ANNALES DE L'INSTITUT FOURIER

KAREL DE LEEUW

H. MIRKIL

Algebras of differentiable functions in the plane

Annales de l'institut Fourier, tome 13, nº 2 (1963), p. 75-89

<http://www.numdam.org/item?id=AIF_1963__13_2_75_0>

Ann. Inst. Fourier, Grenoble
13,2 (1963), 75-90.

ALGEBRAS OF DIFFERENTIABLE FUNCTIONS IN THE PLANE

by K. de LEEUW (Stanford University)
and H. MIRKIL (Dartmouth College) (1)

1. — Introduction.

We denote by C_0 the Banach space of all complex-valued continuous functions on the plane that are zero at infinity, supplied with the supremum norm $||.||$; and by D the dense subspace of C_0 consisting of infinitely differentiable functions with compact support. $\mathfrak{Q}$ is the set of all differential operators of the form

$$\Sigma a_{m,n} \partial^{m+n}/\partial x^m \, \partial y^n, \tag{1.1}$$

where the $a_{m,n}$ are complex constants.

If A is (1.1) its formal adjoint $\tilde{A}$ is the operator

$$\Sigma(-1)^{m+n} a_{m,n} \partial^{m+n}/\partial x^m \, \partial y^n.$$

For f in C_0, the statement « Af is in C_0 » will be interpreted in the sense of the theory of distributions; Af is defined to be the function h in C_0 (unique if it exists) satisfying

$$\int (\tilde{A}g)f = \int hg, \qquad g \in D.$$

For a subset $\mathfrak{A}$ of $\mathfrak{Q}$, we define $C_0(\mathfrak{A})$ to be the space of all f in C_0 which are such that Af is in C_0 for all A in $\mathfrak{A}$. A subspace B of C_0 will be called *a space of differentiable functions* if B is

(1) Dedicated to Professor Charles Loewner on the occasion of his 70 th birthday.

$C_0(\mathcal{A})$ for some subset $\mathcal{A}$ of $\mathcal{Q}$. Each space of differentiable functions is translation-invariant; those that are furthermore invariant under rotations of the plane will be called *rotating spaces of differentiable functions.*

Certain of these spaces are familiar, namely the spaces C_0^N consisting of those functions in C_0 that have all derivatives of order $\leqslant N$ in C_0, and the space C_0^∞, which is $\cap\, C_0^N$. A rotating space of differentiable functions will be called *proper* if it is not C_0^∞ and not one of the C_0^N.

The main result of this paper is Theorem 1.1, which classifies the rotating spaces. A somewhat surprising consequence of the classification is Corollary 1.2, which observes that rotating spaces are automatically closed under pointwise multiplication.

We use the standard notations,

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),$$
$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

THEOREM 1.1. — *If $\mathcal{A}$ is a proper subset of*

$$\{\partial^{m+n}/\partial z^n \partial \bar{z}^m : m + n = N\} = \mathcal{R}_N \tag{1.2}$$

for N *a positive integer, then* $C_0(\mathcal{A})$ *is a proper rotating space of differentiable functions between* C_0^N *and* C_0^{N-1}. *If* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *are distinct proper subsets of* (1.2), *then* $C_0(\mathcal{A}_1)$ *and* $C_0(\mathcal{A}_2)$ *are distinct. Each proper rotating space of differentiable functions lies between some adjacent pair of improper rotating spaces* C_0^N *and* C_0^{N-1}, *and is a* $C_0(\mathcal{A})$ *for some proper subset* $\mathcal{A}$ *of* $\mathcal{R}_N$.

COROLLARY 1.2. — *Let* B *be a rotating space of differentiable functions. Then* B *is an algebra of functions, and* B *is a Banach algebra unless* B *is* C_0^∞.

Theorem 1.1 is proved in section 5, and Corollary 1.2 in section 6. The first three sections are devoted to preliminary material; in sections 2 and 3 we establish the basic properties of spaces of differentiable functions, and in section 4 we classify the spaces of differentiable functions between C_0^N and C_0^{N-1}.

Our work is based on the existence and non-existence of certain supnorm estimates for constant-coefficient differential operators. These results appear in [2]; in section 7 we state the results from that paper, together with certain of their consequences, that will be used here.

An announcement of the results of this paper has appeared in [1]. In a subsequent paper [3] we shall study the analogues of rotating spaces of differentiable functions on Riemann surfaces.

2. — Properties of spaces of differentiable functions.

