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ON ACTIONS OF C^* ON ALGEBRAIC SPACES

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A theorem of Luna [L] says that any torus embedding which is a smooth complete algebraic space (*i.e.* a smooth Moisezon space) is an algebraic variety. This result is a consequence of the following theorem proved in the present paper.

THEOREM. — *Let C^* act on a smooth complete algebraic space X. Let X_1 be the source of the action. If X_1 is an algebraic variety, then X_1 is contained in the set of all schematic points of X.*

As a corollary of the theorem we obtain not only the theorem of Luna, but also a result saying that any smooth and complete algebraic space with an action of a reductive group G, such that there exists only one closed G-orbit in X, is a projective variety.

For basic properties of algebraic spaces see [Kn].

We begin with the following

LEMMA 1. — *Let an algebraic group G act on a complete algebraic space X. Then the action is meromorphic.*

Proof. — Let X_0 be a projective model of the field $C(X)$ of meromorphic functions on X. Then the action of G on X leads to an action of G on $C(X)$ and to the induced action of G on X_0. Moreover by Hironaka Resolution Theorem we may assume that X_0 is smooth and that we have

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a holomorphic G-equivariant map $X_0 \to X$. Let G_1 be a projective variety containing G as an open subset. Since X_0 is projective with an action of G, the action of G on X_0 is meromorphic and the graph of the action $\Gamma_0 \subseteq G \times X_0 \times X_0$ has an analytic subvariety Γ_1 in $G_1 \times X_0 \times X_0$ as its closure. Let Γ be the closure in $G_1 \times X \times X$ of the graph of the action of G on X. Now, the map $X_0 \to X$ induces a map $\Gamma_1 \to \Gamma$. Since the image of a compact analytic subvariety is an analytic subvariety, Γ is an analytic subvariety of $G_1 \times X \times X$ and thus the proof is complete.

Assume now that we have a meromorphic action of C^* on a compact manifold X. Let $X_1 \cup \ldots \cup X_r$ be the decomposition of the fixed point set of the action of C^* on X into connected components. For $i = 1, \ldots, r$, let $X_i^+ = \{x \in X \,;\, \lim_{t\to 0} tx \in X_i\}$. It follows from [B-BS] Appendix to §0, that there exists exactly one $i = 1, \ldots, r$, such that X_i^+ is open and Zariski dense in X. X_i with this property is called the source of the action. Assume that X_1 is the source. Again by [B-BS] Appendix to §0 the map $\tau : X_1^+ \to X_1$ defined by $x \to x_0 = \lim_{t\to 0} tx$ is holomorphic.

We are going to show the following

LEMMA 2. — *Let X be a smooth algebraic space X. Then the map $\tau : X_1^1 \to X_1$ defined above is a holomorphic bundle with fiber being an affine space C^p with a linear action of C^* such that all weights of the action are positive.*

Proof. — Take $x \in X_1$. Then there exists an open neighborhood U of 0 in the tangent space $T_{x,x}$ invariant under the induced action of S^1 (where $S^1 = \{z \in C^* \,;\, |z| = 1\}$) and a S^1-invariant biholomorphic map ϕ of U onto an open neighborhood of $x \in X$ (see e.g. [Ka], Satz 4.4). We may assume that $T_{x,X} = C^n$, ant that the induced action of $C^*$is diagonal

$$t(z_1, \ldots, z_n) = (\kappa_1(t)z_1, \ldots, \kappa_n(t)z_n)\,,$$

where $\kappa_1, \ldots, \kappa_n$ are characters of C^* and hence can be identified with integers. Moreover we may assume that

$$U = \{z = (z_1, \ldots, z_n) \in C^n \,;\, |z_i| \leq \varepsilon, \text{ for } i = 1, \ldots, n\},$$

where ε is a sufficiently small positive real number. Consider an open connected subset V of $C^* \times U$ composed of all such points (t, u) that $tu \in U$. On V we define two holomorphic mappings :

$$(t,u) \mapsto \phi(tu)$$
$$(t,u) \mapsto t\phi(u).$$

For $t = s \in S^1$, we have $\phi(su) = s\phi(u)$. Hence the above mappings are equal on $S^1 \times U$. Since V is connected and the mappings are holomorphic, they coincide. Thus $\phi(tu) = t\phi(u)$, whenever u, $tu \in U$. Now define $\psi : C^* \times U \to X$ by $\psi(t,u) = t\phi(u)$. We claim that $t_1u_1 = tu$ implies $\psi(t_1,u_1) = \psi(t,u)$, i.e. that ψ induces a holomorphic map on C^*U. In order to prove this claim notice that if $t_1u_1 = tu$, then $t^{-1}t_1u_1 = u$ and by the above, $\phi(u) = \phi(t^{-1}t_1u_1) = (t^{-1}t_1)\phi(u_1)$. Hence $\psi(t,u) = t\phi(u) = t(t^{-1}t_1)\phi(u_1) = t_1\phi(u_1) = \psi(t_1,u_1)$.

