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ON CURVES WITH NATURAL COHOMOLOGY AND THEIR DEFICIENCY MODULES

by G. BOLONDI and J.C. MIGLIORE

0. Introduction.

In this paper we study the Hartshorne-Rao module of curves in projective space with natural cohomology. The interest of these curves relies on the fact that, due to semicontinuity, they seems to be «good» points of the Hilbert schemes of curves, and that the components of the Hilbert schemes containing them are candidates to be «good» components. In particular, this notion includes the well-studied notion of maximal rank, taking also into account the speciality of the curve.

We characterize those graded S-modules of finite length which are the Hartshorne-Rao modules of curves with natural cohomology, giving necessary and sufficient numerical conditions on the minimal free resolution, using heavily the fact that if the diameter of the module is bigger than two, then the curve is minimal in its liaison class. A fundamental tool for this result is the work of Martin-Deschamps and Perrin [MP] about minimal curves. In particular, we get that the module is generated in the first two degrees; moreover, we show with an example that the multiplication $S_1 \otimes M(C)_t \to M(C)_{t+1}$ between the first two components of the module need not to be of maximal rank. The case of diameter one is already known (see for instance [BM1]), and the case of diameter two is treated separately. In that case (and also partially for diameter 3) we get also results of smoothness and uniqueness for the components of the Hilbert schemes containing curves with natural cohomology. We point out that the knowledge of the resolution of $M(C)$ actually gives (via the above mentioned results of Martin-Deschamps and Perrin) the resolution of I_C.

Key words : Natural cohomology – Deficiency modules – Liaison.
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When this paper was ready we received a preprint by Floystad [F] where many results about the resolutions of curves with maximal rank and maximal corank are obtained, some of which overlap with the results of the present paper. Other forthcoming results related to this problem, mainly in the case of a module of diameter 3, are contained in a preprint by Martin-Deschamps and Perrin ([MP1], and very recently Ch. Walter announced the construction of smooth curves with natural cohomology in the so-called FW-range.

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1. The Hartshorne-Rao module of a curve in $\mathbb{P}^3$ with natural cohomology.

Throughout this work we let k be an algebraically closed field and $S = k[X_0, X_1, X_2, X_3]$. For a curve C in $\mathbb{P}^3$, we denote by I_C its homogeneous ideal in S, and by $\mathcal{J}_C$ its ideal sheaf. For any sheaf $\mathcal{F}$ on $\mathbb{P}^3$ we sometimes denote by $H^i_*(\mathcal{F})$ the direct sum $\bigoplus_{k\in\mathbb{Z}} H^i(\mathcal{F}(k))$. Two curves C and C' are *directly linked* by a complete intersection X if $I_X : I_C = I_{C'}$ and $I_X : I_{C'} = I_C$. If C and C' are linked in an even number of direct steps, we say that they are *evenly linked*, and the set of all curves evenly linked to a given C is the *even liaison class* of C. In particular the property of being locally Cohen-Macaulay and equidimensional is invariant under liaison, and we make this assumption about all our curves.

An important object for even liaison classes of curves in $\mathbb{P}^3$ is the *Hartshorne-Rao module* $M(C)$ of C, defined by $M(C) = \bigoplus_{k\in\mathbb{Z}} H^1(\mathcal{J}_C(k))$. This is a graded S-module of finite length (by the assumption of locally Cohen-Macaulay and equidimensional) and up to shift is an invariant of the even liaison class of C. The structure of an even liaison class in codimension two has been studied (see [LR], [BBM]) and in particular we know that there is a leftmost shift of the liaison invariant which is actually the Hartshorne-Rao module of a curve. The curves whose modules have this shift can all be deformed one to another, and hence in particular have the same degree and genus, which are strictly less than the degree and genus of any other curve in the even liaison class. We say that such curves are *minimal* in their even liaison class and that the set of minimal curves comprise the *minimal shift* of the even liaison class.

The curves we will be studying in this paper are the curves with *natural cohomology*, that is, curves for which at most one of $h^0(\mathcal{J}_C(k)), h^1(\mathcal{J}_C(k))$ or $h^2(\mathcal{J}_C(k))$ is non-zero for any given k. This contains two notions :

- *maximal rank* (i.e. having at most one of $h^0(\mathcal{J}_C(k))$ or $h^1(\mathcal{J}_C(k))$ non-zero for any k) and, following [MP],
- *maximal corank* (i.e. having at most one of $h^1(\mathcal{J}_C(k))$ or $h^2(\mathcal{J}_C(k))$ non-zero for any k), and in fact it is equivalent to the union of these conditions if C is not arithmetically Cohen-Macaulay.

Notation :

$\alpha(C) = \min\{k \in \mathbb{Z} \mid h^0(\mathcal{J}_C(k)) \neq 0\}$;

$t(C) = \min\{k \in \mathbb{Z} \mid h^1(\mathcal{J}_C(k)) \neq 0\}$;

$r(C) = \alpha(C) - t(C) - 1$;

$m_i = h^1(\mathcal{J}_C(t+i))(t = t(C),\ 0 \leq i \leq r = r(C))$ (so $m_0 > 0$, $m_r \geq 0$ and $m_i = 0$ if $i > r$ for curves of maximal rank);

$e(C) = \max\{k \in \mathbb{Z} \mid h^2(\mathcal{J}_C(k)) \neq 0\}$;

$\operatorname{diam} M(C)$ = number of components of $M(C)$ from degree t to the last non-zero component (for a curve of maximal rank this is either equal to $r(C) + 1$ or $r(C)$).

Remark 1.1. — One can ask about how natural cohomology acts with respect to an even liaison class. If C is a curve with natural cohomology and diam $M(C) \geq 3$ (or more generally $r(C) \geq 2$) then C lies on no surface of degree $e(C) + 3$. Hence it follows from [LR] that C is minimal in its even liaison class. That is, an even liaison class corresponding to a graded S-module of diameter ≥ 3 can have at most one curve with natural cohomology, up to deformation, and this curve occurs in the minimal shift. If diam $M(C) \geq 4$ then [LR] guarantees that in fact C is unique (if it exists at all). Indeed, we shall see that the degree and genus of C are determined by the dimensions of the first three components of $M(C)$, as are the dimensions of the other components. For the question of whether C exists at all for a given M, we give necessary conditions on these dimensions.

On the other hand, in the case of diameter two we will show that for any dimensions of the components and any module structure, there are smooth curves with natural cohomology. And for all shifts there are curves with natural cohomology. The case of diameter 1 is completely treated in [BM1], and for diameter 0 we have arithmetically Cohen-Macaulay curves.

LEMMA 1.2. — *Let C be a curve with $h^2(\mathcal{J}_C(k)) = h^1(\mathcal{J}_C(k+1)) = 0$ for some integer k. Then I_C is generated in degree $\leq k+2$.*

Proof. — This follows from Castelnuovo-Mumford, since :

$$h^3(\mathcal{J}_C(k-1)) = h^2(\mathcal{J}_C(k)) = h^1(\mathcal{J}_C(k+1)) = 0.$$

The fact that $h^3(\mathcal{J}_C(k-1)) = 0$ follows from the usual exact sequence

$$0 \to \mathcal{J}_C(k-1) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(k-1) \longrightarrow \mathcal{O}_C(k-1) \to 0$$

and the fact that $k-1 > -4$. (If C is arithmetically Cohen-Macaulay this last statement is standard. If C is not arithmetically Cohen-Macaulay then the Hartshorne-Rao module is non-zero and hence must end in non-negative degree : see for instance [M].) □

The following lemma is already known [GMa]; we give here a new proof.

LEMMA 1.3. — *Let C be a curve with $h^2(\mathcal{J}_C(k)) = 0$ for some integer k. Then $M(C)$ is generated in degree $\leq k+1$.*

Proof. — Let L_1 and L_2 be general linear forms (so in particular neither contains a component of C, and their intersection is a line λ disjoint from C). Put

$$U_{k+2} = H^0(\mathcal{O}_\lambda(k+2)),$$
$$V_{1,i} = H^1(\mathcal{J}_{C\cap L_1}(i)), \quad V_{2,i} = H^1(\mathcal{J}_{C\cap L_2}(i)),$$

and consider the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & M(C)_k \arrow[r, "L_1"] \arrow[d, "L_2"] & M(C)_{k+1} \arrow[r, "f"] \arrow[d, "L_2"] & V_{1,k+1} \arrow[r] \arrow[d, "g"] & 0 \\
\cdots \arrow[r] & M(C)_{k+1} \arrow[r, "L_1"] \arrow[d] & M(C)_{k+2} \arrow[r, "h"] \arrow[d] & V_{1,k+2} \arrow[r] \arrow[d] & 0 \\
U_{k+2} \arrow[r] & V_{2,k+1} \arrow[r] & V_{2,k+1} \arrow[r] & 0 &
\end{tikzcd}$$

where by $C \cap L_i$ we mean the intersection of C with the plane defined by L_i. Let $x \in M(C)_{k+2}$. If x is in the image of the multiplication by L_1 or L_2, there is nothing to prove, so assume that this is not the case. In particular, by exactness $h(x) \neq 0$ and so there exists a non-zero $y \in H^1(\mathcal{J}_{C\cap L_1}(k+1))$ such that $g(y) = h(x)$. But f is also surjective, so there exists $z \in M(C)_{k+1}$ such that $f(z) = y$, and hence $h(L_2 z) = g(f(z)) = g(y) = h(x) \neq 0$. Hence in particular $L_2 z \neq 0$ and by hypothesis $L_2 z \neq x$. But then $0 \neq x - L_2 z \in \ker h = \operatorname{im} L_1$ so there exists $w \in M(C)_{k+1}$ such that $L_1 w = x - L_2 z$, and x is a linear combination of elements of $M(C)_{k+1}$. $\square$

Remark 1.4. — Notice that two general linear forms suffice to generate all of $M(C)$ in degree $\geq k+2$.

COROLLARY 1.5. — *Let C be a curve with natural cohomology. Let $t = t(C)$, $r = r(C)$. Then I_C is generated in degrees $(t+r+1)$ and $(t+r+2)$, and $M(C)$ is generated in degrees t and $t+1$.* $\square$

Remark 1.6. — Notice that in the special case $r(C) = \operatorname{diam} M(C)$ (i.e. $m_r = 0$), we get that I_C is generated in degree $(t+r+1)$, and that if $h^2(\mathcal{J}_C(t-1)) = 0$ — which may happen — then $M(C)$ is generated in degree t.

Lemma 1.3 says that in degree $i \geq k+2$, multiplication by a general linear form L may not induce a surjection $M(C)_i \to M(C)_{i+1}$, but multiplication by two general linear forms does give a surjection. We now give the «dual» statement.

