ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 43, nº 1 (1993), p. 85-109

<http://www.numdam.org/item?id=AIF_1993__43_1_85_0>

Ann. Inst. Fourier, Grenoble
43, 1 (1993), 85-109

ON THE IRRATIONALITY MEASURE OF $\zeta(2)$

by G. RHIN and C. VIOLA

1. Introduction.

In 1979 Beukers [4] introduced the integral

$$\int_0^1 \int_0^1 \psi^n(x,y) \, \frac{dxdy}{1-xy} \, ,$$

where

$$\psi(x,y) = \frac{xy(1-x)(1-y)}{1-xy} \, ,$$

to give a new proof of the irrationality of $\zeta(2) = \sum_1^\infty n^{-2} = \pi^2/6$. He also used a triple integral for $\zeta(3)$. His method yielded the same sequences of rational approximations to $\zeta(2)$ and $\zeta(3)$ previously obtained by Apéry, and therefore the same irrationality measures of these numbers given in [3].

We recall that μ is said to be an irrationality measure of the irrational number α if for any $\varepsilon > 0$ there exists a constant $q_0(\varepsilon) > 0$ such that

$$\mid \alpha - \frac{p}{q} \mid > q^{-\mu-\varepsilon}$$

Key words : Irrationality measures – Birational transformations – Semi-infinite linear programming.
A.M.S. Classification : 11J82.

for all integers p and q with $q > q_0(\varepsilon)$. The minimum of such μ is denoted by $\mu(\alpha)$. We also say that μ is an effective irrationality measure of α if the above constant $q_0(\varepsilon)$ is effectively computable.

In Section 5 we define a birational transformation τ of $\mathbb{C}^2$ related in a natural way to the above function $\psi(x, y)$, and in Section 6 we employ suitable rational functions automorphic under the transformation group generated by τ. This appears to be a powerful tool to investigate the geometry underlying the behaviour of double integrals of Beukers' type. In fact the transformation τ will play a basic role to obtain good asymptotic estimates for such integrals both on the unit square and on suitable tori in $\mathbb{C}^2$, as is shown in Sections 6 and 9. Moreover, the semi-infinite linear programming method described in Section 7 shows that the best numerical results are obtained when the irreducible algebraic curves of low degree defined by the polynomials occurring in our integrals and equivalent under the action of the transformation group generated by τ all have the same weight (see the remark at the end of Section 7).

We shall prove the following

THEOREM. — 7. 398 537 *is an effective irrationality measure of* $\zeta(2)$.

Thus the above value is an effective irrationality measure of π^2, whence 14. 797 074 is an effective irrationality measure of π.

In 1953, using Padé approximants, Mahler [10] proved $\mu(\pi) \leq 42$. In 1973 Mignotte [11] improved Mahler's result to $\mu(\pi) \leq 20$, and also proved $\mu(\pi^2) \leq 17.8$. In 1978, besides proving the irrationality of $\zeta(3)$ and giving $\mu(\zeta(3)) \leq 13.41782...$, Apéry [3] found $\mu(\zeta(2)) \leq 11.85078...$, whence $\mu(\pi) \leq 23.70156...$. In 1987 Dvornicich and Viola [8], using linear combinations with integer coefficients of Beukers' integrals, proved $\mu(\zeta(2)) \leq 10.0298$ and $\mu(\zeta(3)) \leq 12.7436$. In 1990 Hata [9] proved $\mu(\zeta(2)) \leq 7.5252$ and $\mu(\zeta(3)) \leq 8.83028...$ by showing that the rational approximations to $\zeta(2)$ and $\zeta(3)$ given by suitable integrals close to Beukers' have common prime factors. A similar principle had been already used by Rukhadze [15] in 1987 to give $\mu(\log 2) \leq 3.893$, thus improving earlier results by Alladi and Robinson [1] : $\mu(\log 2) \leq 4.6221...$, by D. and G. Chudnovsky [5] : $\mu(\log 2) \leq 4.1344...$ and by Rhin [14] : $\mu(\log 2) \leq 4.0765$. Finally, D. and G. Chudnovsky repeatedly announced without proofs irrationality measures of π^2 less than 8; in a recent paper they announce $\mu(\pi^2) \leq 7.51...$ ([6], p. 208).

2. Plan of the proof.

We consider integrals

$$I_n = \int_0^1 \int_0^1 \frac{H(x,y)}{(1-xy)^{n+1}}\, dxdy$$

with suitable polynomials $H(x,y) \in \mathbb{Z}[x,y]$. Choosing $H(x,y) = (xy(1-x)(1-y))^n$ one obtains the integral introduced by Beukers. In Section 3 we find conditions on $H(x,y)$ ensuring that $d_n^2 I_n$ converges and has the form $a_n - b_n\zeta(2)$, where a_n, b_n are integers and $d_n = l.c.m.(1,2,...,n)$.

To get an irrationality measure of $\zeta(2)$ we need estimates of $|I_n|$ and $|b_n|$. We show in Section 4 how to relate such estimates to the computation of the stationary points of the rational function $H(x,y)/(1-xy)^n$.

We use polynomials H of the following type :

$$H(x,y) = \prod_{i=1}^{r} P_i(x,y)^{k_i(n)} ,$$

where $k_i(n) = [\alpha_i n]$ and $P_1, ..., P_r$ are polynomials with integer coefficients. To obtain a good irrationality measure of $\zeta(2)$ we choose suitable polynomials P_i associated with the transformation τ, as is shown in Section 6. Then the α_i giving the desired result are found by a method of semi-infinite linear programming. Once we have obtained the α_i, in order to compute all the stationary points of the above rational function we make use of the computer algebra system IBM Scratchpad II, and the system Pari by Batut, Bernardi, Cohen and Olivier.

We are indebted to P. Gianni of the Pisa University for kindly assisting us with the use of Scratchpad, and to the referee for pointing out an oversight in an earlier version of this paper.

3. Some arithmetical lemmas.

LEMMA 1 (Beukers [4]). — *Let k and l be non-negative integers, and $n \geq \max(k,l)$, $n > 0$. If*

$$J = \int_0^1 \int_0^1 \frac{(1-x)^k(1-y)^l}{1-xy}\, dxdy ,$$

then $d_n^2 J \in \mathbb{Z} + \mathbb{Z}\zeta(2)$.

Remark. — If

$$I_n = \int_0^1 \int_0^1 \frac{P_n(x)(1-y)^n}{1-xy}\, dxdy \ ,$$

where $P_n(x)$ is the Legendre polynomial

$$\frac{1}{n!}\,\frac{d^n}{dx^n}(x^n(1-x)^n) \ ,$$

one gets by repeated partial integration

$$I_n = (-1)^n \int_0^1 \int_0^1 \frac{(xy(1-x)(1-y))^n}{(1-xy)^{n+1}}\, dxdy \ ,$$

and in order to prove the irrationality of $\zeta(2)$ it suffices to show that $d_n^2 I_n \to 0$ as $n \to \infty$. We generalize Lemma 1, so that we can replace $(xy(1-x)(1-y))^n$ with a more general polynomial $H(x,y)$.