So we have obtained a holomorphic C^*-invariant map $\psi : C^*U \to X$. Since x belongs to the source of X, the weights κ_i, $i = 1,\ldots,n$, are nonnegative and we may assume that $\kappa_1 \geq \ldots \geq \kappa_p > \kappa_{p+1} = \cdots = \kappa_n = 0$. Since on U the map is an open immersion into X, it is an open immersion of C^*U into X. In fact, assume that $\psi(tu) = \psi(t_1u_1)$. Then, since the weights κ_i are nonnegative, there exists $t_0 \in T$ such that t_0tu, $t_0t_1u_1 \in U$ and $\psi(t_0tu) = t_0\psi(tu) = t_0\psi(t_1u_1) = \psi(t_0t_1u_1)$. Hence $t_0tu = t_0t_1u_1$ and $tu = t_1u_1$.

Let $\pi : C^n \to C^{n-p} \subset C^n$ be the projection map $\pi(z_1,\ldots,z_n) = (0,\ldots,0,z_{p+1},\ldots,z_n)$. Then for $z = (z_1,\ldots,z_n) \in C^*U$, $\psi\pi(z) = \tau\psi(z)$. Thus $\tau|\psi(C^*U)$ is a trivial bundle with fiber C^p. This finishes the proof of the lemma.

The gluing functions of the bundle $X_1^+ \to X_1$ have values in the automorphism group $\mathrm{Aut}_{C^*}(C^p)$ of holomorphic automorphisms of C^p commuting with the action of C^*.

LEMMA 3. — *Let $\tau : X_1^+ \to X_1$ be as in Lemma 2. Then the bundle is algebraic.*

Proof. — By theorem 3 in [Se2] (compare also [Se1]), it is enough to show that $\mathrm{Aut}_{C^*}(C^p)$ is a linear algebraic group. Any $\alpha \in \mathrm{Aut}_{C^*}(C^p)$ is of the form $\alpha(z) = (\alpha_1(z),\alpha_2(z),\ldots,\alpha_p(z))$, where $\alpha_1,\ldots,\alpha_p$ are holomorphic functions in p variables. Moreover since α commutes with action of C^*, α_i, for $i = 1,\ldots,p$, is homogeneous of weight κ_i when we attach weight κ_j to variable x_j, for $j = 1,\ldots,p$.

Since the weights κ_j are strictly positive, α_i for $i = 1,\ldots,p$, is a polynomial and there exists an integer N such that degrees of all polynomials α_i, for all $\alpha \in \mathrm{Aut}_{C^*}(C^p)$, are bounded by N. On the other hand $\alpha \in \mathrm{Aut}_{C^*}(C^p)$ if and only if coefficients of the corresponding polynomials α_i satisfy some fixed polynomial identities. This shows that

$\mathrm{Aut}_{C^*}(C^p)$ is an affine and hence a linear group. The proof is complete.

It follows from Lemma 3 that $X_1^+ - X_1/C^* \to X_1$ is an algebraic bundle with fiber $C^p - \{0\}/C^*$ – *a* weighted projective space.

LEMMA 4. — *Any C^*-invariant meromorphic function on X_1^+ is meromorphic on X.*

Proof. — The field of C^*-invariant meromorphic functions on X_1^+ can be identified with the field $C(X_1^+ - X_1/C^*)$ of meromorphic (hence rational) functions on a complete algebraic variety $X_1^+ - X_1/C^*$. On the other hand the field of C^*-invariant meromorphic functions on X can be identified with a subfield L of $C(X_1^+ - X_1/C^*)$. Since both have transcendence degree over C^* equal to $n-1$, the extension $L \subseteq C(X_1^+ - X_1/C^*)$ is algebraic.

Let $U \subseteq X$ be an open C^*-invariant subset of X composed of all schematic points. Let $U_1 \subseteq U$ be an open C^*-invariant algebraic subvariety such that there exists space of orbits U_1/C^*. Then $U_1 \cap (X_1^+ - X_1)$ is open dense in X and $U_1 \cap (X_1^+ - X_1)/C^*$ is open dense in $(X_1^+ - X_1)/C^*$. Rational C^*-invariant functions on U_1 are meromorphic on X and separate points of U_1/C^*. Hence functions from L separate points belonging to an open dense subset $U_1 \cap (X_1^+ - X_1)/C^* \subseteq (X_1^+ - X_1)/C^*$. This shows that the degree of $C(X_1^+ - X_1)/C^*)$ over L is equal to 1 and hence $L = C\big((X_1^+ - X_1)/C^*\big)$. The proof of the lemma is finished.