LEMMA 1.7. — *Let C be a curve with $h^0(\mathcal{J}_C(k)) = 0$ for some integer k. Let L_1 and L_2 be linear forms neither containing a component of C. Then $[(\ker L_1) \cap (\ker L_2)]_i = 0$ for $i \leq k-2$.*

Proof. — The proof is similar to that of Lemma 3.1 of [GM] and is omitted here. Note that if C is irreducible then L_1 and L_2 can be chosen arbitrarily. □

We need some results from [MDP] ; we may concentrate what we need in the following proposition :

PROPOSITION 1.8. — *Let M be a graded S-module of finite length, and let*

$$0 \to F_4 \xrightarrow{\sigma_4} F_3 \xrightarrow{\sigma_3} F_2 \xrightarrow{\sigma_2} F_1 \xrightarrow{\sigma_1} F_0 \longrightarrow M \to 0$$

be a minimal free resolution of M. Set $F_2 = \bigoplus_{n\in\mathbb{Z}} [b_2(n)S(-n)]$. Then the minimal curve in the even liaison class individuated by M has resolutions

$$0 \longrightarrow \bigoplus_{n\in\mathbb{Z}} [q(n)\mathcal{O}_{\mathbb{P}}(-n)] \longrightarrow \mathcal{N}_0 \longrightarrow \mathcal{J}_C(h) \to 0$$

and

$$0 \to \mathcal{K} \longrightarrow \bigoplus_{n\in\mathbb{Z}} [(b_2(n) - q(n))\mathcal{O}_{\mathbb{P}}(-n)] \longrightarrow \mathcal{J}_C(h) \to 0$$

where $\mathcal{N}_0$ is the sheafification of $N_0 = \operatorname{Ker}\sigma_1$, $\mathcal{K}$ the sheafification of $K = \operatorname{Ker}\sigma_2$, $q(n) \leq b_2(n)$ and there exists a_0 such that $q(n) = b_2(n)$ for $n < a_0$, and $q(a_0) < b_2(a_0)$; moreover, $h = \deg(\mathcal{N}_0) + \sum_{n\in\mathbb{Z}} nq(n)$ (note that a_0 may be equal to $+\infty$).

Proof. — See [MDP, Chap. IV, 2.5, 2.7, 3.4, 4.1, 4.4]. Note moreover that a_0 depends only on σ_2. □

We say that a $(r+1)$-uple of positive integers $(m_0, m_1, \ldots, m_r)$ is admissible if there exist integers $d > 0$ and g such that $(-m_0, -m_1, \ldots, -m_r)$ is exactly the (ordered) $(r+1)$-uple of negative values of the function $\chi_{dg} : \mathbb{Z} \to \mathbb{Z}$ defined by :

$$\chi_{dg}(k) = \binom{k+3}{3} - kd - 1 + g \text{ in the interval } [-2, +\infty[,$$

and such that $\chi_{dg}(-2) \geq 0$, i.e. there exist d, g, and t such that $\chi_{dg}(t+k) = -m_k$, $k = 0, \ldots, r$, and these are the only negative values of χ_{dg} in the interval $[-2, +\infty[$.

For simplicity of notation, we allow $m_r = \chi_{dg}(t+r)$ to be equal to 0.

Note that if $r \geq 2$, then d, g and t are uniquely determined as follows (see also below) :

$$\begin{aligned}
t &= -m_2 + 2m_1 - m_0 - 3,\\
d &= m_1 - m_0 + \frac{1}{2}(2m_1 - m_0 - m_2)(2m_1 - m_0 - m_2 - 1),\\
g &= -\binom{2m_1 - m_0 - m_2}{3} - m_0 + 1\\
&\qquad + (2m_1 - m_0 - m_2 - 3)\left[m_1 - m_0 + \binom{2m_1 - m_0 - m_2}{2}\right].
\end{aligned}$$

We say that a graded S-module of finite length is *numerically admissible* if the dimensions of its homogeneous components form an admissible n−uple, and the first non-zero component is in degree t.

Notation. — Let F be a free S-module : $F = \bigoplus_{i=1}^{r} S(-n_i)$, with $n_1 \leq n_2 \leq \cdots \leq n_r$. We set $\inf F = n_1$ and $\sup F = n_r$.

THEOREM 1.9. — *Let C be a curve with natural cohomology. Assume that* $\operatorname{diam} M(C) \geq 2$. *Then $M(C)$ has a minimal resolution of the form*

$$0 \to F_4 \longrightarrow F_3 \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M(C) \to 0$$

where

$$\begin{aligned}
F_0 &= p_0 S(-t) \oplus q_0 S(-t-1),\\
F_1 &= p_1 S(-t-1) \oplus q_1 S(-t-2),\\
F_2 &= u_2 S(-t-r-1) \oplus v_2 S(-t-r-2) \oplus p_2 S(-t-2) \oplus q_2 S(-t-3),\\
F_3 &= u_3 S(-t-r-2) \oplus v_3 S(-t-r-3),\\
F_4 &= u_4 S(-t-r-3) \oplus v_4 S(-t-r-4)
\end{aligned}$$

for some non-negative integers p_0, p_1, p_2, q_0, q_1, q_2, u_2, u_3, u_4, v_2, v_3, v_4.

Proof. — Of course the case of diameter one is trivial and we omit it. Suppose now that we have a minimal free resolution for $M(C)$:

$$(1) \qquad 0 \to F_4 \xrightarrow{\sigma_4} F_3 \xrightarrow{\sigma_3} F_2 \xrightarrow{\sigma_2} F_1 \xrightarrow{\sigma_1} F_0 \longrightarrow M(C) \to 0.$$

It is a standard fact that for each i we have $\inf F_{i+1} > \inf F_i$. Applying the same fact to the dual module (twisted) $M(C)^{\vee}(4)$ we also get that $\sup F_{i+1} > \sup F_i$. We first consider F_0. If $\operatorname{diam} M(C) = 2$ clearly it is generated in at most two degrees. If $\operatorname{diam} M(C) \geq 3$ then this fact follows from Corollary 1.5 and the fact that C has natural cohomology. For the rest of the proof we will assume that C is minimal in its even liaison class. Indeed, in the case of diameter 3 or more this is automatically true (Remark 1.1). In the case of diameter 2 we will see that the minimal shift of the even liaison class (in the sense of the second paragraph above) always consists of curves with natural cohomology (Proposition 3.1) and clearly if $M(C)$ is shifted then the only change in the minimal resolution of $M(C)$ that we want to prove comes in the value of the integer t which defines the shift.

By Proposition 1.8, F_2 (from the resolution (1)) splits as $F_2 = F \oplus P$ where

$$F = \bigoplus_{n \in \mathbb{Z}} [(b_2(n) - q(n))S(-n)] \quad \text{and} \quad P = \bigoplus_{n \in \mathbb{Z}} [q(n)S(-n)],$$

and they have the following roles : F appears in the minimal free resolution of I_C,

$$0 \to F_4 \xrightarrow{\sigma_4} F_3 \longrightarrow F \longrightarrow I_C \to 0 \tag{2}$$

(F_4, F_3 and σ_4 are the same as in (1)), and P appears in a locally free resolution

$$0 \to P \longrightarrow N_0 \longrightarrow I_C \to 0, \tag{3}$$

where N_0 is the kernel of σ_1.

By definition, $\inf F = \alpha(C) = t + r + 1$, so $\inf F_3 \geq t + r + 2$ and $\inf F_4 \geq t + r + 3$. On the other hand, we claim that $\sup F_4 < \alpha(C) + 4 = t + r + 5$ and hence $\sup F_3 \leq t + r + 3$. Indeed, if we let $K = \operatorname{coker} \sigma_4$, the sheafification of the exact sequence

$$0 \to K \to F \to I_C \to 0, \tag{4}$$

together with maximal rank gives $h^2(\mathcal{K}(i)) = 0$ for all $i \geq t+r+1$ (where $\mathcal{K}$ is the sheafification of K). Then the exact sequence

$$0 \to F_4 \xrightarrow{\sigma_4} F_3 \longrightarrow K \to 0 \tag{5}$$

gives that $H^3(\mathcal{F}_4(i))$ injects into $H^3(\mathcal{F}_3(i))$ for all $i \geq t + r + 1$. Since $\sup F_4 > \sup F_3$, this can only happen if $h^3(\mathcal{F}_4(i)) = 0$ for all $\geq t + r + 1$, i.e. $-\sup F_4 + t + r + 1 > -4$, as claimed. This proves the theorem for F_4 and F_3. Note that $\inf F = t + r + 1$ and $\sup F \leq t + r + 2$.

Then note that a minimal free resolution of $A_C = \bigoplus_{n\in\mathbb{Z}} [H^0(C, \mathcal{O}_C(n))]$ has the form

$$0 \to P \longrightarrow F_1 \longrightarrow F_0 \oplus S \longrightarrow A_C \to 0$$

(see f.i. [MP, II, 4]), and hence $\sup P > \sup F_1 > \sup F_0$, and (considering also the dual sequence) $\inf P > \inf F_1 > \inf F_0 = t$.

But now $h^2(\mathcal{J}_C(t)) = 0$ (by maximal corank) implies that for every $p \geq t$, $H^3(\mathcal{P}(p))$ injects into $H^3(\mathcal{N}_0(p))$, where as usual $\mathcal{P}$ and $\mathcal{N}_0$ are the sheafifications of P and N_0, and $H^3(\mathcal{N}_0(p))$ injects into $H^3(\mathcal{F}_1(p))$. Since $H^3(\mathcal{P}(\sup P - 4)) \neq 0$, while $H^3(\mathcal{F}_1(\sup P - 4)) = 0$, we get that $\sup P - 4 < t$, that is to say $\sup P \leq t + 3$. This implies $\sup F_1 \leq t + 2$; since moreover $F_2 = F \oplus P$, this completes the proof. □

Remark 1.10. — With this theorem in fact we have a clear view of the possible resolutions for the ideal sheaves of curves with natural cohomology. If C is as above and $r(C) \geq 2$, then C is minimal in its even liaison class and locally free resolutions of I_C are :

$$0 \to \widetilde{K} \longrightarrow u_2\mathcal{O}_{\mathbb{P}^3}(-t-r-1) \oplus v_2\mathcal{O}_{\mathbb{P}^3}(-t-r-2) \longrightarrow \mathcal{J}_C \to 0,$$

$$0 \to p_2\mathcal{O}_{\mathbb{P}^3}(-t-2) \oplus q_2\mathcal{O}_{\mathbb{P}^3}(-t-3) \longrightarrow \widetilde{F} \longrightarrow \mathcal{J}_C \to 0.$$

The first one is clearly linked to the «west-side» approach of [EH], and in fact from it we get a free resolution for $\mathcal{J}_C$:

$$\begin{aligned} 0 \to u_4\mathcal{O}_{\mathbb{P}^3}(-t-r-3) &\oplus v_4\mathcal{O}_{\mathbb{P}^3}(-t-r-4) \\ &\longrightarrow u_3\mathcal{O}_{\mathbb{P}^3}(-t-r-2) \oplus v_3\mathcal{O}_{\mathbb{P}^3}(-t-r-3) \\ &\qquad \longrightarrow u_2\mathcal{O}_{\mathbb{P}^3}(-t-r-1) \oplus v_2\mathcal{O}_{\mathbb{P}^3}(-t-r-2) \longrightarrow \mathcal{J}_C \to 0. \end{aligned}$$

