

## ON THE CHARACTERISTIC POWER SERIES OF THE U OPERATOR

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Let  $p$  be a prime number, and let  $k$  be an integer. Atkin's  $U$  operator acts in a completely continuous manner on the  $p$ -adic space of overconvergent modular forms of weight  $k$ . The goal of this note is to show that the "Fredholm" characteristic power series of  $U$  varies " $p$ -adically continuously" in the weight  $k$ , in the following sense. If  $a_m(k)$  is the  $m$ -th coefficient of the characteristic power series of  $U$  acting on overconvergent forms of weight  $k$ , we show that if  $k_1 \equiv k_2 \pmod{p^n(p-1)}$  then  $a_m(k_1) \equiv a_m(k_2) \pmod{p^{n+1}}$  for every  $m \geq 0$ . We then extend this to "higher order differences" of the function  $k \mapsto a_m(k)$ , in the spirit of [Ser2], Thm. 14.

Our  $p$ -adic continuity result leads us to hope that there is a notion of "overconvergent  $p$ -adic modular form of weight  $k$ " not only for rational integers  $k$ , but for  $k$  in the  $p$ -adic space

$$\mathcal{X} = \varprojlim_n \mathbb{Z}/(p-1)p^n\mathbb{Z},$$

and that the  $U$  operator preserves overconvergence and is completely continuous (and therefore has a spectral theory) for all  $k \in \mathcal{X}$ . If so, our result would suggest that this spectral theory is uniformly continuous in  $k$ . At present, however, it is not evident to us how to define overconvergence for  $p$ -adic modular forms of general  $p$ -adic weights.

The methods we use are a direct extension of those in [Gou2], and our main result answers one of the "Further Questions" posed there. This

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paper fits into the general research project outlined in [Gou], and we refer our readers to that paper for further discussion of motivation.

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## 1. Introduction.

To describe our main result precisely, let  $p$  be a prime number, and assume  $p \geq 5$ . Fix a "tame level"  $N$  not divisible by  $p$ ; we will be working with  $p$ -adic modular forms of integral weight on  $\Gamma_1(N)$ . Let  $B$  be a  $p$ -adically complete and separated ring, and let  $r \in B$ . We will let  $M_k(N, B; r)$  denote the space of  $r$ -overconvergent  $p$ -adic modular forms of weight  $k$  on  $\Gamma_1(N)$  defined over  $B$  (for definitions and properties of these spaces, whose importance was first realized by Dwork, we refer to the accounts in [Kat] and [Gou2]). If  $B$  is a discrete valuation ring and  $K$  is its field of fractions, we write  $M_k(N, K; r) = M_k(N, B; r) \otimes K$ ; this is a  $p$ -adic Banach space over  $K$  with respect to the norm determined by making  $M_k(N, B; r)$  the unit ball. This space contains the classical spaces considered in [GM].

We fix a discrete valuation ring  $B$ , let  $K$  be its field of fractions, and write, for simplicity,  $M_k(r) = M_k(N, K; r)$ . When  $0 < \text{ord}(r) < p/(p+1)$ , the Atkin  $U$  operator is a completely continuous linear operator on the  $p$ -adic Banach space  $M_k(r)$ , and hence has a spectral theory. In particular we can consider the characteristic power series  $P_k(t) = \det(1 - tU | M_k(r))$  and, for each rational number  $\alpha$ , the "slope  $\alpha$  subspace"  $M_{k,\alpha}$  which is spanned by all the forms  $f \in M_k(r)$  such that  $(U - \lambda)^m(f) = 0$  for some integer  $m > 0$  and some  $\lambda \in \overline{K}$  such that  $\text{ord}(\lambda) = \alpha$ . It is a basic result in the spectral theory of the  $U$  operator that the space  $M_{k,\alpha}$  is finite-dimensional and independent of the choice of  $r$  (provided  $0 < \text{ord}(r) < p/(p+1)$ ).

We can now state our main result.