We say that a complex valued function g defined on a space Y with an action of C^* is C^*-semi-invariant if, for any $y \in Y$ and $t \in C^*$, $g(ty) = \kappa(t)g(y)$, where $\kappa : C^* \to C^*$ is a character of C^* Then κ is called the weight of the semi-invariant function g.

LEMMA 5. — *Let f be a C^*-semi-invariant meromorphic function on X_1^+. Then f is a meromorphic function on X.*

Proof. — Let U be as in the proof of Lemma 4. Then the field $C(U)$ of rational functions on U coincides with the field of meromorphic functions on X. One can find a function $g \in C(U)$ of the same weight as f. Then f/g is C^*-invariant and meromorphic on X_1^+. Hence by Lemma 4 f/g is meromorphic on X. Since g is meromorphic on X, f is meromorphic on X.

Proof of the theorem. — Let $x \in X_1$. In order to prove that x is a schematic point in X it is sufficient to show that in the local ring of holomorphic functions at x there exists a system of parameters composed of functions meromorphic on X (compare [L]). It follows from Lemma 5 that it suffices to find such a system of parameters composed of C^*-semi-invariant function meromorphic on X_1^+. Since X_1^+ is an algebraic variety, there exists a system of parameters at x composed of C^*-semi-invariant functions which are regular at x and hence rational on X_1^+. The functions are then meromorphic on X and thus the proof is finished.

COROLLARY 6. — *Let a compact and smooth algebraic space X be a torus embedding of a torus T. Then X is an algebraic variety.*

Proof follows from the theorem and the fact that (since X is a torus embedding) any fixed point of the action of T on X is a source of the induced action of a one parameter subgroup $C^* \to T$ (this can be seen similarly as in the proof of Lemma 1 by considering a T-invariant birational morphism of a smooth projective variety $X_0 \to X$). Notice also that any T-orbit contains a fixed point in its closure.

COROLLARY 7. — *Let X be a smooth and compact algebraic space with an action of a reductive group G. Assume that there exists only one closed G-orbit in X. Then X is a projective variety.*

Proof. — By Sumihiro Theorem [Su] any point of a normal algebraic variety X with an action of a connected algebraic group is contained in an invariant open quasi-projective subset. Hence if this variety is complete and contains only one closed orbit it has to be projective (the only open invariant subset containing a point from the closed orbit is the whole space). Thus it suffices to show that the space X is an algebraic variety. Since any G-orbit contains a closed orbit in its closure and the set of schematic points is open G-invariant, it suffices to show that the only closed G-orbit in X contains a schematic point. Therefore it follows from the theorem that it suffices to prove that the source of a one parameter subgroup in G is contained in the closed G-orbit.

Let T be a maximal torus in G. Let $C^* = T_0 \subseteq T$ be a subtorus of T such that the sets of fixed points of T and of T_0 coincide. Let P be the parabolic subgroup corresponding to $C^* = T_0$. Let x belongs to the source of the action of T_0 on X. Then x belongs to the source of the action of T_0 on the closure of Gx in X. Hence the opposite P^- of the parabolic P has to be contained in the stabilizer subgroup of x. Thus the stabilizer is parabolic and the orbit Gx is projective. Hence Gx is the only closed orbit in X. It means that source of T_0 in X is contained in the only closed orbit and the proof is complete.

COROLLARY 8. — *Let X be a smooth algebraic space with an action of C^*. Let X_1 be the source of the action. If $x \in X_1$ is schematic in X_1, then it is schematic in X.*

Proof. — Assume that $x \in X_1$ is schematic in X_1. If $X = X_1$, then the corollary is trivial. Assume that $X \neq X_1$. Then (by [M]) there exists $\rho_1 : Y_1 \to X_1$, where Y_1 is a smooth algebraic variety and ρ_1 is a composition of blow ups of ideals on X_1 and its transforms supported by smooth centers not containing x. Let $\rho : Y \to X$ be the composition of the blow-ups of the corresponding ideals on X and its transforms. Then Y is smooth with the induced action of C^* and Y_1 is the source of the action. Since Y_1 is an algebraic variety, any point of Y_1 (by the theorem) is schematic in Y. In particular x is schematic in Y. Since ρ restricted to a Zariski open neighborhood of x is an isomorphism, x is schematic in X.

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