Ann. Inst. Fourier, Grenoble
43, 2 (1993), 365-368

ON INDUCED ACTIONS OF ALGEBRAIC GROUPS

by Andrzej BIALYNICKI-BIRULA(*)

Let H be a subgroup of an algebraic group G. Let Y be an algebraic space with an action of H (shortly an algebraic H-space). The aim of this note is to study some properties of $G \times_H Y$, defined as a quotient of $G \times Y$ by the action of H determined by $h(g,y) = (gh^{-1}, hy)$, for all $h \in H$, $g \in G$ and $y \in Y$. Left translations by elements of G on G determine an action of G on $G \times_H H$. The importance of the space $G \times_H Y$ follows from the fact that the map $Y \to G \times_H H$, which to $y \in Y$ attaches the image of $(1, y)$ in $G \times_H H$ solves the universal problem of H-equivariant morphisms of Y into G-spaces. In analogy with the theory of modules and representations we can say that the space $G \times_H Y$ is induced from the H-space Y by the group extension $H \subset G$ and that the action of G on $G \times_H Y$ is induced by the action of H on Y. In applications the notion is used for constructing a space with an action of G, when a space with an action of its subgroup H is given. Properties of $G \times_H Y$ in the case where Y is quasi-projective were studied in the classical paper [Se]. Though results presented here are perhaps predictable or even known, we hope that the paper will be useful as a reference.

In order to make our arguments more lucid, we are going to start with considering more general situations.

1. Let X and Y be two algebraic H-spaces. Then $X \times_H Y$ is defined as a quotient of the product $X \times Y$ by the action of H defined by $h(x,y) = (hx, hy)$, for $h \in H$. In general, neither the meaning of the notion

(*) supported by Polish KBN Grant GR-87.
Key words : Principal fiber bundles - Actions of algebraic groups - Algebraic spaces.
A.M.S. Classification : 14L30.

of quotient, nor its existence (when the meaning of the quotient has been already fixed) is clear. In the note we consider only the case when H is affine and X is a principal locally isotrivial H-fibration in the category of algebraic spaces. In this case we require the quotient $X \times_H Y$ to be an algebraic space, and the map $X \times Y \to X \times_H Y$ to be affine and a geometric quotient in the sense of [GIT]. If $X = G$, where G is an affine algebraic group containing H as its subgroup with an action of H by right translations, then by [Se] the above assumptions concerning H and X are satisfied.

THEOREM 1. — *Let H, X, Y be as above. Moreover assume that X is normal. Then $S \times_H Y$ exists in the category of algebraic spaces. If, moreover, X is an algebraic variety, Y is normal and can be covered by H-invariant open quasi-projective subsets, then $X \times_H Y$ is an algebraic variety.*

Proof. — The theorem will be proved in several steps.

1st step. Assume that the H-fibration on X is trivial *i.e.* $X = H \times U$, where U is an algebraic space. Then $X \times Y = H \times U \times Y \to U \times Y$ defined by $(h, x, y) \mapsto (x, h^{-1}y)$ satisfies desired conditions. Thus $X \times_H Y = U \times Y$.

2nd step. Assume that the H-fibration on X is isotrivial with the base space U. Since X is normal, U is normal and then there exists a Galois ramified cover $Z \to U$ such that $X \times_U Z$ is trivial. It follows from the 1st step that there exists $(X \times_U Z) \times_H Y$ and by Deligne's theorem [K] p. 183-4 there exists its quotient by the action of the Galois group (induced by the action on Z) in the category of algebraic spaces. The quotient can be identified with $X \times_H Y$. If moreover Y is normal quasi-projective and U is affine, then $(X \times_U Z) \times_H Y$ is normal quasi-projective. Because the quotient of a normal quasi-projective variety by an action of a finite group is quasi-projective, hence $X \times_H Y$ is also quasi-projective.

