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GAPS BETWEEN CONSECUTIVE DIVISORS OF FACTORIALS

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1. Introduction and main results.

Given a positive integer N, let $\tau = \tau(N)$ be its number of divisors, denoted by

$$1 = d_1 < d_2 < \ldots < d_\tau = N \ .$$

Erdős [3] has defined a family of arithmetic functions

$$(1) \qquad F_\varepsilon(N) = \sum_{i=1}^{\tau-1} \left(\frac{d_{i+1}}{d_i} - 1 \right)^{1+\varepsilon}, \qquad \varepsilon > 0,$$

and conjectured that

$$\liminf_{N\to\infty} F_\varepsilon(N) < \infty, \qquad \varepsilon > 0.$$

(See also [4] for related results and problems.) The conjecture was proved by Vose [9], who was able to construct a sequence $(N_n)_{n=1}^{\infty}$ such that

$$(2) \qquad F_\varepsilon(N_n) = \mathrm{O}_\varepsilon(1).$$

It is clear that to obtain small values for $F_\varepsilon(N)$ one needs numbers N with "many" divisors. In fact, the sequence (N_n) constructed by Vose is a *divisor sequence*, i.e., $N_n | N_{n+1}$ for each n. This was anticipated in [2] and [3], where Erdős specifically suggested the sequences

$$(3) \qquad \mathcal{F} = (n!),$$

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$$\mathcal{L} = ([1,2,\dots,n]) \tag{4}$$

(where $[1,2,\dots,n]$ denotes the least common multiple of all positive integers not exceeding n) and

$$\mathcal{R} = \left(\prod_{i=1}^{n-1} p_i\right) \tag{5}$$

(where $p_1 < p_2 < \dots$ is the sequence of all primes) as candidates to satisfy (2). This was established by Tenenbaum [8], whose results actually apply to a large class of sequences satisfying a few conditions. Moreover, instead of the sum in (1), he considered the more general sum

$$F(N;h) = \sum_{i=1}^{\tau-1} h\left(\frac{d_{i+1}}{d_i} - 1\right), \tag{6}$$

where the function $h : [0,\infty) \longrightarrow [0,\infty)$ belongs to a certain class, containing in particular the functions $x \mapsto x^{1+\varepsilon}$ tackled by Vose.

The main portion of Tenenbaum's paper establishes, for each of the sequences $\mathcal{F}, \mathcal{L}$ and $\mathcal{R}$, a good upper bound on the ratios between consecutive divisors of elements of the sequence. By symmetry, it suffices to deal with the divisors below $\sqrt{N_n}$. For divisor sequences, as the interval $[1, \sqrt{N_n}]$ is covered by the intervals $[\sqrt{N_{j-1}}, \sqrt{N_j}]$, $j \le n$, one needs only consider the interval $[\sqrt{N_{n-1}}, \sqrt{N_n}]$. Tenenbaum proved (for each of the three sequences) that, given any $\beta < \frac{1}{2}$, for any sufficiently large n and $\sqrt{N_{n-1}} \le z \le \sqrt{N_n}$ there exist (many) divisors d of N_n with $z \le d \le (1+n^{-(\ln n)^\beta})z$. (Some aspects of his general approach are treated in more detail in [6].)

In this paper we deal, prompted by a question of Erdős [5], with the density of the divisors of $n!$, mainly near the "center" $\sqrt{n!}$. To state our results, let us introduce the following definitions and notations. A *factorization ratio* for $l \in \mathbf{N}$ is a number of the form $\frac{y}{x}$ where x, y are positive integers and $l = xy$. The *gap* for $n \in \mathbf{N}$ is $M_n = \min\{\alpha - 1 :$ α a factorization ratio for $n!$, $\alpha > 1\}$. Finally, we shall write $\lg t$ for $\log_2 t$ (i.e., $\frac{\ln t}{\ln 2}$). (In particular, $\lg e = \frac{1}{\ln 2}$.)

THEOREM 1. — *For* $n \ge 2^{16}$,

$$M_n \le 10^8 \left(\frac{\lg n}{n}\right)^{\frac{\lg n - \lg(\lg n)+1}{2} + \lg e} \le \left(\frac{1}{n}\right)^{\frac{\lg n}{2} - \lg(\lg n)}.$$

We will in fact prove a more general result of a type suggested by Tenenbaum's work.

THEOREM 2. — *For $n \geq 2^{16}$ and $\sqrt{(n-1)!} \leq D \leq \sqrt{n!}$, there is a divisor x of $n!$ such that*

$$\left|\frac{x}{D} - 1\right| \leq 5 \cdot 10^7 \left(\frac{\lg n}{n}\right)^{\frac{\lg n - \lg(\lg n)+1}{2} + \lg e} \leq \left(\frac{1}{n}\right)^{\frac{\lg n}{2} - \lg(\lg n)}.$$

Theorem 1 is almost the special case in which $D = \sqrt{n!}$. We shall prove the theorems simultaneously.

