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ON THE COMPLEXITY OF SUMS OF DIRICHLET MEASURES

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1. Introduction.

Let E be a metrizable compact space. We denote by $\mathcal{M}(E)$ (resp. $\mathcal{M}_1(E)$) the set of all non-negative (resp. probability) Borel measures on E. Recall that $\mathcal{M}_1(E)$ is a metrizable compact space for the weak* topology, which is the topology of the duality with the set $\mathcal{C}(E)$ of all continuous functions on E; in the following, the topological complexity of a subset M of $\mathcal{M}(E)$ actually means the topological complexity of $M \cap \mathcal{M}_1(E)$. $\mathcal{P}(E)$ (resp. $\mathcal{K}(E)$) denotes the set of all subsets (resp. compact subsets) of E. Let $\mathcal{C}$ be a closed under countable intersections subset of $\mathcal{P}(E)$. We denote by $\mathcal{M}(\mathcal{C})$ the set of all non-negative Borel measures concentrated on an element of $\mathcal{C}$: $\mathcal{M}(\mathcal{C}) = \bigcup_{X\in\mathcal{C}} \mathcal{M}(X)$. Let $\mathcal{C}^\sigma$ (resp. $\mathcal{C}^\uparrow$) denote the set of all unions of sequences (resp. increasing sequences) of elements of $\mathcal{C}$. Note that $\mathcal{M}(\mathcal{C}^\uparrow)$ is equal to the norm-closure $\overline{\mathcal{M}(\mathcal{C})}$ of $\mathcal{M}(\mathcal{C})$ and that $\mathcal{M}(\mathcal{C}^\sigma)$ is the convex norm-closure of $\mathcal{M}(\mathcal{C})$. We denote $\mathcal{C}^\perp$ the set of all measures which annihilate all elements of $\mathcal{C}$. We have the following algebraic decomposition : $\mathcal{M}(E) = \mathcal{M}(\mathcal{C}^\sigma) \oplus \mathcal{C}^\perp$. Recall that $\mathcal{K}(E)$ is a metrizable compact space in the Hausdorff topology. If $\mathcal{C}$ is a Borel subset of $\mathcal{K}(E)$, then $\mathcal{M}(\mathcal{C})$, $\mathcal{M}(\mathcal{C}^\uparrow)$ and $\mathcal{M}(\mathcal{C}^\sigma)$ are analytic subsets of $\mathcal{M}_1(E)$ and $\mathcal{C}^\perp$ is a coanalytic subset of $\mathcal{M}_1(E)$.

Let $\mathbf{T}$ be the unit circle $\mathbf{R}/\mathbf{Z}$. We are interested in the four following subsets of $\mathcal{K}(\mathbf{T})$.

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A compact subset K of $\mathbf{T}$ is a *set of type* D or a *Dirichlet set* if for all $\varepsilon > 0$ and $N \in \mathbf{N}$ there exists $n \geq N$ such that $|\sin 2\pi nx| < \varepsilon$ for all $x \in K$.

A compact subset K of $\mathbf{T}$ is a *set of type* H if there exist a non empty interval I of $\mathbf{T}$ and a strictly increasing sequence $(n_k)_{k\in\mathbf{N}}$ of integers such that $n_k K \cap I = \varnothing$ for each integer k.

A compact subset K of $\mathbf{T}$ is a *set of type* L or a *lacunary set* if there exist a sequence $\varepsilon_n \longrightarrow 0^+$, a sequence $\alpha_n \longrightarrow +\infty$ and for each integer n a finite sequence (I_k) of intervals such that $|I_k| \leq \varepsilon_n$ for each k, $\mathrm{d}(I_k, I_{k'}) \geq \alpha_n \varepsilon_n$ for each $k \neq k'$ and $K \subseteq \bigcup I_k$.

A compact subset K of $\mathbf{T}$ is a *set of type* L_0 if there exist a sequence $\varepsilon_n \longrightarrow 0^+$, $\alpha > 0$ and for each integer n a finite sequence (I_k) of intervals such that $|I_k| \leq \varepsilon_n$ for each k, $\mathrm{d}(I_k, I_{k'}) \geq \alpha\varepsilon_n$ for each $k \neq k'$ and $K \subseteq \bigcup I_k$.

Note that both H and L are supersets of D and subsets of L_0. The classes D and L are $\mathcal{G}_\delta$ subsets of $\mathcal{K}(\mathbf{T})$ and H and L_0 are $\mathcal{K}_{\sigma\delta}$ subsets [1].

A measure concentrated on a $D^\uparrow$-set is called a *Dirichlet measure.* For every $\mu \in \mathcal{M}(\mathbf{T})$ and $n \in \mathbf{N}$, we denote $\hat{\mu}(n) = \int e^{2\pi inx}\,\mathrm{d}\mu(x)$ and $\tilde{\mu}(n) = \int |\sin 2\pi nx|\,\mathrm{d}\mu(x)$. For every $\mu \in \mathcal{M}(\mathbf{T})$, the following conditions are equivalent :

$$\begin{cases} (1)\ \mu \in \mathcal{M}(D^\uparrow) \\ (2)\ \limsup\limits_{n\to\infty} |\hat{\mu}(n)| = \displaystyle\int \mathrm{d}\mu \\ (3)\ \liminf\limits_{n\to\infty} \tilde{\mu}(n) = 0. \end{cases}$$

Note that $\mathcal{M}(D^\uparrow)$ is a norm-closed $\mathcal{G}_\delta$ subset of $\mathcal{M}_1(\mathbf{T})$.

