $L^p(\Omega)$ with respect to Lebesgue volume measure over Ω. Let $\mathcal{B}(\Omega)$ denote the usual holomorphic Bloch space over Ω with norm:

$$\|f\|_{\mathcal{B}(\Omega)} = \sup\{\delta(z)|\nabla f(z)| : z \in \Omega\} + \int_\Omega |f|\, dV < \infty$$

where $\delta(z) = \text{dist}(z, \partial\Omega)$. We say that a holomorphic function $f \in \mathcal{B}_0(\Omega)$ (the little Bloch space) if $f \in \mathcal{B}(\Omega)$ and

$$\lim_{z\to\partial\Omega} \delta(z)|\nabla f(z)| = 0.$$

We will prove the following theorems.

THEOREM 1.1. — *Let Ω be a bounded convex domain of finite type in $\mathbb{C}^n$. Then the dual of $\mathcal{H}^1(\Omega)$ is $BMOA(\Omega)$. Namely, if $g \in BMOA(\Omega)$, then the linear functional on $\mathcal{H}^1(\Omega)$ defined by $l_g(f) = \int_{\partial\Omega} f(w)\overline{g(w)}\, d\sigma(w)$ is bounded, and every bounded linear functional on $\mathcal{H}^1(\Omega)$ arises in this way. Moreover, the BMOA norm of g is comparable to the operator norm of l_g:*

$$C^{-1}\|g\|_* \le \sup\{|l_g(f)| : \|f\|_{\mathcal{H}^1} \le 1\} \le C\|g\|_*.$$

THEOREM 1.2. — *Let Ω be a bounded convex domain of finite type in $\mathbb{C}^n$. Then the dual of $A^1(\Omega)$ is $\mathcal{B}(\Omega)$. Namely, if $g \in \mathcal{B}(\Omega)$, then the linear functional on $A^1(\Omega)$ defined by $l_g(f) = \int_\Omega f(w)\overline{g(w)}\, dV(w)$ is bounded, and every bounded linear functional on $A^1(\Omega)$ arises in this way. Moreover, the $\mathcal{B}$ norm of g is comparable to the operator norm of l_g:*

$$C^{-1}\|g\|_{\mathcal{B}} \le \sup\{|l_g(f)| : \|f\|_{A^1} \le 1\} \le C\|g\|_{\mathcal{B}}.$$

And

THEOREM 1.3. — *Let Ω be a bounded convex domain of finite type in $\mathbb{C}^n$. Then the dual of $\mathcal{B}_0(\Omega)$ is $A^1(\Omega)$. Moreover, the $A^1(\Omega)$ norm of g is comparable to the operator norm of l_g:*

$$C^{-1}\|g\|_{A^1} \le \sup\{|l_g(f)| : \|f\|_{\mathcal{B}_0} \le 1\} \le C\|g\|_{A^1}.$$

We make an effort in this paper to isolate the particular properties of a domain, and of its canonical kernels, that are needed to prove Theorems 1.1–1.3.

In this spirit, in Section 3, we will prove the sufficiency of Theorem 1.1 on a class of "admissible" domains which include smoothly bounded convex domain of finite type in $\mathbb{C}^n$, strictly pseudoconvex domains in $\mathbb{C}^n$, and pseudoconvex domains of finite type in $\mathbb{C}^2$. See [KL1] for related results.

In Section 4, we prove that Theorem 1.2 holds for a class of domains which include the above mentioned three types of domains. The proof of Theorem 1.1 is completed in Section 5. Finally, in Section 6, we prove that Theorem 1.3 holds for a class of domains including the above mentioned domains.

2. The quasimetric for a finite type, convex domain.

For a convex domain of finite type in $\mathbb{C}^n$, let d be the quasimetric defined in [MS1]. For convenience, we now recall it: Let $p \in \partial\Omega$ be fixed with type m, and let U be sufficiently small neighborhood of p. Choose a local defining function r of $\partial\Omega \cap U$ with the property that the sets $\{z \in U : r(z) < \eta\}$ for $-\eta_0 < \eta < \eta_0$, $\eta_0 > 0$ are all convex. For each small $\delta > 0$ and $q \in U$, let $\tau_1(q, \delta)$ be the distance from q to the set $\{z : r(z) = r(q) + \delta\}$. If $p_1 \in \{z : r(z) = r(q) + \delta\}$ such that $\tau_1(q, \delta) = \text{dist}(q, p_1)$, parameterize the complex line from q to p_1 in such a way that q corresponds to the origin (in parametric space) and p_1 lies on the positive real axis of the parameter, this parameter is called coordinate z_1. In complex directions orthogonal to z_1, compute the largest distance: $\tau_2(q, \delta)$ from q to $\{z : r(z) = r(q) + \delta\}$. Let $p_2 \in \{z : r(z) = r(q) + \delta\}$ such that $\tau_2(q, \delta) = \text{dist}(q, p_2)$, we parameterize the complex line from q to p_2 as the coordinate z_2. Taking the orthogonal complement of the span of $\{z_1, z_2\}$ and determining the largest remaining complex distance, we obtain the number $\tau_3(q, \delta)$ and the coordinate z_3. This process may be continued to obtain a full orthogonal coordinate system $(z_1, \cdots, z_n)$ and positive numbers $\tau_1(q, \delta), \cdots, \tau_n(q, \delta)$. We may extend the definition of $\tau_j(q, \delta)$ for δ large and q not close to the boundary by letting $\tau_j(q, \delta) = 1$ for all $1 \leq j \leq n$. The polydisc with respect to q and δ is defined as follows:

$$P_\delta(q) = \{z : |z_i| < \tau_i(q, \delta),\ i = 1, \cdots, n\}$$

where the components of z are measured in terms of the coordinates constructed above. McNeal's construction shows that there are constants $0 < c < C$ such that, if U is a sufficiently small Euclidean ball in $\mathbb{C}^n$, then there is $p \in U \cap \Omega$, and $\epsilon = \text{dist}(p, \partial\Omega)$,

$$P_{c\epsilon}(p) \subseteq U \cap \Omega$$

and

$$P_{C\epsilon}(p) \supseteq U \cap \Omega.$$

The family of balls on $\partial\Omega$ is defined as follows: Let $p \in \partial\Omega$ and $\epsilon > 0$. Then $B(p, \epsilon)$ is defined to be $\{z \in \partial\Omega : z \in P_{C\epsilon}(p)\}$. We define the quasimetric given in [MS1] on $\partial\Omega$ as follows:

$$d(z, w) = \inf\{t : z, w \in B(z, t) \text{ and } z \in B(w, t)\}.$$

One may extend the definition of d to $\overline{\Omega}$ by letting

$$d(z, w) = |r(z)| + |r(w)| + d(\pi(z), \pi(w)), \quad z, w \in \Omega.$$

It is demonstrated in [MCN1], [MNC2], [MS1] that the quasimetric described here is the right one for the study of holomorphic function theory on a finite type convex domain in $\mathbb{C}^n$. In dimension 2, this quasimetric reduces to quasimetrics that are well known (see, for instance, [NRSW]).