COROLLARY 1.11. — *Let C be a curve with natural cohomology and let $M = M(C)$. Then the dual module $M^\vee$ is generated in the first two degrees.*

Proof. — There are two easy proofs of this fact, either from the fact that F_4 is generated in two degrees (and then use the fact that the minimal resolution of $M^\vee$ is a twist of the dual of the minimal resolution of M), or else from the fact that the linked curve C' has maximal corank (and then use Lemma 1.3). □

Now we would like to say something about the ranks of the modules occurring in these resolutions. We suppose that C is a curve with natural cohomology and $\operatorname{diam}(C) \geq 3$, and we suppose henceforth that we have $h^3(\mathcal{J}_C(p)) = 0$ for $p \geq t-2$, or equivalently $t \geq -1$ (this is not of course an heavy assumption, and it is automatically verified if C is smooth connected, but it guarantees that $m_i = -\chi(\mathcal{J}_C(t+i))$ for every $i \leq r$). Recall that by Riemann-Roch we have :

$$\chi(\mathcal{J}_C(p)) = \binom{p+3}{3} - p\deg(C) - 1 + p_a(C).$$

Hence for every p

$$\chi(\mathcal{J}_C(p+2)) - 2\chi(\mathcal{J}_C(p+1)) + \chi(\mathcal{J}_C(p)) = p+3$$

and

$$\begin{aligned}\chi(\mathcal{J}_C(p+3)) - 3\chi(\mathcal{J}_C(p+2)) & \\ + 3\chi(\mathcal{J}_C(p+1)) - \chi(\mathcal{J}_C(p)) &= 1.\end{aligned} \tag{3}$$

If we let $p = t = t(C)$, we get $-m_0 + 2m_1 - m_0 = t + 3$, and hence :

$$t(C) = -m_2 + 2m_1 - m_0 - 3.$$

In the same way, one sees that

$$\chi(\mathcal{J}_C(t+1)) - \chi(\mathcal{J}_C(t)) = \tfrac{1}{2}(t+3)(t+2) - \deg(C),$$

and hence

$$\deg(C) = m_1 - m_0 + \tfrac{1}{2}(2m_1 - m_0 - m_2)(2m_1 - m_0 - m_2 - 1),$$

and finally :

$$\begin{aligned}p_a(C) = -\binom{2m_1 - m_0 - m_2}{3} - m_0 + 1 & \\ + (2m_1 - m_0 - m_2 - 3)\left[m_1 - m_0 + \binom{2m_1 - m_0 - m_2}{2}\right].&\end{aligned}$$

Since moreover $\chi(\mathcal{J}_C(p))$ is a polynomial in p which depends on $\deg(C)$ and $p_a(C)$, we see that

$$m_i = -\chi(\mathcal{J}_C(t+i)) \qquad (3 \leq i \leq r)$$

is determined by m_0, m_1 and m_2. Moreover, since $\chi(\mathcal{J}_C(t-1)) \geq 0$, from (1) we get :

$$-m_2 + 3m_1 - 3m_0 \geq 1.$$

Remark 1.12. — This simple numerical argument shows that the Hartshorne-Rao module of a curve with natural cohomology is not at all symmetric, at least for $\operatorname{diam}(C) \geq 4$. If there exists a curve C with natural cohomology and whose Hartshorne-Rao module has homogeneous components of dimensions $m_0, \dots, m_r$ (where $r \geq 3$), then there cannot exist a curve D with natural cohomology and Hartshorne-Rao module with homogeneous components of dimensions $m_r, \dots, m_0$. In fact, for $r \geq 3$, we have :

$$\text{(2)} \qquad \begin{aligned} m_3 - 3m_2 + 3m_1 - m_0 &= -\chi\big(\mathcal{J}_C(t+3)\big) + 3\chi\big(\mathcal{J}_C(t+2)\big) \\ &\qquad - 3\chi\big(\mathcal{J}_C(t+1)\big) + \chi\big(\mathcal{J}_C(t)\big) \\ &= -1. \end{aligned}$$

If D is a curve with natural cohomology and Hartshorne-Rao module with homogeneous components of dimensions $m_r, \dots, m_0$, and $m_0 = h^1(\mathcal{J}_D(p))$, then analogously :

$$-\chi\big(\mathcal{J}_D(p)\big) + 3\chi\big(\mathcal{J}_D(p-1)\big) - 3\chi\big(\mathcal{J}_D(p-2)\big) + \chi\big(\mathcal{J}_D(p-3)\big) = -1.$$

But the first member is $m_0 - 3m_1 + 3m_2 - m_3$, and this is equal to $+1$, due to (2).

As a consequence, if in an even liaison class (with diam ≥ 4) there exists a curve Y with natural cohomology, then in the other half of the liaison class there cannot exist a curve with natural cohomology. Since moreover Y satisfies $e(Y) < s(Y) + 4$ (the «strong» condition of Lazarsfeld and Rao) one can say that Y is the only curve with natural cohomology in the *whole* liaison class.

Now we come back to our purpose of determining the integers appearing in the resolution of Theorem 1.9. Our goal is to prove the following :

PROPOSITION 1.13. — *Let C be a curve with natural cohomology and* $\operatorname{diam}(C) \geq 3$. *Then (using the notation of* 1.9) :

$$
\begin{aligned}
&-m_2 + 3m_1 - 3m_0 \geq 1,\\
&p_0 = m_0,\\
&p_1 = 4m_0 + q_0 - m_1\\
&p_2 = q_1 + 6m_0 - 4m_1 + m_2,\\
&u_2 = -3m_r + 3m_{r-1} - m_{r-2} + 1,\\
&4m_{r-1} - 6m_r - m_{r-2} \leq u_3 \leq 10m_r + 4u_4 - m_{r-2},\\
&\max\{0, m_{r-1} - 4m_r\} \leq u_4 \leq m_{r-1}\,;
\end{aligned}
$$

$$
\begin{aligned}
&\max\{0, m_1 - 4m_0\} \leq q_0 \leq m_1,\\
&4m_1 - 6m_0 - m_2 \leq q_1 \leq 2m_0 + 2q_0 + 2m_1 - m_2 \leq 10m_0 + 4q_0 - m_2,\\
&q_2 = -3m_0 + 3m_1 - m_2 - 1,\\
&v_2 = u_3 + 6m_r - 4m_{r-1} + m_{r-2},\\
&v_3 = 4m_r + u_4 - m_{r-1},\\
&v_4 = m_r
\end{aligned}
$$

(*note some symmetry*).

Proof.

Module F_0. — First of all, clearly we have

$$p_0 = m_0, \text{ and } p_1 = 4m_0 + q_0 - m_1 \quad \text{(see [BB, 4.2])}$$

and trivially :

$$\max\{0, m_1 - 4m_0\} \leq q_0 \leq m_1.$$

Module F_4. — On the other hand, we get from (5) of 1.9 above that :

$$v_4 = h^2(\widetilde{K}(t+r)) = m_r.$$

Moreover,

$$m_{r-1} - 4m_r \leq u_4 \leq m_{r-1}$$

(consider the minimal resolution of $M(C)^\vee(4)$).

Module F_3. — Again from (5) and from $h^3(\widetilde{K}(t+r-1)) = 0$ we get that

$$v_3 = 4m_r + u_4 - m_{r-1}$$

and that $0 \leq h^3(\widetilde{K}(t+r-2)) = u_3 + 6m_r - 4m_{r-1} + m_{r-2}$, from which we have :

$$u_3 \geq 4m_{r-1} - 6m_r - m_{r-2}.$$

But again considering the minimal free resolution for $M(C)^{\vee}(4)$, we get :

$$u_3 \leq 10m_r + 4u_4 - m_{r-2}$$

(see also the argument for q_1 below, which is similar).

Module F_2. — Since (2) of 1.9 is minimal :

$$\begin{aligned} u_2 = a &= h^0\big(\mathcal{J}_C(t+r+1)\big) \\ &= \chi\big(\mathcal{J}_C(t+r+1)\big) \\ &= -3m_r + 3m_{r-1} - m_{r-2} + 1 \end{aligned}$$

(since $\chi(\mathcal{J}_C(t+r+1) - 3\chi(\mathcal{J}_C(t+r))$
$+ 3\chi(\mathcal{J}_C(t+r-1)) - \chi(\mathcal{J}_C(t+r-2)) = 1)$;

$$\begin{aligned} v_2 = b &= h^3\big(\widetilde{K}(t+r-2)\big) \\ &= m_{r-2} + u_3 + 4v_3 - 4v_4 - 10u_3 \\ &= u_3 + 6m_r - 4m_{r-1} + m_{r-2}. \end{aligned}$$

For q_2, we know that $q_2 = f = h^2(\mathcal{J}_C(t-1)) = \chi(\mathcal{J}_C(t-1))$; but since as usual $1 = \chi(\mathcal{J}_C(t+2)) - 3\chi(\mathcal{J}_C(t+1)) + 3\chi(\mathcal{J}_C(t)) - \chi(\mathcal{J}_C(t-1))$, we get :

$$q_2 = -3m_0 + 3m_1 - m_2 - 1.$$

For p_2, we easily see that

$$\begin{aligned} p_2 = e &= h^3\big(\widetilde{F}(t-2)\big) + h^2\big(\mathcal{J}_C(t-2)\big) - 4h^2\big(\mathcal{J}_C(t-1)\big) \\ &= q_1 + \chi\big(\mathcal{J}_C(t-2)\big) - 4\chi\big(\mathcal{J}_C(t-1)\big), \end{aligned}$$

and since

$$\begin{aligned} \chi\big(\mathcal{J}_C(t-2)\big) - 4\chi\big(\mathcal{J}_C(t-1)\big) + 6\chi\big(\mathcal{J}_C(t)\big) \\ - 4\chi\big(\mathcal{J}_C(t+1)\big) + \chi\big(\mathcal{J}_C(t+2)\big) = 0, \end{aligned}$$

we have :

$$\begin{aligned} p_2 &= q_1 - 6\chi(\mathcal{J}_C(t)) + 4\chi(\mathcal{J}_C(t+1)) - \chi(\mathcal{J}_C(t+2)) \\ &= q_1 + 6m_0 - 4m_1 + m_2. \end{aligned}$$

