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CHERN NUMBERS OF A KUPKA COMPONENT

by O. CALVO-ANDRADE and M. SOARES

0. Introduction.

A codimension one foliation $\mathcal{F}$ in the projective space $\mathbb{P}^n$ has *degree* k, if it is represented by a section $\omega \in H^0\big(\mathbb{P}^n, \Omega^1(k)\big)$, such a section, may be given by a dicritical 1–form in $\mathbb{C}^{n+1}$, homogeneous of degree $(k-1)$.

The *singular set* of the foliation $\mathcal{F}$ is the set $S(\mathcal{F}) = \{p \,|\, \omega(p) = 0\,\}$, and the *Kupka set* is the set $K(\mathcal{F}) = \{p\,|\omega(p) = 0 \quad d\omega(p) \neq 0\}$.

In [CL] it is shown that if $\mathcal{F}$ has a compact Kupka component K which is a complete intersection, then $\mathcal{F}$ has a meromorphic first integral, and in the same paper, they prove that the twisted cubic $C \subset \mathbb{P}^3$, cannot be the Kupka set of any foliation. This motivates the following question :

Is the Kupka set a complete intersection? The aim of this work is to consider this question, our main result in this direction is :

4.4. THEOREM. — *Let $\mathcal{F}$ be a codimension one holomorphic foliation of degree k in $\mathbb{P}^n$ with a Kupka component K. Then K is numerically a complete intersection.*

This result implies an affirmative answer to the question in some cases. Let K be a compact Kupka component of a degree k foliation $\mathcal{F}$. We will show that the transversal type is $\eta = pxdy - qydx$ where p, q are relatively prime positive integers, $1 \leq p < q$ or $p = q = 1$. Then we have

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4.5. Corollary. — *Let $\mathcal{F}$ be as above. If one of the conditions below is satisfied :*

(1) *If* $k \leqq 2(n+1)$ *and* $n \geqq 6$;

(2) $k \leqq 2(n-1)$ *and* $(n \geqq 3)$;

(3) $\min\left\{\dfrac{p\cdot k}{p+q}, \dfrac{q\cdot k}{p+q}\right\} \leqq n-1$;

then K is a complete intersection.

The proof uses a *Koszul resolution* of the sheaf of ideals $\mathcal{J}_K$ of K and the Baum–Bott residues. First, we will show that the Kupka set is *subcanonically embedded*, that is :

$$\Omega_K^{n-2} = \left(\Omega_{\mathbb{P}^n}^n \otimes \mathcal{O}(k)\right)|_K$$

by a Serre's construction (Theorem 2.1), the normal bundle of the Kupka set can be extended to a rank–2 holomorphic vector bundle $E \to \mathbb{P}^n$, and K can be viewed as the zero locus of a global section σ of E. Many properties of K are strictly related with the properties of E. The section σ gives the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\cdot\sigma} E \to \mathcal{J}_K(k) \to 0,$$

and E has total Chern class given by $c(E) = 1 + k\cdot\mathbf{h} + \deg(K)\cdot\mathbf{h}^2$. The second Chern class $c_2(E) = \deg(K)$, is computed with the Baum–Bott formula; thus it depends on the transversal type at K, and the degree of the foliation. Then, K is a complete intersection if and only if E splits in a direct sum of line bundles.

The vector bundle E associated to K is unstable and has Chern classes compatible with the splitting, which implies Theorem 4.4. The proof of the Corollary follows by an application of some well known results.

1. Rank two vector bundles on $\mathbb{P}^n$.

In this section, we will give some general results on rank–2 holomorphic vector bundles on $\mathbb{P}^n$. By a *sheaf* we mean a coherent analytic sheaf of $\mathcal{O}$ modules. No distinction will be made between holomorphic vector bundles and locally free analytic sheaves. $\mathcal{O}(1)$ is the line bundle having a holomorphic section vanishing on a hyperplane. $E(m) := E \otimes \mathcal{O}(1)^{\otimes m}$.

Recall that cohomology ring $H^*(\mathbb{P}^n, \mathbb{Z})$ is isomorphic to the ring $\mathbb{Z}[\mathbf{h}]/\mathbf{h}^{n+1}$ where $\mathbf{h}$ is the Poincaré dual of a hyperplane. The total Chern class of a rank–r vector bundle E over $\mathbb{P}^n$ will be denoted by $c(E) = 1 + c_1(E)\cdot\mathbf{h} + \cdots + c_r(E)\cdot\mathbf{h}^r$ where $c_i(E) \in \mathbb{Z}$. If $\mathcal{S}$ is a sheaf on $\mathbb{P}^n$, then we will denote by $h^i(\mathbb{P}^n, \mathcal{S}) = \dim_{\mathbb{C}} H^i(\mathbb{P}^n, \mathcal{S})$.