THEOREM 1.1. — *There does not exist a Borel subset $\mathcal{B}$ of $\mathcal{M}_1(\mathbf{T})$ such that $\mathcal{B} \cap {L_0}^\perp = \varnothing$ and $\mathcal{M}(D^\uparrow) + \mathcal{M}(D^\uparrow) \subset \mathcal{B}$.*

For all $M \subset \mathcal{M}(\mathbf{T})$ and $n \in \mathbf{N}$, we denote $M^{(n)}$ the set of all sums of n elements of M.

COROLLARY 1.2. — *The sets $\mathcal{M}(\mathcal{C}^\uparrow)^{(n)}$, $\overline{\mathcal{M}(\mathcal{C}^\uparrow)^{(n)}}$ and $\mathcal{M}(\mathcal{C}^\sigma)$ are analytic non Borel for all $n \geq 2$ and $\mathcal{C} = D$, H, L or L_0.*

We obtain also the following property which has been studied successively by Host, Louveau and Parreau [3], Kechris and Lyons [3] and Kaufman [2].

COROLLARY 1.3. — *$\mathcal{C}^{\perp}$ is a coanalytic non Borel set for $\mathcal{C} = D$, H, L or L_0.*

COROLLARY 1.4. — *None of the sets in the two previous corollaries can be pairwise separated by a Borel set.*

We prove also that the sets $\mathcal{M}(\mathcal{C}^{\uparrow})^{(n)}$, for $n \geq 2$ and $\mathcal{C} = D$, H, L or L_0, are not norm-closed.

THEOREM 1.5. — *There exists a measure in $\overline{\mathcal{M}(D^{\uparrow}) + \mathcal{M}(D^{\uparrow})}$ which is not a finite sum of measures in $\mathcal{M}({L_0}^{\uparrow})$.*

THEOREM 1.6. — *For every $n \geq 3$, there exists a measure in $\overline{\mathcal{M}(D^{\uparrow}) + \mathcal{M}(D^{\uparrow})}$ which is the sum of n measures in $\mathcal{M}(D^{\uparrow})$ and is not the sum of $n-1$ measures in $\mathcal{M}({L_0}^{\uparrow})$.*

2. Kaufman's reduction.

We follow Kaufman's construction used to prove that $H^{\perp}$ is not a Borel set [2]. Let $\mathbf{N}$ be the set of positive integers, $[\mathbf{N}]$ be the set of all infinite subsets of $\mathbf{N}$, $\mathbf{N}^{<\mathbf{N}}$ be the set of all finite sequences of positive integers and $\mathcal{T}$ be the set of trees on $\mathbf{N}$, i.e., $\mathcal{T} \subset \mathcal{P}(\mathbf{N}^{<\mathbf{N}})$ and $T \in \mathcal{T}$ if and only if all initial segments of $s \in T$ are also in T. We say that $T \in \mathcal{T}$ is a well founded tree if T has no infinite branch, i.e., there does not exist $\sigma \in \mathbf{N}^{\mathbf{N}}$ all whose initial segments belong to T. The set of all well founded trees is denoted by WF. Recall that $\mathcal{T}$ is a Polish space in the product topology on $\mathcal{P}(\mathbf{N}^{<\mathbf{N}})$ and WF is the classical example of a coanalytic non Borel set.

We denote $2^{\mathbf{N}}$ the compact, metrizable space $\{0,1\}^{\mathbf{N}}$. If $x \in 2^{\mathbf{N}}$, $x = (x(n))_{n \in \mathbf{N}}$ with $x(n) = 0$ or 1. Let λ be the Lebesgue measure on $2^{\mathbf{N}}$. Let Σ be the Polish space of all Borel sets on $2^{\mathbf{N}}$ with metric $d(A,B) = \lambda(A \triangle B)$, quotiented by the relation $d(A,B) = 0$; Σ can be viewed as a closed subspace of $L^1(2^{\mathbf{N}})$. Consider the sets

$$\mathcal{X} = \{\, (A_n)_{n \in \mathbf{N}} \in \Sigma^{\mathbf{N}};\ \lambda\big(\bigcap_R A_n\big) = 0 \text{ for all } R \in [\mathbf{N}] \,\}$$

and

$$\mathcal{Y} = \{\, (A_n)_{n \in \mathbf{N}} \in \Sigma^{\mathbf{N}};\ \lambda\big((\liminf_R A_n) \cup \liminf_S A_n)\big) = 1 \text{ for some } (R,S) \in [\mathbf{N}]^2 \,\},$$

where $\liminf_R A_n = \bigcup_{m\in\mathbf{N}} \bigcap_{n\geq m, n\in R} A_n$. Note that $\mathcal{X}$ is a coanalytic subset of $\Sigma^{\mathbf{N}}$ [2] and that $\mathcal{Y}$ is an analytic subset of $\Sigma^{\mathbf{N}}$.

LEMMA 2.1. — *There is a continuous mapping Φ from $\mathcal{T}$ to $\Sigma^{\mathbf{N}}$ such that $\Phi(WF) \subset \mathcal{X}$ and $\Phi(WF^c) \subset \mathcal{Y}$. Therefore, there is no Borel subset $\mathcal{B}$ of $\Sigma^{\mathbf{N}}$ such that $\mathcal{Y} \subset \mathcal{B}$ and $\mathcal{X} \cap \mathcal{B} = \varnothing$.*

