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UNIFORMIZATION OF THE LEAVES OF A RATIONAL VECTOR FIELD

by A. CANDEL(*) and X. GÓMEZ-MONT(**)

In the present paper we prove uniformization theorems for one dimensional holomorphic foliations with hyperbolic linearizable singularities of a compact complex manifold M. Each leaf of such a foliation is a Riemann surface. If they are hyperbolic (i.e., conformally covered by the unit disc), each has a canonical metric of constant curvature -1, compatible with the complex structure. The uniformization function η ([7], [11]) compares this canonical metric with a metric induced by restriction to the leaves of a Hermitian metric on the ambient manifold M. The main results are :

THEOREM 2.2. — *Let M be a compact complex manifold, $\mathcal{F}$ a holomorphic foliation by curves with linearizable hyperbolic singularities and with hyperbolic leaves. Then the uniformization function η is continuous on $M - \mathrm{Sing}(\mathcal{F})$ and has a continuous extension to M as $\eta(\mathrm{Sing}(\mathcal{F})) = 0$.*

THEOREM 2.3. — *Let $\mathcal{F}$ be a holomorphic foliation by curves with hyperbolic singularities in $\mathbf{CP}(2)$ without leaves with algebraic closure, then the uniformization function η is continuous on $\mathbf{CP}(2)$ with $\eta = 0$ on the singular set of $\mathcal{F}$.*

This last theorem may be applied to the generic foliation in $\mathbf{CP}(2)$. Note that the statement of this theorem contains the fact that if a holomorphic foliation of $\mathbf{CP}(2)$ (with hyperbolic singularities) has an euclidean leaf then it also has an algebraic leaf. It would be interesting

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to know if the uniformization map is continuous when there are algebraic leaves, and also to understand how it varies with the foliation.

We would like to mention that some of the problems discussed here are posed as open questions in the article [6] by D. Cerveau.

Two recent papers discussing uniformization of leaves of holomorphic foliations are [8] and [9], by Glutsuk and Lins Neto, respectively. The method in those papers appears to be a global one, rather than the local approach of this one. We thank the referee for this reference.

1. Uniformization of the local linear hyperbolic foliation.

Let $\Delta^n \subset \mathbf{C}^n$ be the product of the unit disc Δ in $\mathbf{C}$ provided with the norm $|(a_1, \ldots, a_n)| = \max_j(|a_j|)$, where $|\ |$ is the usual norm in the complex numbers $\mathbf{C}$. A diagonal vector field

$$X = \sum_{j=1}^{n} \lambda_j z_j \frac{\partial}{\partial z_j}$$

in $\mathbf{C}^n$ such that any pair of distinct eigenvalues λ_j, λ_k are linearly independent over the real numbers will be called a linear hyperbolic vector field. Let $\mathcal{F}_X$ be the holomorphic foliation by curves defined by X in Δ^n. It has an isolated singularity at 0. Given a point $p = (p_1, \ldots, p_n)$ in $\Delta^n - \{0\}$, the solution of the differential equation $dz/dt = X(z)$ with initial condition $z(0) = p$ is given by

$$z(T) = (p_1 \exp(\lambda_1 T), ..., p_n \exp(\lambda_n T)). \tag{1.1}$$

In the T-plane, there is associated to p a set of real lines $\{\Lambda_p^j\}_{j=1}^n$ formed by those points T such that $|z_j(T)| = 1$. In parametric form the lines are :

$$\left(\frac{1}{|\lambda_j|^2} \log(\frac{1}{|p_j|}) + ir\right) \bar{\lambda_j} \qquad r \in \mathbf{R}. \tag{1.2}$$

The line Λ_p^j has direction vector $i\bar{\lambda_j}$ and the distance from Λ_p^j to the origin is

$$-\frac{1}{|\lambda_j|} \log\left(\frac{1}{|p_j|}\right).$$

If p_j tends to 0, the corresponding lines are parallel, but the distance to the origin is increasing to infinity. If $p_j = 0$, the corresponding line disappears. Conversely, the set of lines determine the absolute values $|p_j|$. Denote by $\mathcal{P}(p)$ the connected component of $\mathbf{C} - \bigcup_{j=1}^{n} \Lambda_p^j$ containing 0. This set $\mathcal{P}(p)$ is a convex (and hence simply connected) domain with boundary a polygon formed by line segments parallel to $i\bar{\lambda}_j$. Not all sides appear in given polygon, reflecting the fact that the corresponding coordinate p_j is comparatively small (i.e. Λ_p^j is far from 0).