Recall that the total Chern class of a rank–2 vector bundle $E(m)$ may be calculated in terms of the Chern classes of E, and the Chern class of the bundle $\mathcal{O}(m)$ by the formula

$$(1) \qquad c\big(E(m)\big) = 1 + \big(c_1(E) + 2m\big)\cdot\mathbf{h} + \big(c_2(E) + mc_1(E) + m^2\big)\cdot\mathbf{h}^2$$

and the Chern class of the bundle $\mathcal{O}(a) \oplus \mathcal{O}(b)$ is given by

$$(2) \qquad c\big(\mathcal{O}(a) \oplus \mathcal{O}(b)\big) = 1 + (a+b)\cdot\mathbf{h} + (ab)\cdot\mathbf{h}^2,$$

we will use the following result [OSS] p. 39.

1.1. Theorem. — *Let $E \to \mathbb{P}^n$ be a holomorphic rank–2 vector bundle, then the following conditions are equivalent :*

(1) *E splits as a sum of line bundles.*

(2) *E has a holomorphic line subbundle.*

(3) *$h^j(\mathbb{P}^n, E(m)) = 0$ for all $m \in \mathbb{Z}$ and $j = 1, \ldots, n-1$.*

Also note that since E has rank–2, the natural map $E \otimes E \to \bigwedge^2 E \simeq \mathcal{O}(c_1)$ is a perfect pairing whence $E^* \simeq E(-c_1)$, this implies the following theorem :

1.2. Theorem. — *A rank–2 holomorphic vector bundle E over $\mathbb{P}^3$ splits if and only if $h^1\big(\mathbb{P}^3, E(m)\big) = 0$ for all $m \in \mathbb{Z}$.*

Proof. — By Theorem 1.1, it suffices to prove that $H^2\big(\mathbb{P}^3, E(m)\big) \simeq 0$ for all $m \in \mathbb{Z}$.

Let $c_1 = c_1(E)$ then by Serre duality we get :

$$\begin{aligned} H^2\big(\mathbb{P}^3, E(m)\big) &\simeq H^1\big(\mathbb{P}^3, E^*(-m-4)\big) \\ &\simeq H^1\big(\mathbb{P}^3, E(-c_1-m-4)\big) \simeq 0. \end{aligned}$$ □

A rank–2 vector bundle is *normalized* if the first Chern class is 0 or -1. Given a rank–2 bundle E its normalization E_{norm} is defined by

$$E_{\text{norm}} := \begin{cases} E(-\frac{c_1}{2}) & \text{if } c_1 \text{ is even} \\ E(-\frac{c_1+1}{2}) & \text{if } c_1 \text{ is odd.} \end{cases}$$

A rank–2 vector bundle E is *stable* (resp. *semistable*) if $H^0(\mathbb{P}^n, E_{\text{norm}}) = 0$ (resp. $H^0(\mathbb{P}^n, E_{\text{norm}}(-1)) = 0$).

Let E be a rank–2 bundle, then

$$\Delta_E = c_1(E)^2 - 4c_2(E).$$

Observe that this number is an invariant under twisting, $\Delta_E = \Delta_{E(m)}$ for all $m \in \mathbb{Z}$.

1.3. THEOREM [B]. — *Let E be a rank–2 holomorphic vector bundle over $\mathbb{P}^n$. If $\Delta_E \geq 0$ then E is not stable.*

We end this section with the following :

CONJECTURE I [HS]. — *A rank–2 bundle on $\mathbb{P}^n$, $n \geqq 5$, which is not stable, splits.*

We will see below that the affirmative solution to this conjecture implies that a Kupka component is a complete intersection in dimension ≥ 5.

2. Codimension two subcanonical submanifolds of $\mathbb{P}^n$.

A codimension two smooth submanifold $X \subset \mathbb{P}^n, n \geq 3$, is called *subcanonical* if its canonical bundle Ω_X^{n-2} is a multiple of the hyperplane bundle on $\mathbb{P}^n$, that is

$$\Omega_X^{n-2} = \mathcal{O}_X\big(e(X)\big).$$

In this situation, from the exact sequence

$$0 \longrightarrow \tau_X \longrightarrow \tau_{\mathbb{P}^n}|_X \longrightarrow \nu_X \longrightarrow 0$$

we have that

$$\wedge^2 \nu_X = \mathcal{O}(n+1-e)|_X$$

and then we have the following result.

2.1. THEOREM. — *Let $X \subset \mathbb{P}^n$ be a codimension two, smooth submanifold with sheaf of ideals $\mathcal{J}_X$. If $\wedge^2\nu_X = \mathcal{O}(c)|_X$ then there exists* *a holomorphic two bundle $E \to \mathbb{P}^n$ with a section $\sigma \in H^0(\mathbb{P}^n, \mathcal{O}(E))$ such that $(\sigma = 0) = (X, \mathcal{O}_X)$ and induces the exact sequence*

$$0 \to \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\cdot\sigma} E \to \mathcal{J}_X(c) \to 0$$
$$\text{with} \quad c(E) = 1 + c \cdot \mathbf{h} + deg(X) \cdot \mathbf{h}^2.$$

Moreover, X is a complete intersection if and only if E splits.

The proof of this theorem will be given in the Appendix in a more general context.

Remark.

1. The sequence

$$0 \to \mathcal{O}(-c_1) \to E(-c_1) \to \mathcal{J}_X \to 0,$$

where c_1 is the first Chern class of E, is the Koszul resolution of the sheaf of ideals $\mathcal{J}_X$ of X, and E is an extension of the normal bundle ν_X of $X \subset \mathbb{P}^n$ [OSS] p. 90.