**THEOREM 1.** — *Let  $p \geq 5$  be a prime number,  $N$  an integer not divisible by  $p$ ,  $B$  a  $p$ -adically complete and separated discrete valuation ring, and  $K$  its field of fractions. Choose any  $r \in B$  satisfying  $0 < \text{ord}(r) < p/(p + 1)$ . Let  $P_k(t)$  be the characteristic power series of the  $U$  operator acting on the space  $M_k(N, K; r)$  of  $r$ -overconvergent  $p$ -adic modular forms of weight  $k$  and level  $N$ . Write  $P_k(t) = \sum a_m(k)t^m$ . If  $k_1$  and  $k_2$  are integers such that*

$$k_1 \equiv k_2 \pmod{p^n(p - 1)},$$

*then we have, for each  $m$ ,*

$$a_m(k_1) \equiv a_m(k_2) \pmod{p^{n+1}}.$$

Much of the technical complication in the proof of such a result is due to the fact that there are two natural topologies on the Banach spaces  $M_k(r)$ . For the first topology, recall that elements of  $M_k(r)$  can be interpreted as functions of “not too supersingular” elliptic curves  $E$  defined over some  $p$ -adically complete and separated  $B$ -algebra  $A$ . The restriction on the curve  $E$  is that  $E_{p-1}(E, \omega)$  should be a divisor of  $r \in B$ . (See [Kat] and [Gou2] for details.) A “test-object of level  $N$  and growth condition  $r$ ” is simply such a curve together with a level structure. The first topology is just the natural topology on such “functions” : its norm  $\|\cdot\|_{\text{mod}}$  is characterized by

$$\|f\|_{\text{mod}} \leq 1 \qquad \text{if and only if} \qquad f(E/A, \omega, \iota, Y) \in A$$

for any test-object  $(E/A, \omega, \iota, Y)$  of level  $N$  and growth condition  $r$ . We call this topology the *modular topology*; its unit ball is precisely the space  $M_k(N, B; r)$ . The second topology, which we call the  *$q$ -expansion topology*, is induced by the  $q$ -expansion map; its norm  $\|\cdot\|_{q\text{-exp}}$  can be described by saying that  $\|f\|_{q\text{-exp}} \leq 1$  if and only if all the coefficients of the  $q$ -expansion of  $f$  are in  $B$  (i.e., are integral). It is a basic fact that the modular topology on  $M_k(1)$  (i.e., for  $r = 1$ ) coincides with the  $q$ -expansion topology, so that the unit ball in  $M_k(r)$  with respect to the  $q$ -expansion topology can also be described as the intersection  $M_k(r) \cap M_k(N, B; 1)$ . (A proof can be found in [Kat].) This shows, in particular, that  $M_k(1)$  is isomorphic to Serre’s space of  $p$ -adic modular forms of weight  $k$ , which is defined in [Ser2] in terms of limits of  $q$ -expansions.<sup>(1)</sup> We have an inclusion of the “closed” unit balls

$$M_k(N, B; r) \subset M_k(r) \cap M_k(N, B; 1),$$

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<sup>(1)</sup> In other words, given a sequence of classical forms  $f_i$  whose  $q$ -expansions  $f_i(q)$  converge, coefficient-by-coefficient, to  $f(q) \in B[[q]]$ , there always exists a form  $f$  in

but the set on the right is *unbounded* with respect to the modular topology.

It is sometimes convenient to use the  $q$ -expansion map to identify  $M_k(r)$  with its image in  $K \otimes B[[q]]$ . (Except in the case when  $r$  is a unit in  $B$ , the image will not be closed with respect to the “natural” topology on  $K \otimes B[[q]]$ .) From this point of view, the “unit ball with respect to the  $q$ -expansion topology” is just the intersection  $M_k(r) \cap B[[q]]$ .