Let $\mathcal{P}$ be the open subset of $[\Delta^n - \{0\}] \times \mathbf{C}$ formed by points (p, T) such that T belongs to the polygon $\mathcal{P}(p)$. The projection $\pi : [\Delta^n - \{0\}] \times \mathbf{C} \to \Delta^n - \{0\}$ is a locally trivial fibre bundle with simply connected fibres.

Recall that a uniformization of a Riemann surface S is a holomorphic map from one of the three simply connected Riemann surfaces to S which is also a covering map. It follows from the uniformization theorem for Riemann surfaces [2] that there is a map

$$v : [\Delta^n - \{0\}] \ \times \Delta \to \mathcal{P} \subset [\Delta^n - \{0\}] \times \mathbf{C}$$

over $\Delta^n - \{0\}$ such that $v(p, -) : p \times \Delta \to p \times \mathcal{P}(p)$ is the unifomization map of $\mathcal{P}(p)$, normalized by the conditions $v(p, 0) = (p, 0)$ and $(\partial v/\partial z)(p, 0) > 0$.

We claim that v is a homeomorphism. To obtain an expression for v, recall that the Schwarz–Christoffel formula [1], p. 228, gives an explicit parametrization by a disc Δ of a polygon of n sides as a function of the form :

$$C \int_0^w \prod_{k=1}^{n} (w - w_k)^{-\beta_k} dw + C' \tag{1.3}$$

where β_k are the exterior angles of the polygon, the w_k are points in the unit circle and C and C' are constants. If $\mathcal{P}(p)$ consists of n sides, then continuous coordinates for the polygons with parallel sides are the points $\{w_k(p)\}$ on the unit circle and the constant $C(p)$ in the expression (1.3), since the exterior angles are fixed [1], p. 228. The normalization of the uniformizing map implies that $C' = 0$ and

$$C = r(p) \prod_{k=1}^{n} (-w_k(p))^{\beta_k},$$

where $r(p)$ is a strictly positive real number that depends continuously on p. In conclusion, we obtain the expression

$$(1.4) \qquad v(p,w) = r(p)\left(\prod_{k=1}^{n}(-w_k(p)\right)^{\beta_k}\int_0^w \prod_{k=1}^{n}(w-w_k(p))^{-\beta_k}dw.$$

If w_j and w_k tend to the point w_l, then the corresponding polygons have one side which becomes smaller and smaller until it disappears. We also have

$$(w-w_j)^{-\beta_j}(w-w_k)^{-\beta_k} \to (w-w_l)^{-(\beta_j+\beta_k)}.$$

Since the exterior angles of the polygon have the additivity property $\beta_j+\beta_k=\beta_l$, the integrand in (1.4) is continuous if the polygon has n sides, or in the limiting case when it has m sides but the remaining $n-m$ lines pass through the vertices of the corresponding polygon. For an arbitrary point p in $\Delta^n-\{0\}$ we can increase the components of those coordinates whose line Λ_p^j does not form part of the polygon $\mathcal{P}(p)$ until the corresponding line passes through a vertex of $\mathcal{P}(p)$ to obtain a point q. Then $\mathcal{P}(q)=\mathcal{P}(p)$ and so for expression (1.4) corresponding to the uniformization of p we use the expression for q. This shows that v is a homeomorphism.

The function $z(T)$ in (1.1) restricted to $\mathcal{P}(p)$ is a parametrization of the leaf $\mathcal{L}_p \subset \Delta^n-\{0\}$ of $\mathcal{F}_X$ passing through p. It is a bijective correspondence, except if only one coordinate of p is non–vanishing, in which case the leaf is a punctured disk and the map is the universal covering map.

The function

$$u:[\Delta^n-\{0\}]\times\Delta \longrightarrow [\Delta^n-\{0\}]$$

defined by

$$u(p,w)=(u_{p,1}(w),\ldots,u_{p,n}(w))=(p_1\exp(\lambda_1 v(p,w)),\ldots,p_n\exp(\lambda_n v(p,w))$$

is then the expression of the uniformization of the leaves of $\mathcal{F}_X$ in $\Delta^n-\{0\}$. If s is a positive number and $u_{p,k}$ is a coordinate of a uniformization, then it is identically zero (if the leaf is contained in $z_k=0$) or it is non–vanishing. In the first case $(u_{p,k})^s := 0$ and otherwise $(u_{p,k}^s = \exp(s\log(u_{p,k}))$. We will also use the notation $(u_p)^s=((u_{p,1})^s,\ldots,(u_{p,n})^s):\Delta\to\Delta^n-\{0\}$.