If $r = 2$, of course (F_2) has the form :

$$F_2 = p_2 S(-t-2) \oplus (q_2 + u_2) S(-t-3) \oplus v_2 S(-t-4).$$

Module F_1. — We have :

$$\begin{aligned} 0 \leq h^0(\widetilde{F}(t+2)) = 4p_1 + q_1 - 10m_0 - 4q_0 + m_2 \\ \Longrightarrow \quad 4m_1 - 6m_0 - m_2 \leq q_1 \, ; \end{aligned}$$

since moreover q_1 is smaller or equal of the dimension of the homogeneous component of degree $t+2$ of the kernel of the map

$$m_0 S(-t) \oplus q_0 S(-t-1) \longrightarrow M,$$

and this dimension (as a k-vector space) is $10m_0 + 4q_0 - m_2$, we get :

$$q_1 \leq 10m_0 + 4q_0 - m_2.$$

This inequality will be used later on, but in fact we can prove more. Let us call R this kernel, and R_i its homogeneous components. Hence we have :

$$\begin{aligned} \dim R_t &= 0, \\ \dim R_{t+1} &= 4p_0 + q_0 - m_1 = p_1, \\ \dim R_{t+2} &= 10p_0 + 4q_0 - m_2. \end{aligned}$$

Since q_1 is the number of generators of R in degree $t+2$, we have that $q_1 = \dim R_{t+2} - \dim[\mathrm{im}(R_{t+1} \otimes S_1 \to R_{t+2})]$. Notice moreover that R is the image of the map :

$$\Phi : p_1 S(-t-1) \oplus q_1 S(-t-2) \longrightarrow p_0 S(-t) \oplus q_0 S(-t-1).$$

Let $n_1, \dots, n_{p_0}$ be minimal generators of $M(C)$ in degree t. Then an element of R_{t+1} may be viewed as a p_0-uple of linear forms $(L_1, \dots, L_{p_0})$ such that $L_1 n_1 + \cdots + L_{p_0} n_{p_0} = 0$. Let $(M_1, \dots, M_{p_0})$ be another element of R_{t+1}. Let A and B be linear forms such that :

$$A(L_1, \dots, L_{p_0}) - B(M_1, \dots, M_{p_0}) = 0.$$

Assume $(L_1, \dots, L_{p_0})$ and $(M_1, \dots, M_{p_0})$ are independent (under k). In particular :

$$AL_1 = BM_1, \dots, AL_{p_0} = BM_{p_0}.$$

If $A = \lambda B$ for some scalar λ, then $(M_1, \dots, M_{p_0}) = \lambda(L_1, \dots, L_{p_0})$ contradicting independence of $(L_1, \dots, L_{p_0})$ and $(M_1, \dots, M_{p_0})$.

So without loss of generality we may assume A and B independent. Then we may write

$$L_1 = a_1 B, \dots, L_{p_0} = a_{p_0} B,$$

and not all a_i are zero. Hence :

$$\begin{aligned} A(a_1 B) = bM_1 \quad &\Longrightarrow \; M_1 = a_1 A, \\ &\dots \\ A(a_{p_0} B) = BM_{p_0} \; &\Longrightarrow \; M_{p_0} = a_{p_0} A. \end{aligned}$$

This means :

$$\begin{aligned} (a_1 B)n_1 + \dots + (a_{p_0} B)n_{p_0} = 0 &\Longrightarrow B(a_1 n_1 + \dots + a_{p_0} n_{p_0}) = 0, \\ (a_1 A)n_1 + \dots + (a_{p_0} A)n_{p_0} = 0 &\Longrightarrow A(a_1 n_1 + \dots + a_{p_0} n_{p_0}) = 0. \end{aligned}$$

For general A and B this is impossible (Lemma 1.7). Therefore

$$R_{t+1} \otimes \langle A, B \rangle \text{ injects into } R_{t+2},$$

hence :

$$\dim\big[\mathrm{im}(R_{t+1} \otimes S_1 \to R_{t+2})\big] \geq 2 \dim R_{t+1} = 2p_1 = 8p_1 + 2q_0 - 2m_1.$$

Therefore :

$$q_1 \leq 10p_0 + 4q_0 - m_2 - (8p_0 + 2q_0 - 2m_1) = 2m_0 + 2q_0 + 2m_1 - m_2. \quad \square$$

Example 1.14. — We shall see in the section 3 that for diameter two, any module structure admits curves with natural cohomology. For diameter ≥ 3, though, one might hope that the number of minimal generators for $M(C)$ would be the «expected» number; that is, that the homomorphism $S_1 \otimes M(C)_t \to M(C)_{t+1}$ would have maximal rank (where $t = t(C)$ and S_1 is the vector space of linear forms) and so q_0 would be determined. We now show that is too much to hope for. Our tool is liaison addition (cf. [Sw], [SV] or [BM4]). Let C_1 be a general set of four skew lines, and let C_2 be a general set of two skew lines. Choose general polynomials $F_1 \in I_{C_2}$ of degree 2 and $F_2 \in I_{C_1}$ of degree 3. The scheme Z defined by the (saturated) ideal $F_1 I_{C_1} + F_2 I_{C_2}$ has degree $4 + 2 + 6 = 12$ and Hartshorne-Rao module with components of dimensions 3, 5 and 2 in degrees 2, 3 and 4 respectively. In particular, there is a minimal generator in degree 3 (coming from the module of C_2). But using the exact sequence [BM4]

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-5) \longrightarrow \mathcal{J}_{C_1}(-2) \oplus \mathcal{J}_{C_2}(-3) \longrightarrow \mathcal{J}_Z \to 0,$$

one checks that Z has natural cohomology. In this case, one has for $M(Z)$ $q_0 = 1$, which is not the minimal possible value, and the multiplication $S_1 \otimes M(C)_2 \to M(C)_3$ has not maximal rank.

Remark 1.15. — We want to stress the fact that most of these numerical conditions just follow from the form of the resolution, and not from the fact that M is the Hartshorne-Rao module of a curve with natural cohomology.

Remark 1.16. — Note a nice consequence of these numerical relations : if $m_1 - 4m_0$ is ≥ 0 and q_0 is the minimal possible, i.e. $q_0 = m_1 - 4m_0$, then q_1 is forced to be the minimal possible too, i.e. $q_1 = 4m_1 - 6m_0 - m_2$, and $p_1 = 0 = p_2$. See also section 3.

2. Necessary and sufficient conditions.

One may ask how far is Theorem 1.9 from giving necessary and sufficient conditions for a finite length graded module M to be the HR-module of a curve with natural cohomology. The answer is again contained in the description given by [MP] of the resolution of M and in Proposition 1.13, and in particular in the integer a_0 defined in Proposition 1.8. In what follows, let C be a curve and $e(C)$, $s(C)$, $t(C)$ and $r(C)$ be as in § 1, and let $m_i = h^1(\mathcal{J}_C(t+i))$. We suppose for Propositions 2.1 and 2.2 that $r \geq 3$. If $r < 2$, then every graded module of diameter r is, up to shifting, the Hartshorne-Rao module of a curve (even smooth) with natural cohomology. Actually a stronger result is true : every graded module of finite length of diameter r which is the Hartshorne-Rao module of some curve is in fact the Hartshorne-Rao module of a curve with natural cohomology (see § 3). The case $r = 2$ will be stated in Propositions 2.4 and 2.5.

PROPOSITION 2.1. — *Let C be a curve with natural cohomology, $t = t(C)$ and let $M = M(C)$ be its HR-module. Then M is numerically admissible and its minimal free resolution has the form :*

$$\begin{aligned} 0 \longrightarrow & v_4S(-t-r-4) \oplus u_4S(-t-r-3) \\ & \xrightarrow{\sigma_4} v_3S(-t-r-3) \oplus u_3S(-t-r-2) \\ & \qquad \xrightarrow{\sigma_3} v_2S(-t-r-2) \oplus u_2S(-t-r-1) \\ & \qquad\qquad\qquad \oplus q_2S(-t-3) \oplus p_2S(-t-2) \\ & \qquad\quad \xrightarrow{\sigma_2} q_1S(-t-2) \oplus p_1S(-t-1) \\ & \qquad\qquad \xrightarrow{\sigma_1} q_0S(-t-1) \oplus p_0S(-t) \to M \to 0, \end{aligned}$$

and $a_0 \geq t+r+1$.

Proof. — The fact that M is numerically admissible follows immediately from the definition and the fact that C has natural cohomology; the form of the resolution is Theorem 1.9. So we have to prove only that $a_0 \geq t+r+1$. Now, just look to the sequence

$$0 \to \mathcal{K} \longrightarrow \bigoplus_{n\in\mathbb{Z}} \big[\big(b_2(n) - q(n)\big)\mathcal{O}_{\mathbb{P}}(-n)\big] \longrightarrow \mathcal{J}_C(h) \to 0$$

from Proposition 1.8. Then the definition of a_0, joint with the facts that $h^0(\mathbb{P}^3, \mathcal{K}(t+r+1)) = 0$ and that C has natural cohomology, immediately gives that $a_0 \geq t+r+1$. □

Now we prove the converse.

PROPOSITION 2.2. — *Let M be a numerically admissible graded S-module of finite length, having a minimal free resolution of the form*

$$\text{(A)}\quad \begin{aligned} 0 \longrightarrow & v_4S(-t-r-4) \oplus u_4S(-t-r-3) \\ & \xrightarrow{\sigma_4} v_3S(-t-r-3) \oplus u_3S(-t-r-2) \\ & \qquad \xrightarrow{\sigma_3} v_2S(-t-r-2) \oplus u_2S(-t-r-1) \\ & \qquad\qquad\qquad \oplus q_2S(-t-3) \oplus p_2S(-t-2) \\ & \qquad\quad \xrightarrow{\sigma_2} q_1S(-t-2) \oplus p_1S(-t-1) \\ & \qquad\qquad \xrightarrow{\sigma_1} q_0S(-t-1) \oplus p_0S(-t) \longrightarrow M \longrightarrow 0, \end{aligned}$$

with $a_0 \geq t+r+1$. Then it is the HR-module of a curve with natural cohomology.

Proof. — From Proposition 1.8, the minimal curve in the even liaison class individuated by M has a resolution :

$$0 \to p_2\mathcal{O}_{\mathbb{P}}(-t-2) \oplus q_2\mathcal{O}_{\mathbb{P}}(-t-3) \oplus \Big(\bigoplus_{n \geq t+r+1} [q(n)\mathcal{O}_{\mathbb{P}}(-n)]\Big) \to \mathcal{N}_0 \to \mathcal{J}_C(h) \to 0.$$

We will show that C has natural cohomology; for this, we need to show that $q(n) = 0$ for $n \geq t+4$. From the exact sequence

$$(*) \qquad 0 \to \mathcal{N}_0 \longrightarrow q_1\mathcal{O}_{\mathbb{P}}(-t-2) \oplus p_1\mathcal{O}_{\mathbb{P}}(-t-1) \longrightarrow q_0\mathcal{O}_{\mathbb{P}}(-t-1) \oplus p_0\mathcal{O}_{\mathbb{P}}(-t) \to 0,$$

we know rank $\mathcal{N}_0$, which is :

$$\begin{aligned} q_1 + p_1 - q_0 - p_0 &= q_1 + 4m_0 + q_0 - m_1 - q_0 - m_0 \\ &= 3m_0 + q_1 - m_1 \,; \end{aligned}$$

but this number is exactly equal to $p_2 + q_2 + 1$ (we need the computations of Proposition 1.13, which follows only from the fact that M has a resolution with the form (A)).