Proof. — **Construction of Φ.** To each $s \in \mathbf{N}^{<\mathbf{N}}$, we attach subsets $E(s)$ and $F(s)$. Let $<\ ,\ >$ be a one-to-one mapping from $\mathbf{N}^2$ to $\mathbf{N}$. We define $E(s)$ and $F(s)$ by induction on the length $|s|$ of s. Let $E(\varnothing) = 2^{\mathbf{N}}$ and $F(\varnothing) = \varnothing$. If $s \in \mathbf{N}^{<\mathbf{N}}$ has length $|s| = k-1$ and $n_k \in \mathbf{N}$, put

$$E(s^\frown n_k) = \Big\{ x \in 2^{\mathbf{N}};\ \big(x \in E(s) \text{ and } \exists i \in [kn_k, k(n_k+1)[,\ x(<k,i>) = 0\big) \text{ or } \big(x \in F(s) \text{ and } \forall i \in [kn_k, k(n_k+1)[,\ x(<k,i>) = 1\big) \Big\}$$

and

$$F(s^\frown n_k) = \Big\{ x \in 2^{\mathbf{N}};\ \big(x \in F(s) \text{ and } \exists i \in [kn_k, k(n_k+1)[,\ x(<k,i>) = 0\big) \text{ or } \big(x \in E(s) \text{ and } \forall i \in [kn_k, k(n_k+1)[,\ x(<k,i>) = 1\big) \Big\}.$$

We have $E((n_1)) = \{x \in 2^{\mathbf{N}};\ x(<1,n_1>) = 0\}$ and $F((n_1)) = \{x \in 2^{\mathbf{N}};\ x(<1,n_1>) = 1\}$ if $n_1 \in \mathbf{N}$. Note that $E(s) = F(s)^c$ and $\lambda(E(s)) = \lambda(F(s)) = \frac{1}{2}$ for all $s \in \mathbf{N}^{<\mathbf{N}} \setminus \{\varnothing\}$. Let $\sigma \in \mathbf{N}^{\mathbf{N}}$. The length k initial segment of σ is denoted by $\sigma\lceil k$. We have

$$\lambda\Big(\bigcap_{k\geq n} E(\sigma\lceil k)\Big) \geq \lambda\big(E(\sigma\lceil n)\big) \times \prod_{k>n}(1-2^{-k})$$

for each $n \in \mathbf{N}$. But $\lim_{n\to+\infty} \prod_{k>n}(1-2^{-k}) = 1$, whence

$$\lambda\big(\liminf E(\sigma\lceil k) \cup \liminf F(\sigma\lceil k)\big) = 1.$$

Let us enumerate $\mathbf{N}^{<\mathbf{N}} = \{s_n;\ n \in \mathbf{N}\}$ and consider the mapping $\Phi : \mathcal{T} \longrightarrow \Sigma^{\mathbf{N}}$, $T \mapsto \big(\Phi_n(T)\big)_{n\in\mathbf{N}}$ defined by

$$\Phi_n(T) = \begin{cases} E(s_p) & \text{if } n = 2p \text{ and } s_p \in T, \\ F(s_p) & \text{if } n = 2p+1 \text{ and } s_p \in T, \\ \varnothing & \text{otherwise.} \end{cases}$$

Clearly, Φ is continuous and $\Phi(WF^c) \subset \mathcal{Y}$.

To complete the proof of Lemma 2.1, it remains only to show that $\Phi(WF) \subset \mathcal{X}$. Let $T \in \mathcal{T}$ such that there exists $R \in [\mathbf{N}]$ with $\lambda\Big(\bigcap_R \Phi_n(T)\Big) > 0$. Let us suppose that $R \cap 2\mathbf{N}$ is infinite (the case

$R \cap (2\mathbf{N}+1)$ infinite is similar). Let $P \in [\mathbf{N}]$ such that $2P \subset R$. We have $\lambda\Big(\bigcap_P E(s_p)\Big) > 0$. Let $s_p = (n_1^p, n_2^p, \cdots, n_{|s_p|}^p)$ for each $p \in P$. Let us prove that $\{\, n_k^p;\, p \in P\,\}$ is finite for all $k \in \mathbf{N}$. Otherwise, there exist $k \in \mathbf{N}$, $s \in \mathbf{N}^{<\mathbf{N}}$ and an infinite subset P' of P such that $s_p = s^\frown n_k^{p\frown} t_p$ with $t_p \in \mathbf{N}^{<\mathbf{N}}$ for all $p \in P'$ and $n_k^p \neq n_k^{p'}$ for distinct $p,\, p' \in P'$. Let $p \in P'$. For all $x \in 2^{\mathbf{N}}$, we have

$$x \in E(s_p) \iff \left\{ \begin{array}{ll} & \left\{\begin{array}{l} \exists i \in [kn_k^p, k(n_k^p+1)[, \\ x(<k,i>) = 0 \end{array}\right. \quad \textit{and} \quad \left\{ \textit{or} \begin{array}{l} x \in E(s) \cap E_k(t_p) \\ x \in F(s) \cap F_k(t_p) \end{array}\right. \\ \textit{or} & \\ & \left\{\begin{array}{l} \forall i \in [kn_k^p, k(n_k^p+1)[, \\ x(<k,i>) = 1 \end{array}\right. \quad \textit{and} \quad \left\{ \textit{or} \begin{array}{l} x \in E(s) \cap F_k(t_p) \\ x \in F(s) \cap E_k(t_p) \end{array}\right. \end{array}\right.$$

where $E_k(t)$ and $F_k(t)$ can be defined by induction as follows : $E_k(\varnothing) = 2^{\mathbf{N}}$ and $F_k(\varnothing) = \varnothing$; if $t \in \mathbf{N}^{<\mathbf{N}}$ has length $|t| = j-1$ and $m_j \in \mathbf{N}$, put

$$\begin{aligned} E_k(t^\frown m_j) = \big\{x \in 2^{\mathbf{N}};\ &\big(x \in E_k(t) \\ &\quad \textit{and } \exists i \in [(k+j)m_j, (k+j)(m_j+1)[, x(<k+j,i>) = 0\big) \\ &\textit{or } \big(x \in F_k(t) \\ &\quad \textit{and } \forall i \in [(k+j)m_j, (k+j)(m_j+1)[, x(<k+j,i>) = 1\big)\big\} \end{aligned}$$

and

$$\begin{aligned} F_k(t^\frown m_j) = \big\{x \in 2^{\mathbf{N}};\ &\big(x \in F_k(t) \\ &\quad \textit{and } \exists i \in [(k+j)m_j, (k+j)(m_j+1)[, x(<k+j,i>) = 0\big) \\ &\textit{or } \big(x \in E_k(t) \\ &\quad \textit{and } \forall i \in [(k+j)m_j, (k+j)(m_j+1)[, x(<k+j,i>) = 1\big)\big\}. \end{aligned}$$