LEMMA 2. — *Let k and ν be positive integers and $n \geq \max(k,\nu)$. If $\nu \leq k$ let*

$$J_1 = \int_0^1 \int_0^1 \frac{(1-x)^k}{(1-xy)^{\nu+1}}\, dxdy \ ;$$

if $k < \nu$ let

$$J_2 = \int_0^1 \int_0^1 \frac{(1-x)^k(1-y)^{\nu-k}}{(1-xy)^{\nu+1}}\, dxdy \ .$$

Then $d_n^2 J_1$ and $d_n^2 J_2$ are integers.

Proof. — Integrating with respect to y we get

$$\begin{aligned} J_1 &= \int_0^1 (1-x)^k \frac{1}{\nu x}\Big(\frac{1}{(1-x)^\nu} - 1\Big)\, dx \\ &= \int_0^1 \frac{1}{\nu x}\big((1-x)^{k-\nu} - (1-x)^k\big)\, dx \ . \end{aligned}$$

Since $\nu \leq n$ and the degree of the polynomial

$$\frac{1}{x}\big((1-x)^{k-\nu} - (1-x)^k\big)$$

is $k-1 \leq n-1$, we have $d_n^2 J_1 \in \mathbb{Z}$. For the second integral, let

$$\gamma(y) = y^{k+1} \int_0^1 \frac{(1-x)^k}{(1-xy)^{\nu+1}}\, dx = y \int_0^1 \frac{(y-xy)^k}{(1-xy)^{\nu+1}}\, dx \ .$$

Using $y - xy = (1 - xy) - (1 - y)$ and expanding, we obtain

$$\gamma(y) = y \sum_{h=0}^{k} \binom{k}{h} (-1)^{k-h} (1-y)^{k-h} \int_0^1 \frac{dx}{(1-xy)^{\nu-h+1}} \ .$$

Since $k < \nu$ we have $\nu - h \geq \nu - k > 0$, whence

$$y \int_0^1 \frac{dx}{(1-xy)^{\nu-h+1}} = \frac{1}{\nu - h}\Big(\frac{1}{(1-y)^{\nu-h}} - 1\Big)$$

and

$$\begin{aligned}\gamma(y) &= \sum_{h=0}^{k} \binom{k}{h} \frac{(-1)^{k-h}}{\nu - h} \Big((1-y)^{k-\nu} - (1-y)^{k-h}\Big) \\ &= (1-y)^{k-\nu} g(y) \ ,\end{aligned}$$

where g is a polynomial satisfying $g(0) = 0$. Hence

$$g(y) = \int_0^y g'(t) dt$$

and

$$\begin{aligned}g'(t) &= \sum_{h=0}^{k} \binom{k}{h} (-1)^{k-h} (1-t)^{\nu-h-1} \\ &= (1-t)^{\nu-k-1} \sum_{h=0}^{k} \binom{k}{h} (-1)^{k-h} (1-t)^{k-h} \\ &= (1-t)^{\nu-k-1}\, t^k \ .\end{aligned}$$

Therefore

$$\gamma(y) = (1-y)^{k-\nu} \int_0^y (1-t)^{\nu-k-1}\, t^k \, dt \ .$$

We remark that the polynomial

$$g(y) = \int_0^y (1-t)^{\nu-k-1}\, t^k \, dt$$

is divisible by y^{k+1} and satisfies $d_n g(y)/y^{k+1} \in \mathbb{Z}[y]$ since $\nu \leq n$. Also $(1-y)^{\nu-k}\gamma(y) = g(y)$. It follows that

$$\begin{aligned} J_2 &= \int_0^1 \int_0^1 \frac{(1-x)^k(1-y)^{\nu-k}}{(1-xy)^{\nu+1}}\, dxdy \\ &= \int_0^1 (1-y)^{\nu-k}\, dy \int_0^1 \frac{(1-x)^k}{(1-xy)^{\nu+1}}\, dx \\ &= \int_0^1 (1-y)^{\nu-k} y^{-k-1}\gamma(y)\, dy \\ &= \int_0^1 \frac{g(y)}{y^{k+1}}\, dy\ , \end{aligned}$$

whence $d_n^2 J_2 \in \mathbb{Z}$.

LEMMA 3. — *Let k, l, ν be non-negative integers such that $k+l \geq \nu$, and $n \geq \max(k,l,\nu)$, $n > 0$. Let*

$$K = \int_0^1 \int_0^1 \frac{(1-x)^k(1-y)^l}{(1-xy)^{\nu+1}}\, dxdy\ .$$

Then $d_n^2 K \in \mathbb{Z} + \mathbb{Z}\zeta(2)$.

Proof. — Let $m = \min\ (k,\ l,\ \nu,\ k+l-\nu)$. If $m = 0$, the result follows from Lemmas 1 and 2 (possibly interchanging x and y in J_1). If $m > 0$, we show that K is a linear combination with integer coefficients of integrals J, J_1 and J_2. Let $u = 1-x$, $v = 1-y$ and $q = 1-xy$, whence $u+v-q = uv$. Then

$$K = -\int_0^1 \int_0^1 \frac{u^{k-1}v^{l-1}}{q^{(\nu-1)+1}}\, dxdy + \int_0^1 \int_0^1 \frac{u^k v^{l-1}}{q^{\nu+1}}\, dxdy + \int_0^1 \int_0^1 \frac{u^{k-1}v^l}{q^{\nu+1}}\, dxdy$$

$$= -K_1 + K_2 + K_3\ .$$

If we denote

$$K_i = \int_0^1 \int_0^1 \frac{u^{k_i}v^{l_i}}{q^{\nu_i+1}}\, dxdy \qquad\qquad (1 \leq i \leq 3)\ ,$$

it is plain that for each i at least two of the integers in $(k_i,\ l_i,\ \nu_i,\ k_i+l_i-\nu_i)$ are smaller than the corresponding integers in $(k,\ l,\ \nu,\ k+l-\nu)$, and none is larger. Applying this argument repeatedly, after finitely many steps we obtain the desired result.

PROPOSITION 1. — *Let $H \in \mathbb{Z}[s,p]$ satisfy the following conditions :*

(i) $\deg_s H \leq n$

(ii) $\deg\ H \leq 2n$

(iii) *there exists an integer $m \geq n/2$ such that $(p-s+1)^m$ divides H.*

Then

$$d_n^2 I_n \;=\; d_n^2 \int_0^1 \int_0^1 \frac{H(x+y,\ xy)}{(1-xy)^{n+1}}\ dxdy \;=\; a_n - b_n\zeta(2)\ ,$$

with a_n and b_n integers.

Remark. — In the above integral we have $s = x+y$ and $p = xy$, whence $p-s+1 = (1-x)(1-y)$. Also, here and in the sequel we use the notation $\deg H$ to mean the total degree of H with respect to s and p.