Since on the other hand

$$\operatorname{rank} \mathcal{N}_0 = \operatorname{rank}\Big[p_2\mathcal{O}_{\mathbb{P}}(-t-2) \oplus q_2\mathcal{O}_{\mathbb{P}}(-t-3) \oplus \Big(\bigoplus_{n \geq t+r+1} [q(n)\mathcal{O}_{\mathbb{P}}(-n)]\Big)\Big] + 1,$$

it follows $q(n) = 0$ for every $n \geq t+4$, hence

$$h = \deg(\mathcal{N}_0) + \sum_{n \in \mathbb{Z}} nq(n) = 0,$$

and C has natural cohomology, since

$$h^0(\mathcal{J}_C(t+r)) = 0 \text{ and } h^2(\mathcal{J}_C(t)) = 0. \qquad \square$$

Remark 2.3. — The condition $a_0 \geq t+r+1$ can be translated, following [MP, Chap. IV, 6.4] as follows. Let us denote σ_x the restriction of σ_2 to $p_2\mathcal{O}_{\mathbb{P}}(-t-2) \oplus q_2\mathcal{O}_P(-t-3)$. Then, in our situation,

$a_0 \geq t+r+1$

$\iff$ σ_x is injective and with a cokernel without torsion

$\iff$ the (p_2+q_2) minors of σ_x have only trivial common factors

(remember that σ_x can be seen as a $(p_2+q_2) \times (p_1+q_1)$ matrix). Note moreover that $q(t+3) = q_2 = -3m_0 + 3m_1 - m_2 - 1$.

The case $r = 2$ is similar, but we have to state it separately since now q_2 and u_2 patch together.

PROPOSITION 2.4. — *Let C be a curve with natural cohomology, and let M be its Hartshorne-Rao module, having homogeneous components in degrees t, $t+1$, $t+2$ of dimensions m_0, m_1, m_2 respectively (m_0 and m_1 are > 0, m_2 may be 0), and all the other components are zero. Then M is numerically admissible, its minimal free resolution has the form :*

$$
\begin{aligned}
0 \longrightarrow\ & v_4 S(-t-6) \oplus u_4 S(-t-5) \\
&\xrightarrow{\sigma_4} v_3 S(-t-5) \oplus u_3 S(-t-4) \\
&\quad\xrightarrow{\sigma_3} v_2 S(-t-4) \oplus w_2 S(-t-3) \oplus p_2 S(-t-2) \\
&\qquad\xrightarrow{\sigma_2} q_1 S(-t-2) \oplus p_1 S(-t-1) \\
&\qquad\quad\xrightarrow{\sigma_1} q_0 S(-t-1) \oplus p_0 S(-t) \to M \longrightarrow 0,
\end{aligned}
$$

with $a_0 \geq t+3$ and $q(t+3) = -3m_0 + 3m_1 - m_2 - 1$.

Proof. — The fact that M is numerically admissible and the form of the resolution follows, as above, from Theorem 1.9, and Proposition 1.8 gives as usual two resolutions of $\mathcal{J}_C$:

$$
\begin{aligned}
(*)\quad 0 \to \mathcal{K} \longrightarrow\ & \big[v_2 - q(t+4)\big]\mathcal{O}_{\mathbb{P}}(-t-4) \oplus \big[w_2 - q(t+3)\big]\mathcal{O}_{\mathbb{P}}(-t-3) \\
& \oplus \big[p_2 - q(t+2)\big]\mathcal{O}_{\mathbb{P}}(-t-2) \to \mathcal{J}_C \to 0,
\end{aligned}
$$

$$
\begin{aligned}
(**)\quad 0 \to\ & q(t+4)\mathcal{O}_{\mathbb{P}}(-t-4) \oplus q(t+3)\mathcal{O}_{\mathbb{P}}(-t-3) \\
& \oplus q(t+2)\mathcal{O}_{\mathbb{P}}(-t-2) \to \mathcal{N}_0 \to \mathcal{J}_C \to 0.
\end{aligned}
$$

Since $H^0(\mathbb{P}^3, \mathcal{K}(t+2)) = 0 = H^0(\mathbb{P}^3, \mathcal{J}_C(t+2))$, it follows that $p_2 = q(t+2)$, and this implies by definition that $a_0 > t+2$.

Again the definition of q, joint with simple computations left to the reader, gives :

$$
\begin{aligned}
q(t+4) &= 0, \\
q(t+3) &= \operatorname{rank} \mathcal{N}_0 - 1 - p_2 = -3m_0 + 3m_1 - m_2 + 1. \qquad \square
\end{aligned}
$$

Conversely,

PROPOSITION 2.5. — *Let M be a numerically admissible graded S-module of finite length of diameter 3, having a minimal free resolution of* *the form*

$$
\text{(A)}\qquad
\begin{aligned}
0 \to v_4S(-t-6) &\oplus u_4S(-t-5)\\
&\xrightarrow{\sigma_4} v_3S(-t-5)\oplus u_3S(-t-4)\\
&\xrightarrow{\sigma_3} v_2S(-t-4)\oplus w_2S(-t-3)\oplus p_2S(-t-2)\\
&\xrightarrow{\sigma_2} q_1S(-t-2)\oplus p_1S(-t-1)\\
&\xrightarrow{\sigma_1} q_0S(-t-1)\oplus p_0S(-t)\to M\to 0,
\end{aligned}
$$

with $a_0 \geq t+3$ *and* $q(t+3) = -3m_0 + 3m_1 - m_2 - 1$. *Then it is the* HR*-module of a curve with natural cohomology.*

Proof. — The minimal curve in the liaison class individuated by M has resolutions :

$$
(*)\qquad 0\to \mathcal{K}\to \big[v_2 - q(t+4)\big]\mathcal{O}_{\mathbb{P}}(-t-4) \oplus \big[w_2 - q(t+3)\big]\mathcal{O}_{\mathbb{P}}(-t-3)\to \mathcal{J}_C(h)\to 0,
$$

$$
(**)\qquad 0\to q(t+4)\mathcal{O}_{\mathbb{P}}(-t-4)\oplus q(t+3)\mathcal{O}_{\mathbb{P}}(-t-3) \oplus p_2\mathcal{O}_{\mathbb{P}}(-t-2)\to \mathcal{N}_0\to \mathcal{J}_C(h)\to 0.
$$

We first prove that $h^0(\mathbb{P}^3, \mathcal{J}_C(t+2+h)) = 0 = h^2(\mathbb{P}^3, \mathcal{J}_C(t+h))$ (again we let details to the reader).

The beginning of a minimal free resolution of $\mathcal{N}_0$ gives that $h^0(\mathbb{P}^3, \mathcal{N}_0(t+2)) = p_2$, and hence $h^0(\mathbb{P}^3, \mathcal{J}_C(t+2+h)) = 0$.

As in § 1, we have $p_2 = h^0(\mathbb{P}^3, \mathcal{N}_0(t+2)) = q_1 + 6m_0 - 4m_1 + m_2$. Moreover, $\operatorname{rank}\mathcal{N}_0 = q(t+4) + 3m_0 - m_1 + q_1$ and from

$$
0\to \mathcal{N}_0 \longrightarrow q_1\mathcal{O}_{\mathbb{P}}(-t-2)\oplus p_1\mathcal{O}_{\mathbb{P}}(-t-1) \longrightarrow q_0\mathcal{O}_{\mathbb{P}}(-t-1)\oplus p_0\mathcal{O}_{\mathbb{P}}(-t)\to 0,
$$

we have

$$
\operatorname{rank}\mathcal{N}_0 = q_1 + p_1 - q_0 - p_0 = 3m_0 - m_1 + q_1,
$$

hence $q(t+4) = 0$. But now again (**) gives that $0 = h^2(\mathbb{P}^3, \mathcal{J}_C(t+h))$. Thus C has natural cohomology and $M(C) = M(-h)$. But C is minimal and M is numerically admissible, so $h = 0$ and $M = M(C)$. □

3 The case of diameter two.

Recall (see [LR], [BM4]) that given a curve X, we define a *basic double link* Y of X as follows : choose a surface F_1 of degree b containing X and a general surface F_2 of degree f. Then the ideal $F_2 I_X + (F_1)$ is the saturated ideal of a curve Y, basic double link of X. We denote this procedure by $X : (b, f) \to Y$. The curve Y is linked to X in two steps, and the cohomology of the ideal sheaf of Y is determined (numerically) by that of X and the degrees b and f.

PROPOSITION 3.1. — *Let $\mathcal{L}$ be an even liaison class whose corresponding Hartshorne-Rao module has diameter two. Then every shift of $\mathcal{L}$ contains curves with natural cohomology, and there exists in $\mathcal{L}$ a smooth curve with natural cohomology.*

Proof. — We first show that if the minimal shift of $\mathcal{L}$ contains no curve with natural cohomology then no curve in $\mathcal{L}$ has natural cohomology. This follows from the Lazarsfeld-Rao Property (cf. [BBM]). Indeed, the cohomology of the ideal sheaf of a curve in $\mathcal{L}$ is the same, numerically, as the cohomology of a curve obtained from a minimal curve by a sequence of basic double links, and it suffices to consider basic double links with $f = 1$ (cf. [BM4, Cor. 3.9]). Now, if C is a curve and we perform a basic double link $C : (f, 1) \to Y$, then we have :

$$t(Y) = t(C) + 1\,;$$

$$\alpha(Y) = \begin{cases} \alpha(C) & \text{if } f = \alpha(C), \\ \alpha(C) + 1 & \text{if } f > \alpha(C)\,; \end{cases}$$

$$e(Y) = \begin{cases} e(C) + 1 & \text{if } f \leq e(C) + 3, \\ f - 3 & \text{if } f > e(C) + 3. \end{cases}$$

(The first two are clear, and the third comes from [BM3, Lemma 1.14].) So one quickly checks that if C fails to have natural cohomology then so does Y, regardless of the f chosen. If now C is a curve in $\mathcal{L}^h$ with natural cohomology and we perform a basic double link $C : (t(C) + 3, 1) \to Y$, then Y is a curve in $\mathcal{L}^h + 1$ with natural cohomology. If we start from a graded module M of finite length and diameter 2 (and we suppose for simplicity that the non zero components are in degree 0 and 1 and of dimensions m_0 and m_1), and we follow as usual Rao'construction as in

Theorem 1.9, we get a locally free sheaf $\widetilde{F}$ whose cohomology is as follows :

$\widetilde{F}$	0	1	2	3
h^0	0	0		$*$
h^1	m_0	m_1	0	0
h^2	0	0	0	0
h^3	0	0	0	0

Then by Castelnuovo-Mumford, $\widetilde{F}(3)$ is globally generated, and $p = \operatorname{rank}(\widetilde{F}) - 1$, general sections of $\widetilde{F}(3)$ drop rank along a curve Y whose Hartshorne-Rao module (shifted) is isomorphic to M, and which has natural cohomology, as one can see from the exact sequence

$$0 \to p\mathcal{O}_{\mathbb{P}^3}(-3) \longrightarrow \widetilde{F} \longrightarrow \mathcal{J}_Y(t) \to 0$$

and the diagram

$\mathcal{J}_C$	$t-1$	t	$t+1$	$t+2$	$t+3$
h^0	0	0	0		
h^1	0	m_0	m_1	0	0
h^2		0	0	0	0.