3. Duality on admissible domains.

Let Ω be a bounded domain in $\mathbb{C}^n$ with smooth boundary $\partial\Omega$ and let d be a quasimetric on $\partial\Omega$. Let $K(z, w)$ be the Bergman kernel with associated Bergman projection P from $L^2(\Omega)$ to the Bergman space $A^2(\Omega)$. Let S denote the Szegö projection from $L^2(\partial\Omega)$ to $\mathcal{H}^2(\Omega)$. Let $B(z_0, \delta)$ denote the ball in $\partial\Omega$ with center at z_0 and radius δ with respect to d, i.e., the set of points $w \in \partial\Omega$ with $d(w, z_0) < \delta$. For a subset X of $\partial\Omega$, $|X|$ denotes its $2n - 1$ dimensional area measure.

We say that $\partial\Omega$ has a *strongly homogeneous structure* with respect to d if there are constants $\beta > 1$ and $0 < \gamma << 1$ such that the followings conditions are satisfied:

(1) If $B(z_1, r_1) \cap B(z_2, r_2) \neq \emptyset$ and $r_1 \geq r_2$, then $B(z_2, r_2) \subset B(z_1, \beta r_1)$;

(2) $\beta^{-1} d(z, w) \leq |z - w| \leq \beta d(z, w)^\gamma$;

(3) $\beta r^\gamma \geq |B(z, r)| > \beta^{-1} r^{1/\gamma}$ for all $z \in \partial\Omega$ and all $1 > r > 0$;

(4) $|B(z, 2r)| \leq \beta |B(z, r)|$ for all $z \in \partial\Omega$ and all $r > 0$.

We say a bounded domain Ω in $\mathbb{C}^n$ has a *strongly homogeneous structure* if there is a quasimetric d on $\partial\Omega \times \partial\Omega$ such that $\partial\Omega$ has a strongly homogeneous structure with respect to d, moreover, we may extend d to be defined on $\overline{\Omega} \times \overline{\Omega}$ by letting

$$d(z,w) = d(\pi(z),\pi(w)) + \delta(z) + \delta(w)$$

so that

$$u(z)V(B(z,\delta(z)/\beta)) \leq C \int_{B(z,\delta(z)/\beta)} u(w)dV(w)$$

for all plurisubharmonic functions on Ω, where $B(z,\delta(z)/\beta) = \{w \in \Omega : d(w,z) < \delta(z)/\beta\}$–the hyperbolic ball centered at $z \in \Omega$.

If Ω is a smoothly bounded convex domain of finite type in $\mathbb{C}^n$, and if d is the quasimetric defined in Section 2, then it is known that conditions (1)–(4) hold with $\gamma \leq 1/(2nm)$, where m is the maximum type of any boundary point of the domain (see [ST2], [NSW], [MS1]). In other words, $\partial\Omega$ has a strongly homogeneous structure when Ω is a smoothly bounded convex domain of finite type in $\mathbb{C}^n$.

By applying the sub-mean value property in each value separately, we see that the sub-mean value property holds on the polydiscs P_ϵ.

Therefore Ω has strongly homogeneous structure if Ω is a smooth convex domain of finite type in $\mathbb{C}^n$. The concept of homogeneous structure that we introduce here is closely related to, indeed is inspired by, the concept of "space of homogeneous type" as introduced in [CW].

For $z_0 \in \partial\Omega$ and $\delta > 0$, the Carleson region $\mathcal{C}(z_0,\delta)$ is defined by

$$\mathcal{C}(z_0,\delta) = \{z \in \Omega : \pi(z) \in B(z_0,\delta), |r(z)| \leq \delta\} \approx \{z \in \Omega : d(z,\pi(z_0)) < \delta\}.$$

For convenience, from now on, we shall use C to denote a positive constant depending only on β, γ and the domain Ω, but this constant does not always have the same value at each occurrence.

DEFINITION 3.1. — *Let Ω be a bounded domain in $\mathbb{C}^n$ with smooth boundary. We say that Ω is* strongly admissible *if Ω has a strongly homogeneous structure and if the Bergman kernel K for Ω satisfies the following "homogeneity condition":*

$$|K(z,w)| \leq C\big(d(z,w) \cdot |B(\pi(z),d(z,w))|\big)^{-1}. \tag{3.1}$$

Note: If Ω is strongly admissible, then Ω is admissible (the definition of an admissible domain was given in [KL1]).

The main point of this section is to prove:

THEOREM 3.2. — *Let Ω be a strongly admissible domain in $\mathbb{C}^n$ such that $S : L^p(\partial\Omega) \to \mathcal{H}^p(\Omega)$ and $P : C^\infty(\overline{\Omega}) \to W^{1,2}(\Omega)$ are continuous. If $f \in BMOA(\Omega)$, then l_f is linear functional on $\mathcal{H}^1(\Omega)$ satisfying:*

$$\|l_f\|_\infty \leq C\|f\|_{BMO(\partial\Omega)}.$$

We need the following theorem that was proved by the authors in [KL1]:

THEOREM 3.3. — *Let $\Omega \subset \mathbb{C}^n$ be an admissible domain. Suppose that the Szegö projection S maps $L^p(\partial\Omega)$ boundedly onto $\mathcal{H}^p(\Omega)$, for all $p \in [2,\infty)$ and that the Bergman projection $P : C^\infty(\overline{\Omega}) \to W^{1,2}(\Omega)$ is continuous. Then $|\nabla f|^2(z)\delta(z)\,dV(z)$ is a Carleson measure for every $f \in BMOA(\Omega)$.*

As an application of the above theorem, we shall prove the following proposition.

PROPOSITION 3.4. — *Let Ω be a smoothly bounded, convex domain of finite type in $\mathbb{C}^n$. Then*

(a) *Ω is a strongly admissible domain;*

(b) *if $f \in BMOA(\Omega)$, then $|\nabla f(z)|^2\delta(z)\,dV(z)$ is a Carleson measure.*

Proof. — We remarked before that Ω has a strongly homogeneous structure respect to the quasimetric defined in Section 2 or in [MS1]. It was shown in [MCN2] that the Bergman kernel admits the following estimate:

$$|K(z,w)| \leq C\prod_{i=1}^{n}\tau_i(z,\delta)^{-2}$$

where $\delta = d(z,w)$. It is not difficult to see that:

$$\prod_{i=1}^{n}\tau(z,\delta)^{-2} \approx [V(P(z,\delta)\cap\Omega)]^{-1} \approx \Big\{d(z,w)|B(\pi(z),d(z,w))|\Big\}^{-1}.$$

This shows that Ω is a strongly admissible domain, i.e., (a) holds. Now we prove (b). Since Ω is a smoothly bounded convex domain of finite type, we know by a theorem of Catlin [C] that Ω satisfies Condition R.

In order to apply Theorem 3.3, we need the following result from [MS2].