2. The first Chern class c of E is related with $e = e(X)$ by $c = e - n - 1$.

Recall that a smooth curve $X \subset \mathbb{P}^n$ is *k–normal* if the hypersurfaces of degree k cut out the complete linear series $|\mathcal{O}_X(k)|$, and $X \subset \mathbb{P}^n$ is *projectively normal* if it is k–normal for all $k \in \mathbb{Z}$.

It may be shown that a curve $X \subset \mathbb{P}^3$ is projectively normal if and only if

$$H^1(\mathbb{P}^3, \mathcal{J}_X(k)) = 0 \quad \text{for all} \quad k \in \mathbb{Z}.$$

This characterization implies the following result.

2.2. THEOREM (Gherardelli). — *Let $X \subset \mathbb{P}^3$ be a subcanonical projectively normal curve. Then X is a complete intersection.*

Proof. — Let E be the associated vector bundle to X and let c be its first Chern class. We need to show that E splits. To do this, it is sufficient by Theorem 1.2, to show that $H^1(\mathbb{P}^3, E(m)) = 0$ for all $m \in \mathbb{Z}$.

Since X is a projectively normal curve, then $H^1(\mathbb{P}^3, \mathcal{J}_X(m)) = 0$ for all $m \in \mathbb{Z}$. The exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\cdot\sigma} E \to \mathcal{J}_X(c) \to 0$$

gives, for all $m \in \mathbb{Z}$, the isomorphism

$$H^1(\mathbb{P}^3, E(m)) \simeq H^1(\mathbb{P}^3, \mathcal{J}_X(m+c)) \simeq 0. \qquad \square$$

2.3. Theorem. — *Let $X \subset \mathbb{P}^n$ be a codimension two subcanonical smooth submanifold and let E be the rank–2 bundle associated to X. If E is unstable and has positive first Chern class, then X is contained in a hypersurface of degree $\leq c/2 = c_1(E)/2$.*

Proof. — Assume that c is even. From the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\cdot\sigma} E \to \mathcal{J}_X(c) \to 0$$

we get the exact sequence

$$0 \to \mathcal{O}(-c/2) \xrightarrow{\cdot\sigma} E_{\text{norm}} \to \mathcal{J}_X(c/2) \to 0$$

and from the associated long cohomology sequence we get

$$0 \neq H^0(\mathbb{P}^n, E_{\text{norm}}) \simeq H^0(\mathbb{P}^n, \mathcal{J}_X(c/2)). \qquad \square$$

One of the unsolved problems in Algebraic Geometry is Hartshorne's conjecture [H] which states that any projective smooth variety $X \subset \mathbb{P}^n$ such that $3\dim X > 2n$ is a complete intersection. In this paper, we shall consider the following assertion which for $n \geq 7$ is a special case of the above.

Conjecture. — *If $n \geq 6$ then any codimension 2 smooth subvariety $X \subset \mathbb{P}^n$ is a complete intersection.*

Some partial results in the solution of this conjecture were given in [R] and [BCh]. An improvement of those results is stated in the following theorem.

2.4. Theorem [HS]. — *Let X be a codimension two subcanonical submanifold. Let c_1, c_2 be the Chern classes of the associated bundle E and $e = e(X)$. If one of the following conditions is satisfied :*

(1) $6 \leq n$ *and* $e \leq (n+1)$;

(2) $c_1 \geq \dfrac{c_2}{n-1} + (n-1)$;

(3) $6 \leq n$ *and* $c_2 \leq (n-1)(n+5)$;

then X is a complete intersection.

The main idea in the proof of this theorem, is that the associated vector bundle is unstable (or semistable) and it follows that X is contained in a hypersurface of degree $d \leq c_1/2$. With these facts, it is possible to find a line subbundle and the conclusion follows from Theorem 1.1.

We say that a submanifold $X \subset \mathbb{P}^n$ is a *numerically complete intersection* if X has the same Chern classes as a complete intersection. Another open question is the following

CONJECTURE III. — *Any codimension 2 submanifold $X \subset \mathbb{P}^n$, $n \geq 6$, is a numerically complete intersection.*

When $n \geq 6$, any codimension two smooth submanifold of $\mathbb{P}^n$ is subcanonically embedded. In this case, an affirmative answer to Conjectures I and III implies that Conjecture II is also true.

Example [OSS] p. 109-110. — There exists a two dimensional smooth complex torus X, of degree 10 in $\mathbb{P}^4$. By Lefschetz's Theorem, this surface can not be a complete intersection.

From the exact sequence

$$0 \to \tau_X \to \tau_{\mathbb{P}^4}|_X \to \nu_X \to 0,$$

and because the tangent bundle of X is trivial, we get

$$\wedge^2 \nu_X = \mathcal{O}(5) \quad \Omega_X^2 = \mathcal{O}|_X$$

thus, the associated vector bundle E has Chern class

$$c(E) = 1 + 5 \cdot \mathbf{h} + 10 \cdot \mathbf{h}^2,$$

and this bundle cannot split, thus X cannot be a complete intersection.