For a subset $\mathcal{A}$ of $\mathcal{Q}$, we shall consider $C_0(\mathcal{A})$ to be a topological linear space under the topology given by the semi-norms

$$f \to ||f||$$

and

$$f \to ||Af||, \qquad A \in \mathcal{A}.$$

Since $\mathcal{Q}$ is countable-dimensional, the topology can be given by a countable number of semi-norms and is thus metrizable. If $\mathcal{A}$ is finite dimensional, $C_0(\mathcal{A})$ is normable.

Proposition 2.1. — *The topological linear space* $C_0(\mathcal{A})$ *is complete.*

Proof. — Let $\{f_n\}$ be a Cauchy sequence in $C_0(\mathcal{A})$. Then in particular there is a function f in C_0 with $f_n \to f$ uniformly. Let $A \in \mathcal{A}$. $\{Af_n\}$ is uniformly Cauchy so there is a function h in C_0 with $Af_n \to h$ uniformly. It suffices to show that $Af = h$. But A is continuous in the distribution topology, and in that topology $f_n \to f$, $Af_n \to h$. This proves Proposition 2.1.

$C_0(\mathcal{A})$ is metrizable, so by Proposition 2.1 is a Frechet space, and even a Banach space if $\mathcal{A}$ is finite dimensional. In particular, the closed graph theorem is applicable to $C_0(\mathcal{A})$.

Suppose now that B is a space of differentiable functions, $B = C_0(\mathcal{A}_1)$ and $B = C_0(\mathcal{A}_2)$. The topologies we have given $C_0(\mathcal{A}_1)$ and $C_0(\mathcal{A}_2)$ are both stronger than pointwise convergence, so by the closed graph theorem, they must be the same. Thus we may speak of *the topology of* B without reference to any $\mathcal{A}$ for which $B = C_0(\mathcal{A})$. This topology will be denoted by $\tau(B)$.

PROPOSITION 2.2. — *Let* B *be a space of differentiable functions. Then* D *is dense in* B.

Proof. — Let $B = C_0(\mathfrak{A})$. We may assume the identity operator in $\mathfrak{A}$. Let $f \in B$, $\varepsilon > 0$ and $A_1, \ldots, A_n$ in $\mathfrak{A}$. We must find a function g in D with

$$||A_i g - A_i f|| < \varepsilon, \qquad i = 1, \ldots, n.$$

Choose m in D, positive, $\int m = 1$, with support so close to 0 that

$$(2.1) \qquad ||m * (A_i f) - A_i f|| < \varepsilon, \qquad i = 1, \ldots, n,$$

where $*$ is convolution. If $h = m * f$, then h is infinitely differentiable, it and all of its derivatives vanish at infinity, and (2.1) becomes

$$||A_i h - A_i f|| < \varepsilon, \qquad i = 1, \ldots, n.$$

Now let k be a function in D, identically 1 near 0. For $r > 0$ define k_r by

$$k_r(x, y) = k\left(\frac{x}{r}, \frac{y}{r}\right).$$

Then as $r \to \infty$, each derivative of $k_r h$ converges uniformly to the corresponding derivative of h, so we may take $g = k_r h$ for r sufficiently large.

PROPOSITION 2.3. — *Let* B_1 *and* B_2 *be spaces of differentiable functions. Then the following are equivalent:*

1° $B_1 \subset B_2$;

2° *Restricted to* D, *the topology* $\tau(B_1)$ *is stronger than the topology* $\tau(B_2)$.

Proof. — (1° implies 2°). By the closed graph theorem, the injection $B_1 \to B_2$ is continuous.

(2° implies 1°). The identity map, from D in $\tau(B_1)$ to D in $\tau(B_2)$, being continuous, extends to the completions, which are B_1 and B_2 by Propositions 2.1 and 2.2. The resulting map is clearly the injection of B_1 into B_2. Hence Proposition 2.3 is proved.

Suppose that $B_1 = C_0(\mathcal{A}_1)$ and $B_2 = C_0(\mathcal{A}_2)$. Then 2° in the above can be rephrased as follows. For each A in $\mathcal{A}_2$ there are $A_1, \ldots, A_n$ in $\mathcal{A}_1$ so that

$$(2.2) \quad \|Ag\| \leqslant K(\|g\| + \|A_1 g\| + \cdots + \|A_n g\|), \qquad g \in D.$$

Thus problems of classification of space of differentiable functions are more or less equivalent to questions of the existence of estimates of the form (2.2). The results that we shall need concerning such estimates are given in section 7.

Each space of differentiable functions is a $C_0(\mathcal{A})$ for many $\mathcal{A}$. It is convenient to have a notation for the largest such $\mathcal{A}$. If B is a space of differentiable functions, we shall denote by $\mathcal{A}_B$ the subspace of $\mathcal{Q}$ consisting of all A for which Af is in C_0, all f in B. Clearly $B = C_0(\mathcal{A}_B)$; and any other $\mathcal{A}$ satisfying $B = C_0(\mathcal{A})$ must be a subset of $\mathcal{A}_B$. Note that B is a rotating space of differentiable functions if and only if $\mathcal{A}_B$ is rotation invariant.