Remark 1. — The first inequality in each theorem is true with a slightly larger constant for $n \geq 2^8$, but we confine formal statements for small n to Lemma 1 and Proposition 1.

Remark 2. — Up to a bounded power of $\lg n$, the theorems above are the best our methods can produce without radical modification. The reason for this is that our approach yields upper bounds of the form $2^{(k^2-k)/2}(k-1)!n^{-k}$, which attains its minimum (for fixed n) when $2^k k \approx n$, and then has approximately the value given above. The large constants arise because we are not quite able to obtain the bound with the optimum value of k. (Again, see Proposition 1.)

We may view Theorem 2 as a "topological" statement regarding the density of divisors of $n!$. One may also inquire about the "measure theoretical" analogue, namely how the finite sequence of all divisors of $n!$ is distributed (after appropriate normalization) as n becomes large. This question was pursued by Vose [11] (following another paper [10], discussing aspects of this problem for general divisor sequences). As one might expect, his results cannot be used to recover the results of this paper (or even Tenenbaum's). He finds the limiting distribution of the discrete probability measures obtained from the set of logarithms of all divisors of $n!$, along with some estimate on the error. Roughly speaking, for this to imply that a certain interval contains a divisor of $n!$, the interval needs to be larger than the error. However, the error term does not decay sufficiently fast, so that the results of [9] do not imply even the existence of a constant c such that the interval $\left[\frac{\sqrt{n!}}{c}, c\sqrt{n!}\right]$ must contain a divisor of $n!$. Of course, our results do not imply those of [11].

One should note that our approach to proving Theorems 1 and 2 is totally different from Tenenbaum's. While improving his upper bounds for

$n!$ (and being able to provide similar bounds for other sequences consisting of "regular" products), our method does not apply at all to the other sequences, $\mathcal{L}$ and $\mathcal{R}$, for which his method worked more easily than for $\mathcal{F}$. Moreover, the improvement we obtain regards being able to find a divisor of $n!$ in a smaller interval; he finds, in a bigger interval, many more divisors. On the other hand, throughout the proofs of Theorems 1 and 2 only "special" divisors of $n!$, i.e., divisors of the form $a_1a_2\dots a_l$, where $1 \le a_1 < a_2 < \dots < a_l \le n$, are used, so we actually prove the existence of such divisors within the specified ranges. We also mention that our results can probably be exploited to extend the class of functions h for which Tenenbaum proved the boundedness of (6) along the three sequences he considered (of course, we could do it only for $\mathcal{F}$), but we shall not pursue this direction here.

Another question Erdős asked [5] was about lower bounds for the gap between consecutive divisors of $n!$ near $\sqrt{n!}$. It will be convenient to formulate our results in this direction with slightly different notations from those for the upper bounds. Thus, consider for each n the "most balanced" factorization of $n!$ into a product of two positive integers,

$$n! = l_n h_n \ , \qquad (l_n \le \sqrt{n!} \le h_n,\ h_n - l_n \text{ minimal }).$$

A sequence $(x_n)_{n=1}^{\infty}$ *converges in density* to x, and we write $x_n \xrightarrow{D} x$, if $x_n \longrightarrow x$ but for a subsequence of zero asymptotic density.

THEOREM 3. — *For any $a \in \mathbf{N}$ the set $\{n \in \mathbf{N} : h_n - l_n = a\}$ has zero asymptotic density. In other words*

$$h_n - l_n \xrightarrow{D} \infty.$$

THEOREM 4. — $h_n - l_n \ge \sqrt{n}$ *for infinitely many positive integers* n*, and* a fortiori

$$h_n - l_n = \Omega(\sqrt{n}).$$

The upper and lower bounds we obtain are very far from each other. The behaviour of the number of divisors of $n!$ as a function of n, which is "almost exponential", seems to hint that neither of these theorems is close to the best possible.

In Section 2 we prove Theorems 1 and 2, and in Section 3 we prove Theorems 3 and 4.

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2. Upper bounds.

