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INFRARED CATASTROPHE IN A MASSLESS FEYNMAN FUNCTION (*)

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In our recent article [KS3] we examined the effect of using Q-couplings (in the sense of [S2]) in some Feynman functions, and found that the resulting functions have singularities of the physically required degree. This basic property of Q-coupled functions is not enjoyed by ordinary Feynman functions, some of which exhibit stronger singularities. We will verify this latter fact by a detailed study of the Feynman function associated with the following diagram D' :

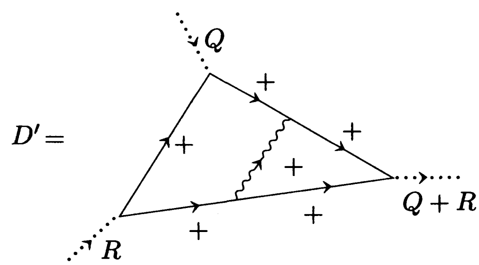

Figure 1

Here the wiggly line represents a massless propagator $1/(k^2 + i0)$, and each straight segment with $+$ sign represents a massive propagator $1/(p^2 - m^2 + i0)$. As we are concerned with the singularity structure of the integrals in question, we ignore the numerator factor in the integrands; i.e., several terms relevant to the γ-matrices are replaced by 1 here and in what follows. In connection with this, the external lines of the diagram are

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simplified so that one external line represents a pair of external lines, and the vertex to which it is connected represents a pair of vertices connected by a line that represents a charged particle that is far from its mass-shell, and hence a non-singular function.

One can easily verify that the Feynman function $F_{D'}(Q, R)$ is singular along the triangle-diagram singularity surface; i.e., along the strictly positive-α Landau-Nakanishi singularity surface $L_0^+(D) = \{\varphi = 0\}$ (hereafter referred to as the Landau surface for short) determined by the triangle diagram D :

$D =$

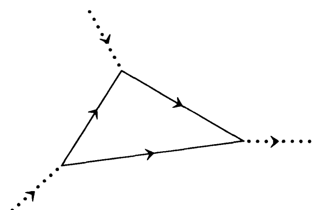


Figure 2

Our principal conclusion is that $F_{D'}$ contains a singularity of the form $(\log \varphi)^2$, despite the well-known fact that F_D behaves like $\log \varphi$ near $L_0^+(D)$. Such an increase of the degree of singularities would be as catastrophic as a divergence (i.e. non-well-definedness as a (hyper)function of (Q, R)) of the function involved, because it would be contradictory to the known large-distance fall-off properties for stable charged particles.

Note that in our analysis we do not associate a fictitious mass to a photon, as is usual in the treatment of mass singularities of Feynman functions. (See Kinoshita [Ki], for example.) The integral is nevertheless well defined after ultraviolet cut-off. (In what follows we assume that all integrals are defined with suitable ultraviolet cut-off factors.) We also note that we really encounter a (non-trivial) function of this sort in QED as a consequence of replacing each external line of D' by a pair of photon lines connected to two vertices separated by a far-off-mass-shell charged particle line. (This makes the number of photons incident upon the closed loop even, so that the function does not vanish by virtue of Furry's theorem.)

Our result pertains only to the single indicated Feynman diagram. It prevents the vertex part of that diagram from being defined as essentially a finite correction factor to the basic triangle diagram function, in the way that is normally done in theories in which all particles are massive. Renormalization does not help.

We have not confirmed that a summation over all diagrams of the same order in e^2 could not lead to a cancellation of the $(\log \varphi)^2$ singularity, but see no reason for such cancellation to occur. A rigorous proof of noncancellation would require a generalization of the present proof that would yield not only the $(\log \varphi)^2$ term, but its coefficient as well. Here it is only shown that the coefficient is nonzero for the single diagram, apart from a trace factor that arises from the numerator. For the case of an *odd* number of photons this trace factor is odd under $p \mapsto -p$, $k \mapsto -k$, and hence would (as noted by Furry) vanish under integration over p and k. For our case, with an even number of photons, this trace factor is even, and will not vanish in general.