Hence in $\mathcal{L}$ there is a smooth curve with natural cohomology; this implies that all curves in $\mathcal{L}^0$ have natural cohomology and therefore every shift of $\mathcal{L}$ contains curves with natural cohomology. □

Remark 3.2.

(a) One cannot strengthen this result by saying that every shift of $\mathcal{L}$ contains smooth curves with natural cohomology. For instance, if M is annihilated by the maximal ideal of S (i.e. $\mathcal{L}$ is a Buchsbaum even liaison class), then there are no smooth curves in the minimal shift (cf. [BM2, Thm 2.12]).

(b) For a curve with natural cohomology and module of diameter two, the degree and arithmetic genus of the curve are easily computed in terms of the shift (i.e. the integer $t(C)$), as we did in Section 1. This is independent of the module structure.

PROPOSITION 3.3. — *If Y is a curve with natural cohomology, with diameter 2, then Y is non-obstructed.*

Proof. — We prove that $H^1(\mathcal{N}_Y) = \operatorname{Ext}^2(\mathcal{J}_Y, \mathcal{J}_Y) = 0$, from which the thesis follows. Let us consider the cohomology of $\mathcal{J}_Y$:

$\mathcal{J}_Y$	$t-1$	t	$t+1$	$t+2$	$t+3$
h^0	0	0	0		
h^1	0	m_0	m_1	0	0
h^2		0	0	0	0.

From Castelnuovo-Mumford it follows that I_Y is generated in degree $t+2$ and $t+3$, and hence a minimal resolution of $\mathcal{J}_Y$ has the form :

$$(*) \qquad 0 \to \mathcal{P} \longrightarrow p\mathcal{O}_{\mathbb{P}^3}(-t-2) \oplus q\mathcal{O}_{\mathbb{P}^3}(-t-3) \longrightarrow \mathcal{J}_Y \to 0.$$

We now consider a minimal free resolution of $M(Y)$:

$$\begin{tikzcd}[column sep=small] 0 \arrow[dr] & & 0 & & & & 0 \arrow[dr] & & 0 \\ & K \arrow[ur] \arrow[dr] & & & & & & F \arrow[ur] \arrow[dr] & \\ \cdots \arrow[r] & p_2S(-t-2) \oplus q_2S(-t-3) \arrow[u] \arrow[rrrrrr] & & & & & & p_1S(-t-1) \oplus q_1S(-t-2) \arrow[u] & \end{tikzcd}$$

$$\longrightarrow p_0S(-t) \oplus q_0S(-t-1) \longrightarrow M(Y) \to 0$$

from which we get the two following exact sequences :

$$(**) \qquad 0 \to \widetilde{K} \to p_2\mathcal{O}_{\mathbb{P}^3}(-t-2) \oplus q_2\mathcal{O}_{\mathbb{P}^3}(-t-3) \to \widetilde{F} \to 0,$$

$$(***) \qquad 0 \to \widetilde{F} \to p_1\mathcal{O}_{\mathbb{P}^3}(-t-1) \oplus q_1\mathcal{O}_{\mathbb{P}^3}(-t-2) \\ \to p_0\mathcal{O}_{\mathbb{P}^3}(-t) \oplus q_0\mathcal{O}_{\mathbb{P}^3}(-t-1) \to 0.$$

From Rao's construction, we have that $\mathcal{P} \cong \widetilde{K} \oplus \mathcal{R}$ where $\mathcal{R} = \bigoplus_i \mathcal{O}_{\mathbb{P}^3}(-a_i)$ and $a_i \geq t+2$ for every i (this follows from (*)).

Then we apply to (*) the functor $\operatorname{Ext}^i(\cdot\,, \mathcal{J}_Y)$, and we get :

$$\begin{aligned} &\operatorname{Ext}^1(\widetilde{K}, \mathcal{J}_Y) \oplus \operatorname{Ext}^1(\mathcal{R}, \mathcal{J}_Y) \\ &\quad \longrightarrow \operatorname{Ext}^2(\mathcal{J}_Y, \mathcal{J}_Y) \\ &\qquad \longrightarrow p\operatorname{Ext}^2(\mathcal{O}_{\mathbb{P}^3}(-t-2), \mathcal{J}_Y) \oplus q\operatorname{Ext}^2(\mathcal{O}_{\mathbb{P}^3}(-t-3), \mathcal{J}_Y). \end{aligned}$$

But now

$$\operatorname{Ext}^1(\mathcal{R}, \mathcal{J}_Y) \cong \oplus H^1(\mathcal{J}_Y(a_i)) = 0 \quad \text{since } a_i \geq t+2,$$

$$\operatorname{Ext}^2(\mathcal{O}_{\mathbb{P}^3}(-t-2), \mathcal{J}_Y) \cong H^2(\mathcal{J}_Y(t+2)) = 0,$$

$$\operatorname{Ext}^2(\mathcal{O}_{\mathbb{P}^3}(-t-3), \mathcal{J}_Y) \cong H^2(\mathcal{J}_Y(t+3)) = 0,$$

hence it is enough to study $\operatorname{Ext}^1(\widetilde{K}, \mathcal{J}_Y)$. From (**) we get :

$$p_2 \operatorname{Ext}^1(\mathcal{O}_{\mathbb{P}^3}(-t-2), \mathcal{J}_Y) \oplus q_2 \operatorname{Ext}^1(\mathcal{O}_{\mathbb{P}^3}(-t-3), \mathcal{J}_Y) \longrightarrow \operatorname{Ext}^1(\widetilde{K}, \mathcal{J}_Y) \longrightarrow \operatorname{Ext}^2(\widetilde{F}, \mathcal{J}_Y).$$

Here we have

$$\operatorname{Ext}^1(\mathcal{O}_{\mathbb{P}^3}(-t-2), \mathcal{J}_Y) \cong H^1(\mathcal{J}_Y(t+2)) = 0,$$
$$\operatorname{Ext}^1(\mathcal{O}_{\mathbb{P}^3}(-t-3), \mathcal{J}_Y) \cong H^1(\mathcal{J}_Y(t+3)) = 0,$$

hence we look to $\operatorname{Ext}^2(\widetilde{F}, \mathcal{J}_Y)$. From (***) we get :

$$p_1 \operatorname{Ext}^2(\mathcal{O}_{\mathbb{P}^3}(-t-1), \mathcal{J}_Y) \oplus q_1 \operatorname{Ext}^2(\mathcal{O}_{\mathbb{P}^3}(-t-2), \mathcal{J}_Y) \longrightarrow \operatorname{Ext}^2(\widetilde{F}, \mathcal{J}_Y) \longrightarrow p_0 \operatorname{Ext}^3(\mathcal{O}_{\mathbb{P}^3}(-t), \mathcal{J}_Y) \oplus q_0 \operatorname{Ext}^3(\mathcal{O}_{\mathbb{P}^3}(-t-1), \mathcal{J}_Y)$$

and since

$$\operatorname{Ext}^2(\mathcal{O}_{\mathbb{P}^3}(-t-1), \mathcal{J}_Y) \cong H^2(\mathcal{J}_Y(t+1)) = 0,$$
$$\operatorname{Ext}^2(\mathcal{O}_{\mathbb{P}^3}(-t-2), \mathcal{J}_Y) \cong H^2(\mathcal{J}_Y(t+2)) = 0,$$
$$\operatorname{Ext}^3(\mathcal{O}_{\mathbb{P}^3}(-t), \mathcal{J}_Y) \cong H^3(\mathcal{J}_Y(t)) = 0,$$
$$\operatorname{Ext}^3(\mathcal{O}_{\mathbb{P}^3}(-t-1), \mathcal{J}_Y) \cong H^3(\mathcal{J}_Y(t+1)) = 0,$$

we get $\operatorname{Ext}^2(\widetilde{F}, \mathcal{J}_Y) = 0$, hence $\operatorname{Ext}^1(\widetilde{K}, \mathcal{J}_Y)$, and therefore

$$\operatorname{Ext}^2(\mathcal{J}_Y, \mathcal{J}_Y) = 0. \qquad \square$$

Remark 3.4. — This result can be also proved, in a simpler way, by using the fact that the corresponding variety of module structures (see [BB]) is smooth, and the results of [MP] about the map from (a suitable covering of) this variety to (a suitable covering of) the Hilbert scheme, and this is perhaps a way for extending it to other more general situations. R.M. Mirò-Roig pointed out to us that she has obtained this same result. Our proof also shows that $h^1(\mathcal{N}_Y) = 0$, which implies that the dimension of the Hilbert scheme is 4d, i.e. as low as possible.

Now we come back, in the case of diameter 2, to the problem of determining the ranks of the free modules appearing in the minimal resolution of the Hartshorne-Rao module of Y, as we did in Proposition 1.13. If the resolution is

$$\begin{aligned} 0 \to u_4S(-t-4) \oplus v_4S(-t-5) &\longrightarrow u_3S(-t-3) \oplus v_3S(-t-4) \\ \longrightarrow c_2S(-t-2) \oplus d_2S(-t-3) &\longrightarrow p_1S(-t-1) \oplus q_1S(-t-2) \\ \longrightarrow p_0S(-t) \oplus q_0S(-t-1) &\longrightarrow M \to 0, \end{aligned}$$

then arguments similar to those used in 1.13 show that

$$\begin{aligned} p_0 &= m_0, & p_1 &= 4m_0 + q_0 - m_1, \\ c_2 &= q_1 - 4m_1 + 6m_0, & d_2 &= u_3 + 6m_1 - 4m_0, \\ v_3 &= 4m_1 + u_4 - m_0, & v_4 &= m_1, \end{aligned}$$

and that

$$\max\{0, m_1 - 4m_0\} \leq q_0 \leq m_1, \quad 4m_1 - 6m_0 \leq q_1 \leq 10m_0 + 4q_0,$$

$$\max\{0, m_0 - 4m_1\} \leq u_4 \leq m_0, \quad 4m_0 - 6m_1 \leq u_3 \leq 10m_1 + 4u_4$$

(provided that $t \geq -1$). With this remark, we are able to prove the following

THEOREM 3.5. — *Let Y and X be curves with natural cohomology, $d = d(Y) = d(X)$, $g = p_a(Y) = p_a(X)$, and $\operatorname{diam}(Y) = 2$. Then Y and X are contained in the same irreducible component of $\mathrm{Hilb}_{d,g}$.*

