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## THE MARTIN COMPACTIFICATION OF A PLANE DOMAIN

by Nikolai S. NADIRASHVILI

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In this note we prove the following

**THEOREM.** — *The Martin compactification of a plane domain is homeomorphic to a subset of the two-dimensional sphere.*

**ASSUMPTIONS.** — *If  $\Omega$  be a plane domain and  $\mathbb{R}^2 \setminus \Omega$  is polar then any positive harmonic function on  $\Omega$  is a constant. In this case we define the Martin compactification of  $\Omega$  as a one point set. So we assume from now on that  $\mathbb{R}^2 \setminus \Omega$  is non-polar. We may also assume without loss of generality that  $\bar{B}_1 \subset \Omega$  where  $B_1$  is the unit disk in  $\mathbb{R}^2$  with the center at 0.*

**Remark.** — *If a simply connected domain is a proper subset of the plane then by Riemann mapping theorem its Martin compactification is homeomorphic to a closed disk.*

**CONJECTURE 1.** — *The Martin compactification of a subdomain of a compact Riemannian surface is homeomorphic to a subset of this surface.*

**CONJECTURE 2.** — *Any compact metrizable space can be represented as the Martin boundary of a certain (generally of infinite genus) Riemannian surface.*

### 1. The Martin compactification.

Let  $G(x, y)$  be the Green function of the Dirichlet Laplacian on  $\Omega$ , with the pole at  $x$ . Let us denote  $g_x(y) = G(x, y)/G(x, 0)$  for  $x \neq 0$  and

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$g_0 \equiv 0$ . Let  $\tilde{g}_x(y)$  be the restriction of the function  $g_x(y)$  on  $y \in B_1$ . So we have a map

$$\gamma : x \rightarrow \tilde{g}_x \in L^2(B_1).$$

The Martin metric on  $\Omega$  can be defined as the metric inducted on  $\Omega$  by the map  $\gamma : \Omega \rightarrow L^2(B_1)$ , (cf. [1]). Compactification of  $\Omega$  in the Martin metric we denote as  $\Omega^M$ .

*Canonical map.*

We set

$$f : x \rightarrow \nabla_y g_x(0)$$

and  $f(0) = \infty$  by the definition. We claim that the introduced canonical map  $f$  has the uniformly continuous inverse map from  $f(\Omega)$  to  $\Omega^M$ .

*Proof of the theorem.*

**1.1.** Let  $G \subset \mathbb{R}^2$  be a domain and  $Q$  a disk such that  $\bar{Q} \subset G$ . Also, let  $a_i \in \partial Q, i = 1, \dots, 2n$ , be distinct points on  $\partial Q$ . We assume that the  $a_i$  are indexed in the order in which they are encountered when traversing  $\partial Q$ . Let  $f$  be a continuous function in  $G \setminus \bar{Q}$  such that  $f(a_i)f(a_{i+1}) < 0$  for all  $i = 1, \dots, 2n - 1$ . We denote by  $G_i \subset G \setminus \bar{Q}$  the domain where  $f$  does not change sign, such that  $a_i \in \bar{G}_i$ .

**Lemma** ([2]). — *At least  $n + 1$  of the domains  $G_i, i = 1, \dots, 2n$ , are distinct.*

**1.2.** Let  $x_1, x_2 \in \Omega$ . We prove that if  $f(x_1) = f(x_2)$  then  $x_1 = x_2$ .

Let  $u = g_{x_1} - g_{x_2}$ . Then  $\Delta u = 0$  in  $\Omega \setminus (\{x_1\} \cup \{x_2\})$ ,  $u(0) = \nabla u(0) = 0$ . Let  $\Gamma$  be the nodal set of  $u$ ,  $\Gamma = \{x \in \Omega, u(x) = 0\}$ . If  $u \not\equiv 0$  then in a neighborhood of 0,  $\Gamma$  consists of  $n$  smooth curves intersected at the point 0, where  $n$  is an order of vanishing of the function  $u$  at 0 (cf. [2]). By Lemma  $\Gamma$  splits the domain  $\Omega$  at least on three distinct subdomains. By maximum principle each of those subdomains should contain a pole of the function  $u$ . Since function  $u$  has only two poles  $x_1, x_2$ , it follows that  $u \equiv 0$ .

**2.** Now we prove that the map

$$F : z = f(x) \in f(\Omega) \rightarrow \tilde{g}_x$$

is uniformly continuous.

**2.1.** Let  $\bar{B}_1 \subset B, \bar{B} \subset \Omega$ . By Harnak inequality for any  $x \in \Omega \setminus B, \tilde{g}_x < C$ , where  $C > 0$  is some constant.