LEMMA 1.1. — *For $p\in\Delta^n-\{0\}$, s a positive real number, u_p the uniformization of the leaf $\mathcal{L}_p$, then the function $(u_p)^s$ is a uniformization of the leaf $\mathcal{L}_{p^s}$.*

Proof. — The distance l_p^j of the lines Λ_p^j to 0 is given by $-\frac{1}{|\lambda_j|}\log\left(\frac{1}{|p_j|}\right)$. Therefore the distance from $\Lambda_{p^s}^j$ to 0 satisfies :

$$l_{p^s}^j = -\frac{1}{|\lambda_j|}\log\left(\frac{1}{|p_j|^s}\right) = -s\frac{1}{|\lambda_j|}\log\left(\frac{1}{|p_j|}\right) = sl_p^j. \tag{1.5}$$

This means that $\mathcal{P}(p^s) = s\mathcal{P}(p)$ and by the uniqueness of the uniformizing map we have $v(p^s, w) = sv(p, w)$. Thus

$$\begin{aligned} u_j(p^s, w) &= p_j^s \exp(\lambda_j v(p^s, w)) \\ &= p_j^s \exp(\lambda_j sv(p, w)) \\ &= (p_j \exp(\lambda_j v(p, w)))^s = u_j(p, w)^s. \end{aligned}$$

PROPOSITION 1.2. — *Let $\{p_m\}$ be a sequence of points in $\Delta^n - \{0\}$ converging to 0, and $u_{p_m} : \Delta \to \Delta^n - \{0\}$ be the uniformizing maps of $\mathcal{L}_{p_m}$, then :*

(1) *$\{u_{p_m}\}$ converges uniformly in compact sets of Δ to 0.*

(2) $\lim_{m\to\infty} (\partial u/\partial w)(p_m, 0) = 0.$

Proof. — The family $\{u_{p_m}\}$ is a normal family, so by taking a subsequence, assume that it converges uniformly in compact sets to a function u_∞. Let $s_m = \frac{\log(1/2)}{\log|p_m|}$ and $q_m = (p_m)^{s_m}$. The points q_m have norm 1/2 and $\lim_{m\to\infty} s_m = 0$.

By Lemma 1.1 the uniformization of $\mathcal{L}_{q_m}$ is $(u_{p_m})^{s_m}$. By taking a subsequence, we may assume that $\lim_{m\to\infty}(u_{p_m})^{s_m} := u'_\infty$ uniformly in compact sets.

From this convergence we obtain that for every $R \in (0,1)$, every $\epsilon > 0$ there exist $R' \in (0,1)$, $N > 0$ such that if $|w| \leqslant R$, $m > N$ then

$$|u_m(w)^{s_m}| \leqslant R', \qquad |u'_\infty(w)| \leqslant R', \qquad |u_m(w)^{s_m} - u'_\infty(w)| < \epsilon. \tag{1.6}$$

Hence we obtain

$$|u_m(w)| \leqslant (R')^{\frac{1}{s_m}}.$$

Since s_m tends to zero, we obtain that u_m converges uniformly to 0 on $|w| \leqslant R$. Hence $u_\infty = 0$. This proves part (1).

To prove part (2), note that from the continuity of the parametrization (1.3) as a function of the polygon it follows that $\lim_{m\to\infty} r(q_m) = r(u'(0)) := r_0$. This means

$$\forall \epsilon > 0 \quad \exists M \quad \text{such that} \quad m > M \Rightarrow \quad |r(q_m) - r_0| < \epsilon.$$

But $r(q_m) = r((p_m)^{s_m}) = s_m r(p_m)$ due to (1.5). Hence $|s_m r(p_m) - r_0| < \epsilon$. Multiplying this inequality by $|p_m|/s_m$ and doing some manipulations, one obtains :

$$|p_m| r(p_m) \leqslant (r_0 + \epsilon)\frac{|p_m|}{s_m} = (r_0 + \epsilon)\frac{|p_m| \log(|p_m|)}{\log(\frac{1}{2})}.$$

Hence

$$\lim_{m\to\infty} |p_m| r(p_m) = 0. \tag{1.7}$$

From (1.4), the definition of u and some elementary calculations one obtains

$$\frac{\partial u}{\partial w}(p, 0) = r(p)(\lambda_1 p_1, \ldots, \lambda_n p_n).$$

Using (1.7) one obtains (2).