Proof. — We have

$$H(x+y,\ xy) \;=\; (uv)^m \sum_{k=0}^{n-m} \sum_{j=0}^{2n-m-k} a_{kj}\ (u+v)^k\ q^j\ ,$$

where the a_{kj} are integers, $u = 1-x$, $v = 1-y$ and $q = 1-xy$. Then

$$H(x+y,\ xy) \;=\; \sum_{k=0}^{n-m} \sum_{j=0}^{2n-m-k} \sum_{l=0}^{k} a_{kj} \binom{k}{l} u^{m+l} v^{m+k-l} q^j\ .$$

Hence it suffices to prove that the integral

$$I = \int_0^1 \int_0^1 u^{m+l} v^{m+k-l} q^{j-n-1} dxdy$$

satisfies $d_n^2 I \in \mathbb{Z} + \mathbb{Z}\zeta(2)$. If $j \geq n+1$ then $u^{m+l}v^{m+k-l}q^{j-n-1}$ is a polynomial in x and y, with degree in x equal to $m+l+j-n-1 \leq n-1$, and degree in y equal to $m+k-l+j-n-1 \leq n-1$. Hence $d_n^2 I \in \mathbb{Z}$. If $j \leq n$ then I satisfies the assumptions in Lemma 3.

4. Computing an irrationality measure of $\zeta(2)$.

LEMMA 4. — *Let $\alpha \in \mathbb{R}$ and let (p_n), (q_n) be sequences of integers satisfying*

$$\limsup_{n\to\infty}\ \frac{1}{n}\ \log|\,q_n\,|\ \leq\ \rho$$

and

$$\lim_{n\to\infty} \frac{1}{n} \log | p_n - q_n\alpha | = -\sigma$$

for some positive numbers ρ and σ. Then

$$\mu(\alpha) \le \frac{\rho}{\sigma} + 1 .$$

This lemma is a special case of [7], Lemma 3.5.

By Proposition 1, we have to estimate $|I_n|$ and $|b_n|$. The following lemma gives b_n explicitly.

LEMMA 5. — *Under the assumptions of Proposition 1, we have*

$$b_n = \frac{d_n^2}{(2\pi i)^2} \int_{C_1} \int_{C_2} \frac{H(x+y,\ xy)}{(1-xy)^{n+1}}\, dxdy ,$$

where $y \in \mathbb{C}$ describes a circumference C_1 with center 0 and radius $\rho_1 > 0$, and $x \in \mathbb{C}$ describes a circumference C_2 with center $1/y$ and radius $\rho_2 > 0$.

Proof. — As in the proof of Proposition 1

$$a_n - b_n\zeta(2) = d_n^2 \int_0^1 \int_0^1 \frac{H(x+y,\ xy)}{(1-xy)^{n+1}}\, dxdy$$

can be expressed as the sum of an integer and a linear combination with integer coefficients of quantities of the type $d_n^2 K^{(j)}$, where $K^{(j)}$ is an integral as in Lemma 3. By the proof of Lemma 3, $K^{(j)}$ is a linear combination with integer coefficients of integrals J, J_1 and J_2. Hence, by Lemmas 1 and 2,

$$a_n - b_n\zeta(2) = (\text{integer}) + \sum_{j=1}^{R} A_j d_n^2 J^{(j)} ,$$

with $A_j \in \mathbb{Z}$ and

$$\begin{aligned} J^{(j)} &= \int_0^1 \int_0^1 \frac{(1-x)^{k_j}(1-y)^{l_j}}{1-xy}\, dxdy \\ &= \sum_{h=0}^{k_j} \sum_{m=0}^{l_j} (-1)^{h+m} \binom{k_j}{h} \binom{l_j}{m} \int_0^1 \int_0^1 \frac{x^h y^m}{1-xy}\, dxdy . \end{aligned}$$

By [4],

$$d_n^2 \int_0^1 \int_0^1 \frac{x^h y^m}{1-xy}\, dxdy$$

is an integer if $h \neq m$, while

$$d_n^2 \int_0^1 \int_0^1 \frac{x^h y^h}{1-xy}\, dxdy \;=\; (\text{integer}) \;+\; d_n^2\, \zeta(2)\ .$$

Therefore

$$a_n - b_n\zeta(2) \;=\; (\text{integer}) \;+\; d_n^2\, \zeta(2) \sum_{j=1}^{R} A_j \sum_{h=0}^{\lambda_j} \binom{k_j}{h} \binom{l_j}{h}\ ,$$

where $\lambda_j \;=\; \min(k_j, l_j)$. Hence

$$b_n \;=\; -\, d_n^2 \sum_{j=1}^{R} A_j \sum_{h=0}^{\lambda_j} \binom{k_j}{h} \binom{l_j}{h}\ .$$

Let now

$$\tilde{J}_1 = \frac{1}{(2\pi i)^2} \int_{C_1} \int_{C_2} \frac{(1-x)^k}{(1-xy)^{\nu+1}}\, dxdy$$

be the complex integral corresponding to the integral J_1 in Lemma 2, and similarly for $\tilde{J}_2$. By Cauchy's integral formula we have

$$\begin{aligned}
\tilde{J}_1 &= \frac{1}{2\pi i} \int_{C_1} \frac{dy}{(-y)^{\nu+1}}\, \frac{1}{2\pi i} \int_{C_2} \frac{(1-x)^k}{(x-1/y)^{\nu+1}}\, dx \\
&= \frac{1}{2\pi i} \int_{C_1} \frac{(-1)^{\nu+1}}{\nu!} \left[\frac{d^\nu}{dx^\nu} (1-x)^k \right]_{x=1/y} \frac{dy}{y^{\nu+1}} \\
&= -\binom{k}{\nu} \frac{1}{2\pi i} \int_{C_1} \frac{(y-1)^{k-\nu}}{y^{k+1}}\, dy \;=\; 0\ ,
\end{aligned}$$

since $\nu > 0$. Similarly

$$\tilde{J}_2 \;=\; \frac{1}{2\pi i} \int_{C_1} \frac{(-1)^{\nu+1}}{\nu!} \left[\frac{d^\nu}{dx^\nu} (1-x)^k \right]_{x=1/y} \frac{(1-y)^{\nu-k}}{y^{\nu+1}}\, dy \;=\; 0\ ,$$

since $k < \nu$ here. Therefore, the same linear decomposition used for

$$d_n^2 \int_0^1 \int_0^1 \frac{H(x+y,\ xy)}{(1-xy)^{n+1}}\, dxdy$$

yields now

$$\frac{d_n^2}{(2\pi i)^2}\int_{C_1}\int_{C_2}\frac{H(x+y,\ xy)}{(1-xy)^{n+1}}\,dxdy \;=\; \sum_{j=1}^{R} A_j d_n^2 \tilde{J}^{(j)}\ ,$$

with the same $A_j \in \mathbb{Z}$ as above, and with

$$\begin{aligned}\tilde{J}^{(j)} &= \frac{1}{(2\pi i)^2}\int_{C_1}\int_{C_2}\frac{(1-x)^{k_j}(1-y)^{l_j}}{1-xy}\,dxdy\\ &= \sum_{h=0}^{k_j}\sum_{m=0}^{l_j}(-1)^{h+m}\binom{k_j}{h}\binom{l_j}{m}\frac{1}{(2\pi i)^2}\int_{C_1}\int_{C_2}\frac{x^h y^m}{1-xy}\,dxdy\ .\end{aligned}$$