To obtain a divisor of $n!$ close to a given number D, we shall construct a factorization ratio α close to $\dfrac{n!/D}{D}$: then $\alpha = \dfrac{x}{y}$ where y is a divisor of $n!$ close to D. Given the target value $A\left(=\dfrac{n!}{D^2}\right)$ of the ratio, we estimate

$M_{n,A} = \min\left\{\left|\ln\left(\dfrac{\alpha}{A}\right)\right| : \alpha \text{ a factorization ratio for } n!\right\}.$

It will be convenient to use the notation

$$a_{1,t} = \frac{t}{t-1}\,, \qquad t > 1,$$

and

$$a_{k+1,t} = \frac{a_{k,t-2^k}}{a_{k,t}}, \qquad k \in \mathbf{N},\ t > 2^{k+1}.$$

The factorization ratios will be constructed by a variant of the greedy algorithm, using the numbers $a_{k,n}$. The point is that if α is a factorization ratio for $(n-2^k)!$ ($n, k \in \mathbf{N}$, $n \geq 2^k$) then $\alpha a_{k,n}$ and $\alpha a_{k,n}^{-1}$ are factorization ratios for $n!$. The greedy algorithm is to find α as close as possible to the target and then take either $\alpha a_{k,n}$ or $\alpha a_{k,n}^{-1}$ as the "good" factorization ratio for $n!$. Since $\dfrac{1}{n} \leq \ln a_{1,n} \leq \dfrac{1}{n-1}$ and $\sum \dfrac{1}{n}$ diverges it is easy to use $a_{1,(.)}$ to show that $M_{n,A} = O\left(\dfrac{1}{n}\right)$ as $n \to \infty$ for every fixed $A > 0$. In particular, one can show that $M_{n,1} \leq \dfrac{1}{n-1}$ for $n \geq 7$. The first difficulty is that the result is not uniform in A. In the proof of Theorem 2, A could be as big as n, but the greedy method already breaks down for $A \approx \sqrt{n}$, since then

$$\sum_{\substack{2 \le m \le n \\ m \equiv n \pmod 2}} \ln a_{1,m} \approx \frac{1}{2} \ln n \not> \ln A.$$

LEMMA 1. — *If $n \ge 4$ and $1 \le A \le n+1$, then :*

$$M_{n,A} \le \frac{2}{n}. \tag{7}$$

Proof. — The reader may readily list the divisors of and factorization ratios for 4! and 5! to confirm the result for $n \in \{4, 5\}$. (It is also easy to see that the conclusion fails for $n = A = 3$.)

Take $n > 5$ and assume that for $m \in \{n-2, n-1\}$ and $1 \le A \le m+1$, $M_{m,A} \le \dfrac{2}{m}$. Suppose $A \le n-1$. Then there is a factorization ratio α for $(n-2)!$ with $\left|\ln \dfrac{\alpha}{A}\right| \le \dfrac{2}{n-2}$, so there is $\beta \in \{\alpha a_{1,n}, \alpha a_{1,n}^{-1}\}$ with $\left|\ln \dfrac{\beta}{A}\right| = \left|\ln \dfrac{\alpha}{A} \pm \ln a_{1,n}\right| \le \max\left\{\dfrac{1}{n-1}, \dfrac{2}{n-2} - \dfrac{1}{n}\right\} \le \dfrac{2}{n}$.

Now suppose $n-1 < A \le n+1$. Choose a factorization ratio α for $n-1$ with $|\ln \alpha| = M_{n-1,1}$. Then $n\alpha$ and $n\alpha^{-1}$ are factorization ratios for n. For some $\beta \in \{n\alpha, n\alpha^{-1}\}$, $\left|\ln \dfrac{\beta}{A}\right| = \left|\ln \dfrac{n}{A} \pm \ln \alpha\right| \le \max\left\{\left|\ln \dfrac{n}{A}\right|, |\ln \alpha|\right\}$. Since $|\ln \dfrac{n}{A}| \le \dfrac{2}{n}$, it remains only to check that $M_{n-1,1} \le \dfrac{2}{n}$. For $n \ge 8$ this is clear since $M_{n-1,1} \le \dfrac{1}{n-2}$, by the remarks above. To complete the proof, observe that $M_{5,1} = \ln \dfrac{12}{10} < \dfrac{1}{5} < \dfrac{2}{6}$ and $M_{6,1} = \ln \dfrac{30}{24} < \dfrac{1}{4} < \dfrac{2}{7}$.

Set :

$$c_k = 2^{(k^2-k)/2}(k-1)!$$

for $k \in \mathbf{N}$. (Note that $c_{k+1} = 2^k k c_k$.)

LEMMA 2. — *For $k \in \mathbf{N}$ and $t \ge 2^{k+1}k$:*

$$0 \le \ln a_{k,t} - c_k t^{-k} \le c_{k+1} t^{-k-1}. \tag{8}$$

A routine calculation yields the first two terms in the Laurent series, showing that $\ln a_{k,t} = c_k t^{-k} + \left(2^{k-1} - \dfrac{1}{2}\right) k c_k t^{-k-1} + O(t^{-k-2})$. Since $c_{k+1} = 2^k k c_k$, it follows that the lemma is essentially sharp.