## 2. Proof of Theorem 1.

As usual, there are Hecke operators  $T_\ell$  for each prime number  $\ell \neq p$  which act on  $M_k(\mathbb{N}, B; r)$ ; these have the expected action on  $q$ -expansions. (See [Kat] or [Gou2] for the definitions.) For  $\ell = p$ , however, the relevant operator is not  $T_p$  (even though  $p \nmid \mathbb{N}$ ), but Atkin’s  $U$  operator, which acts on  $q$ -expansions by the formula

$$U\left(\sum a_n q^n\right) = \sum a_{np} q^n.$$

This is defined on  $M_k(\mathbb{N}, K; r)$  as  $1/p$  times the trace of the Frobenius operator  $\text{Frob}$ , which acts on  $q$ -expansions as

$$\text{Frob}\left(\sum a_n q^n\right) = \sum a_n q^{np}.$$

The theory of these two operators is described in detail in Chapter II of [Gou2]. We will recall here only the most important points for our purposes. To begin with,

PROPOSITION 1. — *If  $\text{ord}(r) < 1/(p+1)$ , then we have*

$$U(M_k(\mathbb{N}, K; r)) \subset M_k(\mathbb{N}, K; r^p).$$

See [Gou2] for a proof; we refer to this result by the code phrase “ $U$  improves overconvergence.” As Dwork was the first to point out, the fact that  $U$  improves overconvergence implies that  $U$  is a *completely continuous* endomorphism of  $M_k(\mathbb{N}, K; r)$  for any  $r$  satisfying  $0 < \text{ord}(r) < p/(p+1)$ . What this means is that for any integer  $n$  one can find a *finite-dimensional*

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some  $M_k(1)$  (here  $k$  may be a  $p$ -adic weight) whose  $q$ -expansion is  $f(q)$ . Conversely, any such form is obtained in this way. A form defined by such a limit *may or may not be overconvergent*, since it is an element of  $M_k(1)$ , which properly contains  $M_k(r)$ , and there seems to be no direct way of deciding if it is from the existence of such a construction.

subspace  $V_n \subset M_k(N, k; r)$  such that the image of the unit ball  $M_k(N, B; r)$  is contained in  $V_n + p^n M_k(N, K; r)$ . In our case, one can find  $V_n$  quite explicitly : it is generated by the  $p$ -adic modular forms obtained as quotients  $f/E_{p-1}^i$ , where  $f$  is a classical modular form of level  $N$  and weight  $k+i(p-1)$ , for  $0 \leq i < (n+1)/((p-1)\text{ord}(r))$ . It is straightforward to estimate that we have  $\dim V_n = O(n^2)$  as  $n$  tends to infinity.<sup>(2)</sup>

The fact that  $U$  is overconvergent implies that it has a spectral theory, as explained in [Ser] and [Mon] (see also the discussion in [Gou2]). In particular, we emphasize the following three facts :

- (1) The  $U$  operator has a characteristic power series

$$P_k(t) = \det(1 - tU|M_k(r)) \in B[[t]]$$

which is independent of  $r$  and defines a  $p$ -adic entire function whose reciprocal roots are the eigenvalues of  $U$  on  $M_k(r)$  and form a sequence tending to zero in  $B$ . In particular, we can write

$$P_k(t) = \prod_i (1 - \lambda_i t)$$

with  $\lambda_i$  ranging through the nonzero eigenvalues of  $U$  (taken in the algebraic closure of  $K$ ). We know that  $\text{ord}(\lambda_i) \geq 0$  and  $\lambda_i \rightarrow 0$ .

- (2) It is possible to define the exterior powers  $\bigwedge^n U$  of any completely continuous operator; they are again completely continuous, hence have traces. Then, if we write

$$P_k(t) = \sum a_n(k)t^n,$$

we have

$$a_n(k) = \text{trace}(\bigwedge^n U).$$

See [Ser2], [Lan], Chapt. 15, §5 and [Gou2] for more information on this.