In what follows we use the same symbols and notations as in [KS2]; for example :

- the function $R(k)$ denotes the retarded propagator $1/\big((k_0 + i0)^2 - \vec{k}^2\big)$ and it is diagramatically represented by $\underset{+}{\dashrightarrow}$;
- the function $A(k)$ denotes the advanced propagator $1/\big((k_0 - i0)^2 - \vec{k}^2\big)$ and it is diagramatically represented by $\underset{-}{\dashrightarrow}$;
- the symbol $-\!\!+\!\!\rightarrow\!-$ represents

$$(-2\pi i)\delta^+(p^2 - m^2) = (-2\pi i)\theta(p_0)\delta(p^2 - m^2).$$

We also freely use several (microlocal) properties of functions related to the retarded and advanced propagators that are obtained in [KS2]; in particular,

$$\frac{1}{k^2 + i0} = (-2\pi i)\delta^+(k^2) + A(k) \tag{1.a}$$

$$= (+2\pi i)\delta^-(k^2) + R(k). \tag{1.b}$$

(See [KS2] for the definition of $\delta^\pm(k^2)$.)

The main result of this paper is the following

THEOREM 1. — *Let D be the triangle diagram, and let $L_0^+(D)$ denote the strictly positive-α Landau surface. Let (Q^0, R^0) be a point in $L_0^+(D)$, and let D' denote the diagram given in Fig.1. Then, on a neighborhood of (Q^0, R^0), $F_{D'}(Q, R)$ has the form*

$$a(Q,R)\big(\log(\varphi(Q,R) + i0)\big)^2 + b(Q,R)\log\big(\varphi(Q,R) + i0\big) + c(Q,R)$$

with a, b and c being analytic on the neighborhood of (Q^0, R^0). Furthermore a does not vanish identically.

Proof. — Using (1.b), we first decompose $F_{D'}$ into the sum of two terms, F_{D_0} and F_{D_1}, where D_0 and D_1 are respectively given as follows :

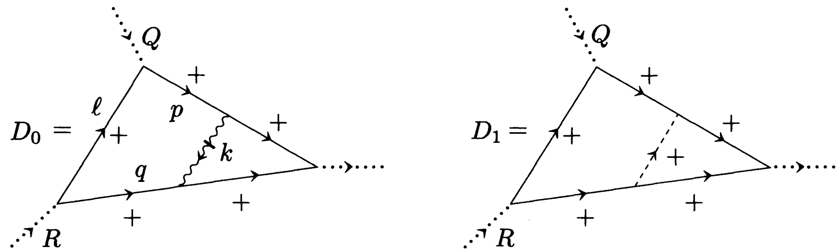


Figure 3

Our purpose is to show the $(\log\varphi)^2$-character of the integral F_{D_0} and the $(\log\varphi)$-character of F_{D_1}. Let us begin our computation with F_{D_0}. Since the integrand of F_{D_0} contains $\delta^-(k^2)$, we find

$$(3)\qquad \frac{1}{p^2-m^2+i0}\,\frac{1}{(p-k)^2-m^2+i0} = \Big(\frac{1}{(p-k)^2-m^2+i0}-\frac{1}{p^2-m^2+i0}\Big)\frac{1}{2pk-i0},$$

$$(4)\qquad \frac{1}{q^2-m^2+i0}\,\frac{1}{(q+k)^2-m^2+i0} = \Big(\frac{1}{q^2-m^2+i0}-\frac{1}{(q+k)^2-m^2+i0}\Big)\frac{1}{2qk+i0}$$

in the integrand of F_{D_0}.

Let us introduce the polar coordinate (r,Ω) in k-space. With $k=r\Omega$ (Ω, a unit Euclidean vector) neither $p\Omega$ nor $q\Omega$ vanishes for massive p and q when $\Omega^2=0$. Hence we find F_{D_0} is expressed as

$$(5)\qquad \int d^4\Omega\,\delta(\Omega\tilde{\Omega}-1)\delta^-(\Omega^2) \times\int_0^{\kappa}\frac{dr}{r}\Big[\{F_\Delta(Q,R)-F_\Delta(Q,R+r\Omega)\} - \{F_\Delta(Q-r\Omega,R)-F_\Delta(Q-r\Omega,R+r\Omega)\}\Big]$$