Proof. — Put $F_k(t) = \ln a_{k,t}$. A standard induction yields :

$$F_k(t) = (-1)^{k-1} \sum_{m=0}^{2^k-1} (-1)^{\sigma(m)} \ln(t-m), \qquad k \geq 1,\ t > 2^k,$$

where $\sigma(m)$ is the sum of digits in the binary representation of m. (Alternatively, we may reduce $\sigma(m)$ modulo 2 and obtain the well-known Thue-Morse sequence – see, for example, [7, p.73].) Setting

$$\lambda_k(n) = \sum_{m=0}^{2^k-1} (-1)^{\sigma(m)} m^n, \qquad n \geq 0,$$

we obtain

$$\begin{aligned} F_k(t) &= (-1)^{k-1} \sum_{m=0}^{2^k-1} (-1)^{\sigma(m)} \left(\ln t + \ln\left(1 - \frac{m}{t}\right)\right) \\ &= (-1)^{k-1} \sum_{m=0}^{2^k-1} (-1)^{\sigma(m)} \left(\ln t - \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{m}{t}\right)^n\right) \\ &= (-1)^{k-1}\lambda_k(0)\ln t + (-1)^k \sum_{n=1}^{\infty} \frac{\lambda_k(n)}{n} t^{-n}. \end{aligned}$$

To calculate the coefficients $\lambda_k(n)$ we employ generating functions :

$$\begin{aligned} \sum_{n=0}^{\infty} \lambda_k(n)\frac{x^n}{n!} &= \sum_{m=0}^{2^k-1} (-1)^{\sigma(m)} \sum_{n=0}^{\infty} \frac{m^n x^n}{n!} \\ &= \sum_{m=0}^{2^k-1} (-1)^{\sigma(m)} e^{mx} \\ &= \prod_{j=0}^{k-1} \left(1 - e^{2^j x}\right) \\ &= (-1)^k \prod_{j=0}^{k-1} \sum_{h=1}^{\infty} \frac{(2^j x)^h}{h!}. \end{aligned}$$

Opening up we find that $\lambda_k(n) = 0$ for $n < k$ and

$$F_k(t) = \sum_{n=k}^{\infty} \gamma_k(n) t^{-n},$$

where

$$\begin{aligned}
\gamma_k(n) &= \frac{1}{n}\cdot n! \sum_{\substack{h_0+h_1+\ldots+h_{k-1}=n \\ \min h_j \geq 1}} \frac{1}{h_0!h_1!\ldots h_{k-1}!} \prod_{j=0}^{k-1} 2^{jh_j} \\
&= (n-1)! \sum_{\substack{u_0+u_1+\ldots+u_{k-1}=n-k \\ \min u_j \geq 0}} \frac{1}{u_0!u_1!\ldots u_{k-1}!} \prod_{j=0}^{k-1} \frac{2^{j(u_j+1)}}{u_j+1} \\
&= \frac{(n-1)!}{(n-k)!} \sum_{\substack{u_0+u_1+\ldots+u_{k-1}=n-k \\ \min u_j \geq 0}} \binom{n-k}{u_0,u_1,\ldots,u_{k-1}} 2^{\frac{k(k-1)}{2}} \prod_{j=0}^{k-1} \frac{2^{ju_j}}{u_j+1} \\
&\leq \frac{(n-1)!}{(n-k)!} 2^{\frac{k(k-1)}{2}} \sum_{\substack{u_0+u_1+\ldots+u_{k-1}=n-k \\ \min u_j \geq 0}} \binom{n-k}{u_0,u_1,\ldots,u_{k-1}} \frac{\prod_{j=0}^{k-1} 2^{ju_j}}{\sum_{j=0}^{k-1} u_j+1} \\
&= \frac{(n-1)!}{(n-k+1)!} 2^{\frac{k(k-1)}{2}} (1+2+\ldots+2^{k-1})^{n-k} \\
&= \frac{(n-1)!}{(n-k+1)!} 2^{\frac{k(k-1)}{2}} (2^k-1)^{n-k}.
\end{aligned}$$

Setting $z = \dfrac{2^k-1}{t}$, we obtain :

$$\begin{aligned}
\sum_{n=k+1}^{\infty} \gamma_k(n) t^{-n} &\leq \sum_{n=k+1}^{\infty} \frac{(n-1)!}{(n-k+1)!} 2^{\frac{k(k-1)}{2}} (2^k-1)^{n-k} t^{-n} \\
&= 2^{\frac{k(k-1)}{2}} (2^k-1)^{-k} \sum_{n=k+1}^{\infty} \frac{(n-1)!}{(n-k+1)!} z^n .
\end{aligned}$$