Proof. — This proof, much simpler than our original proof, is due to the referee. For diameter 1 this result is contained in [B]. We use the notations of [MP]. Let $H_{\gamma\rho}$ the scheme of curves with the same cohomology as X and Y, which is an open subset of H_{dg} by semicontinuity. Let $\widehat{E}_\rho$ the corresponding scheme of module structure, which is irreducible since for diameter 2 it is an affine space. The map $\widehat{\Phi} : \widehat{H}_{\gamma\rho} \to \widehat{E}_\rho$ is smooth and with irreducible fibers (see [MP, VII 1.1 and VII 3.5] for definitions of these objects and proof of these facts), hence $\widehat{H}_{\gamma\rho}$ and $H_{\gamma\rho}$ are irreducible. □

Example 3.6. — From the point of view of [EH], we illustrate with an example this relationship between the resolution of the Hartshorne-Rao module and the resolution of the ideal sheaf. This example also shows how the « actual » construction of graded modules with good properties (which is the difficult part of this approach) can help in finding « good » curves. Let us consider the following module structure $\xi \in \mathcal{V}(2,1)$ (the notation is taken from [BB]) given by the following multiplication maps, where b_1 and b_2 are fixed generators of the homogeneous component of degree 1 and c is the generator of the homogeneous component of degree 2 of ξ :

$$
\begin{aligned}
&x_0 b_1 = c, && x_1 b_1 = x_2 b_1 = x_3 b_1 = 0, \\
&x_0 b_2 = x_2 b_2 = x_3 b_2 = 0, && x_1 b_2 = c.
\end{aligned}
$$

Hence the minimal free resolution of ξ begins with $2S(-1) \to \xi \to 0$. Let N be the kernel of this map; N has $4m_1 - m_0 = 7$ generators in degree 2, namely $(x_0, -x_1), (x_1, 0), (x_2, 0), (x_3, 0), (0, x_0), (0, x_2), (0, x_3)$ for instance. But these generate all of $2S(-1)$ (and hence all of N) in degree 3. Hence the resolution continues :

$$7S(-2) \to 2S(-1) \to \xi \to 0.$$

Let E be the kernel of the first arrow. We consider now the resolution of $\xi^\vee \in \mathcal{V}(1,2)$, which, up to shifting, is :

$$
\begin{tikzcd}
0 \arrow[r] & 2S(1) \arrow[r] & 7S(2) \arrow[rr] \arrow[dr] & & F_2 \arrow[rr] \arrow[dr] & & F_1 \arrow[r] & F_0 \arrow[r] & \xi^\vee(4) \arrow[r] & 0 \\
& & & E^\vee \arrow[ur] \arrow[dr] & & F \arrow[ur] \arrow[dr] & & & & \\
& & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 & & &
\end{tikzcd}
$$

where the cohomology of $\widetilde{E}^\vee$ (the sheafification of $E^\vee$ as usual) is

$\widetilde{E}^\vee$	-8	-7	-6	-5	-4	-3	-2
h^0	0	0	0	0	0	0	7
h^1	0	0	0	0	0	0	0
h^2	0	0	1	2	0	0	0
h^3	$*$	8	0	0	0	0	0

It is easy to check that $F_0 = S(6)$, that $F_1 = 2S(5) \oplus 3S(4)$ and hence that $F_2 = S(4) \oplus 8S(3)$. We are now in the situation

$$
\begin{tikzcd}
0 \arrow[r] & 2S(1) \arrow[r] & 7S(2) \arrow[rr] \arrow[dr] & & S(4) \oplus 8S(3) \arrow[rr] \arrow[dr] & & 2S(5) \oplus 3S(4) \arrow[r] & S(6) \arrow[r] & \xi^\vee(p) \arrow[r] & 0 \\
& & & E^\vee \arrow[ur] \arrow[dr] & & F \arrow[ur] \arrow[dr] & & & & \\
& & 0 \arrow[ur] & & 0 \quad 0 \arrow[ur] & & 0 & & &
\end{tikzcd}
$$

Thanks to [MP], the minimal curve in the liaison class individuated by $\xi^\vee$ has resolutions

$$\begin{tikzcd} 0 \arrow[r] & 2\mathcal{O}_{\mathbb{P}^3}(1) \arrow[r] & 7\mathcal{O}_{\mathbb{P}^3}(2) \arrow[r] & 6\mathcal{O}_{\mathbb{P}^3}(4) \arrow[r] & \mathcal{J}_C(t) \arrow[r] & 0 \\ & & \widetilde{E}^\vee \arrow[ur] & & & \end{tikzcd}$$

and $0 \to 2\mathcal{O}_{\mathbb{P}^3}(3) \oplus \mathcal{O}_{\mathbb{P}^3}(4) \to \widetilde{F} \to \mathcal{J}_C(t) \to 0$, and it can be computed that $t = 6$. Therefore the cohomology of $\mathcal{J}_C$ is

$\mathcal{J}_C$	-1	0	1	2	3
h^0	0	0	0	0	$*$
h^1	0	1	2	0	0
h^3	2	0	0	0	0

and $d(C) = 4$, $p_a(C) = -1$.

Then one finds easily that C can be chosen as the disjoint union of a line and a skew cubic. Note that this is the «missing» curve «à résolution linéaire dominante» of type (2,7) (see [EH, Thm 7.1]). It is not connected, but it can be chosen smooth. The unicity (and the generic smoothness) of the component of $\mathrm{Hilb}_{d,g}$ containing them follows from the results of this chapter. □

4. Comments on diam $\geq$ 3.

Curves with natural cohomology were introduced since, due to semicontinuity, they are «general», as dimensions of the cohomology groups, in their irreducible component of the Hilbert scheme, having the «minimal» (dimensionally) cohomology allowed by the Euler characteristic of the sheaves $\mathcal{J}_C(t)$, determined by d, p_a and t.

Proposition 1.13 suggests another condition of minimality, that is to say we can consider curves with natural cohomology having the Hartshorne-Rao module with the minimal possible integers q_0, q_1, u_3, u_4.

Let us consider, for instance, $r = 2$, that is to say a case with diameter at most 3, and suppose that Y is curve with natural cohomology, $r(Y) = 2$ and $p_2 = 0$ (in the resolution of $M(Y)$). This means $q_1 + 6m_0 - 4m_1 + m_2 = 0$, that is to say $q_1 = 4m_1 - 6m_0 - m_2$ (the minimal possible value if $4m_1 - 6m_0 + m_2 \geq 0$). Then Y is non obstructed. In fact, we can repeat the proof of 3.3, considering the sequences

$$(1) \qquad 0 \to \mathcal{K} \longrightarrow u_2\mathcal{O}_{\mathbb{P}^3}(-t-3) \oplus v_2\mathcal{O}_{\mathbb{P}^3}(-t-4) \longrightarrow \mathcal{J}_Y \to 0,$$

$$(2) \qquad 0 \to \mathcal{K} \longrightarrow p_2\mathcal{O}_{\mathbb{P}^3}(-t-2) \oplus (q_2+u_2)\mathcal{O}_{\mathbb{P}^3}(-t-3) \oplus v_2\mathcal{O}_{\mathbb{P}^3}(-t-4) \longrightarrow \mathcal{F} \to 0,$$

$$(3) \qquad 0 \to \mathcal{F} \longrightarrow p_1\mathcal{O}_{\mathbb{P}^3}(-t-1) \oplus q_1\mathcal{O}_{\mathbb{P}^3}(-t-2) \longrightarrow p_0\mathcal{O}_{\mathbb{P}^3}(-t) \oplus q_0\mathcal{O}_{\mathbb{P}^3}(-t-1) \to 0$$

(where with $\mathcal{K}$ and $\mathcal{F}$ we denote here the sheafification of K and F respectively). Recall the cohomology table of $\mathcal{J}_Y$:

$\mathcal{J}_Y$	$t-1$	t	$t+1$	$t+2$	$t+3$
h^0	0	0	0	0	
h^1	0	m_0	m_1	m_2	0
h^2		0	0	0	0

From (3) one gets

$$\begin{array}{c} p_1 \operatorname{Ext}^2(\mathcal{O}_{\mathbb{P}^3}(-t-1), \mathcal{J}_Y) \oplus q_1 \operatorname{Ext}^2(\mathcal{O}_{\mathbb{P}^3}(-t-2), \mathcal{J}_Y) \\ \cong p_1 H^2(\mathbb{P}^3, \mathcal{J}_Y(t+1)) \oplus q_1 H^2(\mathbb{P}^3, \mathcal{J}_Y(t+2)) \\ \downarrow \\ \operatorname{Ext}^2(\mathcal{F}, \mathcal{J}_Y) \\ \downarrow \\ p_0 \operatorname{Ext}^3(\mathcal{O}_{\mathbb{P}^3}(-t), \mathcal{J}_Y) \oplus q_0 \operatorname{Ext}^3(\mathcal{O}_{\mathbb{P}^3}(-t-1)) \\ \cong p_0 H^2(\mathbb{P}^3, \mathcal{J}_Y(t)) \oplus q_0 H^3(\mathbb{P}^3, \mathcal{J}_Y(t+1)), \end{array}$$

which gives $\operatorname{Ext}^2(\mathcal{F}, \mathcal{J}_Y) = 0$. From (2) one gets

$$\begin{array}{c} p_2 \operatorname{Ext}^1(\mathcal{O}_{\mathbb{P}^3}(-t-2), \mathcal{J}_Y) \oplus (q_2+u_2) \operatorname{Ext}^1(\mathcal{O}_{\mathbb{P}^3}(-t-3), \mathcal{J}_Y) \\ \oplus v_2 \operatorname{Ext}^1(\mathcal{O}_{\mathbb{P}^3}(-t-4), \mathcal{J}_Y) \\ \cong p_2 H^1(\mathbb{P}^3, \mathcal{J}_Y(t+2)) \oplus (q_2+u_2) H^1(\mathbb{P}^3, \mathcal{J}_Y(t+3)) \\ \oplus v_2 H^1(\mathbb{P}^3, \mathcal{J}_Y(t+4)) \\ \downarrow \\ \operatorname{Ext}^1(\mathcal{K}, \mathcal{J}_Y) \\ \downarrow \\ \operatorname{Ext}^2(\mathcal{F}, \mathcal{J}_Y), \end{array}$$

which gives ($p_2 = 0\,!$) that $\operatorname{Ext}^1(\mathcal{K}, \mathcal{J}_Y) = 0$. From (1) one gets

$$\begin{array}{c}
\operatorname{Ext}^1(\mathcal{K}, \mathcal{J}_Y) \\
\downarrow \\
\operatorname{Ext}^2(\mathcal{J}_Y, \mathcal{J}_Y) \\
\downarrow \\
u_2 \operatorname{Ext}^2\big(\mathcal{O}_{\mathbb{P}^3}(-t-3), \mathcal{J}_Y\big) \oplus v_2 \operatorname{Ext}^2\big(\mathcal{O}_{\mathbb{P}^3}(-t-4), \mathcal{J}_Y\big) \\
\qquad \cong u_2 H^2\big(\mathbb{P}^3, \mathcal{J}_Y(t+3)\big) \oplus v_2 H^2\big(\mathbb{P}^3, \mathcal{J}_Y(t+4)\big),
\end{array}$$

and hence $H^1(\mathcal{N}_Y) \cong \operatorname{Ext}^2(\mathcal{J}_Y, \mathcal{J}_Y) = 0$. Therefore Y is non-obstructed. Note, however, that there is no guarantee that for a particular choice of m_0, m_1, m_2 (for which there exist curves with natural cohomology) there will exist one with $p_2 = 0$.