Let g_P be the 'metric' on the tangent bundle to $\mathcal{F}_X$ on $\Delta^n - \{0\}$ made up by putting together the metrics of constant curvature -1 on each leaf of the foliation, and let g_L be the metric on the same bundle obtained by restricting a (Hermitian) metric on Δ^n. Since both metrics are conformal on each leaf, we have $g_L = \eta g_P$ for a positive function on $\Delta^n - \{0\}$, smooth on each leaf, which is called the uniformization map of $\mathcal{F}_X$. Then we have $\eta(p) = |(\partial u/\partial w)(p, 0)|$, and so Proposition 1.2 has the following consequence :

COROLLARY 1.3. — *Let $\eta : \Delta^n - \{0\} \to (0, \infty)$ be the uniformization map of a linear hyperbolic vector field. Then η is continuous and has a continuous extension to 0 with $\eta(0) = 0$.*

2. Uniformization of foliations with linearizable hyperbolic singularities.

Let M be a compact complex manifold with a Hermitian metric, $\mathcal{F}$ a holomorphic foliation by curves with isolated singularities, such that at each singular point the foliation is described by a linear hyperbolic vector field. The leaves of the foliation are the leaves in $M - \mathrm{Sing}(\mathcal{F})$ (we do not attach a singular point to any separatrix passing through it). Assume further that all the leaves of $\mathcal{F}$ in $M - \mathrm{Sing}(\mathcal{F})$ are conformally covered by the unit disc (i.e. hyperbolic) and let $\eta : M - \{0\} \to (0, \infty)$ be the uniformization function comparing the complete hyperbolic metric on the leaves g_P with the metric g_L induced on the leaves by the Hermitian metric on $M : g_L = \eta g_P$.

Proposition 2.1. — *Let M be a compact complex manifold, $\mathcal{F}$ a holomorphic foliation by curves with linearizable hyperbolic singularities and with hyperbolic leaves. If $\{u_m\}$ is a family of uniformizations of leaves of $\mathcal{F}$ converging uniformly on compact sets to u, then u is a uniformization of a leaf or a constant in* $\mathrm{Sing}(\mathcal{F})$.

Proof. — Assume first that $u(0) = q$, a singular point of $\mathcal{F}$. Let U be a neighborhood of q with coordinates such that $\mathcal{F}$ is described by a linear hyperbolic vector field as in section 1. The foliation $\mathcal{F}$ restricted to U will be denoted by $\mathcal{G}$. Let $v_m : \Delta \to U$ be the uniformization of the leaf of $\mathcal{G}$ with $v_m(0) = u_m(0)$. Since the leaves of $\mathcal{G}$ are contained in the leaves of $\mathcal{F}$, and due to the explicit nature of $\mathcal{G}$, one sees that there is a conformal map (holomorphic and injective) $\alpha_m : \Delta \to \Delta$ such that $\alpha_m(0) = 0$ and

$$v_m = u_m \circ \alpha_m. \tag{2.1}$$

Since $\{v_m\}$ and $\{\alpha_m\}$ are normal families, by taking a subsequence, one may suppose that $\{v_m\}$ converges uniformly to $v = 0$ (by Proposition 1.2) and α_m to $\alpha : \Delta \to \Delta$ satisfying $v = u \circ \alpha$.

Let $R < 1$ be such that the image by u of the closed disc $\{z \in \Delta;\ |z| \leqslant R\}$ is contained in U. By uniform convergence this also holds for u_m, for m sufficiently large (so assume it is true for every m). Let $\beta : \Delta \to \Delta$ be defined by $\beta(z) = Rz$. Since the image of β is contained in the image of α_m, and α_m is a conformal map, there exists $\gamma_m : \Delta \to \Delta$ such that

$\beta = \alpha_m \circ \gamma_m$. Taking derivatives at 0 we obtain

$$\alpha'_m(0) = \frac{R}{\gamma'_m(0)}.$$

By Schwarz's lemma we have $|\gamma'_m(0)| < 1$, as γ_m cannot be a surjection. Hence

$$R < |\alpha'_m(0)| \leqslant 1 \tag{2.2}$$

and so $|\alpha'(0)| \geqslant R$. This implies that α is non–constant. Since $v = u \circ \alpha$ we conclude that u takes the value q in a neighbourhood of 0, and consequently u is identically q by analytic continuation.