Now :

$$\begin{aligned}
\sum_{n=k+1}^{\infty} \frac{(n-1)!}{(n-k+1)!} z^n &= (k-1)! z^{k-1} \sum_{n=k+1}^{\infty} \frac{(n-1)!}{(k-1)!(n-k+1)!} z^{n-k+1} \\
&= (k-1)! z^{k-1} \int_0^z \sum_{n=k+1}^{\infty} \frac{(n-1)!}{(k-1)!(n-k)!} x^{n-k} \mathrm{d}x \\
&= (k-1)! z^{k-1} \int_0^z \big((1-x)^{-k}-1\big) \mathrm{d}x \\
&= (k-1)! z^{k-1} G_k(z) ,
\end{aligned}$$

where

$$G_k(y) = \int_0^y \big((1-x)^{-k}-1\big) \mathrm{d}x .$$

By Taylor's Theorem,

$$\begin{aligned} G_k(z) &= G_k(0) + zG_k'(0) + \frac{z^2}{2}G_k''(x) \\ &= 0 + z \cdot 0 + \frac{z^2}{2}k(1-x)^{-k-1} \end{aligned}$$

for some x between 0 and z. Since $(1-x)^{-k-1} < (1-z)^{-k-1} < 2$, it follows that

$$\begin{aligned} \sum_{n=k+1}^{\infty} \gamma_k(n)t^{-n} &\leq 2^{\frac{k(k-1)}{2}}(2^k-1)^{-k} \sum_{n=k+1}^{\infty} \frac{(n-1)!}{(n-k+1)!}z^n \\ &< 2^{\frac{k(k-1)}{2}}(2^k-1)^{-k}(k-1)!z^{k-1}kz^2 \\ &= 2^{\frac{k(k-1)}{2}}(2^k-1)^{-k}k!z^{k+1} \\ &= 2^{\frac{k(k-1)}{2}}(2^k-1)^{-k}k!\left(\frac{2^k-1}{t}\right)^{k+1} \\ &= 2^{\frac{k(k-1)}{2}}(2^k-1)k!t^{-k-1} < c_{k+1}t^{-k-1}. \end{aligned}$$

This completes the proof of Lemma 2.

PROPOSITION 1. — *Suppose $k, n \in \mathbf{N}$, $n \geq n_k = 2^{k+5}k$, and $1 \leq A \leq n+1$. Then $M_{n,A} \leq 2c_k n^{-k}$.*

Proof. — We will use induction on k. Lemma 1 is a better result for $k = 1$. The induction step will be given in detail for $k \geq 3$ and we shall sketch the modifications required for $k \in \{1, 2\}$. Take $k \geq 3$ and assume that for $n \geq n_k$ and $1 \leq A \leq n+1$, $M_{n,A} \leq 2c_k n^{-k}$. The first step is to show that a similar inequality also holds for $n \leq A \leq n^2$ provided $n \geq n_k + 1$. Indeed for such A and n there is a factorization ratio α for $(n-1)!$ with $\left|\ln \frac{\alpha}{A/n}\right| \leq 2c_k(n-1)^{-k}$. Of course it follows that $n\alpha$ is a factorization ratio for $n!$ with $\left|\ln \frac{n\alpha}{A}\right| = \left|\ln \frac{\alpha}{A/n}\right| \leq 2c_k(n-1)^{-k} \leq 3c_k n^{-k}$.

Now take $n \geq n_{k+1}$ and suppose $1 \leq A \leq n+1$. Choose m as small as possible with $m \geq \frac{n}{2}$ and $m \equiv n \pmod{2^k}$. Since $m \geq n_k + 1$ and $A \leq m^2$, there is a factorization ratio α_0 of $m!$ with $\left|\ln \frac{\alpha_0}{A}\right| \leq 3c_k m^{-k}$. Define α_l (a factorization ratio for $(m + 2^k l)!$) recursively for $l \in \mathbf{N}$ by

$$\alpha_l = \begin{cases} \alpha_{l-1}a_{k,m+2^k l}, & \text{if } \alpha_{l-1} < A, \\ \alpha_{l-1}/a_{k,m+2^k l}, & \text{if } \alpha_{l-1} \geq A. \end{cases}$$

Then for $l_0 = (n-m)2^{-k}$, $\beta_0 = \alpha_{l_0}$ is a factorization ratio for $n!$. The plan is to modify β_0 to obtain a factorization ratio suitably close to A. The obvious modification is to replace a term of the form $a_{k,m+2^k l}^{\pm 1}$ in the expansion for α_{l_0} by its reciprocal, but this is too drastic and must be moderated by the opposite action on another term. If the terms are adjacent then this adjustment is exactly multiplication or division by $a_{k,m'}^2$.