$$= \int d^4\Omega\,\delta(\Omega\tilde{\Omega}-1)\delta^-(\Omega^2) \times\int_0^{\kappa}\frac{dr}{r}\Big[\{F_\Delta(Q,R)-F_\Delta(Q,R+r\Omega)\} - \{F_\Delta(Q-r\Omega,R)-F_\Delta(Q,R)\} + \{F_\Delta(Q-r\Omega,R+r\Omega)-F_\Delta(Q,R)\}\Big],$$

where $\check{\Omega} = (\Omega_0, -\vec{\Omega})$, and $F_\Delta(Q,R)$ is an integral of the form

$$(6) \qquad \int A(\ell,Q,R,k)\,\frac{1}{\ell^2-m^2+i0} \times \frac{1}{(\ell+Q)^2-m^2+i0} \times \frac{1}{(R-\ell)^2-m^2+i0}\,\mathrm{d}^4\ell$$

with an analytic factor $A(\ell,Q,R,k)$ (if $(\ell+Q)^2 \neq 0$ and $(R-\ell)^2 \neq 0$). Hence F_Δ is of the form

$$(7) \qquad B(Q,R,k)\log\bigl(\varphi(Q,R)+i0\bigr)+C(Q,R,k)$$

with B and C analytic if (Q,R) is close to (Q^0,R^0). Hence our first task is to calculate the r-integral

$$(8) \qquad \int_0^\kappa \frac{\mathrm{d}r}{r}\Bigl(\log\bigl(\varphi(Q,R+r\Omega)+i0\bigr)-\log\bigl(\varphi(Q,R)+i0\bigr)\Bigr)$$

etc. Using the Landau-Nakanishi equations, we find

$$\frac{\partial}{\partial r}\varphi(Q,R+r\Omega)\Big|_{r=0} = \frac{\partial\varphi(Q,R)}{\partial R}\cdot\Omega = \alpha_1(R-\ell)\Omega$$

on $\varphi(Q,R)=0$ with $\alpha_1>0$. Since $(R-\ell)^2=m^2$ holds, to realize the triangle singularity, and since $\Omega^2=0$, we find $(R-\ell)\Omega \neq 0$, i.e., $\dfrac{\partial\varphi(Q,R+r\Omega)}{\partial r}\Big|_{r=0} \neq 0$. Hence the implicit function theorem guarantees the unique existence of a non-vanishing holomorphic function $\chi_1(Q,R,r,\Omega)$ on a neighborhood of (Q^0,R^0) for sufficiently small r so that

$$(9) \qquad \varphi(Q,R+r\Omega)=\chi_1(Q,R,r,\Omega)\Bigl(r+\frac{\varphi(Q,R)}{\chi_1(Q,R,0,\Omega)}\Bigr).$$

Similarly we can find non-vanishing holomorphic functions $\chi_j(Q,R,r,\Omega)$ $(j=2,3)$ so that the following hold :

$$(10) \qquad \varphi(Q-r\Omega,R)=\chi_2(Q,R,r,\Omega)\Bigl(r+\frac{\varphi(Q,R)}{\chi_2(Q,R,0,\Omega)}\Bigr),$$

$$(11) \qquad \varphi(Q-r\Omega,R+r\Omega)=\chi_3(Q,R,r,\Omega)\Bigl(r+\frac{\varphi(Q,R)}{\chi_3(Q,R,0,\Omega)}\Bigr).$$

Using these normalizations of $\varphi(Q,R+r\Omega)$ etc. we can reduce the computation of the r-integration in (5) to the computation of the integral discussed in the following

LEMMA 2. — *Let $I = I_c(t,\kappa)$ denote the following integral:*

$$\text{(12)} \qquad \int_0^\kappa \frac{\mathrm{d}r}{r}\big(\log(t+cr+i0) - \log(t+i0)\big)$$

where c is a non-zero real constant. Then $I_c(t,\kappa)$ is well-defined and it has the form

$$\text{(13)} \qquad \tfrac{1}{2}\big(\log(t+i0)\big)^2 + a\log(t+i0) + b(t) \quad \textit{near } t=0$$

with a constant a and a holomorphic function $b(t)$ defined near $t=0$. Here a and b depend on c and κ.