THEOREM 3.5. — *Let Ω be a smoothly bounded convex domain of finite type. Then the Szegö projection is bounded from $L^p(\partial\Omega)$ onto $\mathcal{H}^p(\partial\Omega)$ for $1 < p < \infty$.*

We note that Theorem 3.5 also follows from the $T(1)$ theorem that we develop in Section 5 of the present paper - in particular, see Remark 1 of that section. Thus the arguments presented here are self-contained.

Therefore all the conditions of Theorem 3.3 are satisfied. This proves that $\delta(z)|\nabla f(z)|^2\, dV(z)$ is a Carleson measure over Ω, i.e., (b) holds, and the proof of Proposition 3.4 is therefore complete. □

Let $u(z) \in L^1(\partial\Omega)$. We define a Hardy-Littlewood extension function $M(u)$ of u from $\partial\Omega$ to $\overline{\Omega}_\epsilon$ by

$$M[u](z) = \sup\left\{\frac{1}{|B(z_0,r)|}\int_{B(z_0,r)} |u(\xi)|\, d\sigma(\xi) : B(\pi(z), r(z)) \subset B(z_0,r) \subset \partial\Omega\right\}$$

for $z \in \bar{\Omega}_\epsilon$.

Let $u \in L^1(\Omega)$. We shall use $N(u)(z)$ to denote the radial maximal function on $\partial\Omega$ of u, defined by

$$N(u)(z) = \sup\left\{|u(z + t\nu(z))| : 0 < t < \epsilon\right\}, \quad z \in \partial\Omega.$$

Then we have the following lemma:

LEMMA 3.6. — *Let Ω be a strongly admissible domain in $\mathbb{C}^n$. If $u \in \mathcal{H}^1(\Omega)$, then we have*

$$|u(z)|^{1/2} \leq CM[N(|u|^{1/2})](z) + C\|u\|_{\mathcal{H}^p}, \qquad \text{for all } z \in \Omega_\epsilon$$

where C is a constant depending only on Ω.

Proof. — The assertion is obvious when z is far away from boundary. Thus we may assume that $z \in \Omega$ is near to $\partial\Omega$. From the construction of $B(z)$, we have $B(z) \subset B(\pi(z), \beta\delta(z)) \times [0, \beta r(z)]$. By the plurisubharmonicity of $|u(z)|^{1/2}$ and Ω is strongly admissible, we have

$$\begin{aligned}
|u(z)|^{1/2} &\leq \frac{C}{|B(z)|}\int_{B(z)}|u(w)|^{1/2}\,dV(w)\\
&\leq \frac{C}{|B(z)|}\int_{B(\pi(z),\beta|r(z)|)\times[0,\beta|r(z)|]} N(|u|^{1/2})(w)d\sigma(w)dt\\
&\leq \frac{C\beta}{|B(z)|}|r(z)|\int_{B(\pi(z),\beta|r(z)|)} N(|u|^{1/2})(w)d\sigma(w)\\
&\leq \frac{C}{|r(z)||B(\pi(z),|r(z)|)|}|r(z)|\int_{B(\pi(z),\beta|r(z)|)} N(|u|^{1/2})(w)d\sigma(w)\\
&\leq \frac{C}{|B(\pi(z),\beta|r(z)|)|}\int_{B(\pi(z),\beta|r(z)|)} N(|u|^{1/2})(w)d\sigma(w)\\
&\leq CM[N(|u|^{1/2})](z).
\end{aligned}$$

This completes the proof of Lemma 3.6. □

Now we are ready to prove Theorem 3.2.

Proof. — We shall follow the argument given by Fefferman and Stein in [FS]. For fixed $z_0 \in \Omega$, we let $G(z, z_0)$ be the Green's function for the ordinary Laplacian on Ω. For $u \in \mathcal{H}^1(\Omega)$ we have, by Green's formula, that

$$\begin{aligned}
|\langle f,u\rangle| &= \left|\int_{\partial\Omega}\overline{f}u\,d\sigma(z)\right|\\
&= \left|\int_\Omega \Delta(\overline{f}u)G(z,z_0)dV(z) - \int_\Omega \overline{f}u\Delta G(z,z_0)dV(z)\right|\\
&\leq \left|\int_\Omega 4\sum_{j=1}^n \overline{\frac{\partial f}{\partial z_j}}\frac{\partial u}{\partial z_j}G(z,z_0)dV(z)\right| + C|f(z_0)||u(z_0)|\\
&\leq C\Big(\int_\Omega |\nabla f|^2 r(z)|u(z)|dV(z)\Big)^{1/2}\Big(\int_\Omega |\nabla u|^2|u|^{-1}r(z)dV(z)\Big)^{1/2}\\
&\quad + C(1+|\nabla f|(z_0)|\nabla u|(z_0) + |f(z_0)||u(z_0)|)\\
&\leq C\Big(\int_\Omega |\nabla f|^2|r(z)||u(z)|dV(z)\Big)^{1/2}\Big(\int_\Omega |\nabla u|^2|u|^{-1}|r(z)|dV(z)\Big)^{1/2}\\
&\quad + C|r(z_0)|^{-2n}(1+\|f\|_{\mathcal{H}^1}\|u\|_{\mathcal{H}^1}).
\end{aligned}$$

Now

$$
\begin{aligned}
&\int_{\Omega} |u(z)|^{-1}|\nabla u|^2|r(z)|\,dV(z)\\
&= \int_{\Omega} \Delta|u(z)||r(z)|\,dV(z)\\
&= \int_{\partial\Omega} |u(z)|\,d\sigma(z) + \int_{\Omega} |u(z)|\Delta r(z)\,dV(z)\\
&\le \|u\|_1 + C\|u\|_{\mathcal{H}^1} \le C\|u\|_{\mathcal{H}^1}.
\end{aligned}
$$

We then apply Theorem 2.4' in [H] and Corollary in [ST2], p.40 to obtain the estimate

$$
\begin{aligned}
&\int_{\Omega} |\nabla f|^2(z)|r(z)||u(z)|\,dV(z)\\
&\le \int_{\Omega} |\nabla f|^2(z) r(z) M(N(|u|^{1/2}))^2(z)\,dV(z)\\
&\le C\|f\|_{BMOA}\|N(|u|^{1/2}))\|^2_{L^2(\partial\Omega)}\\
&\le C\|f\|_{BMOA}\||u|^{1/2}\|^2_{L^2(\partial\Omega)}\\
&= C\|f\|_{BMOA}\|u\|_{\mathcal{H}^1}.
\end{aligned}
$$

Thus $|\langle f, u\rangle| \le C\|h\|_{BMOA}\|u\|_1$. Therefore, $l_f \in \left(\mathcal{H}^1(\Omega)\right)^*$; more simply, $f \in \left(\mathcal{H}^1(\Omega)\right)^*$ and $\|l_f\|_\infty \le C\|f\|_{BMOA}$. Therefore, the proof of Theorem 3.2 is complete. □

As a corollary of Theorem 3.2 and Proposition 3.4, we have completed the proof of the sufficiency of Theorem 1.1. □

4. The proof of Theorem 1.2.

In this section we shall prove a general result that includes Theorem 1.2 as a special case. First, let us introduce the following definition.