Since $\ln a_{k,t} \geq c_k t^{-k}$, $\left|\ln \frac{\alpha_l}{A}\right| \leq \ln a_{k,m+2^k l}$ for $l \geq 3$. (This follows because $\ln a_{k,m+2^k} + \ln a_{k,m+2\cdot 2^k} + \ln a_{k,m+3\cdot 2^k} \geq 3 \ln a_{k,m+3\cdot 2^k} \geq 3c_k(m+3\cdot 2^k)^{-k} \geq 3e^{-1/10} c_k m^{-k} > 2c_k m^{-k}$. This and similar estimates are used throughout the proof.) For $l \geq 1$, not all three numbers $\ln \alpha_l$, $\ln \alpha_{l+1}$, and $\ln \alpha_{l+2}$ can be on the same side of $\ln A$. Set $S_+ = \{l \in \{2,3,4,\dots,l_0\} : \alpha_l = \alpha_{l-2} a_{k,m+2^k(l-1)}^{-1} a_{k,m+2^k l}\}$ and let $S_- = \{l \in \{2,3,4,\dots,l_0\} : \alpha_l = \alpha_{l-2} a_{k,m+2^k(l-1)} a_{k,m+2^k l}^{-1}\}$. Define factorization ratios β_J of $n!$ recursively as follows : Suppose $\beta_J = \beta_0 \prod_{j=1}^{J} a_{k+1,m+2^k l_j}^{2\sigma(l_j)}$ where $\sigma(l) = 1$ (resp. $\sigma(l) = -1$) if $l \in S_+$ (resp. $l \in S_-$) and $l_{j+1} \geq 2 + l_j$. If $\beta_J > A$ and there is $l \in S_- \cap \{l_J+2, l_J+3, \dots\}$ with $a_{k+1,m+2^k l} < \frac{\beta_J}{A}$ then $\beta_{J+1} = \beta_J a_{k+1,m+2^k l_{J+1}}^{-2}$ where l_{J+1} is the least such l. Similarly if $\beta_J < A$ and there is $l \in S_+ \cap \{l_J+2, l_J+3, \dots\}$ with $a_{k+1,m+2^k l} < \frac{A}{\beta_J}$ then $\beta_{J+1} = \beta_J a_{k+1,m+2^k l_{J+1}}^{2}$ where l_{J+1} is the least such l. Let J_0 be the largest J for which l_J can be found. We claim that β_{J_0} is a factorization ratio for $n!$ with $\left|\ln \frac{\beta_{J_0}}{A}\right| < 2c_k n^{-k}$.

An induction shows that for $J \in \{1,2,\dots,J_0\}$ there is a factorization ratio $\tilde{\alpha}_{l_J}$ of $(m+2^k l_J)!$ such that $\beta_J = \tilde{\alpha}_{l_J} \frac{\alpha_{l_0}}{\alpha_{l_J}}$. It follows that β_J is a factorization ratio for $n!$ for every J. The rest of the proof verifies the upper bound for $\left|\ln \frac{\beta_{J_0}}{A}\right|$.

The approximate alternation between multiplication and division in the construction of β_0 from α_0 implies that S_+ and S_- cannot be very sparse. More precisely, each of them has a point in $\{2,3,4,5,6\}$, and for every $l \in S_+ \cup S_-$, if $l \leq l_0 - 5$ then S_+ and S_- both intersect $\{l+2, l+3, l+4, l+5\}$.

If $J_0 = 0$ then $\frac{A}{\beta_0} \leq a_{k+1,m+2^k l}$ for every $l \in S_+$ and $\frac{\beta_0}{A} \leq a_{k+1,m+2^k l}$ for every $l \in S_-$. Since (by inequalities below) this leads to the estimate claimed, assume $J_0 > 0$.

Suppose that for every $J \in \{1, 2, \ldots, J_0\}$, $\left|\ln \frac{\beta_J}{A}\right| > a_{k+1,m+2^k l_J}$. Then the numbers $\ln \frac{\beta_J}{A}$ all have the same sign : assume first that they are all positive. Since $a_{k+1,(.)}$ is a decreasing function, $\left|\ln \frac{\beta_J}{A}\right| > a_{k+1,m+2^k l}$ for $l \geq l_J$, so by the construction, $\{l_1, l_2, \ldots, l_{J_0}\} = S_- \cap \{l_1, l_1+1, \ldots, l_0\}$. Suppose that $S_- \cap \{2, 3, \ldots, l_1 - 1\}$ is nonempty and let l be the greatest element. Then $\ln \frac{\beta_0}{A} \leq \ln a_{k+1,m+2^k l} < 3 \ln a_{k+1,m+2^k l_1}$ (since $l_1 \leq l + 5$), contrary to assumption. It follows that $\{l_1, l_2, \ldots, l_{J_0}\} = S_-$, so

$$\begin{aligned}
\ln \frac{\beta_0}{A} &\geq \sum_{J=1}^{J_0} \ln a^2_{k+1,m+2^k l_J} \\
&= 2 \sum_{l \in S_-} \ln a_{k+1,m+2^k l} \\
&\geq 2 \sum_{l \in S_-} c_{k+1}(m + 2^k l)^{-k-1} \\
&\geq 2 \sum_{\substack{6 \leq l \leq l_0 \\ l \equiv 1 \pmod 5}} c_{k+1}(m + 2^k l)^{-k-1} \\
&\geq \frac{2}{5} \int_6^{l_0} c_{k+1}(m + 2^k t)^{-k-1} \mathrm{d}t \\
&= \frac{2c_k}{5}\left((m + 6 \cdot 2^k)^{-k} - n^{-k}\right) \\
&\geq \frac{2c_k}{5}\left(\left(\frac{n}{2} + 7 \cdot 2^k\right)^{-k} - n^{-k}\right) \\
&> 2c_k n^{-k} \\
&> \ln a_{k,n} .
\end{aligned}$$

This contradicts the estimates for α_l. The assumption that the numbers $\ln \frac{\beta_J}{A}$ are all negative leads to a similar contradiction.