Proof. — We may suppose $c > 0$ without loss of generality. By decomposing the integral (12) into the sum of two terms

$$\text{(14)} \qquad \int_0^{t/2c} \frac{\mathrm{d}r}{r}\big(\log(t+cr) - \log t\big) + \int_{t/2c}^{\kappa} \frac{\mathrm{d}r}{r}\big(\log(t+cr) - \log t\big),$$

we can readily verify the well-definedness of the integral in the domain $\operatorname{Im} t > 0$; in fact, the Taylor expansion of $\log(1+cr/t)$ in the first term of (14) shows the well-definedness of the term, while the well-definedness of the second term of (14) is clear. As in Appendix I of [KS3], where more general cases are discussed, we find

$$\text{(15)} \qquad \begin{aligned} t\frac{\mathrm{d}}{\mathrm{d}t}I &= \int_0^\kappa t\frac{\mathrm{d}r}{r}\Big(\frac{1}{t+cr+i0} - \frac{1}{t+i0}\Big) \\ &= \int_0^\kappa \frac{-c}{t+cr+i0}\,\mathrm{d}r = \log(t+i0) - \log(t+c\kappa+i0). \end{aligned}$$

Hence we conclude that $t\frac{\mathrm{d}}{\mathrm{d}t}\big(I - \frac{1}{2}(\log(t+i0))^2\big)$ is holomorphic near $t=0$. This means that $I - \frac{1}{2}(\log(t+i0))^2$ is of the form $a\log(t+i0) + b(t)$ with a constant a and a holomorphic function b near $t=0$. This completes the proof of Lemma 2.

Returning to the computation of the integral (5), we use Lemma 2 to find

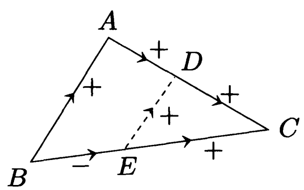
$$\begin{aligned} F_{D_0} = &\int \mathrm{d}^4\Omega\, \delta(\Omega\tilde{\Omega}-1)\delta^-(\Omega^2)\big(-\tfrac{1}{2}\big)B(Q,R,0)\big(\log(\varphi(Q,R)+i0)\big)^2 \\ &+ \int \mathrm{d}^4\Omega\, \delta(\Omega\tilde{\Omega}-1)\delta^-(\Omega^2)\big\{C_1(Q,R,\Omega,\kappa)\log(\varphi(Q,R)+i0) \\ &\qquad\qquad + C_0(Q,R,\Omega,\kappa)\big\}, \end{aligned}$$

with C_0 and C_1 holomorphic. Here we have used the fact that the part of $B(Q,R,k)$ (and C) in (5) which contains a factor of k cancels the $1/r$-factor in (5), hence leading to $\log\varphi$ singularities at worst. Since

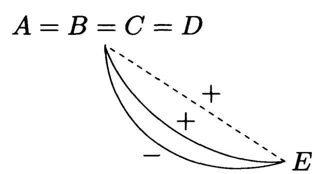
$$\int d^4\Omega\,\delta(\Omega\tilde{\Omega}-1)\delta^-(\Omega^2)\neq 0,$$

we find that the singularity of F_{D_0} consists of $(\log\varphi)^2$ and $\log\varphi$ and that $(\log\varphi)^2$-singularity does not vanish identically.

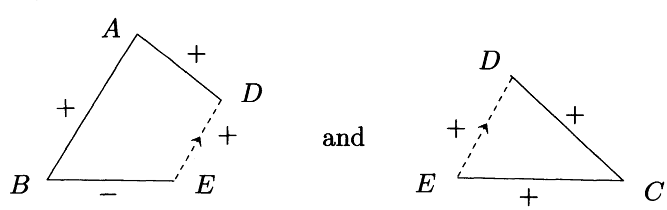
Next we show the $(\log\varphi)$-character of F_{D_1} in Fig. 3. One important fact in the analysis of F_{D_1} is the following : the cotangential component of the singularity spectrum of the retarded propagator is confined to the positive light cone even at $k=0$. (Cf. [KS2], (1.3); see [K³] for the basic notions in microlocal analysis such as singularity spectrum etc.) We shall refer to this fact as the light-cone property for short. Using this fact, we first show that the following diagram D_2 does not have the so-called $u=0$ points. (See [I] and [S1] for the definition of a $u=0$ points.)