Since

$$\begin{aligned}\frac{1}{(2\pi i)^2}\int_{C_1}\int_{C_2}\frac{x^h y^m}{1-xy}\,dxdy &= \frac{1}{2\pi i}\int_{C_1}\frac{y^m}{-y}\,dy\ \frac{1}{2\pi i}\int_{C_2}\frac{x^h}{x-1/y}\,dx\\ &= -\frac{1}{2\pi i}\int_{C_1} y^{m-h-1}\,dy\\ &= \begin{cases}-1 & \text{if } h=m\\ 0 & \text{if } h\neq m,\end{cases}\end{aligned}$$

we obtain

$$\tilde{J}^{(j)} \;=\; -\sum_{h=0}^{\lambda_j}\binom{k_j}{h}\binom{l_j}{h}\ ,$$

whence

$$\frac{d_n^2}{(2\pi i)^2}\int_{C_1}\int_{C_2}\frac{H(x+y,\ xy)}{(1-xy)^{n+1}}\,dxdy \;=\; -\,d_n^2\sum_{j=1}^{R}A_j\sum_{h=0}^{\lambda_j}\binom{k_j}{h}\binom{l_j}{h} \;=\; b_n\ .$$

For every $\rho_1 > 0,\ \rho_2 > 0$ we get from Lemma 5

$$|\,b_n\,| \;\leq\; d_n^2 \max_{|y|=\rho_1}\ \max_{|x-1/y|=\rho_2}\ |\,g_n(x,y)\,|\ ,$$

where

$$g_n(x,y) = \frac{H(x+y,\ xy)}{(1-xy)^n}\ .$$

If $1-xy$ does not divide $H(x+y,\ xy)$ and if $\deg_p H > n$, which we shall henceforth assume, then it is easily seen that the function

$$f(\rho_1, \rho_2) = \max_{|y|=\rho_1} \max_{|x-1/y|=\rho_2} | g_n(x,y) |$$
$$= \frac{1}{(\rho_1\rho_2)^n} \max_{|y|=\rho_1} \max_{|x-1/y|=\rho_2} | H(x+y,\ xy) |$$

is large outside a compact subset of $\{\rho_1 > 0,\ \rho_2 > 0\}$, and has therefore a minimum $f(\rho_1^{(0)}, \rho_2^{(0)})$. For any $\rho_1 > 0,\ \rho_2 > 0$ let $(X(\rho_1,\rho_2), Y(\rho_1,\rho_2))$ be a maximal point for $|g_n|$ on the torus

$$T(\rho_1,\rho_2) = \{(x,y) \in \mathbb{C}^2 \ :\ |y| = \rho_1,\ |x - 1/y| = \rho_2\} ,$$

i.e. such that $|g_n(X,Y)| = f(\rho_1,\rho_2)$, and let $X_0 = X(\rho_1^{(0)}, \rho_2^{(0)})$, $Y_0 = Y(\rho_1^{(0)}, \rho_2^{(0)})$, whence $|b_n| \le d_n^2 \mid g_n(X_0,Y_0) \mid$. If $X(\rho_1,\rho_2)$, $Y(\rho_1,\rho_2)$ were continuous functions in a neighbourhood of $(\rho_1^{(0)}, \rho_2^{(0)})$, which is not true in general, the partial derivatives $\partial g_n/\partial x$ and $\partial g_n/\partial y$ would vanish at (X_0,Y_0), as is clear by a simple local argument.

Let $E_n \subset \mathbb{C}^2$ be the set of points (x,y) satisfying

$$\frac{\partial g_n}{\partial x} = \frac{\partial g_n}{\partial y} = 0 , \quad g_n(x,y) \ne 0 ,$$

and let $(x_0,y_0) \in E_n$ be such that

$$| g_n(x_0,y_0) | = \max_{(x,y)\in E_n} | g_n(x,y) | .$$

For each choice of $H(x+y,\ xy)$ we shall find a torus $T(r_1,r_2)$ such that $f(r_1,r_2)$ is very close to $|g_n(x_0,y_0)|$. Moreover, if $H(x+y,\ xy) \ge 0$ in the unit square

$$\mathcal{U} = \{0 < x < 1,\ \ 0 < y < 1\} ,$$

then to estimate

$$I_n = \int_0^1 \int_0^1 \frac{H(x+y,\ xy)}{(1-xy)^{n+1}}\, dxdy$$

from above and from below we seek the maximum of $g_n(x,y)$ in $\mathcal{U}$, and hence the points of E_n belonging to $\mathcal{U}$.

To compute the above quantities explicitly we choose H as follows. Let $P_1, ..., P_r$ $(r \ge 2)$ be polynomials in $s = x+y$ and $p = xy$ with integer coefficients and not divisible by $1-p$. Let two of the P_i be p and $p-s+1$. We define

$$H=\prod_{i=1}^{r} P_i^{2[\frac{\alpha_i}{2}n]},$$

where the exponents $\alpha_1,\dots,\alpha_r$ are positive, the α_i corresponding to $p-s+1$ is $>1/2$, and

$$\begin{cases} \sum_{i=1}^{r} \alpha_i \deg_s P_i = 1 \\ \sum_{i=1}^{r} \alpha_i \deg_p P_i > 1 \\ \sum_{i=1}^{r} \alpha_i \deg P_i = 2, \end{cases}$$

so that the assumptions of Proposition 1, as well as the later assumptions on H, are fulfilled. In particular we have $H \geq 0$ for real s and p, and $I_n > 0$.

Defining

$$g(x,y) = \frac{\prod_{i=1}^{r} P_i^{\alpha_i}}{1-xy}$$

and

$$a = \max_{(x,y)\in\mathcal{U}} \log |\, g(x,y)\,|$$

(note that $|g(x,y)|=0$ on the border of $\mathcal{U}$ since p and $p-s+1$ occur among the P_i, and $|g(x,y)| \to 0$ as $(x,y)\to(1,1)$ in $\mathcal{U}$ since the exponent of $p-s+1$ is $>1/2$), we have for any $\varepsilon>0$ and $n>n_0(\varepsilon)$

$$\exp((a+2-\varepsilon)n) \leq d_n^2\, I_n \leq \exp((a+2+\varepsilon)n),$$

since $d_n = \exp\,(n+o(n))$ by the prime number theorem. Similarly, for any $\varepsilon>0$ and $n>n_1(\varepsilon)$ we have

$$|\, b_n\,| \leq \exp((b+2+\varepsilon)n),$$

where

$$b = \max_{(x,y)\in T(r_1,r_2)} \log |\, g(x,y)\,| .$$

All the above computations can be made effective. By Lemma 4, we have an effective irrationality measure

$$\mu(\zeta(2)) \leq \frac{a-b}{a+2},$$

provided $a < -2$.