As far as we know, this is essentially the only one non–splitting bundle over $\mathbb{P}^4$ which is known.

We will show later that this torus can not be the Kupka set of any foliation in $\mathbb{P}^4$.

3. Kupka type singularities.

3.1. DEFINITION. — A *codimension one holomorphic foliation $\mathcal{F}$* (with singularities) in a complex manifold M is an equivalence class of sections

$\omega \in H^0(M, \Omega^1(L))$ where L is a holomorphic line bundle, ω does not vanish on any connected component of M and satisfies the integrability condition $\omega \wedge d\omega = 0$.

The *singular set* of the foliation $\mathcal{F}$ is the set of points $S(\mathcal{F}) = \{p \in M | \omega(p) = 0\}$. We will assume that it has codimension ≥ 2. The *leaves of the foliation* are the leaves of the non–singular foliation in $M - S(\mathcal{F})$. When a leaf $\mathcal{L}$ of $\mathcal{F}$ is such that its closure $\overline{\mathcal{L}}$ is a closed analytic subspace of M of codimension 1, we will also call $\overline{\mathcal{L}}$ a leaf of $\mathcal{F}$.

3.2. DEFINITION. — Let $\mathcal{F}$ be a codimension one holomorphic foliation represented by $\omega \in H^0(M, \Omega^1(L))$. The *Kupka singular set* $K(\mathcal{F}) \subset S(\mathcal{F})$ is defined by

$$K(\mathcal{F}) := \{p \in M \,|\, \omega(p) = 0 \quad d\omega(p) \neq 0\}.$$

We say that $\mathcal{F}$ has a *Kupka component* K if K is a compact, connected component of the Kupka set $K(\mathcal{F})$.

The *degree* of a codimension one holomorphic foliation, represented by a section $\omega \in H^0(M, \Omega^1(L))$ is the first Chern class $c_1(L) \in H^2(M, \mathbb{Z})$ of the line bundle L. If $M = \mathbb{P}^n$, a foliation is represented by an integrable section $\omega \in H^0(\mathbb{P}^n, \Omega^1(k))$, and in this case the degree is k. Such a section, is given by a dicritical 1–form in $\mathbb{C}^{n+1}$, homogeneous of degree $k - 1$.

An integrable section $\omega \in H^0(M, \Omega^1(L))$, is represented by a family of integrable 1-forms ω_α defined on an open cover $U = \{U_\alpha\}$ of M, satisfying $\omega_\alpha = \lambda_{\alpha\beta} \cdot \omega_\beta$ in $U_\alpha \cap U_\beta$, and $\lambda_{\alpha\beta}$ are the defining cocycles of the line bundle L. The Kupka set is well defined, and independent of the choice of the section ω. We will use the following result.

3.3. THEOREM. — *Let $\omega \in H^0(M, \Omega^1(L))$ be a codimension one holomorphic foliation.*

(1) [M]. *Given a connected component $K \subset K(\mathcal{F})$ there exists a germ at $0 \in \mathbb{C}^2$ of a holomorphic 1–form $\eta = A(x,y)\,dx + B(x,y)\,dy$ with an isolated singularity at 0, an open covering $\{U_\alpha\}$ of a neighborhood of $K \subset M$, and a family of submersions $\varphi_\alpha : U_\alpha \to \mathbb{C}^2$ such that*

$$\varphi_\alpha^{-1}(0) = K \cap U_\alpha \quad \text{and} \quad \omega_\alpha = \varphi_\alpha^* \eta.$$

(2) [GML]. *If the first Chern class of the normal bundle ν_K of a compact connected component $K \subset K(\mathcal{F})$ is non–zero, then the 1–form η has the form $\eta = pxdy - qydx$ for $1 \leq p < q$ relatively prime integers or $p = q = 1$.*

Remark. — The 1–form η is called the *transversal type* of the component K. Moreover, it is well defined up to biholomorphism and multiplication by non–vanishing holomorphic functions.

3.4. THEOREM. — *Let $\mathcal{F}$ be a codimension one holomorphic foliation represented by a section $\omega \in H^0(M, \Omega^1(L))$. Then*

(1) *Let $K \subset K(\mathcal{F})$ be a compact connected component. Then K is subcanonically embedded. i. e.*

$$\Omega_K^{n-2} = \left(\Omega_M^n \otimes L\right)|_K.$$

(2) *If the line bundle L is positive then the transversal type of the Kupka component K is given by $\eta = pxdy - qydx$ with p, q positive relatively prime integers or $p = q = 1$.*

Proof.