$D_2 =$ 

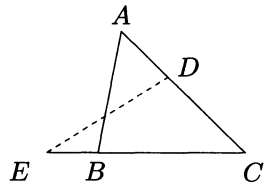
In fact, the existence of a $u=0$ point would entail the realization of the diagram



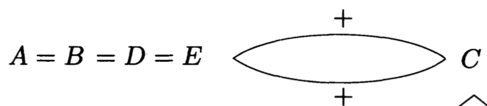
but this would contradict the light-cone property. Furthermore the light-cone property also entails that there is no contribution to the singularity near (Q^0, R^0) coming from D_2. In fact, the closed loop conditions for both

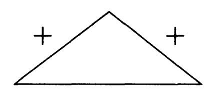


that is, the realization of the configuration



with the segment $\overrightarrow{ED}$ being a forward light-cone vector, would imply that the segment $\overrightarrow{BA}$ should be outside the forward light-cone, leading to a contradiction. Hence one of the above two closed loops should collapse. However, the collapsing of the first loop forces D_2 to become



and that of the second forces D_2 to become 

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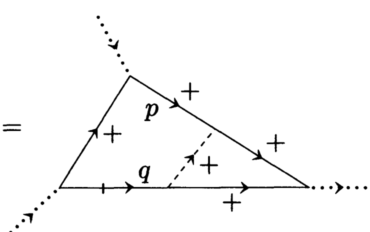
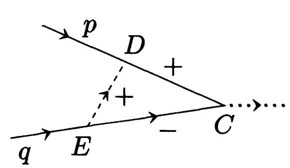
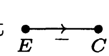
Neither of them contributes to the singularity near (Q^0, R^0) that lies in $L_0^+(D)$. Hence we may consider the following D_3 in place of D_1, as we are interested only in the region where $|k|$ is sufficiently small.

$D_3 =$ 

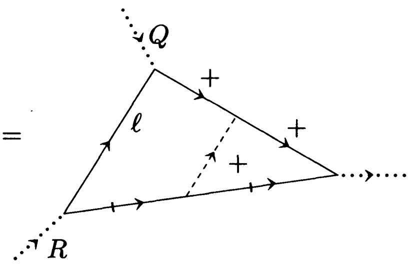
Here —+→— represents $(-2\pi i)\delta^+(q^2 - m^2)$, as usual. To analyze the integral associated with D_3, let us consider the function $I(p, q)$ associated with the following diagram D_4 with keeping $q^2 = m^2$, $q_0 > 0$:

$D_4 =$ 

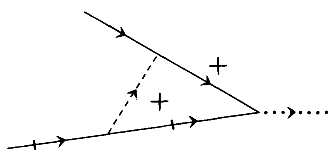
Then the propagator associated with the segment 

reduces to $-1/(2qk - k^2 + i0)$. Hence the closed-loop condition for the triangle DCE

is $w + \alpha(p+k) + \beta(q-k) = 0$ with $\alpha, \beta \geqq 0$ and w a forward light-cone vector. As $|k| \ll 1$ by the assumption, this closed-loop condition forces the triangle DCE to collapse. Otherwise stated, $I(p,q)$ is analytic if q is confined to the mass-shell manifold $q^2 = m^2$. Since q is confined to the mass-shell manifold in the diagram D_3, we may replace D_3 by

$D_5 =$ 

On the other hand, the integral associated with

$D_6 =$ 

is known to be analytic. (See [KS2], (6.2); to be precise the integral considered here is the integral (6.2) with the deletion of $r^2 P^{\mu\nu}\Omega_\mu\Omega_\nu$ in the numerator.) Therefore the integral F_{D_3} has the form

$$(16) \quad \int A_1(Q,R,\ell)\,\frac{1}{\ell^2 - m^2 + i0}\,\frac{1}{(Q+\ell)^2 - m^2 + i0}\,\delta^+\big((R-\ell)^2 - m^2\big)\,\mathrm{d}^4\ell$$

with A_1 being analytic if $\ell^2 \sim m^2$, $(Q+\ell)^2 \sim m^2$ and $(R-\ell)^2 = m^2$. As it is well-known that the integral (16) is of the form

$$a_1(Q,R)\log\big(\varphi(Q,R) + i0\big) + b_1(Q,R)$$

with a_1 and b_1 being analytic functions near $L_0^+(D)$, we have verified the $(\log\varphi)$-character of F_{D_1}. This completes the proof of Theorem 1.

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