Note that $E_k(t) = F_k(t)^c$ for all $t \in \mathbf{N}^{<\mathbf{N}}$. Moreover in the probability space $(2^{\mathbf{N}}, \lambda)$, the conditions $\{\, x \in E(s)\,\}$, $\{\, \exists i \in [(k+j)m_j, (k+j)(m_j + 1)[, x(<k+j,i>) = 0\,\}$ and $\{\, x \in E_k(t)\,\}$ are independent, because the mappings $x \mapsto x(j)$, $j \in \mathbf{N}$, are independent. The conditions $\{\, \exists i \in [kn_k^p, k(n_k^p+1)[, x(<k,i>) = 0\,\}$ and $\{\, \exists i \in [kn_k^{p'}, k(n_k^{p'}+1)[, x(<k,i>) = 0\,\}$ are also independent for distinct $p, p' \in P'$. So we can explicitely calculate $\lambda\Big(\bigcap_{p \in I} E(s_p)\Big)$ for any finite subset I of P'. We have

$$\lambda\Big(\bigcap_{p\in I} E(s_p)\Big) = \sum_{i=0}^{|I|} \alpha_i (2^{-k})^i (1-2^{-k})^{|I|-i}$$

where $\alpha_0 = \lambda\Big(\big[E(s) \cap \bigcap_{p\in I} E_k(t_p)\big] \cup \big[F(s) \cap \bigcap_{p\in I} F_k(t_p)\big]\Big)$ and $\alpha_i \geq 0$, $\sum_{i=0}^{|I|} \alpha_i = 1$. So $\alpha_0 \leq \frac{1}{2}$, whence

$$\lambda\Big(\bigcap_{p\in I} E(s_p)\Big) \leq \frac{1}{2}(1-2^{-k})^{|I|} + \frac{1}{2}2^{-k}(1-2^{-k})^{|I|-1} = \frac{1}{2}(1-2^{-k})^{|I|-1}.$$

Thus $\lambda\Big(\bigcap_{p\in P'} E(s_p)\Big) = 0$ which is a contradiction, and proves that $\{ n_k^p ; \, p \in P \}$ is finite for all $k \in \mathbf{N}$. So the tree $T' = \{ s \in \mathbf{N}^{<\mathbf{N}} ; \, \exists p \in P$, *s is an initial segment of* $s_p \}$ is an infinite tree (P is infinite) with finite branching, so $T' \notin WF$, whence $T \notin WF$. □

3. The abstract case.

We introduce a subset I of $\mathcal{K}(2^{\mathbf{N}})$ which plays the role of D in this simpler case.

A compact subset K of $2^{\mathbf{N}}$ is a *set of type* I if for all $N \in \mathbf{N}$ there exists $n \geq N$ such that $x(n) = 0$ for all $x \in K$. Note that I is a $\mathcal{G}_\delta$ subset of $\mathcal{K}(2^{\mathbf{N}})$.

For each $A \in [\mathbf{N}]$, put

$$K_A = \{ x \in 2^{\mathbf{N}} ; \, \forall n \in A, \, x(n) = 0 \},$$
$$K_A^\uparrow = \{ x \in 2^{\mathbf{N}} ; \, \exists m \in \mathbf{N}, \, \forall n \in A \cap [m, +\infty[, \, x(n) = 0 \}$$

and let μ_A be the Haar measure on the subgroup K_A of $2^{\mathbf{N}} \cong (\mathbf{Z}/2\mathbf{Z})^{\mathbf{N}}$. More precisely, μ_A is the product measure $\bigotimes_{n\in\mathbf{N}} \nu_n$ with $\nu_n = \delta_0$ if $n \in A$ and $\nu_n = \frac{1}{2}(\delta_0 + \delta_1)$ otherwise.

We will use the following elementary, but fundamental fact.

LEMMA 3.1. — *Let A and $B \in [\mathbf{N}]$. If $B \setminus A$ is finite, then $\mu_A(K_B^\uparrow) = 1$. If $B \setminus A$ is infinite, then $\mu_A(K_B^\uparrow) = 0$.*

Note that

$$I = \{ K \in \mathcal{K}(2^{\mathbf{N}}) ; \, \exists A \in [\mathbf{N}], \, K \subset K_A \}$$

and

$$I^\uparrow = \{ K \in \mathcal{K}(2^{\mathbf{N}}) ; \, \exists A \in [\mathbf{N}], \, K \subset K_A^\uparrow \}.$$

Let $\tilde{\mu}(n) = \int x(n)\, d\mu(x)$. We have

$$\mathcal{M}(I^{\uparrow}) = \{\, \mu \in \mathcal{M}(2^{\mathbf{N}});\ \liminf \tilde{\mu}(n) = 0 \,\}.$$

Note that $\mathcal{M}(I^{\uparrow})$ is a $\mathcal{G}_\delta$ subset of $\mathcal{M}_1(2^{\mathbf{N}})$.

Following Kaufman's ideas [2], we assign to each sequence $\bar{A} = (A_n)_{n\in\mathbf{N}} \in \Sigma^{\mathbf{N}}$a mapping A from $2^{\mathbf{N}}$ to $\mathcal{P}(\mathbf{N})$, defined by $A(x) = \{\, n \in \mathbf{N};\ x \in A_n \,\}$, and a measure $\nu_{\bar{A}}$ defined by $\nu_{\bar{A}} = \int \mu_{A(x)}\, d\lambda(x)$. Let Θ be the mapping from $\Sigma^{\mathbf{N}}$ to $\mathcal{M}_1(2^{\mathbf{N}})$ defined by $\Theta(\bar{A}) = \nu_{\bar{A}}$. Note that Θ is continuous.

