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On actions of $\mathbb{C}^*$ on algebraic spaces


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ON ACTIONS OF $C^*$ ON ALGEBRAIC SPACES

by Andrzej BIALYNICKI-BIRULA

A theorem of Luna [L] says that any torus embedding which is a smooth complete algebraic space (i.e. a smooth Moisezon space) is an algebraic variety. This result is a consequence of the following theorem proved in the present paper.

**THEOREM.** — Let $C^*$ act on a smooth complete algebraic space $X$. Let $X_1$ be the source of the action. If $X_1$ is an algebraic variety, then $X_1$ is contained in the set of all schematic points of $X$.

As a corollary of the theorem we obtain not only the theorem of Luna, but also a result saying that any smooth and complete algebraic space with an action of a reductive group $G$, such that there exists only one closed $G$-orbit in $X$, is a projective variety.

For basic properties of algebraic spaces see [Kn].

We begin with the following

**LEMMA 1.** — Let an algebraic group $G$ act on a complete algebraic space $X$. Then the action is meromorphic.

**Proof.** — Let $X_0$ be a projective model of the field $C(X)$ of meromorphic functions on $X$. Then the action of $G$ on $X$ leads to an action of $G$ on $C(X)$ and to the induced action of $G$ on $X_0$. Moreover by Hironaka Resolution Theorem we may assume that $X_0$ is smooth and that we have

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a holomorphic $G$-equivariant map $X_0 \to X$. Let $G_1$ be a projective variety containing $G$ as an open subset. Since $X_0$ is projective with an action of $G$, the action of $G$ on $X_0$ is meromorphic and the graph of the action $\Gamma_0 \subseteq G \times X_0 \times X_0$ has an analytic subvariety $\Gamma_1$ in $G_1 \times X_0 \times X_0$ as its closure. Let $\Gamma$ be the closure in $G_1 \times X \times X$ of the graph of the action of $G$ on $X$. Now, the map $X_0 \to X$ induces a map $\Gamma_1 \to \Gamma$. Since the image of a compact analytic subvariety is an analytic subvariety, $\Gamma$ is an analytic subvariety of $G_1 \times X \times X$ and thus the proof is complete.

Assume now that we have a meromorphic action of $C^*$ on a compact manifold $X$. Let $X_1 \cup \ldots \cup X_r$ be the decomposition of the fixed point set of the action of $C^*$ on $X$ into connected components. For $i = 1, \ldots, r$, let $X_i^+ = \{x \in X ; \lim_{t \to 0} tx \in X_i\}$. It follows from [B-BS] Appendix to §0, that there exists exactly one $i = 1, \ldots, r$, such that $X_i^+$ is open and Zariski dense in $X$. $X_i$ with this property is called the source of the action. Assume that $X_1$ is the source. Again by [B-BS] Appendix to §0 the map $\tau : X_i^+ \to X_1$ defined by $x \to x_0 = \lim_{t \to 0} tx$ is holomorphic.

We are going to show the following

**Lemma 2.** Let $X$ be a smooth algebraic space $X$. Then the map $\tau : X_1^+ \to X_1$ defined above is a holomorphic bundle with fiber being an affine space $C^n$ with a linear action of $C^*$ such that all weights of the action are positive.

*Proof.* Take $x \in X_1$. Then there exists an open neighborhood $U$ of 0 in the tangent space $T_{x,X}$ invariant under the induced action of $S^1$ (where $S^1 = \{z \in C^* ; |z| = 1\}$) and a $S^1$-invariant biholomorphic map $\phi$ of $U$ onto an open neighborhood of $x \in X$ (see e.g. [Ka], Satz 4.4). We may assume that $T_{x,X} = C^n$, ant that the induced action of $C^*$ is diagonal

$$t(z_1, \ldots, z_n) = (\kappa_1(t)z_1, \ldots, \kappa_n(t)z_n),$$

where $\kappa_1, \ldots, \kappa_n$ are characters of $C^*$ and hence can be identified with integers. Moreover we may assume that