DEFINITION 4.1. — *Let Ω be a bounded C^2 domain in $\mathbb{C}^n$. We say Ω satisfies Condition D if the Bergman kernel $K(z,w)$ is also a reproducing kernel for $A^1(\Omega)$ and satisfies the following inequality:*

$$
\int_{\Omega} |\nabla_z K(z,w)|\; dV(w)|r(z)| \le C. \tag{4.1}
$$

The main point of this section is to prove:

THEOREM 4.2. — *Let Ω be either a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ or a bounded convex in $\mathbb{C}^n$ satisfying Condition D. Then $(A^1(\Omega))^* = \mathcal{B}(\Omega)$, and the operator norm of l_g is comparable to $\|g\|_{\mathcal{B}}$.*

The proof of Theorem 4.2 is divided into the following lemmas.

LEMMA 4.3. — *Let Ω be a bounded domain in $\mathbb{C}^n$. Then*

(a) $\left(\mathcal{H}^1(\Omega)\right)^* \subset S(L^\infty(\partial\Omega))$;

(b) $\left(A^1(\Omega)\right)^* \subset P(L^\infty(\Omega))$;

Proof. — This is standard, so we present the proof of (a) and leave (b) for the reader. Let l be a linear functional on $\mathcal{H}^1(\Omega)$. Since $\mathcal{H}^1(\Omega)$ is a closed subspace of $L^1(\partial\Omega)$ then, by the Hahn-Banach Theorem, we can extend l to be a linear functional on $L^1(\partial\Omega)$ with the same norm. Since $L^1(\partial\Omega)^* = L^\infty(\partial\Omega)$, there is an $f \in L^\infty(\partial\Omega)$ such that for each $u \in \mathcal{H}^1(\Omega)$ we have

$$l(u) = \int_{\partial\Omega} \bar{f} u \, d\sigma = \int_{\partial\Omega} \bar{f} S(u) d\sigma = \langle S(u), f\rangle = \langle u, S(f)\rangle = \int_{\partial\Omega} \overline{S(f)} u d\sigma(z).$$

Therefore we may identify $S(f)$ as a linear functional on $\mathcal{H}^1(\Omega)$ and $l(u) = \langle u, S(f)\rangle$. Moreover

$$\|l_g\|_\infty = \|f + \text{kernel}(S)\|_\infty,$$

the proof of Lemma 4.3 is thus complete. □

LEMMA 4.4. — *Let Ω be a bounded domain in $\mathbb{C}^n$ that satisfies Condition D. Then $P(L^\infty(\Omega)) = \mathcal{B}(\Omega)$. Moreover,*

$$\|P(f)\|_{\mathcal{B}} \approx \|f + \text{kernel}(P)\|_\infty.$$

Proof. — Applying some simple calculations, one can show that: $P(L^\infty(\Omega)) \subset \mathcal{B}(\Omega)$, and $\|P(f)\|_{\mathcal{B}(\Omega)} \leq C\|f + \text{kernel}(P)\|_\infty$. To prove the converse inclusion, we let $f \in \mathcal{B}(\Omega)$. By a partition of unity, we may choose an open cover $U_0, U_1, \cdots, U_m$ of Ω so that U_0 is relatively compact in Ω and $U_i \cap \partial\Omega \neq \emptyset$. Moreover, there are functions $\mathcal{X}_i$ with support in U_i and

$\sum_{i=0}^{m} \mathcal{X}_i = 1$ on $\overline{\Omega}$, and $\mathcal{X}_0$ has compact support in Ω. Thus

$$\begin{aligned} f(z) &= \int_\Omega K(z,w) f(w) dV(w) \\ &= \sum_{i=0}^{m} \int_\Omega \mathcal{X}_i(z) f(z) K(w,z) \frac{1}{\partial_{z_{k_i}} r(z)} \partial_{z_{k_i}} r(z)\, dV(z) \\ &= -\sum_{i=1}^{m} \Big[\int_\Omega \mathcal{X}_i(z) \frac{1}{\partial_{z_{k_i}} r(z)} \partial_{z_{k_i}} f(z) K(w,z) r(z)\, dV(z) \\ &\quad - \int_\Omega \partial_{z_{k_i}} \Big(\frac{\mathcal{X}_i(z)}{\partial_{z_{k_i}} r(z)} \Big) r(z) f(z) K(w,z)\, dV(z) \Big] \\ &\qquad\qquad + \int_\Omega \mathcal{X}_0(z) f(z) K(w,z) dV(z) \\ &= \int_\Omega K(w,z) \Big[\mathcal{X}_0(z) f(z) - \sum_{i=1}^{m} \frac{\mathcal{X}_i(z) \frac{\partial f(z)}{\partial z_{k_i}} r(z)}{\partial_{z_{k_i}} r(z_i)} \\ &\qquad\qquad + \partial_{z_{k_i}} \Big(\frac{\mathcal{X}_i(z)}{\partial_{z_{k_i}} r(z)} \Big) r(z) f(z) \Big]\, dV(z) \\ &= \int_\Omega K(w,z) g(z) dV(z), \end{aligned}$$

where

$$g(z) = \mathcal{X}_0(z) f(z) - \sum_{i=1}^{m} \frac{\mathcal{X}_i(z) \frac{\partial f(z)}{\partial z_{k_i}} r(z)}{\partial_{z_{k_i}} r(z_i)} + \partial_{z_{k_i}} \left(\frac{\mathcal{X}_i(z)}{\partial_{z_{k_i}} r(z)} \right) r(z) f(z).$$

Since $f \in \mathcal{B}(\Omega)$, we have $g \in L^\infty(\Omega)$ and $\|g\|_\infty \le C \|f\|_{\mathcal{B}(\Omega)}$. Therefore, the proof of Lemma 4.4 is completes. □

Therefore, combining Lemmas 4.3 and 4.4, we have $(A^1(\Omega))^* \subset \mathcal{B}(\Omega)$.

Now we prove the converse. Let us start with the following lemma.