Proposition 2.4. — *Let* B *be a space of differentiable functions. Then the following are equivalent:*

1° B *is a Banach space*;

2° *For some* N, $C_0^N \subset B$;

3° $\mathcal{A}_B$ *is a finite-dimensional subspace of* $\mathcal{Q}$.

Proof. — (3° implies 1°) Clear since $B = C_0(\mathcal{A}_B)$.

(1° implies 2°) Let $\|.\|_B$ be a norm for B. The injection $C_0^\infty \to B$ is continuous by the closed graph theorem. Thus, by the definition of the topology of C_0^∞, there is a finite subset $\mathcal{A}$ of $\mathcal{Q}$ and a constant K so that

$$\|g\|_B \leqslant K(\|g\| + \sum_{A \in \mathcal{A}} \|Ag\|), \qquad g \in D.$$

It then follows from Proposition 2.3 that B contains $C_0(\mathcal{A})$ and thus B contains C_0^N if N exceeds the order of all the operators in $\mathcal{A}$.

(2° implies 3°) If $\mathcal{A}_B$ is not finite-dimensional, it contains operators of arbitrarily high order. By Corollary 7.2 this cannot occur if B contains C_0^N for some N.

3. — Properties of rotating spaces of differentiable functions.

In what follows we shall denote by $\mathfrak{D}_N$ the subspace of $\mathfrak{D}$ consisting of those differential operators of order $\leqslant N$.

It will be convenient to have a notation for rotation of functions and operators in the plane. Let w be a complex number of modulus 1. For f in C_0, we denote by $R_w f$ the function defined by $R_w f(z) = f(wz)$. For A in $\mathfrak{D}$, $R_w A$ is defined to be the operator in $\mathfrak{D}$ satisfying

$$(R_w A)(g) = A(R_w g), \qquad g \in D.$$

Note that

$$R_w(\partial^{m+n}/\partial z^m \partial \bar{z}^n) = w^{m-n}\partial^{m+n}/\partial z^m \partial \bar{z}^n. \tag{3.1}$$

LEMMA 3.1. — *Let* A *be an operator in* $\mathfrak{D}$, s *an integer. Then the following are equivalent:*

1° $R_w A = w^s A$ *for all* w *with* $|w| = 1$;

2° A *is a linear combination of the* $\partial^{m+n}/\partial z^m \partial \bar{z}^n$ *with* $m - n = s$.

Proof. — (2° implies 1°) This is immediate from (3.1.)

(1° implies 2°) Let M be a non-negative integer, A_M the homogeneous part of A of degree M. As a consequence of 1°, and since rotation preserves homogeneous parts,

$$R_w A_M = w^s A_M, \qquad |w| = 1. \tag{3.2}$$

But A_M is of the form

$$\sum_{m+n=M} a_{m,n}\partial^{m+n}/\partial z^m \partial \bar{z}^n,$$

so because of (3.1), the equality (3.2) cannot hold unless $a_{m,n} = 0$ for $m - n \neq s$. This proves Lemma 3.1.

The operators A in $\mathfrak{D}$ satisfying the conditions of Lemma 3.1 for some integer s will be called *rotating operators.*

Now let B be a rotating space of differentiable functions, so that $\mathfrak{A}_B$, is a rotation-invariant subspace of $\mathfrak{D}$. We want to show that $\mathfrak{A}_B$ is spanned by the rotating operators it contains. We need the following well known result.

PROPOSITION 3.2. — *Let* G *be a commutative compact group and* $\sigma \to U_\sigma$ *a continuous representation of* G *on a finite dimen-*

sional complex linear space V. *Then* V *is spanned by common eigenvectors*; i.e., *there is a basis* $v_1, \ldots, v_n$ *of* V *and characters* $\chi_1, \ldots, \chi_n$ *of* G *so that*

$$U_\sigma v_i = \chi_i(\sigma) v_i, \qquad \sigma \in G, \qquad i = 1, \ldots, n.$$

Let N be a positive integer. Then $\mathcal{A}_B \cap \mathfrak{Q}_N$ is a finite-dimensional rotation-invariant subspace of $\mathfrak{Q}$. Proposition 3.2 applied to the representation $w \to R_w$ of the circle group $\{w : |w| = 1\}$ on $\mathcal{A}_B \cap \mathfrak{Q}_N$ yields the following.