LEMMA 3.2. — *$\Theta(\mathcal{X}) \subset I^{\perp}$ and $\Theta(\mathcal{Y}) \subset \mathcal{M}(I^{\uparrow}) + \mathcal{M}(I^{\uparrow})$.*

Proof. — Using Lemma 3.1 we have

$$\lambda\big(\liminf_R A_n\big) = \lambda\big(\{\, x \in 2^{\mathbf{N}};\ R \setminus A(x) \textit{ finite} \,\}\big) = \nu_{\bar{A}}\big(K_R^{\uparrow}\big),$$

for all $\bar{A} = (A_n)_{n\in\mathbf{N}} \in \Sigma^{\mathbf{N}}$ and $R \in [\mathbf{N}]$. This remark allows us to finish easily the proof. □

We have an abstract version of Theorem 1.1.

THEOREM 3.3. — *There does not exist a Borel subset $\mathcal{B}$ of $\mathcal{M}_1(2^{\mathbf{N}})$ such that $\mathcal{M}(I^{\uparrow}) + \mathcal{M}(I^{\uparrow}) \subset \mathcal{B}$ and $\mathcal{B} \cap I^{\perp} = \varnothing$.*

Proof. — Such $\mathcal{B}$ insure $(\Phi\circ\Theta)^{-1}(\mathcal{B}) = WF^c$ and cannot be a Borel set, because $\Phi\circ\Theta$ is continuous. □

4. How to go from the abstract case to T.

Every element x of $\mathbf{T}$ can be expressed in the form $x = \sum_{n\in\mathbf{N}} x(n)2^{-n}$ with $x(n)$ either 0 or 1, and $x(n) = 0$ for large enough n if x is rational.

For each $A \in [\mathbf{N}]$, let

$$K_A = \{\, x \in \mathbf{T};\ \forall n \in A,\ x(n) = 0 \,\},$$
$$K_A^{\uparrow} = \{\, x \in \mathbf{T};\ \exists m \in \mathbf{N},\ \forall n \in A \cap [m, +\infty[,\ x(n) = 0 \,\}$$

and

$$\mu_A = \bigotimes_{n\in\mathbf{N}} \left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_{2^{-n}} \right)$$

be the canonical Bernoulli product measure concentrated on K_A. A set A is called colacunary if for each $n \in \mathbf{N}$, there exists $a \in \mathbf{N}$ such that $[a, a+n] \subset A$. Note that $K_A \in D$ if A is colacunary.

Lemma 3.1 still holds with these new notations. Our next goal is to extend this property to the L_0-sets.

LEMMA 4.1. — *Let $K \in L_0$ and $\alpha > 0$ and $(\varepsilon_n)_{n\in\mathbf{N}}$ witnessing this. Let $A \in [\mathbf{N}]$ and $c = \sup(-\lfloor \log_2 \alpha \rfloor, 0) + 2$. If $\limsup_n d\Big(\log_2 \dfrac{1}{\varepsilon_n}, A\Big) \geq c$, then $\mu_A(K) = 0$, where $d(x, A) = \inf\{\, |x - n|\,;\, n \in A\,\}$ $(x \in \mathbf{R})$.*

This property is derived from a result of Lyons [4] whose conclusion is much more precise, but which concerns only the case $K \in H$ and A lacunary. The proof of Lemma 4.1 uses the following simple result ([1] Lemma 2.9).

LEMMA 4.2. — *Let $K \in L_0$ and $\alpha > 0$ and $\varepsilon_n \in]0, \frac{1}{8}[$ witness this. Let $m = -\lfloor \log_2 \varepsilon_n \rfloor$ and $p = \sup(-\lfloor \log_2 \alpha \rfloor, 0)$. For each $(x_i)_{i\in[1,m-2]} \in \{0,1\}^{m-2}$, there exists $(x_i)_{i\in[m-1,m+p+1]} \in \{0,1\}^{p+3}$ such that for each $x \in \mathbf{T}$,*

$$(\forall i \in [1, m+p+1],\ x(i) = x_i) \Longrightarrow x \notin K.$$

Proof of Lemma 4.1. — Let $K \in L_0$ and let $\alpha > 0$ and $(\varepsilon_n)_{n\in\mathbf{N}}$ witness this. Let $p = \sup(-\lfloor \log_2 \alpha \rfloor, 0)$ and $m_n = -\lfloor \log_2 \varepsilon_n \rfloor$ for each $n \in \mathbf{N}$. Without loss of generality, we can suppose that the intervals $[m_n - 1, m_n + p + 1]$, $n \in \mathbf{N}$, are pairwise disjoint and disjoint from A. Let $n \in \mathbf{N}$. There exists, by Lemma 4.2, a mapping φ_n from $\{0,1\}^{[1,m_n-2]}$ to $\{0,1\}^{[m_n-1,m_n+p+1]}$ such that the set B_n of all $x \in \mathbf{T}$ such that

$$\forall s \in \{0,1\}^{[1,m_n-2]}\ \Big(s = \big(x(i)\big)_{i\in[1,m_n-2]} \Longrightarrow \varphi(s) = \big(x(i)\big)_{i\in[m_n-1,m_n+p+1]}\Big)$$

is disjoint from K. But $\mu_A(B_n) = 2^{-p-3}$ and the B_n's, $n \in \mathbf{N}$, are independent events in the probability space $(\mathbf{T}, \mu_A)$, so $\mu_A(K) \leq \mu_A\big(\bigcap {B_n}^c\big) = \prod \mu_A({B_n}^c) = 0$. □

5. Proof of Theorem 1.1.

Let $(a_k)_{k\in\mathbf{N}}$ and $(b_k)_{k\in\mathbf{N}}$ be two sequences of positive integers such that $\lim(b_k - a_k) = +\infty$ and $\lim(a_{k+1} - b_k) = +\infty$. Put $I_k = [a_k, b_k] \subset \mathbf{N}$.