**2.2.** Let  $x_n, z_n \in \Omega, n = 1, 2, \dots$ , and  $g_{x_n} \rightarrow h_1, g_{z_n} \rightarrow h_2$  on any compact in  $\Omega$  as  $n \rightarrow \infty, h_1 \neq h_2$ . Its required to prove that  $\nabla h_1(0) \neq \nabla h_2(0)$ . Let us assume the contrary, namely that  $\nabla h_1(0) = \nabla h_2(0)$ . We denote  $h = h_1 - h_2$  and let  $k$  be an order of vanishing of the function  $h$  at 0,  $k \geq 2$ .

**2.3.** Let  $\Gamma$  be the nodal set of the function  $h$ . There exists such a small  $\rho > 0$  that on  $S_\rho = \partial B_\rho, |\nabla h| > 0$  and the cardinality of the set  $S_\rho \cap \Gamma$  is equal to  $2k$ .

**2.4.** We prove the existence of two bounded non-constant harmonic functions  $v_1, v_2$  in  $\Omega$ , such that  $\nabla v_1(0) \neq 0, \nabla v_2(0) \neq 0, \nabla v_1(0) \neq a \nabla v_2(0)$ , for any  $a \in \mathbb{R}$ .

Let us choose discs  $D_1, D_2, D_3 \subset \mathbb{R}^2$  such that  $D_i \setminus \Omega$  non-polar,  $i = 1, 2, 3$ , and for any points  $x_i \in D_i$  the quadrangle  $0, x_1, x_2, x_3$  is convex. Let  $\mu_i$  be a probability measure on  $D_i \setminus \Omega$  such that the convolution  $\ln |x| * \mu_i$  is bounded from below. We set  $v_1 = \ln |x| * (\mu_1 - \mu_2), v_2 = \ln |x| * (\mu_3 - \mu_2)$ . Then  $v_1, v_2$  are bounded harmonic functions in  $\Omega$  and the  $\nabla v_1(0), \nabla v_2(0)$  have the required property.

For any  $\alpha \in \mathbb{R}^2$  there exists a unique linear combination

$$w_\alpha = \beta_1 v_1 + \beta_2 v_2 - \beta_1 v_1(0) - \beta_2 v_2(0)$$

such that  $\nabla w_\alpha(0) = \alpha, w_\alpha(0) = 0$ . Further, if  $|\alpha| \rightarrow 0$  then  $|w_\alpha| \rightarrow 0$  uniformly in  $\Omega$ .

**2.5.** Let us denote

$$\nabla g_{x_n}(0) - \nabla g_{z_n}(0) = \alpha_n,$$

$$q_n = g_{x_n} - g_{z_n} - w_{\alpha_n}.$$

Then  $q_n(0) = \nabla q_n(0) = 0$  for all  $n = 1, 2, \dots$

From (2.1), (2.2), (2.4) it follows that  $q_n \rightarrow h$  in  $B_1$  and hence also  $q_n \rightarrow h$  in  $C^1(B_\rho)$  as  $n \rightarrow \infty$ . Therefore, if  $\Gamma_n$  is a nodal set of  $q_n$  then

for a sufficiently large  $n \geq N$ ,  $S_\rho \setminus \Gamma_n$  is a union of  $2k$  distinct intervals  $I_n^1, \dots, I_n^{2k}$  and

$$\sup_{n \geq N} \inf_{1 \leq j \leq 2k} \sup_{I_n^j} q_n > a > 0$$

with some constant  $a$ . Since  $w_{\alpha_n} \rightarrow 0$  uniformly in  $\Omega$  as  $n \rightarrow \infty$  then for sufficiently large  $n \geq N' \geq N$ ,  $|w_{\alpha_n}| < a$  in  $\Omega$ . Hence  $|q_n| < a$  on  $\partial\Omega$  for  $n \geq N'$ .

**2.6.** Since  $q_n(0) = \nabla q_n(0) = 0$  then by Lemma the set  $\Omega \setminus \Gamma_n$  contains at least three components  $G_1, G_2, G_3$  such that  $0 \in \bar{G}_i$ ,  $i = 1, 2, 3$ . From (2.5) it follows that for  $n \geq N'$  and  $i = 1, 2, 3$

$$\sup_{G_i \cap B_\rho} |q_n| > \sup_{\partial G_i} |q_n|.$$

By the maximum principle from the last inequality it follows that any of the domains  $G_i$ ,  $i = 1, 2, 3$  contains a pole of function  $q_n$ . Since the function  $q_n$  has only two poles we get a contradiction which proves the theorem.

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