- (3) Fix  $\alpha \geq 0$ , and define  $M_{k,\alpha}$  to be the subspace of  $M_k(r)$  spanned by the forms  $f$  such that we have

$$(U - \lambda)^m(f) = 0$$

for some integer  $m > 0$  and some  $\lambda \in \overline{K}$  with  $\text{ord}(\lambda) = \alpha$ .  $M_{k,\alpha}$  is then a finite-dimensional vector space, and there exists a closed Banach subspace

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(2) After a conversation with G. Stolzenberg, we have come to think of an estimate for  $\dim V_n$  as giving a “modulus of complete continuity” for our operator.

$F_{k,\alpha}$  such that we have a  $U$ -equivariant decomposition of  $M_k(r)$  as a direct sum :

$$M_k(N, K; r) = M_{k,\alpha} \oplus F_{k,\alpha}.$$

We call  $M_{k,\alpha}$  the *slope  $\alpha$  eigenspace* for  $U$  acting on forms of weight  $k$ .

Recall that a  $\mathbb{Z}_p$ -lattice  $D \subset V$  in a  $p$ -adic vector space  $V$  is a free  $\mathbb{Z}_p$ -submodule of  $V$  such that  $D \otimes \mathbb{Q}_p = V$ .

LEMMA 2. — *Let  $\Phi_1$  and  $\Phi_2$  be completely continuous operators on a  $p$ -adic Banach space  $V$ , and let  $D \subset V$  be any  $\mathbb{Z}_p$ -lattice in  $V$ . If  $\Phi_1(D) \subset D$ ,  $\Phi_2(D) \subset D$  and*

$$(\Phi_1 - \Phi_2)(D) \subset p^n D,$$

then

$$P(t, \Phi_1) \equiv P(t, \Phi_2) \pmod{p^n},$$

where we understand the congruence coefficient-by-coefficient.

*Proof.* — Put

$$P(t, \Phi_1) = \sum a_i t^i \quad \text{and} \quad P(t, \Phi_2) = \sum b_i t^i.$$

We have  $a_0 = b_0 = 1$ , and we want to show that  $a_i \equiv b_i \pmod{p^n}$  for each  $i \geq 1$ .

Let  $\Psi = \Phi_1 - \Phi_2$ . Clearly,  $\Psi$  is completely continuous, and  $\Psi(D) \subset p^n D$  implies that every eigenvalue of  $\Psi$  is divisible by  $p^n$ . Hence we have

$$\text{trace}(\Phi_1) - \text{trace}(\Phi_2) = \text{trace}(\Psi) = \sum \lambda \equiv 0 \pmod{p^n},$$

where the sum is over the eigenvalues of  $\Psi$ . Since  $a_1 = \text{trace}(\Phi_1)$  and  $b_1 = \text{trace}(\Phi_2)$ , this proves the first congruence.

For the remaining congruences, recall that we have

$$a_m = \text{trace} \left( \bigwedge^m \Phi_1 \right) \quad \text{and} \quad b_m = \text{trace} \left( \bigwedge^m \Phi_2 \right),$$

so we need to look at  $\Psi = \bigwedge^m \Phi_1 - \bigwedge^m \Phi_2$ . These are operators on  $\bigwedge^m V$ , which contains the  $\mathbb{Z}_p$ -lattice  $D' = \bigwedge^m D$ . Then, noting that

$$\bigwedge^m \Phi_1 - \bigwedge^m \Phi_2 = \left( \bigwedge^{m-1} \Phi_1 \right) \wedge (\Phi_1 - \Phi_2) + \left( \bigwedge^{m-1} \Phi_1 - \bigwedge^{m-1} \Phi_2 \right) \wedge \Phi_2,$$

we prove by induction that  $\Psi(D')$  is contained in  $p^n D'$ . Thus,  $a_m \equiv b_m \pmod{p^n}$ , as claimed. □

Now assume  $k_1 \equiv k_2 \pmod{p^n(p-1)}$ , and let  $\mathcal{E} : M_{k_1}(\mathbb{N}, K; r) \rightarrow M_{k_2}(\mathbb{N}, K; r)$  denote multiplication by  $E_{p-1}^{(k_2-k_1)/(p-1)}$ . This is easily seen to be an isomorphism of Banach spaces. (One needs only check that the inverse map preserves overconvergence; for this, note that if  $f \in M_{k_2}(B, \mathbb{N}; r)$  then one sees directly from the definition that  $r^{(k_2-k_1)/(p-1)} \mathcal{E} f \in M_{k_1}(B, \mathbb{N}; r)$ .)