Corollary 3.3. — *Let* B *be a rotating space of differentiable functions. Then* $\mathcal{A}_B \cap \mathfrak{Q}_N$ *has a basis of rotating operators.*

Let us first consider the case when $\mathcal{A}_B$ is not finite-dimensional. Then by Corollary 3.3, $\mathcal{A}_B$ must contain rotating operators of arbitrarily high order. As a consequence of Corollary 7.6 below, $\mathcal{A}_B$ contains each $\mathfrak{Q}_N$ and thus all of $\mathfrak{Q}$. So we have proved.

Proposition 3.4. — *The only rotating space of differentiable functions* B *having* $\mathcal{A}_B$ *infinite-dimensional is the space* C_0^∞.

So now let B be a rotating space of differentiable functions having $\mathcal{A}_B$ finite-dimensional. Let N be the largest integer so that $\mathcal{A}_B$ contains an operator of order N. By Corollary 3.3. $\mathcal{A}_B = \mathcal{A}_B \cap \mathfrak{Q}_N$ must contain a rotating operator of order N. Thus, by Corollary 7.6, $\mathcal{A}_B$ contains $\mathfrak{Q}_{N-1}$, and as a consequence each function in B has all derivatives of order $< N$ in C_0. We have proved.

Proposition 3.5. — *Let* B *be a rotating space of differentiable functions that is not* C_0^∞. *Then there is a positive integer* N *so that*

$$C_0^N \subset B \subset C_0^{N-1}.$$

4. — Spaces between C_0^N and C_0^{N-1}.

In view of Proposition 3.5, to complete the classification of rotating spaces of differentiable functions, it remains only to study those between C_0^N and C_0^{N-1}. In this section we obtain a classification of all of the spaces (not only the rotating ones) between C_0^N and C_0^{N-1}. The next lemma provides the main tool by showing that when B is such a space, then $\mathfrak{A}_B$ contains none but the expected operators. (And we note in passing that a general space B, not contained between adjacent C_0^N and C_0^{N-1}, is intractable precisely because its $\mathfrak{A}_B$ is not easy to describe).

LEMMA 4.1. — *Let* $B = C_0(\mathfrak{A})$ *be a space of differentiable functions satisfying* $C_0^N \subset B \subset C_0^{N-1}$. *Then* $\mathfrak{A}_B$ *is the linear subspace of* $\mathfrak{Q}$ *spanned by* $\mathfrak{A}$ *and* $\mathfrak{Q}_{N-1}$.

Proof. — Let $A \in \mathfrak{A}_B$. $f \to Af(0)$ is a continuous linear functional on $B = C_0(\mathfrak{A}_B)$, and thus, since $B = C_0(\mathfrak{A})$, there are $A_1, \dots, A_n$ in $\mathfrak{A}$ so that

$$|Ag(0)| \leqslant K(\|A_1 g\| + \cdots + \|A_n g\|), \qquad g \in D.$$

Since $C_0^N \subset C_0(\mathfrak{A})$, the A_i are of order $\leqslant N$ by Corollary 7.2. Because of Theorem 7.1 below, A is of order $\leqslant N$, and the homogeneous part of A of order N is a linear combination of the corresponding homogeneous parts of the A_i. Thus A is in the linear subspace of $\mathfrak{Q}$ spanned by $\mathfrak{A}$ and $\mathfrak{Q}_{N-1}$. Since it is clear that the linear subspace of $\mathfrak{Q}$ spanned by $\mathfrak{A}$ and Q_{N-1} is contained in $\mathfrak{A}_B$, the proof of Lemma 4.1 is complete.

THEOREM 4.2. — *Let* N *be a positive integer. Then the mapping*

$$\mathfrak{A} \to C_0(\mathfrak{A}) \tag{4.1}$$

establishes a one-one correspondence between the linear subspaces $\mathfrak{A}$ *of* $\mathfrak{Q}$ *satisfying*

$$\mathfrak{Q}_{N-1} \subset \mathfrak{A} \subset \mathfrak{Q}_N \tag{4.2}$$

and those spaces B *of differentiable functions satisfying*

$$C_0^N \subset B \subset C_0^{N-1}. \tag{4.3}$$

The inverse of the mapping (4.1) *is*

$$B \to \mathfrak{A}_B. \tag{4.4}$$

Proof. — That $C_0^N \subset C_0(\mathfrak{A}) \subset C_0^{N-1}$ if $\mathfrak{A}$ satisfies (4.2) is clear. Let B be any space of differentiable functions satisfying (4.3). Since $B = C_0(\mathfrak{A}_B)$, to show the mapping (4.1) is onto, it is only necessary to show that $\mathcal{Q}_{N-1} \subset \mathcal{A}_B \subset \mathcal{Q}_N$. That $\mathcal{Q}_{N-1} \subset \mathcal{A}_B$ is clear, and $\mathcal{A}_B \subset \mathcal{Q}_N$ follows from Corollary 7.2 below. Finally, for any linear subspace $\mathcal{A}$ of $\mathcal{Q}$ satisfying (4.2), $\mathcal{A} = \mathcal{A}_{C_0(\mathcal{A})}$, by Lemma 4.1, so the mapping (4.1) is one-one and has (4.4) for its inverse as claimed.