(1) Observe that $\omega_\alpha := \varphi_\alpha^* \eta$ is the defining 1-form of $\mathcal{F}$ in the open set U_α, thus for any $p \in K$, one has the equations

$$\begin{aligned} d\omega_\alpha(p) &= \varphi_\alpha^* \left(\frac{\partial B}{\partial x_\alpha} - \frac{\partial A}{\partial y_\alpha} \right)(0) \cdot dx_\alpha \wedge dy_\alpha \\ d\omega_\alpha(p) &= \lambda_{\alpha\beta}(p) \cdot d\omega_\beta(p). \end{aligned}$$

This shows that $d\omega_\alpha$ is a never vanishing holomorphic section of the line bundle $\wedge^2 \nu_K^* \otimes L|_K$ where ν_K denotes the normal bundle of K in M, thus we have

$$\wedge^2 \nu_K^* = L^{-1}|_K,$$

and from the exact sequence

$$0 \to \tau_K \to \tau_M|_K \to \nu_K \to 0$$

we get the isomorphism

$$\Omega_K^{n-2} \otimes \wedge^2 \nu_K^* = \Omega_K^{n-2} \otimes L^{-1}|_K \simeq \Omega_M^n|_K$$

and this implies

$$\Omega_K^{n-2} \simeq (\Omega_M^n \otimes L)|_K.$$

(2) We have shown that

$$\wedge^2 \nu_K = L|_K,$$

and by assumption, the line bundle L is positive. Let $i : K \to M$ be the inclusion map, then

$$c_1(\nu_K(M)) = i^* c_1(L)$$

and by the naturality of the Chern classes, the first Chern class of the normal bundle of K does not vanish. This implies by the second part of Theorem 3.3 that the transversal type of K is as claimed. □

3.5. Corollary. — *Let $\mathcal{F}$ be a codimension one holomorphic foliation of degree k with a Kupka component K in $\mathbb{P}^n$, $n \geq 3$, then there exists a rank–2 holomorphic vector bundle E with a section σ which induces the exact sequence*

$$0 \to \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\cdot\sigma} E \to \mathcal{J}_K(k) \to 0$$

and the total Chern class of this bundle is

$$c(E) = 1 + k \cdot \mathbf{h} + \deg(K) \cdot \mathbf{h}^2.$$

Proof. — Since the canonical bundle of $\mathbb{P}^n$ is the line bundle $\Omega^n_{\mathbb{P}^n} = \mathcal{O}(-n-1)$, part (1) of the above theorem shows that the canonical bundle of a Kupka component K, of a foliation of degree k is

$$\begin{aligned} \Omega^{n-2}_K &= \mathcal{O}(e_K)|_K \quad \text{and} \quad e_K = k - n - 1 \\ \wedge^2 \nu_K &= \mathcal{O}(k)|_K \end{aligned}$$

and it follows from Theorem 2.1 the existence of the rank–2 vector bundle E as claimed. □

Examples. — Let C be some of the following smooth curves in $\mathbb{P}^3$: The twisted cubic, a quintic of genus 2, a septic of genus 5, or a septic of genus 6. We will show that C cannot be the Kupka set of a codimension one holomorphic foliation in $\mathbb{P}^3$.

All those curves are projectively normal, and none of them is a complete intersection. If $C = K_\omega$ then its canonical bundle would be

$$\Omega^1_C = \mathcal{O}(k-4)|_C \quad k = \deg(\omega).$$

From Theorem 2.2 we conclude that C is a complete intersection, a contradiction.

Remark. — Corollary 3.5 may be generalized to codimension one foliations in projective manifolds (see Appendix).

4. Baum–Bott formula for Kupka sets.

In this section we will compute the second Chern class of the rank–2 vector bundle associated to a Kupka component.

Let M be a compact complex manifold of complex dimension n. Following [BB], a holomorphic foliation with singularities is an *integrable* subsheaf $\Theta_{\mathcal{F}}$ of the tangent sheaf Θ_M. namely :

(1) $\Theta_{\mathcal{F}}$ is *coherent*.

(2) $\Theta_{\mathcal{F}}$ is closed under the bracket operation of vector fields.

The *singular set* consists of those points where the sheaf $Q = \Theta_M/\Theta_{\mathcal{F}}$ is not free. We have the following exact sequence :

$$0 \to \Theta_{\mathcal{F}} \to \Theta_M \to Q \to 0.$$

Let $\Theta_{\mathcal{F}} \subset \Theta_M$ be a full integrable subsheaf of dimension $n-k$ and let $\varphi \in \mathbb{C}[x_1 \dots, x_n]$ be a symmetric and homogeneous polynomial of degree $1 \leq l \leq n-k$. For such a polynomial, there exists a unique polynomial $\widetilde{\varphi}$ such that

$$\varphi = \widetilde{\varphi}(\sigma_1, \dots, \sigma_n)$$

where σ_i are the elementary symetric functions. In this case, $\varphi(Q) := \widetilde{\varphi}(c_1(Q), \dots c_n(Q))$ where $c_i(Q)$ are the Chern classes of the sheaf Q.

Let Z be a compact, connected component of the singular set. Consider μ_* is defined by $\mu_* = \alpha \circ i_*$ where $i_* : H_j(Z, \mathbb{C}) \to H_j(M, \mathbb{Z})$ is induced by the inclusion and $\alpha : H_j(M, \mathbb{C}) \to H^{2n-j}(M, \mathbb{C})$ is the Poincaré isomorphism. The main result in [BB] is the following.