LEMMA 4.5. — *Let Ω be a smoothly bounded domain in $\mathbb{C}^n$ satisfying condition D and $A^p(\Omega)$ is dense in $A^1(\Omega)$ for some $p > 1$. Then $\mathcal{B}(\Omega) \subset A^1(\Omega)^*$.*

Proof. — Let $f \in \mathcal{B}(\Omega)$. Then there is $f_0 \in L^\infty(\Omega)$ such that $P(f_0) = f$. Then l_f on $A^p(\Omega)$ is defined as follows:

$$l_f(u) = \int_\Omega \langle u(w), f(w) \rangle dV(w).$$

By Lemma 4.5, there is an $f_0 \in L^\infty(\Omega)$ such that

$$f(z) = P(f_0)(z).$$

Thus, by Fubini's theorem and condition D, we have

$$\begin{aligned}
|l_f(u)| &= \Big|\int_\Omega \langle u(w), f(z)\rangle dV(z)\Big| \\
&= \Big|\int_\Omega \langle u(w), \int_\Omega K(w,z) f_0(z) dV(z)\rangle dV(w)\Big| \\
&= \Big|\int_\Omega \langle u(w), f_0(w)\rangle dV(w)\Big| \\
&\le \|f_0\|_\infty \|u\|_{A^1} \le C\|f\|_{B(\Omega)}\|u\|_{A^1}.
\end{aligned}$$

This completes the proof of Lemma 4.5. □

LEMMA 4.6. — *Let Ω be a bounded convex domain $\mathbb{C}^n$. Then $\mathcal{H}(\overline{\Omega})$ is dense in $A^p(\Omega)$ for $0 < p < \infty$.*

Proof. — Without loss of generality, we may assume that $0 \in \Omega$. Let

$$\Omega(t) = \{tw : w \in \Omega\}, \quad 0 < t \le 1.$$

It is clear that $\Omega(t_1) \subset \Omega(t_2)$ if $t_1 \le t_2$; and $\Omega = \Omega(1)$. So we have $\lim_{t\to 1^-} \Omega(t) = \Omega$. In other words, $\lim_{t\to 1^-} |\Omega\setminus\Omega_t| = 0$. Thus, for each $f \in A^p(\Omega)$, we let

$$f_t(z) = f(tz), \quad z \in \Omega,\ 0 < t \le 1.$$

It is easy to see that $f_t \in \mathcal{H}(\overline{\Omega})$ for every $0 < t < 1$. Moreover, one can easily prove:

$$\int_\Omega |f_t(z) - f(z)|^p dV(z) \to 0, \quad \text{as } t \to 1^-.$$

This completes the proof of Lemma 4.6. □

Combining Lemmas 4.3–4.6, the proof of Theorem 4.2 is complete. □

If Ω is a smoothly bounded convex domain of finite type in $\mathbb{C}^n$, then Lemma 4 in [MS1] shows that Ω satisfies condition D. Moreover, if Ω is a bounded strictly pseudoconvex domain in $\mathbb{C}^n$, then it satisfies Condition D (see [BEA], [F] or [KL1] for details.) Thus Theorem 4.2 implies Theorem 1.2.

5. The proof of Theorem 1.1.

In this section, we shall complete the proof of Theorem 1.1, that is, we are going to prove that $(\mathcal{H}^1(\Omega))^* \subset BMOA(\Omega)$. By Lemma 4.3 and Theorem 3.2, it suffices to prove: $S(L^\infty(\partial\Omega)) \subset BMOA(\Omega)$.

For convenience, we shall from now on assume that Ω is a bounded, pseudoconvex domain of finite type in $\mathbb{C}^n$ with smooth boundary. By the result of N. Kerzman in [KER] or in [BEL], and the result of Catlin [C] on the regularity of the $\overline{\partial}$-Neumann problem in a domain of finite type, we have, for each $w \in \Omega$, that $K(\cdot, w) \in \mathbb{C}^\infty(\overline{\Omega})$, and Condition R holds.

Let $\nu(z)$ denote the unit inward normal vector to $\partial\Omega$ at $z \in \partial\Omega$. Then we may choose an $\epsilon_0 > 0$ small enough that

$$z = \pi(z) + r(z)\nu(\pi(z)) \tag{5.1}$$

for all $z \in \Omega_{\epsilon_0}$. For each $a \in C^\infty(\overline{\Omega}_{\epsilon_0})$, we define kernels $\widetilde{S}(z,w)$ and $\widetilde{S}_\epsilon(z,w)$ on $\partial\Omega \times \partial\Omega$ by

$$\widetilde{S}(z,w) = \int_0^{\epsilon_0} a(z+t\nu(z))K(z+t\nu(z),w)\,dt, \tag{5.2}$$

and

$$\widetilde{S}_\epsilon(z,w) = \int_\epsilon^{\epsilon_0} a(z+t\nu(z))K(z+t\nu(z),w)\,dt. \tag{5.3}$$

DEFINITION 5.1. — *Let Ω be a smoothly bounded domain of finite type in $\mathbb{C}^n$. We say Ω is a $T1$ domain if there is a $\gamma > 0$ such that:*

$$|\widetilde{S}(z,\xi)| \le C|B(z,d(z,\xi))|^{-1} \qquad z,\xi \in \partial\Omega; \tag{5.4}$$

and

(5.5)
$$|\widetilde{S}(z,\xi) - \widetilde{S}(w,\xi)| + |\widetilde{S}(\xi,z) - \widetilde{S}(\xi,w)| \le C|B(z_0,d(z_0,\xi))|^{-1-\gamma}|B(z_0,\delta)|^\gamma$$

if $z,w \in B(z_0,\delta)$, $\xi \in \partial\Omega - \beta B(z_0,\delta)$ and $z_0 \in \partial\Omega$.

Then we shall prove the following key lemma.

LEMMA 5.2. — *Let Ω be a smoothly bounded convex domain of finite type in $\mathbb{C}^n$. Then Ω is a $T1$ domain.*

Proof. — By Theorem 5.2 [MCN2], we have the estimates

$$|K(z+t\nu(z),\xi)| \le C|B(z,d(z,\xi))|^{-1}d(z,\xi)^{-1}, \quad 0 \le t \le \epsilon$$

and

$$|K(z+t\nu(z),\xi)| \le C|B(z,d(z,\xi))|^{-1}\frac{d(z,\xi)}{t^2}, \quad 0 \le t \le \epsilon$$

for some small positive ϵ depending only on Ω. So we have

$$\begin{aligned}|\widetilde{S}(z,\xi)| \le\ & C\int_0^{d(z,\xi))} |B(z,d(z,\xi)|^{-1}d(z,\xi)^{-1}\,dt \\ & + \int_{d(z,\xi)}^{\epsilon_0} |B(z,d(z,\xi))|^{-1}\frac{d(z,w)}{t^2}\,dt \\ \le\ & C|B(z,d(z,\xi))|^{-1} + C|B(z,d(z,\xi))|^{-1} \\ \le\ & C|B(z,d(z,\xi))|^{-1}.\end{aligned}$$

This completes the proof of (5.4).

Next we prove (5.5). We first consider

$$\begin{aligned}&|\widetilde{S}(z,\xi) - \widetilde{S}(w,\xi)| \\ &= |\int_0^{\epsilon_0} a(z+t\nu(z))K(z+t\nu(z),\xi)\,dt \\ &\quad - \int_0^{\epsilon_0} a(w+t\nu(w))K(w+t\nu(w),\xi)\,dt| \\ &\le \int_0^{\epsilon_0} |a(z+t\nu(z) - a(w+t\nu(w)||K(z+t\nu(z),\xi)|\,dt \\ &\quad + \int_0^{\epsilon_0} |a(w+t\nu(w))|\cdot|K(z+t\nu(z),\xi) - K(w+t\nu(w),\xi)|dt \\ &= I_1(z,w,\xi) + I_2(z,w,\xi).\end{aligned}$$

Observe that

$$\begin{aligned}|a(z+t\nu(z)) - a(w+t\nu(w))| \le\ & C|z-w+t(\nu(z)-\nu(w))| \\ \le\ & C(1+t)|z-w| \\ \le\ & C(1+t)\delta^\gamma \le C\delta^\gamma,\end{aligned}$$

for some $\gamma > 0$ depending only on the type of Ω. (In fact, we may choose $\gamma > 0$ such that $\tau_n(z,\delta) \le C\delta^\gamma$ if $\delta < 1$ and if z is near the boundary.)