The above proof of Theorem 4.2 actually shows somewhat more. We shall denote by C_K the subspace of C_0 consisting of those functions having compact support. If B_1 and B_2 are distinct spaces of differentiable functions, it is still possible that they are the same locally; i.e., that $B_1 \cap C_K = B_2 \cap C_K$. This is the case, for instance, when $B_1 = C_0(\partial^2/\partial x\partial y,\ \partial/\partial x)$ and $B_2 = C_0(\partial^2/\partial x\partial y)$. Indeed, let D_ε resp. $C_\varepsilon(\partial^2/\partial x\partial y)$ consist of the functions in D resp. $C_0(\partial^2/\partial x\partial y)$ having support in the ball of radius $1/\varepsilon$. Then the closure of D_ε in $C_0(\partial^2/\partial x\partial y)$ contains at least $C_{2\varepsilon}(\partial^2/\partial x\partial y)$. Hence we need only compare the norms of $C_0(\partial^2/\partial x\partial y)$ and $C_0(\partial^2/\partial x\partial y,\ \partial/\partial x)$ on each fixed D_ε. And then it is evident that

$$\left|\frac{\partial f}{\partial x}(a,\ b)\right| \leqslant \int_{-1/\varepsilon}^{b} \left|\frac{\partial^2 f}{\partial x\partial y}\right| dy \leqslant \frac{2}{\varepsilon} \sup \left|\frac{\partial^2 f}{\partial x\partial y}\right|.$$

The following proposition shows, however, that such collapsing of norms can not occur on the spaces we are studying in this paper.

Proposition 4.3. — *Let* B_1 *and* B_2 *be distinct spaces of differentiable functions between* C_0^N *and* C_0^{N-1}. *Then* $B_1 \cap C_K$ *and* $B_2 \cap C_K$ *are distinct.*

Proof. — Assume that $B_1 \cap C_K = B_2 \cap C_K$. Let

$$E_j = \{f : f \in B_j,\ f(z) = 0 \quad \text{if} \quad |z| \geqslant 1\}, \qquad j = 1, 2.$$

Then $E_1 = E_2$. E_j is a closed linear subspace of the Banach space B_j and is thus a Banach space in the induced topology. By the closed graph theorem, the B_1 topology on $E_1 = E_2$ is identical with the B_2 topology. Since $B_1 \neq B_2$, then $\mathcal{A}_{B_1} \neq \mathcal{A}_{B_2}$. We may assume that there is an operator A in $\mathcal{A}_{B_1}$ that is not in $\mathcal{A}_{B_2}$. The mapping

$$f \to Af(0), \qquad f \in E_1 = E_2,$$

is a continuous linear functional, and thus there are $A_1, \ldots, A_n$

in $\mathfrak{A}_{B_2}$ and a constant M so that

$$|Ag(0)| \leqslant M(\|A_1 g\| + \cdots + \|A_n g\|)$$

for all g in D with $g(z) = 0$ for $|z| \geqslant 1$. Thus by Theorem 7.1, the homogeneous part of A of degree N is a linear combination of the corresponding homogeneous parts of the A_i. But since $\mathfrak{A}_{B_2}$ is a linear subspace of $\mathfrak{Q}$ containing $\mathfrak{Q}_{N-1}$, A is in $\mathfrak{A}_{B_2}$. Contradiction.

5. — Classification of rotating spaces.

This section is devoted to the proof of Theorem 1.1, which classifies the rotating spaces of differentiable functions.

There are three things to be established :

1° *If* $\mathfrak{A}$ *is a proper subset of* $\mathfrak{R}_N$, *then* $C_0(\mathfrak{A})$ *is a proper rotating space of differentiable functions between* C_0^N *and* C_0^{N-1}.

2° *If* $\mathfrak{A}_1$ *and* $\mathfrak{A}_2$ *are distinct proper subsets of* $\mathfrak{R}_N$, *then* $C_0(\mathfrak{A}_1)$ *and* $C_0(\mathfrak{A}_2)$ *are distinct.*

3° *If* B *is a proper rotating space of differentiable functions, then* $B = C_0(\mathfrak{A})$, *where* $\mathfrak{A}$ *is a proper subset of* $\mathfrak{R}_N$ *for some* N.