For $A \subset \mathbf{N}$, put $\tilde{A} = \bigcup_{k \in A} I_k$. Note that $\tilde{A}$ is colacunary if and only if A is infinite.

To each sequence $\bar{A} = (A_n)_{n \in \mathbf{N}} \in \Sigma^{\mathbf{N}}$, we assign a mapping A from $2^{\mathbf{N}}$ to $\mathcal{P}(\mathbf{N})$ defined by $A(x) = \{ n \in \mathbf{N};\, x \in A_n \}$, and next a measure $\tilde{\nu}_{\bar{A}} = \int \mu_{\widetilde{A(x)}}\, d\lambda(x)$. Let $\tilde{\Theta}$ be the mapping from $\Sigma^{\mathbf{N}}$ to $\mathcal{M}_1(\mathbf{T})$ defined by $\tilde{\Theta}(\bar{A}) = \tilde{\nu}_{\bar{A}}$. Note that $\tilde{\Theta}$ is continuous.

LEMMA 5.1. — *$\tilde{\Theta}(\mathcal{X}) \subset {L_0}^{\perp}$ and $\tilde{\Theta}(\mathcal{Y}) \subset \mathcal{M}(D^{\uparrow}) + \mathcal{M}(D^{\uparrow})$.*

Proof. — Using Lemma 3.1, we have for each $\bar{A} = (A_n)_{n \in \mathbf{N}} \in \Sigma^{\mathbf{N}}$ and each $R \in [\mathbf{N}]$,

$$\begin{aligned}\lambda\big(\liminf_R A_n\big) &= \lambda\big(\{ x \in 2^{\mathbf{N}};\, R \setminus A(x) \textit{ finite} \}\\ &= \lambda\big(\{ x \in 2^{\mathbf{N}};\, \tilde{R} \setminus \widetilde{A(x)} \textit{ finite} \}\\ &= \tilde{\nu}_{\bar{A}}\big(K^{\uparrow}_{\tilde{R}}\big).\end{aligned}$$

But $K^{\uparrow}_{\tilde{R}} \in D^{\uparrow}$, because $\tilde{R}$ is colacunary, whence $\tilde{\Theta}(\mathcal{Y}) \subset \mathcal{M}(D^{\uparrow}) + \mathcal{M}(D^{\uparrow})$.

The previous remark does not allow us to prove that $\tilde{\Theta}(\mathcal{X}) \subset {L_0}^{\perp}$. Let $\bar{A} = (A_n)_{n \in \mathbf{N}} \in \Sigma^{\mathbf{N}}$ such that $\tilde{\Theta}(\bar{A}) \notin {L_0}^{\perp}$, i.e., there exists $K \in L_0$ such that $\tilde{\nu}_{\bar{A}}(K) > 0$. Let $\alpha > 0$ and $(\varepsilon_n)_{n \in \mathbf{N}}$ witness that $K \in L_0$. We have $\lambda(H) > 0$ with $H = \{ x \in 2^{\mathbf{N}};\, \mu_{\tilde{A}(x)}(K) > 0 \}$. Now $H \subset \{ x \in 2^{\mathbf{N}};\, \limsup d\Big(\log_2 \dfrac{1}{\varepsilon_n}, \tilde{A}(x)\Big) < c \}$ by Lemma 4.1. Thus $\limsup d\Big(\log_2 \dfrac{1}{\varepsilon_n}, \tilde{\mathbf{N}}\Big) < c$, because $\lambda(H) > 0$, so $d\Big(\log_2 \dfrac{1}{\varepsilon_n}, \tilde{\mathbf{N}}\Big) \leq c$ for large enough n. Moreover $a_{k+1} - b_k > 2c$ for large enough k, so there exists a unique k_n such that $d\Big(\log_2 \dfrac{1}{\varepsilon_n}, I_{k_n}\Big) < c$ for large enough n $(n \geq n_0)$. Let $R = \{ k_n;\, n \geq n_0 \}$. We have $H \subset \{ x \subset 2^{\mathbf{N}};\, R \setminus A(x) \textit{ finite} \} = \liminf_R A_n$, so there exists $a \in \mathbf{N}$ such that $\lambda\Big(\bigcap_{R \cap [a, +\infty[} A_n\Big) > 0$, whence $\bar{A} \notin \mathcal{X}$. □

Clearly, we can deduce Theorem 1.1 from this.

6. Theorems 1.5 and 1.6 in the abstract case.

We denote $\mathcal{C}^U = \{ X \cup Y;\, (X, Y) \in \mathcal{C}^2 \}$ for $\mathcal{C} \subset \mathcal{P}(E)$ where E is a metrizable, compact set. It is easy to verify that

$$\mathcal{M}(\mathcal{C}^{\uparrow}) + \mathcal{M}(\mathcal{C}^{\uparrow}) = \mathcal{M}\big((\mathcal{C}^{\uparrow})^{\mathcal{U}}\big)$$

and

$$\overline{\mathcal{M}(\mathcal{C}^{\uparrow}) + \mathcal{M}(\mathcal{C}^{\uparrow})} = \mathcal{M}\big((\mathcal{C}^{\mathcal{U}})^{\uparrow}\big).$$