Write  $U_k$  for the U operator acting on forms of weight  $k$ . We consider the operators

$$\Phi = U_{k_1} \quad \text{and} \quad \Psi = \mathcal{E}^{-1} U_{k_2} \mathcal{E},$$

both acting on  $M_{k_1}(\mathbb{N}, K; r)$ . Note, first, that both are completely continuous, because both U operators are. Furthermore, our two series may be computed using them :

$$P_1(t) = \det(1 - tU|M_{k_1}(\mathbb{N}, K; r)) = \det(1 - t\Phi)$$

and, since conjugate operators have the same characteristic series,

$$\begin{aligned} P_2(t) &= \det(1 - tU|M_{k_2}(\mathbb{N}, K; r)) \\ &= \det(1 - t(\mathcal{E}^{-1}U\mathcal{E})|M_{k_1}(\mathbb{N}, K; r)) = \det(1 - t\Psi). \end{aligned}$$

Now we are in position to invoke Lemma 2. We take

$$D = M_{k_1}(\mathbb{N}, K; r) \cap M_{k_1}(\mathbb{N}, B; 1) = \{f \in M_{k_1}(\mathbb{N}, K; r) | f(q) \in B[[q]]\}.$$

This is a lattice in  $M_k(r)$ , since the  $q$ -expansions of modular forms have bounded denominators. To apply the lemma, we need to see that  $(\Phi - \Psi)D \subset p^n D$ .

LEMMA 3. — *Let  $W$  be a vector space over  $K$ , and let  $L$  be a lattice in  $W$ . Suppose  $E : W \rightarrow W$  satisfies  $E = I + p^t T$ , where  $I$  is the identity map and  $T : W \rightarrow W$  is a linear map stabilizing  $L$ . Set  $F = E^{-1}$ .*

*If  $\Upsilon : W \rightarrow W$  is a linear operator mapping  $L$  into  $vL$  for some  $v \in K$ , then the linear operator  $F\Upsilon E - \Upsilon$  maps  $L$  into  $p^t vL$ .*

*Proof.* — Simply note that

$$F\Upsilon E - \Upsilon = F\Upsilon(E - I) + (F - I)\Upsilon,$$

that both  $E - I$  and  $F - I$  map  $L$  to  $p^t L$ , and that  $F$  preserves  $L$ . □

In our situation, we take  $W = K \otimes B[[q]]$ ,  $L = B[[q]]$ ,  $E = \mathcal{E}$  to be multiplication by  $E_{p-1}^{(k_2 - k_1)/(p-1)}$ , and  $\Upsilon = U$ , so that  $v = 1$ . Applying the lemma, we get

$$(\Phi - \Psi)B[[q]] \subset p^n B[[q]].$$

Since we already know that the operator  $\Phi - \Psi$  preserves  $M_{k_1}(r)$ , it follows that  $(\Phi - \Psi)(D) \subset p^n D$ , as claimed.

Thus, the hypotheses of Lemma 2 are satisfied, and this completes the proof of the theorem.

### 3. Higher order differences.

Given what has just been proved, it is natural to ask whether the coefficients  $a_m(k)$  are Iwasawa functions, i.e., if there exist power series  $A_m \in \mathbb{Z}_p[[T]]$  such that we have  $a_m(k) = A_m((1 + p)^k - 1)$ . We cannot yet answer this question. We can, however, move a few more steps in the direction of an answer by obtaining further congruence relations among the coefficients  $a_m(k)$ . In fact, Iwasawa functions can be completely characterized (as in [Ser2], Theorem 14) in terms of congruence properties; what we will show is that at least some of the congruences in Serre's characterization are indeed satisfied.