4.1. Theorem. — *Let M be a compact complex manifold, $\Theta_{\mathcal{F}}, Q, \varphi$ be as above. Then*

(1) *there exists a homology class* $\mathrm{Res}_\varphi(\Theta_{\mathcal{F}}, Z) \in H_{2n-2l}(Z, \mathbb{Z})$ *such that*

(1.1) $\mathrm{Res}_\varphi(\Theta_{\mathcal{F}}, Z)$ *depends only on the polynomial* φ *and the local behavior of the leaves of the foliation* $\Theta_{\mathcal{F}}$ *near* Z.

(1.2) $\sum_Z \mu_* \mathrm{Res}_\varphi(\Theta_{\mathcal{F}}, Z) = \varphi(Q)$.

(2) *Let Z be an irreducible component of the singular set with* $\dim(Z) = n-k-1$. *If $\Theta_{\mathcal{F}}$ is locally generated by*

$$\left\{X(x_1,\ldots,x_k) = \sum_{i=1}^{k} X_i(x_1,\ldots x_k)\frac{\partial}{\partial x_i}, \frac{\partial}{\partial z_{k+1}}, \ldots, \frac{\partial}{\partial z_n}\right\}$$

then

$$\mathrm{Res}_{\varphi}(\Theta_{\mathcal{F}}, Z) = \left[\begin{matrix}\widetilde{\varphi}(DX(0))\, dx_1\cdots dx_k \\ X_1(x)\cdots X_k(x)\end{matrix}\right][Z].$$

If A is a matrix with eigenvalues $(\lambda_1,\ldots,\lambda_n)$ then $\varphi(A) := \widetilde{\varphi}(\lambda_1, \ldots, \lambda_n)$.

If φ has degree k, we set

$$\begin{aligned}\varphi(X,0) = \mathrm{Res}_0&\left[\begin{matrix}\widetilde{\varphi}(DX(0))\, dx_1\cdots dx_k \\ X_1(x)\cdots X_k(x)\end{matrix}\right] \\ &= \left(\frac{1}{2i\pi}\right)^k \int_{\|X_i\|=\epsilon} \frac{\widetilde{\varphi}(DX(0))dx_1\wedge\ldots\wedge dx_k}{X_1(x)\cdots X_k(x)}.\end{aligned}$$

In our case, the sheaf $\Theta_{\mathcal{F}}$ consists of those germs of holomorphic vector fields X such that $\omega(X)=0$ and has dimension $n-1$, the singular set is the zero set of ω. Thus the Baum-Bott formula implies the following.

4.2. THEOREM. — *Let $\mathcal{F}$ be a codimension one holomorphic foliation of degree k in $\mathbb{P}^n$ such that the codimension two component of the singular set consists of a single compact Kupka component K of transversal type $\eta = pxdy - qydx$, then*

$$\deg(K) = \frac{p\cdot k}{p+q}\cdot\frac{q\cdot k}{p+q}.$$

Proof. — Consider the polynomial $\varphi(x_1,\ldots,x_2) = (x_1+x_2)^2$. Then

$$\widetilde{\varphi}(x_1,x_2,0,\ldots,0) = \sigma_1^2(x_1,x_2,0,\ldots,0)$$

and observe that Q is the *normal sheaf* in the regular values, and the tangent sheaf is precisely the annihilator of a local 1-form which defines the section ω. This gives that for a local section of γ of Q, we have

$$\omega(\gamma)\neq 0$$

thus the section $\omega|_{(\mathbb{P}^n - S)}$ is a never vanishing section of the sheaf $\mathcal{O}(k) \otimes Q^*|_{(\mathbb{P}^n - S)} \simeq \mathcal{O}$ and we get

$$Q \simeq \mathcal{O}(k).$$

The Baum-Bott formula implies that

$$\begin{aligned}\frac{(p+q)^2}{p \cdot q} \deg(K) &= \sigma_1^2(Q) \\ &= \sigma_1^2(\mathcal{O}(k)) = k^2.\end{aligned}$$

Hence the degree of the Kupka set is

$$\deg(K) = \frac{k \cdot p}{(p+q)} \cdot \frac{k \cdot q}{(p+q)}. \qquad \square$$

4.3. Corollary. — *Let K be a Kupka component of a degree k foliation in $\mathbb{P}^n$, $n \geq 3$.*

(1) *The rank–2 vector bundle associated to a Kupka component is not stable.*

(2) *K is contained in a hypersurface of degree $\leq k/2$.*

Proof. — (1) Let E be the vector bundle associated to the Kupka set and c_1, c_2 its Chern classes :

$$\begin{aligned}\Delta_E &= c_1^2 - 4c_2 \\ &= k^2 - 4k^2 \cdot \frac{q \cdot p}{(p+q)^2} \\ &= \left(\frac{(p-q) \cdot k}{(p+q)}\right)^2 \geq 0.\end{aligned}$$

(2) Follows from part (1) and Theorem 2.3. $\square$

Now we are in a position to prove the main theorem.