Let now $E \subset \mathbb{C}^2$ be the set of points (x, y) satisfying

$$\frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} = 0, \quad g(x,y) \neq 0,$$

and let

$$\beta = \max_{(x,y)\in E} \log |g(x,y)|.$$

Since, by Lemma 5, b_n can be expressed by an integral over a surface homotopic to $T(r_1, r_2)$, we conjecture that $T(r_1, r_2)$ can be slightly deformed to a surface $\mathcal{S}$ of this type and such that $\max_{(x,y)\in\mathcal{S}} \log |g(x,y)| = \beta$. However, this would improve by less than 10^{-3} the irrationality measure of $\zeta(2)$ given in our theorem.

5. A cyclic group of birational transformations of $\mathbb{C}^2$.

Let $\mathcal{T}$ be the set of points $(x,y) \in \mathbb{C}^2$ satisfying $xy(1-x)(1-y)(1-xy) = 0$. We consider the transformation $\tau : (x,y) \longmapsto (\xi, \eta)$, mapping $\mathbb{C}^2 \setminus \mathcal{T}$ onto itself, defined by the equations

$$\tau : \begin{cases} \xi = \dfrac{1-x}{1-xy} \\ \eta = 1-xy. \end{cases}$$

It is easy to see that τ satisfies the following properties :

(i) τ is a birational diffeomorphism of $\mathbb{C}^2 \setminus \mathcal{T}$, with inverse

$$\tau^{-1} : \begin{cases} x = 1-\xi\eta \\ y = \dfrac{1-\eta}{1-\xi\eta}. \end{cases}$$

Further, τ induces a diffeomorphism of $\mathcal{U}$.

(ii) The cyclic group of transformations of $\mathbb{C}^2 \setminus \mathcal{T}$ generated by τ has order 5.

(iii) The fixed points of τ^k $(1 \leq k \leq 4)$ are

$$F_0 = \Big(\frac{-\sqrt{5}-1}{2}\,,\ \frac{-\sqrt{5}-1}{2}\Big)$$

and

$$F_1 = \Big(\frac{\sqrt{5}-1}{2}\,,\ \frac{\sqrt{5}-1}{2}\Big)\,.$$

(iv) The function

$$\psi(x,y) = \frac{xy(1-x)(1-y)}{1-xy}$$

and the measure

$$\frac{dxdy}{1-xy}$$

are invariant under the action of τ, i.e. $\psi(\xi,\eta) = \psi(x,y)$ for all $(x,y) \in \mathbb{C}^2 \setminus \mathcal{T}$, and

$$\frac{d\xi d\eta}{1-\xi\eta} = \frac{dxdy}{1-xy}\,.$$

Remark. — The integral

$$\int_0^1 \chi^n(x)\,\frac{dx}{1+x}\,,$$

where

$$\chi(x) = \frac{x(1-x)}{1+x}\,,$$

can be used to compute an irrationality measure of $\log\,2$ (see [1]). In this case, one can introduce the transformation ω of $\mathbb{C} \setminus \{0,1,-1\}$ defined by $\xi = (1-x)/(1+x)$, which has order 2 and fixed points $x_0 = -\sqrt{2}-1$ and $x_1 = \sqrt{2}-1$. Moreover, the function $\chi(x)$ and the measure $\mid dx/(1+x) \mid$ are invariant under ω.

6. A class of rational functions invariant under τ.

6.1. We consider rational functions ϕ of the following type :

$$\phi(x,y)=\frac{Q(s,p)}{(1-p)^k} \qquad (s=x+y,\ \ p=xy)\ ,$$

where $Q\in\mathbb{Z}[s,p]$ is not divisible by p, $p-s+1$ or $1-p$, where $k=\deg_s Q$, $2k=\deg\ Q$, and ϕ is invariant under τ, i.e.

$$\frac{Q(\xi+\eta,\ \xi\eta)}{(1-\xi\eta)^k}=\frac{Q(x+y,\ xy)}{(1-xy)^k}$$

for all $(x,y)\in\mathbb{C}^2\setminus\mathcal{T}$, which is equivalent to

(1) $$\eta^k\ Q(\xi+\eta,\ \xi\eta)\ =\ x^k\ Q(x+y,\ xy)\ .$$

We remark that if $Q_1,\dots,Q_\nu$ are polynomials satisfying the above conditions for integers $k_1,\dots,k_\nu$ respectively, so that the functions

$$\phi_j(x,y)=\frac{Q_j(s,p)}{(1-p)^{k_j}} \qquad (1\le j\le\nu)$$

are invariant under τ, then the polynomial

$$H(s,p)=\big(p\ (p-s+1)\big)^{n_0}Q_1(s,p)^{n_1}\dots\ Q_\nu(s,p)^{n_\nu}\ ,$$

where the integers $n_0,\ n_1,\ \dots\ ,\ n_\nu$ are even and such that $n_0>\sum_{j=1}^{\nu}k_jn_j$, satisfies all the assumptions required for H with $n\ =\ n_0+\sum_{j=1}^{\nu}k_jn_j$. Moreover

$$\frac{H(s,p)}{(1-p)^n}=\Big(\frac{p\ (p-s+1)}{1-p}\Big)^{n_0}\prod_{j=1}^{\nu}\Big(\frac{Q_j(s,p)}{(1-p)^{k_j}}\Big)^{n_j}=\psi(x,y)^{n_0}\prod_{j=1}^{\nu}\phi_j(x,y)^{n_j}$$

is invariant under τ, since $\psi(x,y)$ and $\phi_j(x,y)$ are invariant.

Each polynomial Q_j will be the product of several irreducible polynomials P_i . In accordance with the definitions given in Section 4, we take

$n_j = 2\left[\frac{a_j}{2}n\right]$ for $0 \le j \le \nu$, with $a_0 > 1/2$, $a_0 + \sum_{j=1}^{\nu} k_j a_j = 1$, and

$$g(x,y) = \frac{\left(p\,(p-s+1)\right)^{a_0}\prod_{j=1}^{\nu} Q_j(s,p)^{a_j}}{1-p}$$

$$= \psi(x,y)^{a_0}\prod_{j=1}^{\nu}\phi_j(x,y)^{a_j} \; .$$

We recall that $E \subset \mathbb{C}^2$ denotes the set of the stationary points (x,y) of g at which $g \neq 0$. Let E_1 be the set of the points of E belonging to the unit square $\mathcal{U}$. Since $g(x,y)$ is invariant under τ, by property (i) in Section 5 both E and E_1 are stable under the action of τ. Furthermore, by (ii) and (iii), E (resp. E_1) is the union of $m(E)$ (resp. $m(E_1)$) disjoint orbits of τ, each consisting of either just one point F_i ($i = 0$ or 1), or five distinct points at which $|g(x,y)|$ takes on the same value. Thus it suffices to compute $\log|g(x,y)|$ at one point of each orbit to obtain the values a and β defined in Section 4. We describe in Section 7 how to find the points of E and the best exponents a_j.

