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THE TOPOLOGY OF STEIN CR MANIFOLDS AND THE LEFSCHETZ THEOREM

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1. Preliminaries.

An abstract CR manifold is a triple (M, H, J) where M is a connected paracompact smooth real manifold, H is an even dimensional subbundle of the tangent bundle TM, and J a partial pseudocomplex structure on H; *i.e.* $J : H \to H$ is a smooth fiber preserving bundle isomorphism with $J^2 = -I$. We also require that J be formally integrable; *i.e.* that we have

$$[\tau^{0,1}M, \tau^{0,1}M] \subset \tau^{0,1}M, \tag{1.1}$$

where

$$\tau^{0,1}M = \{X + iJX | X \in \Gamma(M, H)\} \subset \Gamma(M, \mathbb{C}TM),$$

with Γ denoting smooth sections.

Let m be the real dimension of M and $2n$ be the real dimension of the fiber of H. Then n is called the CR dimension of M and $k = m - 2n$ is its CR codimension. In this case we say that M is of type (n, k).

A CR map of a CR manifold (M_1, H_1, J_1) into a CR manifold (M_2, H_2, J_2) is a differentiable map $\varphi : M_1 \to M_2$ such that

$$d\varphi(H_1) \subset H_2, \tag{1.3}$$

$$d\varphi(J_1 X) = J_2 d\varphi(X) \quad \text{for } X \in H_1. \tag{1.4}$$

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A CR map $f : M \to \mathbb{C}$ is called a CR function.

If, in addition, φ is a diffeomorphism and φ^{-1} is CR, then we say that φ is a CR isomorphism.

A CR manifold of the form (M, TM, J) is a complex manifold (type $(n, 0)$) by the Newlander-Nirenberg theorem, and one of the form $(M, 0, 0)$ is simply a real differentiable manifold (type $(0, k)$).

Let M be a real submanifold of a complex manifold $\widetilde{M}$, with complex structure $\widetilde{J}$. For $p \in M$ set

$$H_pM = T_pM \cap \widetilde{J}T_pM. \tag{1.5}$$

Then $(M, HM, \widetilde{J}|HM)$ is a CR manifold provided that the spaces H_pM have constant dimension. In this case the embedding map $\iota : M \to \widetilde{M}$ is a CR map of $(M, HM, \widetilde{J}|HM)$ into $(\widetilde{M}, T\widetilde{M}, \widetilde{J})$.

An *embedding* (resp. *immersion*) φ of a CR manifold (M, H, J) into a complex manifold $(\widetilde{M}, T\widetilde{M}, \widetilde{J})$ is a CR map $\varphi : M \to \widetilde{M}$ which is an embedding (resp. immersion).

An immersion (resp. embedding) of a CR manifold (M, H, J) of type (n, k) into a complex manifold $\widetilde{M}$ of complex dimension $n + k$ is said to be *generic*.

DEFINITION 1. — *A Stein* CR *manifold is a* CR *manifold* (M, H, J) *such that*

(1.6) (M, H, J) *has a* CR *embedding as a closed* CR *submanifold of some Stein manifold* X.

Because of the known results about the embedding of Stein manifolds [3] we could just as well replace (1.6) by the equivalent condition

(1.7) (M, H, J) *has a* CR *embedding as a closed* CR *submanifold of* $\mathbb{C}^N$, *for some* N.

For example suppose Ω is a domain of holomorphy in $\mathbb{C}^\ell$, with complex structure $\widetilde{J}$, and let $(M, HM, \widetilde{J}|HM)$ be a closed CR submanifold of Ω. Then it is a Stein CR manifold. Let (M, H, J) be a CR manifold of type (n, k). Let $H^0 \subset T^*M$ be the annihilator bundle of the bundle H. We consider the bundle $T^{(0,1)}M = \{X + iJX | X \in H\}$. Then the Levi form of (M, H, J) at $\omega \in H^0_P$ is the Hermitian form on $T^{(0,1)}_P M$:

$$L(\omega, Z) = \mathrm{i}d\tilde{\omega}(Z, \bar{Z}) = -i\omega([\mathbf{Z}, \widetilde{\mathbf{Z}}]), \tag{1.8}$$

where $\tilde{\omega} \in \Gamma(M, H^0)$ satisfies $\tilde{\omega}(P) = \omega$ and $\mathbf{Z} \in \tau^{0,1}M$ satisfies $\mathbf{Z}(P) = Z$.