$$U = \{z = (z_1, \ldots, z_n) \in C^n ; |z_i| \leq \varepsilon, \text{ for } i = 1, \ldots, n\},$$

where $\varepsilon$ is a sufficiently small positive real number. Consider an open connected subset $V$ of $C^* \times U$ composed of all such points $(t, u)$ that $tu \in U$. On $V$ we define two holomorphic mappings :

$$(t, u) \mapsto \phi(tu)$$

$$(t, u) \mapsto t\phi(u).$$
For $t = s \in S^1$, we have $\phi(su) = s\phi(u)$. Hence the above mappings are equal on $S^1 \times U$. Since $V$ is connected and the mappings are holomorphic, they coincide. Thus $\phi(tu) = t\phi(u)$, whenever $u, tu \in U$. Now define $\psi : C^* \times U \to X$ by $\psi(t, u) = t\phi(u)$. We claim that $t_1u_1 = tu$ implies $\psi(t_1, u_1) = \psi(t, u)$, i.e. that $\psi$ induces a holomorphic map on $C^*U$. In order to prove this claim notice that if $t_1u_1 = tu$, then $t^{-1}t_1u_1 = u$ and by the above, $\phi(u) = \phi(t^{-1}t_1u_1) = (t^{-1}t_1)\phi(u_1)$. Hence $\psi(t, u) = t\phi(u) = t(t^{-1}t_1)\phi(u_1) = t_1\phi(u_1) = \psi(t_1, u_1)$.

So we have obtained a holomorphic $C^*$-invariant map $\psi : C^*U \to X$. Since $x$ belongs to the source of $X$, the weights $\kappa_i$, $i = 1, \ldots, n$, are nonnegative and we may assume that $\kappa_1 \geq \ldots \geq \kappa_p > \kappa_{p+1} = \cdots = \kappa_n = 0$. Since on $U$ the map is an open immersion into $X$, it is an open immersion of $C^*U$ into $X$. In fact, assume that $\psi(tu) = \psi(t_1u_1)$. Then, since the weights $\kappa_i$ are nonnegative, there exists $t_0 \in T$ such that $t_0tu, t_0t_1u_1 \in U$ and $\psi(t_0tu) = t_0\psi(tu) = t_0\psi(t_1u_1) = \psi(t_0t_1u_1)$. Hence $t_0tu = t_0t_1u_1$ and $tu = t_1u_1$.

Let $\pi : C^n \to C^{n-p} \subset C^n$ be the projection map $\pi(z_1, \ldots, z_n) = (0, \ldots, 0, z_{p+1}, \ldots, z_n)$. Then for $z = (z_1, \ldots, z_n) \in C^*U$, $\psi\pi(z) = \tau\psi(z)$. Thus $\tau|\psi(C^*U)$ is a trivial bundle with fiber $C^p$. This finishes the proof of the lemma.

The gluing functions of the bundle $X_1^+ \to X_1$ have values in the automorphism group $\text{Aut}_{C^*}(C^p)$ of holomorphic automorphisms of $C^p$ commuting with the action of $C^*$.

**Lemma 3.** — Let $\tau : X_1^+ \to X_1$ be as in Lemma 2. Then the bundle is algebraic.

**Proof.** — By theorem 3 in [Se2] (compare also [Se1]), it is enough to show that $\text{Aut}_{C^*}(C^p)$ is a linear algebraic group. Any $\alpha \in \text{Aut}_{C^*}(C^p)$ is of the form $\alpha(z) = (\alpha_1(z), \alpha_2(z), \ldots, \alpha_p(z))$, where $\alpha_1, \ldots, \alpha_p$ are holomorphic functions in $p$ variables. Moreover since $\alpha$ commutes with action of $C^*$, $\alpha_i$, for $i = 1, \ldots, p$, is homogeneous of weight $\kappa_i$ when we attach weight $\kappa_j$ to variable $x_j$, for $j = 1, \ldots, p$.