Remark. — Dvornicich and Viola [8] used the polynomials $H(s,p)$ defined by

$$\frac{H(s,p)}{(1-p)^n} = \psi(x,y)^{n_0}\,\left(11\psi(x,y)-1\right)^{n_1}\left(12\psi(x,y)-1\right)^{n_2} \; ,$$

where $n_j = 2\left[\frac{a_j}{2}n\right]$ and $a_0 = 0.\,885856...$, $a_1 = 0.\,066342...$, $a_2 = 0.\,047802...$, to obtain $\mu(\zeta(2)) \le 10.\,02979...$.

6.2. We now give some examples of rational functions ϕ_j leading to good irrationality measures of $\zeta(2)$.

First case ($\nu = 1,\ Q_1 = Q,\ k_1 = 2,\ a_0 = 1-2a_1,\ a_1 < 1/4$). We take the only polynomial $Q(s,p) = (2p-1)(4p-2s+1)(1-2s+6p-2sp+p^2)$. Substituting $x+y$ for s and xy for p we have the following factorization :

$$Q(x+y,\ xy) = (2x-1)(2y-1)(2xy-1)(1-2x+xy)(1-2y+xy) \; ,$$

whence Q satisfies (1) for $k = 2$. Since $\psi(x,y)$ attains its maximum in $\mathcal{U}$ at the point

$$F_1 = \left(\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2} \right),$$

and $x = 1/2$ is a convergent in the continued fraction expansion of $(\sqrt{5}-1)/2$, we are led to the choice of $2x-1$ among the factors of Q. Also, the line $2x-1=0$ is transformed by the powers of τ into the curves defined by the other factors of Q. In this case, F_0 and F_1 belong to E, $m(E) = 5$ and $m(E_1) = 3$. Moreover, each orbit has just one point on the line $x = y$.

The best irrationality measure of $\zeta(2)$ is now 8. 0370 3087, and is obtained when a_1 is approximately 187/1754. With this choice of a_1, the values of $h = \log|g(x,y)|$ at the points of E belonging to the five orbits are :

$x = y =$ 0. 3728 5671...		$h =$ $-2.$ 6625 7736...	$= a$
$x = y =$ 0. 6180 3398...	$= (\sqrt{5}-1)/2$	$h =$ $-2.$ 6625 7740...	
$x = y =$ 0. 7641 6886...		$h =$ $-2.$ 7618 2755...	
$x = y =$ $-0.$ 6903 3230...		$h =$ 0. 9727 5242...	
$x = y =$ $-1.$ 6180 3398...	$= (-\sqrt{5}-1)/2$	$h =$ 2. 6625 7740...	$= \beta$.

With the notation of Section 4, we choose here $r_1 = (\sqrt{5}+1)/2$, $r_2 = 1$. The maximum of h on the torus $T((\sqrt{5}+1)/2,\ 1)$ is attained at $x = y = (-\sqrt{5}-1)/2$, whence $b = \beta$ in this case.

Remark. — If we write

$$\frac{\partial}{\partial x} \log\ g(x,y) = \frac{N_1(x,y) + a_1 N_2(x,y)}{p(p-s+1)(1-p)Q(s,p)}$$

and

$$\frac{\partial}{\partial y} \log\ g(x,y) = \frac{N_3(x,y) + a_1 N_4(x,y)}{p(p-s+1)(1-p)Q(s,p)},$$

when a_1 varies the points of E plainly belong to the curve $N_1(x,y)$ $N_4(x,y) - N_2(x,y)N_3(x,y) = 0$. One finds that the polynomial $N_1N_4 - N_2N_3$ is divisible by $x-y$, $1-2x+x^2y$, $1-2y+xy^2$, $1-x-xy$ and $1-y-xy$.

Second case ($\nu = 1$, $Q_1 = R$, $k_1 = 6$, $a_0 = 1-6a_1$, $a_1 < 1/12$). We now take the polynomial

$$R(x+y,\ xy) = \{(x-y)(1-2x+x^2y)(1-2y+xy^2)(1-x-xy)(1-y-xy)\}^2$$

which satisfies (1) for $k = 6$. The choice of R is justified by the remark above. The irreducible components of the curve $R(x+y,\ xy) = 0$ are transformed into one another by the powers of τ; in particular, they all contain the points F_0 and F_1. In this case we have $m(E) = 4$, $m(E_1) = 2$, and the two orbits in E_1 (as well as the two further orbits in E) are interchanged by the symmetry $\sigma : \{\xi = y,\ \ \eta = x\}$. Since $|g(x,y)|$ is symmetric in x and y, it takes on the same value at the two orbits in E_1 (and respectively at the two other orbits in E).

The best irrationality measure of $\zeta(2)$ is now 7. 690 704, and is obtained when a_1 is approximately 37/986. The values of $h = \log|g(x,y)|$ at the points of E (expressed in s and p) are :

$$
\begin{array}{llllll}
s = & 1.\ 4507\ 1637... & p = & 0.\ 5179\ 1294... & h = & -2.\ 5530\ 6095... \quad = a \\
s = & -1.\ 9420\ 3620... & p = & 2.\ 5143\ 5635... & h = & 1.\ 6995\ 4141... \quad = \beta.
\end{array}
$$

We now choose $r_1 = 23/15 = 1.5333...$, $r_2 = 1$. The maximum of h on $T(23/15,\ 1)$ is $b = 1.\ 7003\ 6709...$, and is attained e.g. at the point

$$
\begin{array}{lllll}
x = & -1.\ 2599\ 0176... & + & 0.\ 7969\ 5689...\ i \\
y = & -1.\ 5333\ 1903... & - & 0.\ 0066\ 2115...\ i.
\end{array}
$$

Third case ($\nu = 2$, $Q_1 = Q$, $Q_2 = R$, $k_1 = 2$, $k_2 = 6$, $a_0 = 1 - 2a_1 - 6a_2$, $a_1 + 3a_2 < 1/4$). We put together the polynomials Q and R occurring in the first two cases. We now have $m(E) = 10$ and $m(E_1) = 6$. As in the previous case, the orbits in E_1 (and in E) are interchanged in pairs by the symmetry σ. Hence it suffices to compute $\log|g(x,y)|$ at five points of E belonging to distinct orbits, no two of which are interchanged by σ.