Now assume that $u(0)$ does not belong to $\mathrm{Sing}(\mathcal{F})$. Then the image of u contains no singular points of $\mathcal{F}$, for otherwise u would be identically this singular point by the preceeding argument. Thus the image of u is contained in the leaf $\mathcal{L}_q$. We claim that u is a uniformization of $\mathcal{L}_q$. Let $\tilde{u}$ be a uniformization of $\mathcal{L}_q$ with $\tilde{u}(0) = q$. Then

$$|u'_m(0)| \to |u'(0)| \leqslant |\tilde{u}'(0)|. \tag{2.3}$$

The argument used to prove the lower semicontinuity of η (see [7], [11], [5]) can be applied here to show that we may transport approximations of $\tilde{u}$ to nearby leaves. It then follows that a strict inequality in (2.3) would contradict that u_m are uniformizations. Therefore $|u'(0)| = |\tilde{u}'(0)|$ and so $u = \tilde{u}$ as we wanted.

THEOREM 2.2. — *Let M be a compact complex manifold with a Hermitian metric, $\mathcal{F}$ a holomorphic foliation by curves with linearizable hyperbolic singularities and with hyperbolic leaves. Then the uniformization function η is continuous on $M - \mathrm{Sing}(\mathcal{F})$ and has a continuous extension to M as $\eta(\mathrm{Sing}(\mathcal{F})) = 0$.*

Proof. — Note that the argument to prove the lower semicontinuity of η ([7], [11], [5]) still works here because it reduces to a local question around a compact piece of a leaf.

The argument in Theorem 4.8 of [5], which is based on [3], shows that η is bounded above, for otherwise one could prove that there is a leaf of $\mathcal{F}$ which is not hyperbolic, contradicting the hypothesis. This implies that the family of uniformizations is equicontinuous, and hence it is a normal family. Therefore, by Proposition 2.1, η is continuous on $M - \mathrm{Sing}(\mathcal{F})$.

To analyze the behaviour of η near the critical points, we take sequences and coordinates as in the proof of Proposition 2.1. By (2.1) and (2.2) we obtain

$$R|u_m'(0)| \leqslant |v_m'(0)| \leqslant |u_m'(0)|. \tag{2.4}$$

By part (2) of Proposition 1.2 we have $\lim_{m\to\infty} v_m'(0) = 0$ and using (2.4) we obtain that $\lim_{m\to\infty} u_m'(0) = 0$.

Hence η extends continuously to the singular points as $\eta = 0$ there and the theorem is proved.

In case M is the projective plane $\mathbf{CP}(2)$, the condition of hyperbolic leaves can be somehow relaxed. We have :

THEOREM 2.3. — *Let $\mathcal{F}$ be a holomorphic foliation by curves with hyperbolic singularities in $\mathbf{CP}(2)$ and with no leaf with algebraic closure. Then the uniformization function η is continuous on $\mathbf{CP}(2)$ with $\eta(\mathrm{Sing}(\mathcal{F})) = 0$.*

Proof. — First we show that there is no invariant transverse measure for $\mathcal{F}$ in $\mathbf{CP}(2) - \mathrm{Sing}(\mathcal{F})$. Near the singular points, the transverse measure has support only at the separatrices, since there is a hyperbolic holonomy on each separatrix. The hypothesis of no algebraic leaves implies that the measure cannot be a Dirac delta on a separatrix. So, the transverse measure has support on the complement of the basin of attraction of the critical points. This set (the support of the measure) is a Riemann surface lamination and it is shown in [4] that there are no invariant transverse measures there.

The proof of the theorem would be complete if we show that every leaf is hyperbolic. Suppose there is an euclidean leaf and let $B(n)$ be the averaging sequence it carries ([5]; Ahlfors' lemma). The sequence of currents in $\mathbf{CP}(2)$,

$$T_n(\cdot) = \frac{1}{\mathrm{Area}(B(n))} \int_{B(n)} (\cdot)$$

converges to a closed current T in $\mathbf{CP}(2)$. This is a non-trivial current. Indeed, if ω denotes the Kaehler form of projective space, then

$$T_n(\omega) = T(\omega) = 1$$

by Wirtinger's theorem.

Although the foliated manifold we are working on is not compact, the foliation current T still defines a measure on compact discs transverse to $\mathcal{F}$ in $\mathbf{CP}(2) - \mathrm{Sing}(\mathcal{F})$, which is invariant by holonomy (see for example the proof of the Riesz representation theorem in [10]). In the non-compact case it is not automatic that T is nontrivial (as a current acting on forms with compact support in $\mathbf{CP}(2) - \mathrm{Sing}(\mathcal{F})$), and that is what we show next. Let C be a projective line in $\mathbf{CP}(2)$ not containing any of the singularities. Then the Poincaré dual of C may be represented by the Kaehler form ω, but also by a closed form σ with support contained in a small tubular neighborhood of C. Since ω and σ are cohomologous, $T(\sigma) = T(\omega) = 1$.

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