Since the weights $\kappa_j$ are strictly positive, $\alpha_i$ for $i = 1, \ldots, p$, is a polynomial and there exists an integer $N$ such that degrees of all polynomials $\alpha_i$, for all $\alpha \in \text{Aut}_{C^*}(C^p)$, are bounded by $N$. On the other hand $\alpha \in \text{Aut}_{C^*}(C^p)$ if and only if coefficients of the corresponding polynomials $\alpha_i$ satisfy some fixed polynomial identities. This shows that
Aut_{C^*}(C^p) is an affine and hence a linear group. The proof is complete.

It follows from Lemma 3 that $X_1^+ - X_1/C^* \to X_1$ is an algebraic bundle with fiber $C^p - \{0\}/C^* - a$ weighted projective space.

**Lemma 4.** — Any $C^*$-invariant meromorphic function on $X_1^+$ is meromorphic on $X$.

*Proof.* — The field of $C^*$-invariant meromorphic functions on $X_1^+$ can be identified with the field $C(X_1^+ - X_1/C^*)$ of meromorphic (hence rational) functions on a complete algebraic variety $X_1^+ - X_1/C^*$. On the other hand the field of $C^*$-invariant meromorphic functions on $X$ can be identified with a subfield $L$ of $C(X_1^+ - X_1/C^*)$. Since both have transcendence degree over $C^*$ equal to $n - 1$, the extension $L \subseteq C(X_1^+ - X_1/C^*)$ is algebraic.

Let $U \subseteq X$ be an open $C^*$-invariant subset of $X$ composed of all schematic points. Let $U_1 \subseteq U$ be an open $C^*$-invariant algebraic subvariety such that there exists space of orbits $U_1/C^*$. Then $U_1 \cap (X_1^+ - X_1)$ is open dense in $X$ and $U_1 \cap (X_1^+ - X_1)/C^*$ is open dense in $(X_1^+ - X_1)/C^*$. Rational $C^*$-invariant functions on $U_1$ are meromorphic on $X$ and separate points of $U_1/C^*$. Hence functions from $L$ separate points belonging to an open dense subset $U_1 \cap (X_1^+ - X_1)/C^* \subseteq (X_1^+ - X_1)/C^*$. This shows that the degree of $C((X_1^+ - X_1)/C^*)$ over $L$ is equal to 1 and hence $L = C((X_1^+ - X_1)/C^*)$. The proof of the lemma is finished.

We say that a complex valued function $g$ defined on a space $Y$ with an action of $C^*$ is $C^*$-semi-invariant if, for any $y \in Y$ and $t \in C^*$, $g(ty) = \kappa(t)g(y)$, where $\kappa : C^* \to C^*$ is a character of $C^*$. Then $\kappa$ is called the weight of the semi-invariant function $g$.

**Lemma 5.** — Let $f$ be a $C^*$-semi-invariant meromorphic function on $X_1^+$. Then $f$ is a meromorphic function on $X$.

*Proof.* — Let $U$ be as in the proof of Lemma 4. Then the field $C(U)$ of rational functions on $U$ coincides with the field of meromorphic functions on $X$. One can find a function $g \in C(U)$ of the same weight as $f$. Then $f/g$ is $C^*$-invariant and meromorphic on $X_1^+$. Hence by Lemma 4 $f/g$ is meromorphic on $X$. Since $g$ is meromorphic on $X$, $f$ is meromorphic on $X$.