Remark 4.1. — Note that if $m_1 \geq 4m_0$, then $4m_1 - 6m_0 - m_2 \geq 0$ (use the fact that $-m_2 + 3m_1 - 3m_0 \geq 1$), and that in this situation it is enough to have $q_0 = m_1 - 4m_0$, see remark 1.15.

An analogous (and in fact slightly better) result is true from the point of view of unicity.

Suppose now that Y and X are curves with natural cohomology, the same degree d and arithmetic genus g, $r(Y) = 3$ and that $p_2(X) = p_2(Y) = 0$. Then they are in the same irreducible component of the Hilbert scheme $H_{d,g}$. In fact, let us consider the minimal resolutions of M_Y and of M_X :

(4)
$$\begin{tikzcd}
0 \arrow[r] & P_4 \arrow[r] & P_3 \arrow[rr] \arrow[dr] & & P_2 \arrow[rr] \arrow[dr] & & P_1 \arrow[r] & P_0 \arrow[r] & M_Y \arrow[r] & 0 \\
& & & K_Y \arrow[ur] \arrow[dr] & & F_Y \arrow[ur] \arrow[dr] & & & & \\
& & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 & & &
\end{tikzcd}$$

(5)
$$\begin{tikzcd}
0 \arrow[r] & R_4 \arrow[r] & R_3 \arrow[rr] \arrow[dr] & & R_2 \arrow[rr] \arrow[dr] & & R_1 \arrow[r] & R_0 \arrow[r] & M_X \arrow[r] & 0 \\
& & & K_X \arrow[ur] \arrow[dr] & & F_Y \arrow[ur] \arrow[dr] & & & & \\
& & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 & & &
\end{tikzcd}$$

and of $\mathcal{J}_Y$ and $\mathcal{J}_X$:

(6)
$$0 \to q_2 \mathcal{O}_{\mathbb{P}^3}(-t-3) \to \mathcal{F}_Y \to \mathcal{J}_Y \to 0$$

$$0 \to q_2\mathcal{O}_{\mathbb{P}^3}(-t-3) \to \mathcal{F}_X \to \mathcal{J}_X \to 0 \tag{7}$$

(remember that $p_2(X) = p_2(Y) = 0$, and that $q_2 = -3m_0 + 3m_1 - m_2 - 1$ is independent from the curve).

One can easily check that the cohomology of $\mathcal{F}_X$ and $\mathcal{F}_Y$ (dimensionally) is

$\mathcal{F}_X, \mathcal{F}_Y$	$t-2$	$t-1$	t	$t+1$	$t+2$	$t+3$
h^0	0	0	0	0	0	
h^1	0	0	m_0	m_1	m_2	0
h^2	0	0	0	0	0	0
h^3			0	0	0	0

(it is seminatural). In fact, one has from (4) and (5) the exact sequences

$$\begin{aligned} (4')\quad 0 \to \mathcal{F}_Y \longrightarrow (4m_0 + q_0(Y) - m_1)\mathcal{O}_{\mathbb{P}^3}(-t-1) \oplus q_1(Y)\mathcal{O}_{\mathbb{P}^3}(-t-2)& \\ \longrightarrow m_0\mathcal{O}_{\mathbb{P}^3}(-t) \oplus q_0(Y)\mathcal{O}_{\mathbb{P}^3}(-t-1) \to 0,& \end{aligned}$$

$$\begin{aligned} (5')\quad 0 \to \mathcal{F}_X \longrightarrow (4m_0 + q_0(X) - m_1)\mathcal{O}_{\mathbb{P}^3}(-t-1) \oplus q_1(X)\mathcal{O}_{\mathbb{P}^3}(-t-2)& \\ \longrightarrow m_0\mathcal{O}_{\mathbb{P}^3}(-t) \oplus q_0(X)\mathcal{O}_{\mathbb{P}^3}(-t-1) \to 0,& \end{aligned}$$

and $q_1(Y) = q_1(X)$, since

$$p_2(Y) = 0 \implies q_1(Y) = 4m_1 - m_2 - 6m_0,$$
$$p_2(X) = 0 \implies q_1(X) = 4m_1 - m_2 - 6m_0.$$

Therefore

$$\begin{aligned} h^0(\mathbb{P}^3, \mathcal{F}_Y(t+2)) &= 4\big[4m_0 + q_0(Y) - m_1\big] + q_1(Y) - 10m_0 \\ &\qquad - 4q_0(Y) + m_2 \\ &= 6m_0 - 4m_1 + q_1(Y) + m_2 = 0 \end{aligned}$$

and analogously for $h^0(\mathbb{P}^3, \mathcal{F}_X(t+2))$. Now, simply adding the addenda $(m_1 - q_0(Y))\mathcal{O}_{\mathbb{P}^3}(-t-1)$ and $(m_1 - q_0(X))\mathcal{O}_{\mathbb{P}^3}(-t-1)$ and the identity maps to (4′) and (5′) we have

$$\begin{aligned} (4'')\quad 0 \to \mathcal{F}_Y \longrightarrow (4m_0)\mathcal{O}_{\mathbb{P}^3}(-t-1) \oplus q_1\mathcal{O}_{\mathbb{P}^3}(-t-2)& \\ \longrightarrow m_0\mathcal{O}_{\mathbb{P}^3}(-t) \oplus m_1\mathcal{O}_{\mathbb{P}^3}(-t-1) \to 0& \end{aligned}$$

$$\begin{aligned} (5'')\quad 0 \to \mathcal{F}_X \longrightarrow (4m_0)\mathcal{O}_{\mathbb{P}^3}(-t-1) \oplus q_1\mathcal{O}_{\mathbb{P}^3}(-t-2)& \\ \longrightarrow m_0\mathcal{O}_{\mathbb{P}^3}(-t) \oplus m_1\mathcal{O}_{\mathbb{P}^3}(-t-1) \to 0,& \end{aligned}$$

that we write briefly :

$$0 \to \mathcal{F}_Y \longrightarrow \mathcal{C} \xrightarrow{\phi_Y} \mathcal{D} \to 0,$$

$$0 \to \mathcal{F}_X \longrightarrow \mathcal{C} \xrightarrow{\phi_X} \mathcal{D} \to 0.$$

If we consider $\mathrm{Hom}(\mathcal{C}, \mathcal{D})$, we see that there is a Zariski open (irreducible) subset T corresponding to morphisms σ whose kernel $\mathcal{K}_\sigma$ is locally free; by semicontinuity, it must have the same cohomology (dimensionally) as $\mathcal{F}_X$ and $\mathcal{F}_Y$ (which have the «minimal» (dimensionally) possible cohomology). Now one can work as in [BB] getting that Y and X are contained in the same irreducible component of the Hilbert scheme.

These results can be translated as follows : if $4m_1 - 6m_0 - m_2 \geq 0$, there is only one irreducible component of the Hilbert scheme containing curves with natural cohomology and minimal q_1, and these curves correspond to smooth point.

But exactly in the same way one can argue if $4m_1 - 6m_2 - m_0 \geq 0$ and u_3 is minimal (that is to say $v_2 = 0$). We have :

$$(4''') \qquad \begin{aligned} 0 \to\ & u_4(Y)\mathcal{O}_{\mathbb{P}^3}(-t-5) \oplus m_2\mathcal{O}_{\mathbb{P}^3}(-t-6) \\ & \longrightarrow u_3(Y)\mathcal{O}_{\mathbb{P}^3}(-t-4) \oplus (4m_2 + u_4(Y) - m_1)\mathcal{O}_{\mathbb{P}^3}(-t-5) \\ & \longrightarrow \mathcal{K}_Y \to 0, \end{aligned}$$

$$(5''') \qquad \begin{aligned} 0 \to\ & u_4(X)\mathcal{O}_{\mathbb{P}^3}(-t-5) \oplus m_2\mathcal{O}_{\mathbb{P}^3}(-t-6) \\ & \longrightarrow u_3(X)\mathcal{O}_{\mathbb{P}^3}(-t-4) \oplus (4m_2 + u_4(X) - m_1)\mathcal{O}_{\mathbb{P}^3}(-t-5) \\ & \longrightarrow \mathcal{K}_X \to 0, \end{aligned}$$

$$(6') \qquad 0 \to \mathcal{K}_Y \longrightarrow (-3m_2 + 3m_1 - m_0 + 1)\mathcal{O}_{\mathbb{P}^3}(-t-3) \longrightarrow \mathcal{J}_Y \to 0,$$

$$(7') \qquad 0 \to \mathcal{K}_X \longrightarrow (-3m_2 + 3m_1 - m_0 + 1)\mathcal{O}_{\mathbb{P}^3}(-t-3) \longrightarrow \mathcal{J}_X \to 0.$$

The cohomology of $\mathcal{K}_Y$ and $\mathcal{K}_X$ (dimensionally) is

$\mathcal{K}_Y, \mathcal{K}_X$	$t-2$	$t-1$	t	$t+1$	$t+2$	$t+3$
h^0	0	0	0	0	0	0
h^1	0	0	0	0	0	0
h^2	0	0	m_0	m_1	m_2	0
h^3			0	0	0	0

since $h^3(\mathbb{P}^3, \mathcal{K}_Y(t)) = h^3(\mathbb{P}^3, \mathcal{K}_X(t)) = 0$.

Moreover, $u_3(Y) = u_3(X) = 4m_1 - 6m_2 - m_0$. Simply adding to the left and center terms of $(4''')$ and $(5''')$ the addenda $(m_2 - u_4(Y))\mathcal{O}_{\mathbb{P}^3}(-t-5)$ and $(m_2 - u_4(X))\mathcal{O}_{\mathbb{P}^3}(-t-5)$ respectively, the proof continues as in the previous cases.

Hence, if $4m_1 - 6m_2 - m_0 \geq 0$ there is only one irreducible component of the Hilbert scheme containing curves with natural cohomology and minimal u_3. Note that $m_1 \geq 3m_2 + 1$, together with the fact that $-3m_2 + 3m_1 - m_0 + 1 \geq 0$ (this comes from $u_2 \geq 0$) implies that $4m_1 - 6m_2 - m_0 \geq 0$.

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