We use again the notations of Part 3. Let $(A_n)_{n\in\mathbf{N}}$ be a sequence of infinite, pairwise disjoint subsets of $\mathbf{N}$. Consider the set

$$X_0 = \bigcup_{n\in\mathbf{N}} \bigcap_{n\geq m} \left(K_{A_{2n}} \cup K_{A_{2n+1}}\right)$$

which belongs to $(I^{\mathcal{U}})^{\uparrow}$. Note that $X_0 \notin (I^{\uparrow})^{\mathcal{U}}$. To all $x \in 2^{\mathbf{N}}$ and $m \in \mathbf{N}$, we attach $C_m(x) = \bigcup\limits_{n\geq m} A_{2n+x(n)}$; note that $K_{C_m(x)} \subset X_0$. Consider the weak*-integral

$$\mu_\infty = \sum_{m\in\mathbf{N}} 2^{-m} \int \mu_{C_m(x)}\, d\lambda(x).$$

Clearly $\mu_\infty \in \mathcal{M}_1(X_0)$ and $\mathcal{M}_1(X_0) \subset \overline{\mathcal{M}(I^{\uparrow}) + \mathcal{M}(I^{\uparrow})}$.

Lemma 6.1. — *μ_∞ is not a finite sum of measures in $\mathcal{M}(I^{\uparrow})$.*

We can immediately deduce an abstract version of Theorem 1.5.

Theorem 6.2. — *There exists a measure in $\overline{\mathcal{M}(I^{\uparrow}) + \mathcal{M}(I^{\uparrow})}$ which is not a finite sum of measures in $\mathcal{M}(I^{\uparrow})$.*

We can generalize the previous construction. Let $(F_m)_{m\in\mathbf{N}}$ be a sequence of finite subsets of $\mathbf{N}$. We define

$$\mu_{(F_m)} = \sum_{m\in\mathbf{N}} 2^{-m} \int \mu_{C(x,F_m)}\, d\lambda(x),$$

where $C(x, F_m) = \bigcup\limits_{n\notin F_m} A_{2n+x(n)}$. Note that $\mu_{(F_m)} \in \mathcal{M}_1(X_0)$. In particular, $\mu_\infty = \mu_{([1,m[)}$.

Let $k \in \mathbf{N}$ and $(F^k_m)_{m\in\mathbf{N}}$ be an enumeration of all subsets of $\mathbf{N}$ containing k elements and

$$\mu_k = \mu_{(F^k_m)}.$$

In particular $\mu_1 = \mu_{(\{m\})}$. Note that μ_k is concentrated on $\bigcup\limits_{n\in F} \left(K_{A_{2n}} \cup K_{A_{2n+1}}\right)$ for each subset F of $\mathbf{N}$ containing $k+1$ elements, whence $\mu_k \in \mathcal{M}(I^{\uparrow})^{(2k+2)}$.

Lemma 6.3. — *$\mu_k \notin \mathcal{M}(I^{\uparrow})^{(2k+1)}$ for each $k \geq 0$.*

We can immediately deduce an abstract version of Theorem 1.6.

THEOREM 6.4. — *For every $n \geq 3$, there exists a measure in $\overline{\mathcal{M}(I^\uparrow) + \mathcal{M}(I^\uparrow)}$ which is the sum of n measures in $\mathcal{M}(I^\uparrow)$ and is not the sum of $n-1$ measures in $\mathcal{M}(I^\uparrow)$.*

PROPOSITION 6.5. — *For every $n \geq 2$, there exists a measure in $\mathcal{M}(I^\uparrow)^{(n)}$ which is not in $\overline{\mathcal{M}(I^\uparrow)^{(n-1)}}$.*

Proof. — Consider $\nu_n = \frac{1}{n}\sum_{k=1}^{n} \mu_{A_k}$. If $B \in [\mathbf{N}]$, there exists at most one k such that $B \setminus A_k$ finite, so, by Lemma 3.1, $\nu_n(K_B^\uparrow) \leq \frac{1}{n}$. If X is a union of $n-1$ $I^\uparrow$-sets, then $\nu_n(X) \leq \frac{n-1}{n}$, whence $\nu_n \notin \overline{\mathcal{M}(I^\uparrow)^{(n-1)}}$. □

We deduce from Theorem 6.4 and Proposition 6.5 the following fact.

COROLLARY 6.6. — *The sets $\mathcal{M}(I^\uparrow)^{(n)}$ and $\overline{\mathcal{M}(I^\uparrow)^{(n)}}$, $n \geq 2$, are all distinct.*

We will now prove Lemmas 6.1 and 6.3.

LEMMA 6.7. — *Let $\mu = \mu_{(F_m)}$ and $X \in I^\uparrow$. Then $\mu_{\lceil X}$ is concentrated on K_{A_p} for some p.*

Proof. — Let $X \in I^\uparrow$ such that $\mu(X) > 0$. There exists $B \in [\mathbf{N}]$ such that $X \subset K_B^\uparrow$. If $B \setminus \bigcup_{p\in\mathbf{N}} A_p$ is infinite, then $B \setminus C(x, F_m)$ is infinite for all $m \in \mathbf{N}$ and $x \in 2^{\mathbf{N}}$. Using Lemma 3.1, we have $\mu_{C(x,F_m)}(K_B^\uparrow) = 0$, so $\mu(K_B^\uparrow) = 0$ which contradicts our hypothesis, whence $B \setminus \bigcup_{p\in\mathbf{N}} A_p$ is finite.

Consider $C = \{p \in \mathbf{N};\, B \cap A_p \neq \varnothing\}$. If C is infinite, $C = \{2n_k + \zeta_k;\, k \in \mathbf{N}, \zeta_k = 0, 1\}$. If $m \in \mathbf{N}$ and $x \in 2^{\mathbf{N}}$ are such that $\mu_{C(x,F_m)}(K_B^\uparrow) > 0$, then $B \setminus C(x, F_m)$ is finite by Lemma 3.1, so $x(n_k) = \zeta_k$ for large enough k, because F_m is finite. But $\lambda(\{x \in 2^{\mathbf{N}};\, x(n_k) = \zeta_k \textit{ for large enough } k\}) = 0$, so $\mu(K_B^\uparrow) = 0$. This contradiction prove that C is finite and $B \cap A_p$ is infinite for some p.