The equality of the last two expressions shows that they do not depend on the choice of $\tilde{\omega}$ and $\mathbf{Z}$, and therefore L is a function defined on the direct sum of the bundles H^0 and $T^{0,1}M$.

DEFINITION 2. — *A weakly q-concave* CR *manifold* $(0 \leq q \leq n)$ *is a* CR *manifold* (M, H, J) *such that, for every* $\omega \in H^0 - \{0\}$, *the Levi form* $L(\omega, \cdot)$ *has at least* q *eigenvalues that are* ≤ 0.

Replacing the requirement ≤ 0 above by < 0 we arrive at the standard definition of a q-concave CR manifold. Hence every q-concave CR manifold is a fortiori weakly q concave. Note that weak 0-concavity involves no condition at all on the manifold. We adopt the convention that a real differentiable manifold (type $(0, k)$) is Stein (Whitney embedding theorem) and is also weakly 0-concave. A complex manifold (type $(n, 0)$) is weakly n-concave. Any Levi flat ($L \equiv 0$) CR manifold of type (n, k) is also weakly n-concave. It follows from the result of [2] that a weakly 1-concave Stein CR manifold cannot be compact.

1. The topology of weakly q-concave CR manifolds.

THEOREM 1. — *Let* (M, H, J) *be a weakly q-concave Stein* CR *manifold of type* (n, k). *Then* M *has the homotopy type of a* CW-*complex of dimension* $\leq 2n + k - q$. *In particular*

(2.1) $$H_j(M; \mathbb{Z}) = 0 \quad \text{for } j > 2n + k - q$$

and

(2.2) $$H_{2n+k-q}(M; \mathbb{Z}) \ \text{ has no torsion.}$$

Note that the above theorem interpolates between two classical results :

1. (type $(0, k)$) Any real k-dimensional differentiable manifold has the homotopy type of a CW-complex of dimension $\leq k$ [4], and

2. (type $(n, 0)$) Any complex n-dimensional Stein manifold has the homotopy type of a CW-complex of dimension $\leq n$. Thus we obtain the classical result of Andreotti-Frankel [1].

Proof. — Assume for a moment the first assertion of the theorem. Then we have (2.1) as well as

(2.3) $$H_j(M; K) = 0 \quad \text{for } j > 2n + k - q,$$

where K is an arbitrary field. The universal coefficient theorem

$$H_j(M;K) = H_j(M;\mathbb{Z}) \otimes K + \mathrm{Tor}[H_{j-1}(M;\mathbb{Z}), K] \tag{2.4}$$

then yields (2.2).

Since M is Stein we can assume that M is embedded as a closed submanifold of $\mathbb{C}^N$. Following Andreotti-Frankel we take, for $P_0 \in \mathbb{C}^N - M$, the square of the Euclidean distance

$$\varphi(P) = |P - P_0|^2, \quad P \in M. \tag{2.5}$$

By a standard argument using Sard's theorem, we choose the point P_0 so that $\varphi(P)$ is a Morse function on M; *i.e.* φ has only isolated nondegenerate critical points. By Morse theory (see [4], p. 20) M has the homotopy type of a CW-complex obtained by attaching an r-cell for each critical point having Morse index r. Hence it will suffice to show that φ has no critical point with Morse index $r > 2n + k - q$.

Let $P \in M$ be a critical point of φ. By an affine orthogonal change of coordinates we may assume $P = 0$ and that M is described in a neighborhood of 0 by equations of the form

$$\begin{cases} y_j = h_j(x_1, \dots, x_k, w_1, \dots, w_n) & 1 \le j \le k \\ \zeta_s = g_s(x_1, \dots, x_k, w_1, \dots, w_n) & 1 \le s \le \ell \end{cases} \tag{2.6}$$

where $x = (x_1, \dots, x_k)$ and $y = (y_1, \dots, y_k)$ are real coordinates, and $z = x + iy$, $w = (w_1, \dots, w_n)$, $\zeta = (\zeta_1, \dots, \zeta_\ell)$ are complex holomorphic coordinates, with $N = k + n + \ell$. Here the smooth functions h_j are real valued, the g_s are complex valued, and all vanish to second order at 0. The tangent space to M at 0 is $T_0M = \{(x, w, 0)\}$. As the line segment $\overline{PP_0}$ is orthogonal to M, the point P_0 has coordinates $P_0 = (ia_1, \dots, ia_k, 0, \dots, 0, \lambda_1, \dots, \lambda_\ell)$ with the $a_j \in \mathbb{R}$ and the $\lambda_s \in \mathbb{C}$. A point $Q \in M$ near $P = 0$ has coordinates $Q = (x + ih(x, w), w, g(x, w))$. We have