Proof of 1°. — Let $\mathfrak{A}$ be a proper subset of $\mathfrak{R}_N$. Since $\mathfrak{A}$ spans a rotation-invariant subspace of $\mathfrak{Q}$, $C_0(\mathfrak{A})$ is a rotating space of differentiable functions. By Corollary 7.5, $C_0(\mathfrak{A}) \subset C_0^{N-1}$. And by Corollary 7.2, $C_0(\mathfrak{A}) \neq C_0^{N-1}$. That $C_0^N \subset C_0(\mathfrak{A})$ is clear. Since $C_0^N \subset C_0(\mathfrak{A}) \subset C_0^{N-1}$, by Lemma 4.1, $\mathfrak{A}_{C_0(\mathfrak{A})}$ is the linear subspace of $\mathfrak{Q}$ spanned by $\mathfrak{A}$ and $\mathfrak{Q}_{N-1}$ and is thus not all of $\mathfrak{Q}_N$, so $C_0(\mathfrak{A}) \neq C_0^N$.

Proof of 2°. — Let $\mathfrak{A}$ be a proper subset of (1.2). We know that $C_0^N \subset C_0(\mathfrak{A}) \subset C_0^{N-1}$, so by Lemma 4.1, $\mathfrak{A}_{C_0(\mathfrak{A})}$ is the linear subspace of $\mathfrak{Q}$ spanned by $\mathfrak{A}$ and $\mathfrak{Q}_{N-1}$. Thus if $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are distinct proper subsets of $\mathfrak{R}_N$, $\mathfrak{A}_{C_0(\mathfrak{A}_1)}$ and $\mathfrak{A}_{C_0(\mathfrak{A}_2)}$ are distinct, so $C_0(\mathfrak{A}_1)$ and $C_0(\mathfrak{A}_2)$ must be distinct.

Proof of 3°. — Let B be a proper rotating space of differentiable functions. Then by Proposition 3.5, there is a positive integer N so that $C_0^N \subset B \subset C_0^{N-1}$. Let $\mathfrak{A}$ be the intersection of $\mathfrak{R}_N$ and $\mathfrak{A}_B$. We will show that $\mathfrak{A}$ is a proper subset of $\mathfrak{R}_N$ and $B = C_0(\mathfrak{A})$. First it is necessary to relate $\mathfrak{A}$ and $\mathfrak{A}_B$.

Let $\mathcal{A}'$ be the linear subspace of $\mathcal{Q}$ spanned by $\mathcal{A}$ and $\mathcal{Q}_{N-1}$. We will show $\mathcal{A}' = \mathcal{A}_B$. First, $\mathcal{A} \subset \mathcal{A}_B$, and $\mathcal{Q}_{N-1} \subset \mathcal{A}_B$ since $B \subset C_0^{N-1}$, so $\mathcal{A}' \subset \mathcal{A}_B$. We have to demonstrate the reverse inclusion. By Corollary 3.2, $\mathcal{A}_B$ has a basis of rotating operators. Let A be one. This A is of order $\leqslant N$, for if not, by Corollary 7.2, $B = C_0(\mathcal{A}_B)$ would not contain C_0^N. Let $A = A_1 + A_2$, where order $A_1 = N$, order $A_2 < N$. Since $\mathcal{Q}_{N-1} \subset \mathcal{A}_B$, $A_2 \in \mathcal{A}_B$, so $A_1 \in \mathcal{A}_B$. But A_1 is a constant multiple of some operator in $\mathcal{A}$, so $A \in \mathcal{A}'$. This shows that $\mathcal{A}_B \subset \mathcal{A}'$, which completes the proof that $\mathcal{A}_B = \mathcal{A}'$. $\mathcal{A}$ is not all of $\mathcal{R}_N$ since $B \neq C_0^{N-1}$. There must be some operator of order N in $\mathcal{A}_B$, for otherwise we would have $\mathcal{A}_B \subset \mathcal{Q}_{N-1}$; and thus $B = C_0(\mathcal{A}_B) \supset C_0(\mathcal{Q}_{N-1}) = C_0^{N-1}$. Thus, since $\mathcal{A}_B = \mathcal{A}'$, $\mathcal{A}$ is non-empty. So we know $\mathcal{A}$ to be a proper subset of $\mathcal{R}_N$ and it remains to show $B = C_0(\mathcal{A})$. Part 1° of the proof shows that $C_0^N \subset C_0(\mathcal{A}) \subset C_0^{N-1}$, so by Lemma 4.1, $\mathcal{A}_{C_0(\mathcal{A})} = \mathcal{A}'$. But $\mathcal{A}' = \mathcal{A}_B$, so $B = C_0(\mathcal{A}_B) = C_0(\mathcal{A}') = C_0(\mathcal{A}_{C_0(\mathcal{A})}) = C_0(\mathcal{A})$. This completes the proof of Theorem 1.1.