Therefore we have

$$\begin{aligned}
I_1(z,w,\xi) \le\ & C\delta^{\gamma}\int_0^{\epsilon_0} |K(z+t\nu(z),\xi)|\,dt \\
\le\ & C\delta^{\gamma}\Big\{\int_0^{d(z,\xi)} |B(z,d(z,\xi))|^{-1}d(z,\xi)^{-1}\,dt \\
& + \int_{d(z,\xi)}^{\epsilon_0} |B(z,d(z,\xi))|^{-1}\frac{d(z,\xi)}{t^2}\,dt\Big\} \\
\le\ & C\delta^{\gamma}|B(z,d(z,\xi))|^{-1} \\
\le\ & C|B(z_0,d(z,\xi))|^{-1}|B(z_0,\delta)|^{\gamma^2} \\
\le\ & C_{\gamma}|B(z_0,d(z_0,\xi))|^{-1-\gamma^2}|B(z,\delta)|^{\gamma^2}.
\end{aligned}$$

Now we turn to the estimate of $I_2(z,w,\xi)$. First we fix $z_0 \in \partial\Omega$. By Theorem 5.2 in [MCN2] there is a neighborhood U of z_0 so that, for all multi-indices α and β, there is a constant C such that

$$|\nabla_z K(z,w)| \le C\sum_{j=1}^{n}\prod_{i=1}^{n}\tau_i(z,d(z,w))^{-2}\tau_j(z,d(z,w))^{-1}.$$

We choose a curve $\phi(s)$ from $[0,1]$ to $\partial\Omega$ such that

$$\phi(0)=z,\ \phi_t(1)=w,\quad \phi(s)=\exp\Big(\sum_{i=1}^{n}\alpha_i(s)z_i\Big)$$

and

$$\int_0^1 |\alpha_j'(s)|\,ds \le C\tau_j(z,\delta),\quad j=1,2,\cdots,n.$$

Therefore

$$\begin{aligned}
&|K(z+t\nu(z),\xi)-K(w+t\nu(w),\xi)| \\
\le\ & \int_0^1\left|\frac{\partial}{\partial s}K(\phi(s)+t\nu(\phi(s)),\xi)\right|\,ds \\
\le\ & \int_0^1\sum_{j=1}^{n}|D_jK(\phi(s)+t\nu(\phi(s))||\alpha_j'(s)|\,ds \\
\le\ & C\sum_{j=1}^{n}\int_0^1\prod_{i=1}^{n}\tau(\phi(s),d(\phi(s),\xi))^{-2}\tau_j(\phi(s),d(\phi(s),\xi))^{-1}|\alpha_j'(s)|\,ds \\
\le\ & C\sum_{j=1}^{n}\prod_{i=1}^{n}\tau_i(z,d(z,\xi))^{-2}\tau_j(z,d(z,\xi))^{-1}\tau_j(z,\delta)).
\end{aligned}$$

If we keep track of t in the above computation, we also have

$$\begin{aligned}
&|K(z+t\nu(z),\xi)-K(w+t\nu(w),\xi)| \\
&\qquad\le C\sum_{j=1}^{n}\prod_{i=1}^{n}\tau_i(z,d(z,\xi))^{-2}t^{-2}\tau_j(z,d(z,\xi))^{-1}\tau_j(z,\delta)d(z,\xi)^2.
\end{aligned}$$

Therefore, since $|B(z,d(z,\xi))| \approx d(z,\xi)\prod_{j=2}^{n}\tau_j(z,d(z,\xi))^2$, we have

$$\begin{aligned} I_2(z,w,\xi) \le\ & C|B(z,d(z,\xi))|^{-1}\sum_{j=1}^{n}\frac{\tau_j(z,\delta)}{\tau_j(z,d(z,\xi))} \\ & \times\left\{\int_0^{d(z,\xi)}\frac{1}{d(z,\xi)}\,dt+\int_{d(z,\xi)}^{\epsilon_0} d(z,\xi)t^{-2}\,dt\right\} \\ \le\ & C|B(z,d(z,\xi))|^{-1}\sum_{j=1}^{n}\frac{\tau_j(z,\delta)}{\tau_j(z,d(z,\xi))} \\ \le\ & C|B(z,d(z,\xi))|^{-1-\gamma^2}|B(z,\delta)|^{\gamma^2}, \end{aligned}$$

for some small positive γ depending only on the type of Ω.

Similar arguments also give the desired estimate for $|\widetilde{S}(\xi,z)-\widetilde{S}(\xi,w)|$. Therefore the proof of Lemma 5.2 is complete. □

To complete the proof of Theorem 1.1, we recall a proposition proved in [KL1] or [NRSW]:

PROPOSITION 5.3. — *Let Ω be a bounded pseudoconvex domain of finite type in $\mathbb{C}^n$ with smooth boundary. Then the Szegö projection has the following property:*

$$S(f) = A(f) + ES(f), \qquad f \in L^2(\partial\Omega),$$

where

$$A = -\sum_{j=1}\frac{\partial r}{\partial z_j} I_{\widetilde{S}_j}, \quad \widetilde{S}_j(z,w) = \int_0^{\epsilon_0}\frac{\partial r}{\partial \bar{z}_j}(z+t\nu(z))K(z+t\nu(z),w)\,dt;$$

and

$$E = -I_{\widetilde{S}_j}\frac{\partial r}{\partial z_j} + \frac{\partial r}{\partial z_j} I_{\widetilde{S}_j} + Q_{\epsilon_0}, \quad Q_{\epsilon_0}(S(f))(z) = S(f)(z+\epsilon_0\nu(z)).$$

PROPOSITION 5.4. — *Let Ω be a $T1$ domain in $\mathbb{C}^n$. Then the singular integral operator A defined above is bounded on $L^p(\partial\Omega)$ for all $1 < p < \infty$.*