$$\begin{cases} \varphi(Q) = \sum_1^k x_j^2 + \sum_1^k (h_j - a_j)^2 \\ \qquad + \sum_1^n |w_r|^2 + \sum_1^\ell |g_s - \lambda_s|^2 \\ \quad = \varphi(0) + \sum_1^k x_j^2 + \sum_1^n |w_r|^2 - 2\sum_1^k a_j h_j \\ \qquad - 2\mathrm{Re} \sum_1^\ell \bar{\lambda}_s g_s + O(3). \end{cases}$$

The g_s are CR functions on M near 0 since they are the restrictions to M of the holomorphic coordinates ζ_s. Therefore the formal Taylor expansion of each g_s, about 0 is an expansion in terms of the $x_j + ih_j$ and the w_r. Thus

$$g_s = \sum_1^k a_s^{ij} x_i x_j + \sum_1^k \sum_1^n b_s^{jr} x_j w_r + \sum_1^n c_s^{rt} w_r w_t + O(3). \tag{2.8}$$

We write

$$\varphi(Q) = \varphi(0) + B(x, w, \bar{w}) + O(3) \tag{2.9}$$

where B is a quadratic form in $x, w, \bar{w}$. To prove our contention, it is enough to show that the quadratic form $B(0, w, \bar{w})$ has at most $2n - q$ negative eigenvalues. In view of (2.8) we can write

$$B(0, w, \bar{w}) = \operatorname{Re} A(w) + L(\omega, w) + \sum_1^n |w_r|^2 \tag{2.10}$$

where A is a holomorphic quadratic form in w, and L is the Levi form of M at $\omega = \pm 2 \sum_1^k a_j dx_j|_0$.

Let W be a maximal real subspace of $\mathbb{C}^n$ on which $B(0, w, \bar{w})$ is negative definite. On $W \cap \sqrt{-1}W$ the quadratic form $L(\omega, w) + \sum_1^n |w_r|^2$ is negative definite. Indeed if w and $\sqrt{-1}w$ belong to W we have

$$\operatorname{Re} A(w) + L(\omega, w) + \sum_1^n |w_r|^2 < 0 \tag{2.11}$$

and

$$-\operatorname{Re} A(w) + L(\omega, w) + \sum_1^n |w_r|^2 < 0, \tag{2.11}$$

as $\operatorname{Re} A(\sqrt{-1}w) = -\operatorname{Re} A(w)$. It follows, using our hypothesis of weak q concavity, that $W \cap \sqrt{-1}W$ has complex dimension $\leq n - q$ and therefore W has dimension $\leq 2n - q$. The proof of the theorem is complete.

3. The Lefschetz theorem on hyperplane sections.

DEFINITION 3. — *A projective* CR *manifold is a* CR *manifold* (M, H, J) *which has a closed* CR *embedding in* $\mathbb{CP}^N$, *for some* N.

It follows from the results of [2] that a weakly 1-concave projective CR manifold, when embedded in $\mathbb{CP}^N$, for some N, intersects every hyperplane. Let Σ be such a hyperplane, and set $M_0 = M \cup \Sigma$. We call the closed subset M_0 a hyperplane section of M.

THEOREM 2. — *Let (M, H, J) be an orientable weakly q-concave projective* CR *manifold of type (n, k), and M_0 be a hyperplane section of M. Then the natural homomorphism*

$$H^j(M; \mathbb{Z}) \to H^j(M_0; \mathbb{Z}) \tag{3.1}$$

is an isomorphism for $j < q - 1$. It is injective for $j = q - 1$.

THEOREM 2'. — *Dropping the assumption of orientability, the same results are valid with $\mathbb{Z}_2$ coefficients.*

Proof. — Since M_0 is closed in M, we have the exact cohomology sequence

$$\begin{aligned} \cdots \to H^j_K(M - M_0; \mathbb{Z}) \to H^j(M; \mathbb{Z}) \to H^j(M_0; \mathbb{Z}) \to \\ \to H^{j+1}_K(M - M_0; \mathbb{Z}) \to \cdots, \end{aligned} \tag{3.2}$$

where the subscript K denotes compact supports.