To state these congruences, let  $k \mapsto a(k)$  be any function from  $\mathbb{Z}$  to  $\mathbb{Z}_p$ . Fix an  $n$ , set  $s = p^n(p - 1)$ , and construct difference functions as follows :

$$\begin{aligned} \delta_1(a, k) &= a(k + s) - a(k) \\ \delta_2(a, k) &= \delta_1(a, k + s) - \delta_1(a, k) \\ &= a(k + 2s) - 2a(k + s) + a(k) \end{aligned}$$

and, in general, for  $i > 1$ ,

$$\delta_i(a, k) = \delta_{i-1}(a, k + s) - \delta_{i-1}(a, k).$$

What Serre shows is that if there exists a power series  $A \in \mathbb{Z}_p[[T]]$  such that  $a(k) = A((1 + p)^k - 1)$  for all  $k \equiv k_0 \pmod{p - 1}$ , then we must have

$$\delta_i(a, k_0) \equiv 0 \pmod{p^{i(n+1)}}.$$

Theorem 1 is the special case of  $a(k) = a_m(k)$  and  $i = 1$ . The basic idea of the proof, however, easily extends to handle the general case, as follows.



**THEOREM 2.** — *Let  $p \geq 5$  be a prime number,  $N$  an integer not divisible by  $p$ ,  $B$  a  $p$ -adically complete and separated discrete valuation ring, and  $K$  its field of fractions. Let  $P_k(t)$  be the characteristic power series of the  $U$  operator acting on the space  $M_k(N, K; r)$  of  $r$ -overconvergent  $p$ -adic modular forms of weight  $k$  and level  $N$ . Write  $P_k(t) = \sum a_m(k)t^m$ . Let  $\delta_i$  be as above; then we have, for each  $m$  and  $k$ ,*

$$\delta_i(a_m, k) \equiv 0 \pmod{p^{i(n+1)}}.$$

*Proof.* — Fix an integer  $m$ , and recall that  $a_m(k)$  is the trace of the exterior power  $\bigwedge^m U$  acting on (the  $m$ -th exterior power of) forms of weight  $k$ . We use this fact to express  $\delta_i(a_m, k)$  as the trace of an operator.

Consider first the case when  $i = 2$ . Let  $E$  be the map  $\bigwedge^m M_k(r) \rightarrow \bigwedge^m M_{k+s}(r)$  which is the  $m$ -th exterior power of the map given by multiplication by  $E_{p-1}^{p^n}$ . Then, as we saw above,

$$\delta_1(a_m, k) = \text{trace} \left( E^{-1} \circ \bigwedge^m U \circ E - \bigwedge^m U \right).$$

Similarly, we have

$$\delta_2(a_m, k) = \text{trace} \left( E^{-2} \circ \bigwedge^m U \circ E^2 - 2E^{-1} \circ \bigwedge^m U \circ E + \bigwedge^m U \right).$$

But since

$$\begin{aligned} E^{-2} \circ \bigwedge^m U \circ E^2 - 2E^{-1} \circ \bigwedge^m U \circ E + \bigwedge^m U \\ = E^{-1} \circ \left( E^{-1} \circ \bigwedge^m U \circ E - \bigwedge^m U \right) \circ E - \left( E^{-1} \circ \bigwedge^m U \circ E - \bigwedge^m U \right), \end{aligned}$$

we can apply Lemma 3 twice : once with

$$\Upsilon = \bigwedge^m U \quad \text{and} \quad v = 1,$$

and once with

$$\Upsilon = E^{-1} \circ \bigwedge^m U \circ E - \bigwedge^m U \quad \text{and} \quad v = p^{n+1}.$$

We conclude that  $E^{-2} \circ \bigwedge^m U \circ E^2 - 2E^{-1} \circ \bigwedge^m U \circ E + \bigwedge^m U$  maps  $\bigwedge^m D$  to  $p^{2(n+1)} \bigwedge^m D$ , and therefore that its trace is congruent to zero modulo  $p^{2(n+1)}$ , as desired.