3rd step. Assume that X and Y are covered by H-invariant open subsets $\{U_i,\, i \in I\}$, $\{V_j,\, j \in J\}$, such that $U_i \times_H V_j$ exist, for all $i \in I$ and $j \in J$ (in the category of algebraic spaces). Then $X \times_H Y$ also exists (in the same category) and $\{U_i \times_H V_j\}$ form an open covering of $X \times_H Y$. Moreover if $U_i \times_H V_j$ are quasi-projective, then $X \times_H Y$ is an algebraic variety. Proof of this step is obvious.

4th step. Now we consider the general case. Notice first that the base space of the H-fibration given on X can be covered by open subsets $\{W_k\}$,

$k \in K$, such that for every $k \in K$, the inverse image U_k of W_k in X, as a principal H-fibration, is isotrivial. Then it follows from the 2nd step that, for every $k \in K$, $U_k \times_H Y$ exists and from the 3rd step that $X \times_H Y$ exists in the category of algebraic spaces. Moreover, if Y can be covered by H-invariant open quasi-projective subsets V_j, where $j \in J$, then by the second part of the 3rd step, we infer that $X \times_H Y$ is an algebraic variety.□

COROLLARY 2. — *Let Y be an algebraic space with an action of an algebraic group H and let G be an affine algebraic group containing H as its subgroup. Then $G \times_H Y$ is an algebraic space with an action of G induced by left translations on G. Moreover, if Y can be covered by H-invariant quasi-projective open subsets, then $G \times_H Y$ is an algebraic variety.*

THEOREM 3. — *Let G be a connected affine algebraic group and let H be its subgroup. Let Y be a normal algebraic space with an action of H. Then $G \times_H Y$ is an algebraic variety if and only if Y can be covered by H-invariant quasi-projective open subsets.*

Proof. — It follows from Corollary 2 that, if Y can be covered by H-invariant open quasi-projective subsets, then $G \times_H Y$ is an algebraic variety. Let us assume now that $G \times_H Y$ is an algebraic variety. Since G is connected, $G \times_H Y$ by Sumihiro Theorem [Su] can be covered by G-invariant open quasi-projective subsets. Intersecting these subsets with $H \times_H Y \subseteq G \times_H Y$ we obtain an H-invariant quasi-projective open covering of $H \times_H Y$. Since $Y \simeq H \times_H Y$ we obtain that Y can be covered by open quasi-projective H-invariant subsets. □

It follows from the above results and Sumihiro Theorem that whenever H is connected, any induced G-space from an algebraic normal H-variety is also an algebraic (normal) variety. However in case where H is a finite subgroup of a connected algebraic group and Y is an algebraic H-variety which can not be covered by H-invariant open quasi-projective open subsets, then the induced algebraic G-space is not an algebraic variety. For example, if two element group Z_2 acts on a normal algebraic variety Y in such a way that a Z_2-orbit is not contained in any affine open subset (see [H] or Chap. 4§ 3 in [GIT] for an example), then for any connected affine group G containing E_2 as a subgroup, $G \times_{Z_2} Y$ is an algebraic space but not an algebraic variety.

BIBLIOGRAPHY

[H] H. HIRONAKA, An example of a non-kahlerian deformation, Ann. of Math., 75 (1962), 190–208.

[K] D. KNUTSON, Algebraic spaces, Lecture Notes in Mathematics, 203, Springer-Verlag, 1971.

[GIT] D. MUMFORD, J. FOGARTY, Geometric Invariant Theory, 2nd edition, Ergeb. Math. 36, Springer-Verlag, 1982.

[Se] J.-P. SERRE, Espaces fibrés algébriques in Anneaux de Chow et Applications, Séminaire Chevalley, E.N.S. Paris, 1958.

[Su] H. SUMIHIRO, Equivariant completions I, J. Math. Kyoto Univ., 14 (1974), 1–28.

Manuscrit reçu le 23 avril 1992.

Andrzej BIALYNICKI-BIRULA,
Instytut Matematyki UW
Banacha 2
02097 Warszawa (Pologne).