By Poincaré duality we obtain $H^j_K(M - M_0; \mathbb{Z}) \cong H_{2n+k-j}(M - M_0; \mathbb{Z}) = 0$ for $j < q$. Hence the result follows by (3.2). For the case where $M - M_0$ is not orientable, we apply Poincaré duality for $\mathbb{Z}_2$ coefficients and argue as above.

THEOREM 3. — *Under the same hypotheses as Theorem 2, the natural homomorphism*

$$H_j(M_0; \mathbb{Z}) \to H_j(M; \mathbb{Z}) \tag{3.3}$$

is an isomorphism for $j < q - 1$ and is surjective for $j = q - 1$. For the homomorphism

$$H_j(M; \mathbb{Z}) \to H_j(M, M - M_0; \mathbb{Z}), \tag{3.4}$$

we obtain an isomorphism if $j > 2n + k - q + 1$, and an injection for $j = 2n + k - q + 1$.

Proof. — We consider the exact homology sequence for the pair (M, M_0)

$$\begin{aligned} \cdots \to H_{j+1}(M, M_0; \mathbb{Z}) \to H_j(M_0; \mathbb{Z}) \to H_j(M; \mathbb{Z}) \to \\ \to H_j(M, M_0; \mathbb{Z}) \to \cdots \end{aligned} \tag{3.5}$$

But the Lefschetz duality theorem asserts that

$$H_j(M, M_0; \mathbb{Z}) \cong H^{2n+k-j}(M - M_0; \mathbb{Z}),$$

and the latter group is zero for $j < q$, again by Theorem 1.

Next we consider the exact homology sequence for the pair $(M, M - M_0)$

$$\begin{aligned} \cdots \to H_j(M - M_0; \mathbb{Z}) \to H_j(M; \mathbb{Z}) \to H_j(M, M - M_0; \mathbb{Z}) \to \\ \to H_{j-1}(M - M_0; \mathbb{Z}) \to \cdots \end{aligned} \tag{3.6}$$

By Theorem 1, $H_j(M - M_0; \mathbb{Z}) = 0$ for $j > 2n + k - q$. Hence the desired conclusion follows.

Remark. — When M is a smooth projective algebraic variety (type $(n, 0)$) we recover the classical Lefschetz theorem on hyperplane sections along the lines of the Morse theoretic proof given by Andreotti-Frankel [1].

4. Homotopy of projective CR manifolds.

Following Milnor [4] we prove

THEOREM 4. — *Let (M, H, J) be a weakly q-concave projective CR manifold of type (n, k), and M_0 be a hyperplane section of M. Then*

$$\pi_j(M, M_0) = 0 \quad \text{for} \quad j < q. \tag{4.1}$$

Proof. — (We fix a base point $x_0 \in M_0$.) We have a closed CR embedding of M in $\mathbb{CP}^N$, for some N, and $M_0 = M \cap \Sigma$ with Σ given by $z_0 = 0$ in homogeneous coordinates. Let $\mathcal{N}(\Sigma, \delta)$ and $\mathcal{N}(M, \varepsilon)$ be tubular neighborhoods of Σ and M, of radius δ and ε, respectively, with respect to the Fubini-Study metric. For $\delta_0 > 0$ sufficiently small, the geodesic flow determines a deformation retract

$$F_\delta : I \times \mathcal{N}(\Sigma, \delta) \to \mathcal{N}(\Sigma, \delta) \tag{4.2}$$

of $\mathcal{N}(\Sigma, \delta) \to \Sigma$ if $0 < \delta < \delta_0$. For $\varepsilon > 0$ sufficiently small, we have a retraction

$$g : \mathcal{N}(M, \varepsilon) \to M. \tag{4.3}$$

Since $\mathcal{N}(M, \varepsilon)$ is open, $F_\delta^{-1}(\mathcal{N}(M, \varepsilon))$ is open in $I \times \mathcal{N}(\Sigma, \delta)$, and contains $I \times M_0$. Then we can find η, $0 < \eta < \delta$, such that $F_\delta(I \times M \cap \mathcal{N}(\Sigma, \eta)) \subset \mathcal{N}(M, \varepsilon)$.