Proof. — Since A is a Calderòn-Zygmund type operator on $\partial\Omega$ by Proposition 5.2, the boundedness of A on $L^p(\partial\Omega)$ for all $1<p<\infty$ can be reduced to proving that A is bounded on $L^2(\partial\Omega)$ (see [KL1] for reference). Since the Szegö projection $S : L^2(\partial\Omega) \to \mathcal{H}^2(\partial\Omega)$ is bounded, and the identity of Proposition 5.3, we have

$$A(f) = S(f) - ES(f), \quad f \in L^2(\partial\Omega).$$

Therefore, A is bounded on $L^2(\partial\Omega)$ if and only if ES is bounded on $L^2(\partial\Omega)$. It is obvious that $Q_{\epsilon_0}S(f)(z) = S(f)(z+\epsilon_0\nu(z))$ for all $z \in \partial\Omega$. So $Q_{\epsilon_0}S$ is bounded on $L^2(\partial\Omega)$. Now we may write the operator $E - Q_{\epsilon_0}$ as an integral operator with kernel given by

$$\sum_{j}^{n} N_j(z,w) = \sum_{j=1}^{n}(r_j(w) - r_j(z))\widetilde{S}_j(z,w) \tag{5.6}$$

and

$$|r_j(w) - r_j(z)| \leq C|z-w|.$$

Since Ω is finite type, then we can easily see that

$$\int_{\partial\Omega} |N_j(z,w)|d\sigma(w) + \int_{\partial\Omega} |N_j(z,w)|d\sigma(z) \leq C$$

for all $z, w \in \overline{\Omega}$. The Schur's lemma (se [BL]) shows that $E - Q_{\epsilon_0}$ is bounded on $L^q(\partial\Omega)$ for all $1 \leq q \leq \infty$. Since S is bounded on $L^2(\partial\Omega)$, we have $E\,S$ is bounded on $L^2(\partial\Omega)$. Therefore A is bounded on $L^2(\partial\Omega)$, and the proof of the proposition is complete. □

Remark. — Note that, as a corollary of Lemma 5.2, Propositions 5.3 and 5.4, we obtain a proof of Theorem 3.5.

We need the following theorem from [KL1].

THEOREM 5.5. — *Let Ω be a $T1$ domain in $\mathbb{C}^n$ and let $I_{\widetilde{S}}$ is bounded on $L^2(\partial\Omega)$. Then $I_{\widetilde{S}}(L^\infty(\partial\Omega)) \subset BMO(\partial\Omega)$, and $\|I_{\widetilde{S}}(f)\|_{BMO} \leq C\|f\|_\infty$.*

In fact, one may modify the proof to get $I_{\widetilde{S}}$ is bounded on $BMO(\partial\Omega)$.

Next we prove the main theorem of this section.

THEOREM 5.6. — *Let Ω be a $T1$ domain in $\mathbb{C}^n$. Then $S(L^\infty(\partial\Omega)) \subset BMOA(\Omega)$, and $\|S(f)\|_{BMO} \leq C\|f + \text{kernel}\,(S)\|_\infty$.*

Proof. — Let $f \in L^\infty(\partial\Omega)$. Since

$$S(f)(z) = A(f)(z) + ES(f)(z), \quad z \in \partial\Omega$$

where A and E are given in Proposition 5.2. By Theorem 5.5, we have $I_{\tilde{S}_j}(f) \in BMO(\partial\Omega)$ and $\|I_{\tilde{S}_j}(f)\|_{BMO} \leq C\|f\|_\infty$. Since $\nabla r(z) \in C^1(\overline{\Omega})$, and any BMO function multiplied by a C^1 function still belongs to BMO, then $A(f) \in BMO(\partial\Omega)$, and $\|Af\|_{BMO} \leq C\|f\|_\infty$. By (5.6) and the fact that Ω is finite type domain in $\mathbb{C}^n$ with $|r_j(w) - r_j(z)| \leq C|z-w|$, we have further that $N_j(z,\cdot) \in L^p(\partial\Omega)$ uniformly in $z \in \partial\Omega$ for some $p = 1+\gamma/4 > 1$. That is, we have

$$\|N_j(z,\cdot)\|_{L^p(\partial\Omega)} \leq C, \qquad \text{for all } z \in \partial\Omega.$$

Since $f \in L^\infty(\partial\Omega)$, by Remark 1, we have $S(f) \in L^{p'}(\partial\Omega)$ where p' is the conjugate exponent to p. Therefore $S(f)(\cdot)N_j(z,\cdot) \in L^1(\partial\Omega)$ uniformly for $z \in \partial\Omega$. So we have $E(S(f) \in L^\infty(\partial\Omega)$ since Q_{ϵ_0} is a smoothing operator. Combining the above estimates, the proof of Theorem 5.6 is complete. □

Combining Theorem 5.6 and Theorem 3.2, the proof of Theorem 1.1 is complete. □

6. Proof of Theorem 1.3.

In this section, we shall prove a general result which contains Theorem 1.3. Thus the main point of this section is to prove:

THEOREM 6.1. — *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ satisfies Condition D and Condition R. Then $\mathcal{B}_0(\Omega)^* = A^1(\Omega)$.*

In order to prove Theorem 6.1, let us start with the following lemma:

LEMMA 6.2. — *Let Ω be a smoothly bounded domain in $\mathbb{C}^n$ satisfying conditions D and R. Then $P(C_0(\Omega)) = \mathcal{B}_0(\Omega)$.*

Proof. — First we show $P(C_0(\Omega)) \subset \mathcal{B}_0(\Omega)$. We need only show that:

$$\lim_{z\to\partial\Omega} \delta(z)|\nabla P(f)(z)| = 0,$$

for any $f \in C_0(\Omega)$. Since

$$\begin{aligned}\nabla P(f)(z) &= \int_\Omega \nabla_z K(z,w)f(w)\,dV(w)\\ &= \int_{\Omega_\delta} + \int_{\Omega\setminus\Omega_\delta} \nabla_z K(z,w)f(w)dv(w)\\ &= J_1(z,\delta) + J_2(z,\delta)\end{aligned}$$

and since Ω satisfies Condition R, we have (see [BB])

$$|J_1(z,\delta)| \leq C\delta^{-m_\Omega}$$

where m_Ω is some positive number depending only on Ω. Since Ω satisfies condition D, we have

$$|J_2(z,\delta)| \leq C\delta(z)^{-1}\max\{|f(w)| : w \in \Omega\setminus\Omega_\delta\}.$$

For any $\epsilon > 0$, since $f \in C_0(\Omega)$, there is a $\delta > 0$ such that

$$\max\{|f(w)| : w \in \Omega \setminus \Omega_\Omega\} < \epsilon.$$

Thus

$$|J_2(z,\delta)|\delta(z) \le C\epsilon.$$

Fix this number δ. Then there is $\delta_0 > 0$ such that when $\delta(z) < \delta_0$, then we have

$$\delta(z)C\delta^{-m_\delta} < \epsilon.$$

Thus

$$J_1(z,\delta)\delta(z) \le \epsilon$$

for all $z \in \Omega$ and $\delta(z) < \delta_0$. This proves that $P(f) \in \mathcal{B}_0(\Omega)$.