The penultimate inequality in the calculation above uses the assumption that $k \geq 3$. For $k \in \{1, 2\}$, it is important that m can be chosen substantially less than $n/2$. (This is possible because the results for small k are better than those stated.) We see that $M_{n,A} \leq 4n^{-2} = 2c_2 n^{-2}$ for $n \geq 160$, $1 \leq A \leq n + 1$ by choosing $m \approx \sqrt{n+1}$ congruent to n modulo 4 in case $k = 1$. In the case $k = 2$, we choose $m \approx \frac{n}{3}$. Note that for large k we could take m substantially greater than $n/2$ and still have

$(m+7\cdot 2^k)^{-k} - n^{-k} > 4n^{-k}$. We leave it to the interested reader to verify that for large k and $n \geq 2^{k+3}k$, if $1 \leq A \leq n+1$ then $M_{n,A} \leq 2c_k n^{-k}$.

By the result above, there is $J \in \{1, 2, \ldots, J_0\}$ such that $\left|\ln \frac{\beta_J}{A}\right| \leq a_{k+1,m+2^k l_J}$. For any such $J < J_0$, the argument used above to show that l_1 is the least element of S_- shows that $\left|\ln \frac{\beta_{J+1}}{A}\right| \leq a_{k+1,m+2^k l_{J+1}}$. Thus by induction

$$\left|\ln \frac{\beta_{J_0}}{A}\right| \leq a_{k+1,m+2^k l_{J_0}} \leq a_{k+1,n-5\cdot 2^k} < 2c_{k+1}n^{k+1}.$$

Proposition 1 follows.

Proof of Theorems 1 and 2. — Let $n \geq 2^{16}$ be given and suppose $\sqrt{(n-1)!} \leq D \leq \sqrt{n!}$. Set $A = \frac{n!}{D^2} \in [1, n]$. By the definitions, there is a factorization ratio α for $n!$ with $\left|\ln \frac{\alpha}{A}\right| = M_{n,A}$. The first step is to estimate $M_{n,A}$.

Define $\tilde{k}(\geq 8)$ by the equation $2^{\tilde{k}+5}\tilde{k} = n$, let k be the integer part of $\tilde{k}$, and set $t = \tilde{k} - k$, so that $2^{k+5}k \leq 2^{k+t+5}k \leq n < 2^{k+6}(k+1)$. Then by Proposition 1,

$$\begin{aligned}
M_{n,A} &\leq 2c_k n^{-k} \\
&= 2\cdot 2^{(k^2-k)/2}(k-1)!\, n^{-k} \\
&\leq 2(2^k)^{(k-1)/2}\sqrt{2\pi}\,(k-1)^{k-\frac{1}{2}}e^{1-k+\frac{1}{12(k-1)}}n^{-k} \\
&\leq 2(k-1)^{-\frac{1}{2}}\left(\frac{n}{2^{t+5}k}\right)^{(k-1)/2}\sqrt{2\pi}\,e^{1-k+\frac{1}{84}}(k-1)^k n^{-k} \\
&\leq 2^{1+\frac{5+t}{2}}7^{-\frac{1}{2}}\sqrt{2\pi}\,e^{1/84}2^{-(t+5)k/2}e^{-k}\left(\frac{k}{n}\right)^{(1-k)/2}k^k n^{-k} \\
&< 2^{\frac{7}{2}+\frac{t}{2}}\sqrt{2\pi/7}\,e^{1/84}(2^{-k})^{\lg e+\frac{t+5}{2}}\left(\frac{k}{n}\right)^{(k+1)/2} \\
&\leq 2^4 2^{\frac{t}{2}}\sqrt{\pi/7}\,e^{1/84}\left(\frac{2^6(k+1)}{n}\right)^{\lg e+\frac{t+5}{2}}e^{-1/2}\left(\frac{k+1}{n}\right)^{(k+1)/2} \\
&< 2^{22}2^{\frac{t}{2}}\sqrt{\pi/7}\,e^{6-\frac{41}{84}}\left(\frac{k+1}{n}\right)^{\frac{k+t+6}{2}+\lg e}
\end{aligned}$$