In this case, the best irrationality measure of $\zeta(2)$ is 7. 417 844, and is obtained approximately for $a_1 = 179/4549$ and $a_2 = 1/50$. The values of $h = \log|g(x,y)|$ at the points of E are :

$$
\begin{array}{lllllll}
s = & 1.\ 5032\ 7422... & p = & 0.\ 5580\ 9535... & h = & -2.\ 6324\ 0610... & = a \\
s = & 1.\ 3772\ 5364... & p = & 0.\ 4665\ 3011... & h = & -2.\ 6624\ 1391... & \\
s = & 1.\ 1488\ 1695... & p = & 0.\ 3276\ 5516... & h = & -2.\ 7417\ 1321... & \\
s = & -1.\ 3439\ 0564... & p = & 0.\ 4910\ 8889... & h = & 0.\ 9675\ 3319... & \\
s = & -2.\ 5276\ 9856... & p = & 2.\ 5637\ 5842... & h = & 2.\ 0584\ 6700... & = \beta.
\end{array}
$$

We choose $r_1 = 118/75 = 1.57333...$, $r_2 = 1$. The maximum of h on $T(118/75,\ 1)$ is $b = 2.\,0586\,8315...$, and is attained e.g. at

$$\begin{aligned} x &= -1.\,4236\,7032... + 0.\,6165\,7309...\, i \\ y &= -1.\,5733\,3139... - 0.\,0024\,6932...\, i. \end{aligned}$$

Fourth case ($\nu = 3,\ Q_1 = Q,\ Q_2 = R,\ Q_3 = S,\ k_1 = 2,\ k_2 = 6,\ k_3 = 2,\ a_0 = 1 - 2a_1 - 6a_2 - 2a_3,\ a_1 + 3a_2 + a_3 < 1/4$). We take again the polynomials Q and R considered above, together with

$$S(x+y,\ xy) = (3x-2)(3y-2)(3xy-1)(1-3x+2xy)(1-3y+2xy)$$

which satisfies (1) for $k = 2$. Note that $1/2$ and $2/3$ are two successive convergents in the continued fraction expansion of $(\sqrt{5}-1)/2$, and the factors of S are obtained from $3x-2$ as is done in the first case with the factors of Q. We now have $m(E) = 18$ and $m(E_1) = 12$. By symmetry, it suffices to compute $\log|g(x,y)|$ at nine points of E.

The best irrationality measure of $\zeta(2)$ is now $7.\,398\,537$, and is obtained approximately for $a_1 = 622/17501$, $a_2 = 701/35002$ and $a_3 = 68/17501$. The values of $h = \log|g(x,y)|$ at the points of E are :

$s=$	$0.\,8077\,1588...$	$p=$	$0.\,1622\,4753...$	$h=$	$-2.\,6357\,2567...$	$=a$
$s=$	$1.\,4874\,1700...$	$p=$	$0.\,5455\,2906...$	$h=$	$-2.\,6357\,2631...$	
$s=$	$1.\,3790\,6475...$	$p=$	$0.\,4675\,7791...$	$h=$	$-2.\,6545\,2797...$	
$s=$	$1.\,1413\,1784...$	$p=$	$0.\,3231\,8641...$	$h=$	$-2.\,7473\,9234...$	
$s=$	$1.\,2229\,8453...$	$p=$	$0.\,3711\,7259...$	$h=$	$-2.\,7592\,2185...$	
$s=$	$1.\,3160\,9155...$	$p=$	$0.\,4329\,6833...$	$h=$	$-2.\,8296\,1228...$	
$s=$	$-1.\,1041\,7984...$	$p=$	$0.\,3325\,2513...$	$h=$	$0.\,3364\,5279...$	
$s=$	$-1.\,3450\,7841...$	$p=$	$0.\,4919\,9469...$	$h=$	$0.\,9872\,5357...$	
$s=$	$-2.\,5270\,2055...$	$p=$	$2.\,5637\,0342...$	$h=$	$2.\,0674\,9747...$	$=\beta.$

We choose $r_1 = 483/307 = 1.573289...$, $r_2 = 1$. The maximum of h on $T(483/307,\ 1)$ is $b = 2.\,0677\,1416...$, and is attained e.g. at

$$\begin{aligned} x &= -1.\,4234\,8195... + 0.\,6168\,3816...\, i \\ y &= -1.\,5732\,8795... - 0.\,0024\,7301...\, i. \end{aligned}$$

In the following three sections we describe the general method yielding the above results. We explicitly discuss only the fourth case, i.e. the one leading to the theorem stated in the Introduction.

7. Choice of the exponents a_j.

Given the polynomials Q_j $(1 \le j \le \nu)$, the exponents a_j yielding the optimal values for a and β, i.e. those for which $\mu_0 = (a-\beta)/(a+2)$ is minimum, are found through a method of semi-infinite linear programming, which we now describe (if $\nu = 1$, one easily gets the optimal a and β by applying to several values of a_1 the method in step (ii) below). See [2] for a general theory.

Owing to the symmetry in x and y, we may use the variables $s = x+y$ and $p = xy$. In the fourth case of Section 6.2, which we now discuss, the part of the unit square $\mathcal{U}$ below the diagonal $x = y$ where $g(x,y) \neq 0$, i.e. $Q_j \neq 0$ $(1 \le j \le 3)$, splits up into 30 connected components $D_1, ..., D_{30}$. The following algorithm determines the optimal exponents a_j.

(i) *Initial values of the exponents.* As an initial choice, we take $a_1 = a_2 = a_3 = 0.01$, whence $a_0 = 0.9$.

(ii) *Computation of* μ_0 . Let

$$Q_0(s,p) = p\,(p-s+1)\ ,$$

$$\omega_j(s,p) = \log\,|Q_j(s,p)| - k_j\,\log\,|Q_0(s,p)| \qquad (1 \le j \le 3)\ ,$$

$$\omega_4(s,p) = \log\,|Q_0(s,p)| - \log\,|1-p|\ ,$$

$$a_4 = 1\ .$$

We compute in each region D_i $(1 \le i \le 30)$ the maximum of the function

$$\log\,|\,g(x,y)\,| = h(s,p) = \sum_{j=1}^{4} a_j\,\omega_j(s,p)\ ,$$

using the "Downhill Simplex Method" due to Nelder and Mead [12] (each region D_i contains just one stationary point of h). We make use of a modified version of the program AMOEBA given in [13]. Thus we find the value $a = -2.\,575201...$.

In accordance with the properties of the transformation τ, we see that h takes on only six distinct values at the maximal points of $D_1, ..., D_{30}$, each value being repeated five times. We keep in the sequel only six regions, say $D_1, ..., D_6$, corresponding to those distinct values of h.

We assume that each orbit of τ in E contains at least one point (x, y) such that $s = x + y$ and $p = xy$ are real (this will be proved in Section 8 by symbolic computation). In each connected region of the real plane (s, p) defined by $Q_j(s, p) \neq 0 \quad (0 \leq j \leq 3)$, $1 - p \neq 0$, we seek the stationary points of h (if there are any) by computing with the program AMOEBA the minimum of $|\partial h/\partial s| + |\partial h/\partial p|$. This yields the value $\beta = h(s_0, p_0) = 2.\ 221224...$ at the stationary point $s_0 = -2.\ 879852...$, $p_0 = 2.\ 591412...$. Associating this with the above value a, we find $\mu_0 = (a - \beta)/(a + 2) = 8.\ 3387$. Besides $D_1, ..., D_6$, we keep in the sequel only the region D_0 containing (s_0, p_0).