6. — Rotating spaces as algebras.

In this section we show that the rotating spaces of differentiable functions are all algebras (and that all except C_0^∞ are Banach algebras). Because of Proposition 3.5 it is enough to show that each space of differentiable functions between C_0^N and C_0^{N-1} is a Banach algebra.

Lemma 6.1. — *Let* A *be an operator in* $\mathcal{Q}$ *of order* N. *Then there are* A_k' *and* A_k'' *in* $\mathcal{Q}$ *of order* $< N$ *so that*

$$Afg = fAg + gAf + \sum_k A_k'fA_k''g \qquad \text{for all} \qquad f,\ g \in D.$$

Proof. — It suffices to prove the assertion for $A = \partial^{m+n}/\partial x^m \partial y^n$. The result is true for $m + n = 1$. And it is simple to establish the induction step: If the assertion is valid for $\partial^{m+n}/\partial x^m \partial y^n$, then it is valid for $\partial^{m+n+1}/\partial x^{m+1} \partial y^n$ and $\partial^{m+n+1}/\partial x^m \partial y^{n+1}$.

Lemma 6.2. — *Let B be a space of differentiable functions between* C_0^N *and* C_0^{N-1}. *Let* $A \in \mathcal{A}_B$. *Then there are* A_k' *and* A_k''

in $\mathfrak{Q}$ of order less than the order of A *so that, for all f and g in* B, *the distribution* Afg *is equal to the continuous function*

$$fAg + gAf + \sum_k A'_k f A''_k g.$$

Proof. — We use the A'_k and A''_k given by Lemma 6.1. We shall prove that

$$(6.1)\quad \int (fAg + gAf + \sum_k A'_k f A''_k g)h = \int fg\tilde{A}h, \qquad h \in D,$$

for all f and g in B. Once (1.1) is established, by the definition of the distribution Lfg, we are done. Fix h in D. By the choice of the L'_K and L''_K, (6.1) holds for f and g in D. D $\times$ D is dense in B $\times$ B by Proposition 2.2. Both sides if (6.1) are continuous in (f, g) in the topology of B $\times$ B, so (6.1) holds for all f and g in B.

Proposition 6.3. — *Let* B *be a space of differentiable functions between* C_0^N *and* C_0^{N-1}. *Then* B *is a Banach algebra.*

Proof. — By Lemma 6.2, B is an algebra. Let $\mathfrak{A}$ be a basis for $\mathfrak{A}_B$. Define the norm $||\cdot||_B$ on B by

$$||f||_B = \sum_{A \in \mathfrak{A}} ||Af||, \qquad f \in B.$$

Then $||\cdot||_B$ gives the topology of B, and it is clear, because of Lemma 6.2, that there is a constant K so that

$$||fg||_B \leqslant K||f||_B\,||g||_B, \qquad f,\ g \in B.$$

This completes the proof of Proposition 6.3.

Corollary 1.2. now follows immediately from Proposition 6.3 and Proposition 3.5.

7. — Sup norm estimates.

In this section we give the results concerning sup norm estimates on which our work is based.

The first theorem is a strengthening of Propositions 1 and 2 of [2].

THEOREM 7.1. — *Let* $A_1, \ldots, A_m$ *be operators in* $\mathfrak{Q}$ *of order* $\leqslant N$. *Let* A *be an operator in* $\mathfrak{Q}$ *for which there is a constant* K *so that*

$$(7.1) \qquad |Ag(0)| \leqslant K(||A_1 g|| + \cdots + ||A_m g||)$$

for all g *in* D *with support in* $\{z : |z| < 1\}$. *Then* A *has order* $\leqslant N$ *and the homogeneous part of* A *of order* N *is a linear combination of the corresponding homogeneous parts of* $A_1, \ldots, A_m$.

Proof. — We denote by D_1 the subspace of D consisting of those g in D with support in $\{z : |z| < 1\}$. Take D_1 as domain for each of the A_k and by the mapping $g \to (A_1 g, \ldots, A_m g)$ embed their joint range in the direct sum $\oplus^m C_0$ of m copies of C_0. Because of (7.1), the functional

$$(A_1 g, \ \ldots, \ A_m g) \to Ag(0)$$

on the embedded joint range is continuous with respect to the natural topology of $\oplus^m C_0$. By Hahn-Banach this functional extends to the whole space $\oplus^m C_0$. And by the Riesz representation we can write

$$(7.2) \qquad Ag(0) = \sum_k \int A_k g d\mu_k, \qquad g \in D_1,$$

for some measures $\mu_1, \ \ldots, \ \mu_m$ of finite total mass. Write A and the A_k as sums of their homogeneous parts

$$A = \sum_e A^e, \qquad A_k = \sum_e A_k^e.$$

Substituting into (7.2) we have

$$(7.3) \qquad \sum_e A^e g(0) = \sum_e \sum_k \int A_k^e g d\mu_k, \qquad g \in D_1.$$