4.4. Theorem. — *Let $\mathcal{F}$ be a codimension one holomorphic foliation of degree k in $\mathbb{P}^n$ with a Kupka component K. Then K is numerically a complete intersection.*

Proof. — Note that the vector bundle E has the Chern class of the bundle

$$\mathcal{O}(a) \oplus \mathcal{O}(b), \quad a = \frac{k \cdot p}{p+q}, \quad b = \frac{k \cdot q}{p+q}.$$

From the exact sequence

$$0 \to \tau_K \to \tau_{\mathbb{P}^n}|_K \to \nu_K \to 0$$

if $i : K \to \mathbb{P}^n$ denotes the inclusion map we get

$$\begin{aligned} c(\tau_K) &= i^* c(\tau_{\mathbb{P}^n}) \cdot c(E)^{-1} \\ &= i^* c(\tau_{\mathbb{P}^n}) \cdot \big(1 + (a+b) \cdot \mathbf{h} + (ab) \cdot \mathbf{h}^2\big)^{-1}. \end{aligned}$$

□

Examples.

(1) The above theorem shows that the complex torus $X \subset \mathbb{P}^4$ cannot be the Kupka set of any foliation, since the Chern classes of the associated bundle are not compatible with the spliting.

(2) For a foliation in $\mathbb{P}^3$ we can calculate the genus of the Kupka set in terms of the transversal type and the degree of the foliation.

Following [H1] let E be the rank–2 bundle associated to a curve $\mathcal{C}$. The genus is given by

$$g = \frac{c_2(c_1 - 4) + 2}{2}$$

from this formula and Theorem 4.4 we have that the quartic of genus 1 and the non–hyperelliptic sextic of genus 3 cannot be the Kupka set of any foliation.

4.5. Corollary. — *Let $\mathcal{F}$ be as above. If one of the conditions below is satisfied :*

(1) *If $k \leq 2(n+1)$ and $n \geq 6$,*

(2) *$k \leq 2(n-1)$ and $(n \geq 3)$,*

(3) $\min\left\{\dfrac{p \cdot k}{p+q}, \dfrac{q \cdot k}{p+q}\right\} \leq n-1,$

then K is a complete intersection.

Proof. — (1) We have that $e_K = k - n - 1$ thus by part (a) of Theorem 2.4 we have

$$e_K \leqq (n+1) \quad \text{if and only if} \quad k \leqq 2 \cdot (n+1).$$

(2) If $k \leq 2(n-1)$ then the second condition of Theorem 2.4 is satisfied.

(3) It is easy to see that the second condition of Theorem 2.4 is satisfied. □

Remark. — In [GS] it is shown that the unstability condition is sufficient for the splitting of the bundle E. If $n \geq 4$, unfortunately their proof seems to be incomplete.

Appendix.

Let M be a projective manifold and let L be a holomorphic line bundle such that

$$H^1\big(M, \mathcal{O}(-L)\big) \simeq H^2\big(M, \mathcal{O}(-L)\big) \simeq 0.$$

This condition is satisfied if for example, M has complex dimension at least 3, and L is a positive line bundle.

THEOREM. — *Let $V \subset M$ be a smooth, codimension two submanifold with sheaf of ideals $\mathcal{J}_V \subset \mathcal{O}_M$. If the determinant of the normal bundle is extendable to a line bundle L over M such that $H^1\big(M, \mathcal{O}(-L)\big) \simeq H^2\big(M, \mathcal{O}(-L)\big) \simeq 0$, then there exists a holomorphic two bundle $E \to M$ with a section $\sigma \in H^0\big(M, \mathcal{O}(E)\big)$ such that $(\sigma = 0) = (V, \mathcal{O}_V)$ and it induces the exact sequence*

$$0 \to \mathcal{O}_M \xrightarrow{\cdot\sigma} E \to \mathcal{J}_V \otimes L \to 0.$$

Moreover, $c_1(E) = c_1(L)$ and $c_2(E) = [V]$ the fundamental class of V.

Proof. — Following [OSS] or [H] p. 1029, if there exists E as claimed, then $\wedge^2 \nu_V^* = L^*|_V$, and E^* is an extension of the sequence

$$0 \to \mathcal{O}_M(-L) \to E^* \to \mathcal{J}_V \to 0.$$

The global Ext–group

$$\mathrm{Ext}^1\big(\mathcal{J}_V, \mathcal{O}(-L)\big)$$

classifies such extensions [GH] p. 725. We will use the lower terms of the spectral sequence relating the global with the local Ext's :

$$\begin{aligned} E_2^{pq} &= H^p\Big(M, \underline{\mathrm{Ext}}^q\big(\mathcal{J}_V, \mathcal{O}(-L)\big)\Big) \\ E_\infty^{p+q} &\Rightarrow \mathrm{Ext}_M^{p+q}\big(\mathcal{J}_V, \mathcal{O}(-L)\big). \end{aligned}$$

From this spectral sequence, we get the following exact sequence [G] p. 265 :

$$\begin{aligned}0 &\to H^1\Big(M, \underline{\mathrm{Hom}}(\mathcal{J}_V, \mathcal{O}(-L))\Big) \to \mathrm{Ext}^1(\mathcal{J}_V, \mathcal{O}(-L))\\ &\to H^0\Big(M, \underline{\mathrm{Ext}}^1(\mathcal{J}_V, \mathcal{O}(-L))\Big) \to H^2\Big(M, \underline{\mathrm{Hom}}(\mathcal{J}_V, \mathcal{O}(-L))\Big)\\ &\to \mathrm{Ext}^2(J_V, \mathcal{O}(-L)) \to \cdots .\end{aligned}$$