If $m \in \mathbf{N}$ and $x \in 2^{\mathbf{N}}$ are such that $\mu_{C(x,F_m)}(K_B^\uparrow) > 0$, then $A_p \subset C(x, F_m)$, so $\mu_{C(x,F_m)}(K_{A_p}) = 1$, whence $\mu_{\lceil K_B^\uparrow}$ is concentrated on K_{A_p}. □

Proof of Lemma 6.1. — Let X be a finite union of $I^\uparrow$-sets. Using Lemma 6.7, we can suppose that $X = \bigcup_{n\in F} K_{A_p}$ for some finite subset F

of $\mathbf{N}$. Let m_0 with $p < 2m_0$ for all $p \in F$. For all $m \geq m_0$ and $x \in 2^{\mathbf{N}}$, $\mu_{C_m(x)}(X) = 0$ by Lemma 3.1. So $\mu_\infty(X^c) > 0$. □

Proof of Lemma 6.3. — Using Lemma 6.7, we have just to prove that μ_k cannot be concentrated on $X = \bigcup_{n \in F} K_{A_p}$ for every F with cardinality $\leq 2k+1$. Let F be a set having this property. Thus $F = \{2n;\, n \in G\} \cup \{2n+1;\, n \in G\} \cup \{2n+\zeta_n;\, n \in H\}$ with ζ_n either 0 or 1. Now G has cardinality $\leq k$, so $G \subset F^k_{m_0}$ for some m_0. Using Lemma 3.1, we have $\mu_{C(x,F^k_{m_0})}(X^c) > 0$ for every $x \in 2^{\mathbf{N}}$ such that $x(n) = 1 - \zeta_n$ for each $n \in H$, whence $\mu_k(X^c) > 0$. □

7. Proof of theorems 1.5 and 1.6.

To prove Theorems 1.5 and 1.6, we follow the ideas and techniques of Part 6. We introduce the same notations and the same lemmas, expect that, in this case, $(A_n)_{n\in\mathbf{N}}$ is a sequence of colacunary subsets of $\mathbf{N}$ such taht for k going to $+\infty$, $d(A_n \cap [k,+\infty[, A_m \cap [k,+\infty[) \to +\infty$ uniformly for all $n \neq m$. Moreover, K_A and μ_A, $A \in [\mathbf{N}]$, are the same as in Part 4. Finally, Lemma 6.7 is replaced by the following result.

LEMMA 7.1. — *Let $\mu = \mu_{(F_m)}$ and $X \in L_0^{\uparrow}$. Then $\mu_{\restriction X}$ is concentrated on K_{A_p} for some p.*

Proof. — We start by proving the result for $K \in L_0$. Let $\alpha > 0$ and $(\varepsilon_k)_{k\in\mathbf{N}}$ witness that $K \in L_0$. Let $p = \sup(-\lfloor \log_2 \varepsilon_k \rfloor, 0)$, $m_k = -\lfloor \log_2 \varepsilon_k \rfloor$ and $J_k = [m_k - 1, m_k + p + 1]$, $k \in \mathbf{N}$. If $\mu(K) > 0$, then $\mu_{C(x,F_m)}(K) > 0$ for some $x \in 2^{\mathbf{N}}$ and $m \in \mathbf{N}$. But $C(x,F_m) \subset \bigcup A_p$, so, using Lemma 4.1, we deduce that J_k meets at least one A_p for large enough k. Now $|J_k|$ is constant and as $k \to +\infty$, $d(A_n \cap [k,+\infty[, A_m \cap [k,+\infty[) \to +\infty$ uniformly for all $n \neq m$, so J_k meets exactly one A_{p_k} for large enough k. If $(p_k)_{k\in\mathbf{N}}$ is unbounded, then $(p_k)_{k\in D}$ is injective for some $D \in [\mathbf{N}]$. Put $p_k = 2n_k + \zeta_k$ with $\zeta_k = 0$ or 1. By Lemma 4.1, if $\mu_{C(x,F_m)}(K) > 0$ for some $x \in 2^{\mathbf{N}}$, then $x(n_k) = \zeta_k$ for large enough k. But $\lambda(\{x \in 2^{\mathbf{N}};\ x(n_k) = \zeta_k \textit{ for large enough } k\}) = 0$, so $\mu(K) = 0$. If $(p_k)_{k\in\mathbf{N}}$ is bounded, there exists p such that $p = p_k$ for infinitely many k. If $m \in \mathbf{N}$ and $x \in 2^{\mathbf{N}}$ are such that $\mu_{C(x,F_m)}(K) > 0$, then $A_p \subset C(x,F_m)$, so $\mu_{C(x,F_m)}(K_{A_p}) = 1$, whence $\mu_{\restriction K}$ is concentrated on K_{A_p}.

Let $X \in L_0{}^{\uparrow}$. There exists a sequence $(K_j)_{j\in\mathbf{N}}$ of L_0-sets such that $X \subset \text{liminf} K_j$. Now, for each j, there exists p_j that $\mu_{\lceil K_j}$ is concentrated on $K_{A_{p_j}}$, so $\mu_{\lceil X}$ is concentrated on $\text{liminf} K_{A_{p_j}}$. As before, $\mu(\text{liminf} K_{A_{p_j}}) = 0$ if $(p_j)_{j\in\mathbf{N}}$ is unbounded. So $\mu_{\lceil X}$ is concentrated on K_{A_p} for some p. □

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