$$\leq 2^{22}\sqrt{\pi/7}\,2^{\frac{t}{2}}e^{4-\frac{t}{2}-\frac{41}{84}}\left(\frac{k+t+5}{n}\right)^{\frac{k+t+6}{2}+\lg e}$$

$$\leq 2^{22}\sqrt{\pi/7}\,e^{4-\frac{41}{84}}\left(\frac{k+t+5}{n}\right)^{\frac{k+t+6}{2}+\lg e}$$

$$< 9.5\cdot 10^{7}\left(\frac{k+t+5}{n}\right)^{\frac{k+t+6}{2}+\lg e}.$$

Now $k+t+6=\tilde{k}+6\geq \lg n-\lg(\lg n)+1$ and $k+t+5=\tilde{k}+5\leq \lg n$, so

$$M_{n,A}\leq 9.5\cdot 10^{7}\left(\frac{\lg n}{n}\right)^{\frac{\lg n-\lg(\lg n)+1}{2}+\lg e}.$$

Moreover, since $n\geq 2^{16}>2\cdot 2^{7}$, $M_{n,A}\leq 4n^{-2}\leq 2^{-30}$, so $e^{M_{n,A}}<1.01$.

Let $x=\sqrt{\frac{n!}{\alpha}}$, which is a divisor of $n!$. Then by the estimates above

$$\begin{aligned}\left|\frac{x}{D}-1\right| &\leq \exp\left(\left|\ln\frac{x}{D}\right|\right)-1\\ &=\exp\left(\frac{1}{2}\left|\ln\frac{\alpha}{A}\right|\right)-1\\ &=e^{M_{n,A}/2}-1\\ &<\frac{M_{n,A}}{2}e^{M_{n,A}}\\ &<5\cdot 10^{7}\left(\frac{\lg n}{n}\right)^{\frac{\lg n-\lg(\lg n)+1}{2}+\lg e}.\end{aligned}$$

This proves Theorem 2.

To prove Theorem 1 proceed as above with $A=1$. We obtain a factorization ratio α with $\ln|\alpha|=M_{n,1}$. Pick $\beta=\alpha$ or α^{-1}, whichever is bigger than 1. Then

$$M_n=\beta-1\leq e^{M_{n,1}}-1\leq e^{M_{n,1}}M_{n,1}<10^{8}\left(\frac{\lg n}{n}\right)^{\frac{\lg n-\lg(\lg n)+1}{2}+\lg e}$$

This proves Theorem 1.

3. Lower bounds.

Proof of Theorem 3. — In [1], it was proved that for any polynomial P with integer coefficients of degree 2 or more, the equation

$$P(x)=n!$$

has only a density 0 set of solutions n. This implies in particular that for any positive integer a the equation

$$x(x+a) = n! \tag{9}$$

has only a density 0 set of solutions n. Hence, after omission of such a set of n's, the minimal a for which (9) has a solution becomes arbitrarily large as n grows. This proves the theorem.

Theorem 4 follows straightforwardly from

PROPOSITION 2. — *If $n \geq 4$ is a perfect square then either $h_{n-1} - l_{n-1} > \sqrt{n-1}$ or $h_n - l_n \geq \sqrt{n}$.*

Proof. — Let $n = m^2$. Put :

$$a = h_{n-1} - l_{n-1}, \qquad b = h_n - l_n.$$

Since $n \geq 4$ there exists a prime $p \in \left(\frac{n-1}{2}, n-1\right]$. For such p we have $p|(n-1)!$ but $p^2 \nmid (n-1)!$. In particular, $(n-1)!$ is not a square, so $a \geq 1$. Now

$$4l_n(l_n + b) = 4n! = 4m^2(n-1)! = 4m^2 l_{n-1}(l_{n-1} + a).$$

This yields :

$$b^2 - m^2a^2 = (2l_n + b - m(2l_{n-1} + a))(2l_n + b + m(2l_{n-1} + a)).$$

If $2l_n + b \neq m(2l_{n-1} + a)$ then

$$\begin{aligned} |b^2 - m^2a^2| &\geq 2l_n + b + m(2l_{n-1} + a) \\ &= 2\frac{h_n + l_n}{2} + 2m\frac{h_{n-1} + l_{n-1}}{2} \\ &\geq 2\sqrt{h_n l_n} + 2m\sqrt{h_{n-1}l_{n-1}} \\ &= 4\sqrt{n!}\,. \end{aligned}$$

Hence in this case either $b \geq 2(n!)^{1/4} > \sqrt{n}$, or $ma \geq 2(n!)^{1/4} > \sqrt{n}$, so that $a > \sqrt{n-1}$. On the other hand, if $2l_n + b = m(2l_{n-1} + a)$ then $b^2 - m^2a^2 = 0$, and therefore $b \geq ma \geq m = \sqrt{n}$. This completes the proof.

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