*Proof of the theorem.* — Let $x \in X_1$. In order to prove that $x$ is a schematic point in $X$ it is sufficient to show that in the local ring of
holomorphically functions at \( x \) there exists a system of parameters composed of functions meromorphic on \( X \) (compare [L]). It follows from Lemma 5 that it suffices to find such a system of parameters composed of \( C^* \)-semi-invariant function meromorphic on \( X_{1}^{+} \). Since \( X_{1}^{+} \) is an algebraic variety, there exists a system of parameters at \( x \) composed of \( C^* \)-semi-invariant functions which are regular at \( x \) and hence rational on \( X_{1}^{+} \). The functions are then meromorphic on \( X \) and thus the proof is finished.

**Corollary 6.** — Let a compact and smooth algebraic space \( X \) be a torus embedding of a torus \( T \). Then \( X \) is an algebraic variety.

Proof follows from the theorem and the fact that (since \( X \) is a torus embedding) any fixed point of the action of \( T \) on \( X \) is a source of the induced action of a one parameter subgroup \( C^* \rightarrow T \) (this can be seen similarly as in the proof of Lemma 1 by considering a \( T \)-invariant birational morphism of a smooth projective variety \( X_{0} \rightarrow X \)). Notice also that any \( T \)-orbit contains a fixed point in its closure.

**Corollary 7.** — Let \( X \) be a smooth and compact algebraic space with an action of a reductive group \( G \). Assume that there exists only one closed \( G \)-orbit in \( X \). Then \( X \) is a projective variety.

Proof. — By Sumihiro Theorem [Su] any point of a normal algebraic variety \( X \) with an action of a connected algebraic group is contained in an invariant open quasi-projective subset. Hence if this variety is complete and contains only one closed orbit it has to be projective (the only open invariant subset containing a point from the closed orbit is the whole space). Thus it suffices to show that the space \( X \) is an algebraic variety. Since any \( G \)-orbit contains a closed orbit in its closure and the set of schematic points is open \( G \)-invariant, it suffices to show that the only closed \( G \)-orbit in \( X \) contains a schematic point. Therefore it follows from the theorem that it suffices to prove that the source of a one parameter subgroup in \( G \) is contained in the closed \( G \)-orbit.

Let \( T \) be a maximal torus in \( G \). Let \( C^* = T_{0} \subseteq T \) be a subtorus of \( T \) such that the sets of fixed points of \( T \) and of \( T_{0} \) coincide. Let \( P \) be the parabolic subgroup corresponding to \( C^* = T_{0} \). Let \( x \) belongs to the source of the action of \( T_{0} \) on \( X \). Then \( x \) belongs to the source of the action of \( T_{0} \) on the closure of \( Gx \) in \( X \). Hence the opposite \( P^- \) of the parabolic \( P \) has to be contained in the stabilizer subgroup of \( x \). Thus the stabilizer is parabolic and the orbit \( Gx \) is projective. Hence \( Gx \) is the only closed orbit
in $X$. It means that source of $T_0$ in $X$ is contained in the only closed orbit and the proof is complete.

**COROLLARY 8.** — Let $X$ be a smooth algebraic space with an action of $C^*$. Let $X_1$ be the source of the action. If $x \in X_1$ is schematic in $X_1$, then it is schematic in $X$.

**Proof.** — Assume that $x \in X_1$ is schematic in $X_1$. If $X = X_1$, then the corollary is trivial. Assume that $X \neq X_1$. Then (by [M]) there exists $\rho_1 : Y_1 \to X_1$, where $Y_1$ is a smooth algebraic variety and $\rho_1$ is a composition of blow ups of ideals on $X_1$ and its transforms supported by smooth centers not containing $x$. Let $\rho : Y \to X$ be the composition of the blow-ups of the corresponding ideals on $X$ and its transforms. Then $Y$ is smooth with the induced action of $C^*$ and $Y_1$ is the source of the action. Since $Y_1$ is an algebraic variety, any point of $Y_1$ (by the theorem) is schematic in $Y$. In particular $x$ is schematic in $Y$. Since $\rho$ restricted to a Zariski open neighborhood of $x$ is an isomorphism, $x$ is schematic in $X$.

**BIBLIOGRAPHY**


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