For $r > 1$ and g in D_1 define g_r in D_1 by

$$g_r(x, \ y) = g(rx, \ ry).$$

Then we have

$$A^e(g_r) = r^e(A^e g)_r, \qquad A_k^e(g_r) = r^e(A_k^e g)_r, \qquad g \in D_1,$$

so by (7.3),

$$(7.4) \qquad \sum_e r^e A^e g(0) = \ \sum_e \int r^e(A_k^e g)_r d\mu_k, \qquad g \in D_1.$$

Let $M = \max \{N, \text{degree } A\}$. Dividing (7.4) by r^M and letting $r \to \infty$, we have

$$A^M g(0) = \sum_k c_k A_k^M g(0), \qquad g \in D_1, \tag{7.5}$$

where c_k is the measure assigned to the origin by μ_k. If $M > N$, (7.5) is impossible, since each of the A_k^M is zero. Thus $M = N$, so

$$A^N = \sum_k c_k A_k^N$$

is a consequence of (7.5).

Corollary 7.2. — *Let* A *be an operator in* $\mathfrak{Q}$ *of order* N. *Then there exists an* f *in* C_0^{N-1} *with* Af *not in* C_0.

Proof. — Suppose that this were not the case. Then we would have $C_0^{N-1} \subset C_0(\{A\})$. By the closed graph theorem, the injection $C_0^{N-1} \to C_0(\{A\})$ is continuous. Thus, by the definition of the topology of C_0^{N-1}, there must be $A_1, \ldots, A_m$ in $\mathfrak{Q}$ of order $\leqslant N - 1$ and a constant K so that

$$\|Ag\| \leqslant K(\|A_1 g\| + \cdots + \|A_m g\|), \qquad g \in D.$$

Since A is of order N, this is impossible by Theorem 7.1.

The next result is half of Proposition 5 of [2].

Theorem 7.3. — *Let* A *be an elliptic operator in* $\mathfrak{Q}$ *of order* N. *If* A_0 *is an operator in* $\mathfrak{Q}$ *of order* $< N$, *then there is a constant* K *so that*

$$\|A_0 g\| \leqslant K(\|Ag\| + \|g\|), \qquad g \in D.$$

Corollary 7.4. — *Let* A *be an elliptic operator in* $\mathfrak{Q}$ *of order* N. *Then* $C_0(\{A\}) \subset C_0^{N-1}$.

Proof. — By Theorem 7.3, for every A_0 in $\mathfrak{Q}$ of order $< N$, there is a constant K so that

$$\|A_0 g\| \leqslant K(\|Ag\| + \|g\|), \qquad g \in D.$$

Thus, by Proposition 2.3, $C_0(\{A\})$ is contained in C_0^{N-1}.

Corollary 7.5. — *Let* $A = \partial^{m+n}/\partial z^m \partial \bar{z}^n$. *Then* $C_0(\{A\}) \subset C_0^{m+n-1}$

Proof. — A is elliptic, so Corollary 7.4 is applicable.

COROLLARY 7.6. — *Let* B *be a space of differentiable functions. If* $\mathcal{A}_B$ *contains a rotating operator* A *of order* N, *then* $\mathcal{A}_B$ *contains* $\mathcal{Q}_{N-1}$.

Proof. — Since A is rotating of order N, its homogeneous part of order N must be a multiple of some $\partial^{m+n}/\partial z^m \partial \bar{z}^n$, for $m + n = N$. Thus A is elliptic, so by Corollary 7.4,

$$C_0(\{A\}) \subset C_0^{N-1}.$$

But $B = C_0(\mathcal{A}_B) \subset C_0(\{A\})$ since $A \in \mathcal{A}_B$, so $B \subset C_0^{N-1}$. Equivalently, $\mathcal{Q}_{N-1} \subset \mathcal{A}_B$.

BIBLIOGRAPHY

[1] K. de LEEUW and H. MIRKIL, Algebras of differentiable functions, *Bull. Amer. Math. Soc.*, 68, 1962, 411-415.

[2] K. de LEEUW and H. MIRKIL, A priori sup norm estimates for differential operators, to appear, *Illinois Jour. of Math.*

[3] K. de LEEUW and H. MIRKIL, Algebras of differentiable functions on Riemann surfaces, to appear.