The general case follows in an analogous way, by repeated application of Lemma 3.  $\square$

#### 4. Open questions.

What about the other congruences given by Serre in [Ser2]? Specifically, we would like to know the answer to the following :

QUESTION. — Let  $c_{ij}$  be defined by the equation

$$Y(Y-1)\cdots(Y-j+1) = \sum c_{ij}Y^i,$$

and, with notations as above, let

$$\gamma_j(a, k_0) = \sum_{i=1}^j c_{ij}p^{-i(n+1)}\delta_i(a, k_0).$$

Is it true that we have

$$\text{ord}_p(\gamma_j(a_m, k)) \geq \text{ord}_p(j!)$$

for every  $m$  and  $k$ ?

The point is that, according to [Ser2], this extra series of congruences, along with the congruences already proven, would be sufficient to guarantee that the  $a_m(k)$  are Iwasawa functions of  $k$ .

There is a connection between Theorem 1 and the conjectures about “ $p$ -adic families” of modular eigenforms which we proposed in [GM]. In that paper, we considered the classical spaces  $\mathbf{M}_k(K, Np)$  of modular forms of weight  $k$  on  $\Gamma_1(N) \cap \Gamma_0(p)$ . On these spaces, there is an action of the  $U$  operator; thus, for each rational number  $\alpha \geq 0$  we can look at the subspace  $\mathbf{M}_{k,\alpha}$  spanned by the eigenforms for the  $U$  operator whose eigenvalues had valuation  $\alpha$ . We write  $d(k, \alpha)$  for the dimension of this space. In [GM], we made the following conjecture :

CONJECTURE 1. — Let  $k_1$  and  $k_2$  be integers. Suppose both  $k_1$  and  $k_2$  are bigger than  $2\alpha + 2$ , and that  $k_1 \equiv k_2 \pmod{p^n(p-1)}$  for some integer  $n \geq \alpha$ . Then  $d(k_1, \alpha) = d(k_2, \alpha)$ .

In attempting to prove this conjecture, it seems natural to embed the classical spaces into the corresponding spaces of overconvergent  $p$ -adic

modular forms, which should be the “correct” context for studying  $p$ -adic properties of modular forms. Recall that we have an inclusion

$$\mathbf{M}_k(K, Np) \hookrightarrow M_k(B, N; r) \otimes K,$$

which therefore gives an inclusion  $\mathbf{M}_{k,\alpha} \hookrightarrow M_{k,\alpha}$  of the slope  $\alpha$  subspaces. Writing  $d_p(k, \alpha) = \dim M_{k,\alpha}$  for the dimension of the  $p$ -adic slope  $\alpha$  subspace, one might then consider a  $p$ -adic variant of our conjecture :

CONJECTURE 2. — *Let  $k_1$  and  $k_2$  be integers such that  $k_1 \equiv k_2 \pmod{p^n(p-1)}$  for some integer  $n \geq \alpha$ . Then  $d_p(k_1, \alpha) = d_p(k_2, \alpha)$ .*

Both of these conjectures are known to be true when  $\alpha = 0$ , by the work of Hida on ordinary modular forms.<sup>(3)</sup> In Hida’s work, proving Conjecture 2 is the first step in the proof of Conjecture 1, so that it is not unreasonable to expect the two conjectures to be similarly connected in general.

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(3) The case  $\alpha = 0$  of Conjecture 2 also follows from the main theorem in this note, as one can see by considering the Newton polygons of characteristic power series in weights  $k_1$  and  $k_2$ . The dimension in question is the length of the first (horizontal) segment of the polygon, which is clearly the same when the coefficients of two power series are congruent modulo  $p$ .

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