Next we show that $\mathcal{B}_0(\Omega) \subset P(C_0(\Omega))$. Let $f \in \mathcal{B}_0(\Omega)$. By a partition of unity, we may choose an open cover $U_0, U_1, \cdots, U_m$ of Ω so that U_0 is relatively compact in Ω and $U_i \cap \partial\Omega \ne \emptyset$. Moreover, there are functions $\mathcal{X}_i$ with support in U_i and $\sum_{i=0}^{m} \mathcal{X}_i = 1$ on $\overline{\Omega}$, and $\mathcal{X}_0$ has compact support in Ω. Furthermore, for each U_i $1 \le i \le m$, we may choose a local holomorphic coordinates such that $|\frac{\partial r(z)}{\partial \overline{z}_{k_i}}| \in [1/2, 1]$ for all $z \in U_i$ and some $1 \le k_i \le n$. With the same argument as the proof of Lemma 4.4, we have:

$$\begin{aligned} f(z) &= \int_\Omega K(w,z)f(z)dV(z) \\ &= \int_\Omega K(w,z)g(z)dV(z) \end{aligned}$$

where

$$g(z) = \mathcal{X}_0(z)f(z) - \sum_{i=1}^{m} \frac{\mathcal{X}_i}{\partial_{z_{k_i}} r(z_i)} \partial_{z_{k_i}} f(z) r(z) - \partial_{z_{k_i}}\Big(\frac{\mathcal{X}_i}{\partial_{z_{k_i}} r(z)}\Big) r(z) f(z).$$

Since $f \in \mathcal{B}_0(\Omega)$, we have $g \in C_0(\Omega)$ and $\|g\|_{C(\Omega)} \le C\|f\|_{\mathcal{B}(\Omega)}$. This completes the proof of Lemma 6.2. □

Proof of Theorem 6.1. — First we prove $A^1(\Omega) \subset \mathcal{B}_0(\Omega)^*$ with $\|l_g\|_\infty \le \|g\|_{A^1}$ for all $g \in A^1(\Omega)$. Let $g \in A^1(\Omega)$ and $f = P(f_0) \in \mathcal{B}_0(\Omega)$ with $f_0 \in C_0^\infty(\Omega)$. Thus

$$\Big|\int_\Omega \langle g(w), P(f_0)(w)\rangle dV(w)\Big| = \Big|\int_\Omega \langle g(w), f_0(w)\rangle dV(w)\Big| \le \|f_0\|_\infty \|g\|_{A^1}.$$

Since $C_0^\infty(\Omega)$ is dense in $C_0(\Omega)$, we have $g \in \mathcal{B}_0(\Omega)$. We shall prove the converse.

Let l be a bounded linear functional on $\mathcal{B}_0(\Omega)$. Then we define a linear function L on $C_0(\Omega)$ as follows:

$$L(f) = l(P(f)), \quad f \in C_0(\Omega).$$

By Lemma 6.2, we know that L is bounded linear functional on $C_0(\Omega)$. By Riesz's representation theorem, there is a regular Borel measure $d\mu$ on Ω such that

$$l(P(f)) = L(f) = \int_\Omega f(w)d\mu(w), \quad f \in C_0(\Omega).$$

From the definition of L, one can see that

$$\int_\Omega f(w)d\mu(w) = l(P(f)) = l(0) = 0$$

for all $f \in C_0(\Omega) \cap A^2(\Omega)^\perp$. We claim that $d\mu$ is absolutely continuous with respect to dV. By the Radon-Nikodym decomposition theorem, we have

$$d\mu = u(z)dv(z) + d\mu_0$$

where $d\mu_0$ is a bounded singular measure. We shall prove that $d\mu_0$ is identically zero. Let F be compact set in Ω with $V(F) = 0$. We shall show that $\mu_0(F) = 0$.

Let $g_k \in C_0(\Omega)$ be such that $0 \leq g_k \leq 1$ and $g = 1$ on F and $\lim_{k\to\infty} g_k = \mathcal{X}_F$, the characteristic function of F. Thus

$$l(P(g_k)) = \int_\Omega g_k d\mu = \int_\Omega u(z)g_k\, dV(z) + \int_\Omega g_k d\mu_0(z) \to \int_\Omega \mathcal{X}_F(z)d\mu_0(z)$$

and

$$\begin{aligned} |l(P(g_k))| &\leq \|l\|_\infty \|P(g_k)\|_{\mathcal{B}} \\ &= \|l\|_\infty \sup_{z\in\Omega}\{\delta(z)\Big| \int_\Omega \nabla_z K(z,w) g_k\, dV(w)\Big| \\ &\leq C\text{dist}(F,\partial\Omega)^{-m_\Omega} \int_\Omega g_k\, dV(z) \\ &= C\text{dist}(F,\partial\Omega)^{-m_\Omega} \|g_k\|_{L^1(\Omega)}. \end{aligned}$$

Since $\|\mathcal{X}_F\|_{L^1(\Omega)} = 0$, we have $\lim_{k\to 0} \|g_k\|_{L^1(\Omega)} = 0$, and so

$$\lim_{k\to\infty} \|l(P(g_k))\|_{\mathcal{B}} = 0.$$

Thus $\mu_0(F) = \lim_{k\to\infty} \int_\Omega g_k(z)d\mu_0 = \lim_{k\to\infty} (P(g_k) = 0$ for all compact subsets F in Ω with $V(F) = 0$. This implies that $\mu_0 = 0$. Therefore $d\mu$ is absolutely continuous with respect to dV. Thus

$$L(f) = \int_\Omega u(z)f(z)\, dV(z), \quad \text{and} \quad \|u\|_{L^1(\Omega)} = \|\mu\|.$$

To complete the proof of Theorem 6.1, it suffices to prove that u is holomorphic in Ω. Since

$$\int_\Omega f(w)u(w)\,dV(w) = 0$$

for all $f \in C_0(\Omega) \cap A^2(\Omega)^\perp$, then for any $\phi \in C_0^\infty(\Omega)$ we have

$$\int_\Omega \frac{\partial\phi(z)}{\partial\bar{z}_j} u(z)\,dV(z) = 0$$

(since $\dfrac{\partial\phi}{\partial\bar{z}_j} \in C_0^\infty(\Omega)$ and

$$\int_\Omega \frac{\partial\phi}{\partial\bar{z}_j} f(z)\,dV(z) = -\int_\Omega \phi(z)\frac{\partial f(z)}{\partial\bar{z}_j}\,dV(z) = 0$$

for all $f \in A^2(\Omega)$). Therefore $\bar{\partial}u = 0$ in the sense of distributions. Thus u is holomorphic in Ω and $u \in A^1(\Omega)$. This completes the proof of Theorem 6.1.

It is known from [BEA], [F], [NRSW] and [MS1] and Lemma 4.6, that Ω satisfies conditions D and R if Ω is one of the following three domains: strictly pseudoconvex domains in $\mathbb{C}^n$ (for this case, L. Chen [CHE] has created a different proof by using an asymptotic expansion of the weighted Bergman kernel), convex domains of finite type in $\mathbb{C}^n$, or pseudoconvex domains of finite type in $\mathbb{C}^2$. As a consequence of Theorem 6.1 and the above remark, the proof of Theorem 1.3 is complete. □

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