The exact sequence $0 \to \mathcal{J}_V \to \mathcal{O}_M \to \mathcal{O}_M/\mathcal{J}_V = \mathcal{O}_V \to 0$ induces the long exact sequence :

$$\begin{aligned}0 &\to \underline{\mathrm{Hom}}(\mathcal{O}_V, \mathcal{O}(-L)) \to \underline{\mathrm{Hom}}(\mathcal{O}_M, \mathcal{O}(-L))\\ &\to \underline{\mathrm{Hom}}(\mathcal{J}_V, \mathcal{O}(-L)) \to \underline{\mathrm{Ext}}^1(\mathcal{O}_V, \mathcal{O}(-L)) \to \cdots\end{aligned}$$

The local Ext–group $\underline{\mathrm{Ext}}^i(\mathcal{O}_V, \mathcal{O}(-L)) = 0$ for $i = 0, 1$ because V is smooth, and has codimension two, so it is a local complete intersection. This gives the isomorphisms :

$$\mathcal{O}(-L) \simeq \mathrm{Hom}(\mathcal{O}_M, \mathcal{O}(-L)) \simeq \mathrm{Hom}(\mathcal{J}_V, \mathcal{O}(-L)).$$

If we put this sequence on the first exact sequence, we obtain :

$$\begin{aligned}0 &\to H^1(M, \mathcal{O}(-L)) \to \mathrm{Ext}^1(\mathcal{J}_V, \mathcal{O}(-L))\\ &\to H^0\big(M, \underline{\mathrm{Ext}}^1(\mathcal{J}_V, \mathcal{O}(-L))\big) \to H^2(M, \mathcal{O}(-L)).\end{aligned}$$

Now, by hypothesis, $0 = H^i(M, \mathcal{O}(-L))$ for $i = 1, 2$, so we get the isomorphism

$$\mathrm{Ext}^1(\mathcal{J}_V, \mathcal{O}(-L)) \cong H^0(M, \underline{\mathrm{Ext}}^1(\mathcal{J}_V, \mathcal{O}(-L))).$$

Consider the exact sequence $0 \to \mathcal{J}_V \to \mathcal{O}_M \to \mathcal{O}_V \to 0$. The Ext–sequence gives the isomorphism

$$\mathrm{Ext}^1(\mathcal{J}_V, \mathcal{O}(-L)) \simeq \mathrm{Ext}^2(\mathcal{O}_V, \mathcal{O}(-L)).$$

Since V is smooth, it is a local complete intersection, thus we have the local fundamental isomorphism [GH] p. 690-692

$$\mathrm{Ext}^2(\mathcal{O}_V, \mathcal{O}(-L)) \simeq \mathrm{Hom}_{\mathcal{O}_V}(\det \mathcal{J}_V/\mathcal{J}_V^2, \mathcal{O}_V(-L))$$

where $\mathcal{O}_V(-L) = \mathcal{O}(-L) \otimes \mathcal{O}_V$ and by assumption, $\det \mathcal{J}_V/\mathcal{J}_V^2 \simeq \mathcal{O}_V(-L)$ so

$$\mathrm{Hom}_{\mathcal{O}_V}\big(\det \mathcal{J}_V/\mathcal{J}_V^2, \mathcal{O}_V(-L)\big) \simeq \mathcal{O}_V,$$

and thus

$$\mathrm{Ext}^1\big(\mathcal{J}_V, \mathcal{O}(-L)\big) \simeq \mathrm{Ext}^2\big(\mathcal{O}_V, \mathcal{O}(-L)\big) \simeq H^0\big(V, \mathcal{O}_V\big).$$

We will consider the extension defined by $1 \in H^0(V, \mathcal{O}_V)$

$$0 \to \mathcal{O}(-L) \to F \to \mathcal{J}_V \to 0.$$

Then F is locally free, and the dual $E = F^*$ is the desired vector bundle. This sequence is the Koszul complex of a section $\sigma \in H^0\big(M, \mathcal{O}(E)\big)$. □

COROLLARY. — *Let $\mathcal{F} \in \mathrm{Fol}(M, L)$ be a codimension one holomorphic foliation with a compact Kupka component K. If L is a positive line bundle then :*

(1) *there exists a rank–2 bundle E over M with a section σ which induces the exact sequence*

$$0 \to \mathcal{O}_M \xrightarrow{\cdot\sigma} E \to \mathcal{J}_K \otimes L \to 0.$$

(2) *The Chern class of E is $c(E) = 1 + c_1(L) + [K]$ where $[K]$ is the fundamental class of K.*

(3) *If the transversal type is $\eta = pxdy - qydx$ then*

$$[K] = \frac{p}{p+q} \cdot c_1(L) \wedge \frac{q}{p+q} \cdot c_1(L).$$

Proof. — (1) and (2) follow from the above theorem.

(3) Follows from the Baum–Bott residue formula. □

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