We have $\mathbb{CP}^N - \Sigma \cong \mathbb{C}^N$ and take φ as in (2.5). Set

$$\psi = \begin{cases} 0 & \text{on } \Sigma \\ \frac{1}{\varphi} & \text{on } M - \Sigma. \end{cases}$$

Then ψ is continuous on M and is a Morse function on $\psi^{-1}([r,\infty))$ for $r > 0$. The critical points of ψ all have Morse index $\geq q$. Therefore M has the homotopy type of $\psi^{-1}([0,r])$ with finitely many cells of dimension $\geq q$ attached. Then, for $j < q$, every continuous pointed map $f : (I^j, \partial I^j) \to (M, M_0)$ can be deformed to a continuous pointed map $f_1 : (I^j, \partial I^j) \to (\psi^{-1}([0,r], M_0)$. If r is sufficiently small, then $\psi^{-1}([0,r]) \subset M \cap \mathcal{N}(\Sigma, \eta)$, and $g \circ F_\delta(t, f_1(p))$ gives a homotopy of f_1 to a continuous pointed map $f_2 : (I^j, \partial I^j) \to (M, M_0)$. The proof is complete.

We consider next the exact homotopy sequence of the pair (M, M_0) :

$$\begin{aligned} &\cdots \to \pi_j(M_0) \to \pi_j(M) \to \pi_j(M, M_0) \to \pi_{j-1}(M_0) \to \cdots \\ &\cdots \to \pi_1(M, M_0) \to \pi_0(M_0) \to \pi_0(M). \end{aligned} \tag{4.5}$$

Therefore if M_0 is a hyperplane section of a weakly q-concave projective CR manifold (M, H, J), then the natural map

$$\pi_j(M_0) \to \pi_j(M) \tag{4.6}$$

is an isomorphism for $j < q-1$, and is surjective for $j = q-1$. In particular, for $q \geq 2$, every hyperplane section of M is arcwise connected (as M is connected), and for $q \geq 3$, every hyperplane section of M has the same fundamental group as M. Finally we remark that a generically chosen hyperplane section M_0 of M is a smooth submanifold.

5. A remark on the embedding dimension of projective CR manifolds.

Let (M, H, J) be a projective CR manifold of type (n, k). Consider a closed CR embedding of it into $\mathbb{CP}^N$, for some N. Then it may be possible to reduce the embedding dimension N as follows :

THEOREM 5. — *With (M, H, J) as above we have*

$(k = 1)$: It has a global closed CR *embedding in $\mathbb{CP}^{2n+2}$, and a global closed* CR *immersion in $\mathbb{CP}^{2n+1}$.*

$(k \geq 2)$: It has a global closed CR *embedding in $\mathbb{CP}^m$, where $m = [2n + (3/2)k]$ (greatest integer in).*

Remark. — For compact Stein CR manifolds of type (n,k), precisely the same results hold, with complex projective space replaced by complex Euclidean space. For non compact Stein CR manifolds of type (n,k), the same results hold, with the word "closed" removed, and the word "embedding" replaced by "one-to-one immersion".

Proof. — Let $M' = \{(p,q) \in M \times \mathbb{CP}^N | q \in \mathbb{CP}^N$ tangent to M at $p\}$. This is a smooth submanifold of $M \times \mathbb{CP}^N$ of real dimension $4n+3k$. The map

$$M' \ni (p,q) \mapsto q \in \mathbb{CP}^N \tag{5.1}$$

is smooth. By Sard's theorem its image has measure zero in $\mathbb{CP}^N$ if $N > 2n + (3/2)k$. Choosing a point $Q_0 \notin \{\text{its range}\} \cup M$, and projecting from this point into a hyperplane Σ not containing Q_0, we obtain a CR closed immersion into a $\mathbb{CP}^{N-1}$.

Next we consider $M'' = \{(p,q,r)|(p,q) \in M \times M - \Delta,\ r \in \mathbb{CP}^N$ and p,q,r are collinear$\}$. It is a smooth manifold of real dimension $4n+2k+2$. The map

$$M'' \ni (p,q,r) \mapsto r \in \mathbb{CP}^N \tag{5.2}$$

is smooth and, again by Sard's theorem, its image has measure zero if $N > 2n + k + 1$. If N satisfies both inequalities, the above CR immersion can be chosen to be globally one-to-one.

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