(iii) *Initial control points.* In each region D_i $(1 \leq i \leq 6)$ we choose a few points (s_k, p_k) $(1 \leq k \leq n_0)$. In our case, three well-spaced points for each region suffice.

(iv) *The upper bound for h in the unit square, and the precision.* We choose a suitable real number $A < -2$, e.g. $A = a$, the value found in step (ii), and the desired precision δ, e.g. $\delta = 10^{-8}$.

(v) *Linear programming.* We solve the following problem of linear programming in a_j $(1 \leq j \leq 4)$:

Minimize

$$\sum_{j=1}^{4} a_j \ \omega_j(s_0, p_0) \ ,$$

where (s_0, p_0) *is the stationary point in* D_0 *found in step (ii), under the following conditions :*

$$a_1 + 3a_2 + a_3 \ < \ 1/4 \ ,$$

$$a_4 \ = \ 1 \ ,$$

$$\sum_{j=1}^{4} a_j \ \omega_j(s_k, p_k) \ \leq \ A \qquad (1 \leq k \leq n_0) \ .$$

We get a solution which we still denote a_j $(1 \leq j \leq 4)$, yielding a value

$$B_1 = \sum_{j=1}^{4} a_j \ \omega_j(s_0, p_0) \ .$$

We repeat the computation in (ii) with the new function

$$h(s,p) = \sum_{j=1}^{4} a_j \, \omega_j(s,p)$$

containing the a_j just found, which gives a value $\mu_0 = m_1(A)$. If we compare this with $M_1(A) = (A - B_1)/(A+2)$, we obviously expect $M_1(A) < m_1(A)$. If $m_1(A) - M_1(A) > \delta$, we denote by (s_k, p_k) $(n_0+1 \leq k \leq n_1 = n_0+6)$ the points, found by AMOEBA in each region D_i $(1 \leq i \leq 6)$, where the new function $h(s,p)$ has its local maxima. Then we go back to the beginning of (v), replacing n_0 with n_1 and (s_0, p_0) with the stationary point of the new $h(s,p)$ contained in D_0. Thus we obtain a new solution a_j giving values $m_2(A)$, $M_2(A)$ such that $m_2(A) - M_2(A) < m_1(A) - M_1(A)$, and so on. This process converges in a few (say r) steps to values satisfying $m_r(A) - M_r(A) < \delta$, yielding a value $\mu_0 = m_r(A) = m(A)$.

(vi) *Variation of A.* We let A vary, to get $\mu_0 = \min_A m(A) = 7.398\,1959\ldots$.

Remark. — The above method is clearly applicable to the polynomials $P_i(s,p)$ $(1 \leq i \leq 11)$ factors of the polynomials $Q_j(s,p)$ $(0 \leq j \leq 3)$, and to their exponents α_i satisfying suitable inequalities. It turns out that the best μ_0 is obtained when the exponents α_i of the polynomials P_i factors of a given Q_j are all equal to a_j, and we find again the same function $g(x,y)$ considered in Section 6.1.

8. Verification of the preceding results by symbolic computation.

The exponents a_j are now chosen to be good simultaneous rational approximations to the values obtained with the method of the previous section, namely $a_1 = 622/17501, a_2 = 701/35002$, $a_3 = 68/17501$. To find the stationary points of g we compute the common zeros of the numerators of the partial derivatives of $\log g$ with respect to s and p. We are led by some elementary simplifications to seek the common zeros of two polynomials $U_1(s,p)$, $U_2(s,p) \in \mathbb{Z}[s,p]$, both having degree 6 in s and 14 in p.

With the aid of computer algebra systems, we compute the resultant of $U_1(s,p)$ and $U_2(s,p)$ with respect to s. This is a polynomial in p of degree

138, and is divisible by $p^{10}(p-1)^{33}(2p-1)^{16}(3p-1)^{13}(4p-1)^{3}(5p-2)^{2}(p^2-3p+1)^6(p^2+2p-1)(2p^2+2p-1)$. These factors correspond to the multiple intersections of the curves $P_i=0$, and hence are independent of the choice of the a_j. The resultant divided by the above factors is a polynomial $V(p)$ of degree 45. The 45 distinct roots of $V(p)$ (of which 33 are real) are the coordinates $p=xy$ of the 90 points $(x,y)\in E$, contained in 18 orbits. For each root of $V(p)$ we compute the common root s of $U_1(s,p)$ and $U_2(s,p)$ (one can also compute the Gröbner basis of the ideal generated by $U_1(s,p)$, $U_2(s,p)$ and $V(p)$, which yields a polynomial $G(s,p)=s+\sum_{l=0}^{44} b_l p^l$, with $b_l\in\mathbb{Q}$, satisfying $G(s,p)=0$ at each solution of the system $U_1(s,p)=U_2(s,p)=0$). Thus we find the 45 stationary points (s,p) of g, of which 33 are real. 30 of these correspond to the 60 points $(x,y)\in E_1$ in the unit square, contained in 12 orbits. The remaining 3 real points (s,p) (the last three listed at the end of Section 6.2) correspond to 6 points (x,y) with complex conjugate coordinates x and y, no two of which are equivalent under the action of the group generated by τ. In other words, each of the 6 orbits in $E\backslash E_1$ contains just one of those points (x,y).

Computing the function $h(s,p)$ at the 45 stationary points (s,p) shows the correctness of all the numerical results given by the program AMOEBA.

9. The minimal torus.

Again in the fourth case of Section 6.2, we consider the torus $T(R_1,R_2)$ containing the point (x_0,y_0) which corresponds to $s=-2.\,5270\,2055...$, $p=2.\,5637\,0342...$. The radii are

$$\begin{cases} R_1 = \sqrt{p} = 1.\,60115... \\ R_2 = \dfrac{p-1}{\sqrt{p}} = 0.\,97660... \end{cases}$$

Using the methods described in Sections 7 and 8, we find exactly ten distinct points on $T(R_1,R_2)$, conjugate in pairs, at which $h=\log|g(x,y)|$ attains its local maxima. Thus we have five distinct maximal values of h, of which the smallest is $\beta=2.\,0674\,9747...$ at (x_0,y_0), and the largest is $2.\,0685\,2383...$. Our aim is to find a torus $T(r_1,r_2)$ close to $T(R_1,R_2)$ such that the maximum b of h on $T(r_1,r_2)$ is as small as possible. To do this, we remark that if we apply to any point (x,y) satisfying $y\neq 0$ and $xy\neq 1$ first the symmetry $\sigma:(x,y)\longmapsto(y,x)$ and then the transformation τ, we have $(\tau \circ \sigma)(x,y) = \tau(y,x) = \left(\dfrac{1-y}{1-xy}\,,\ 1-xy\right)$, whence $(\tau\circ\sigma\circ\tau\circ\sigma)(x,y) = (x,y)$, i.e. $\tau\circ\sigma$ has order 2. Moreover, for any $\rho_1 > 0,\ \rho_2 > 0$,

$$(\tau\circ\sigma)T(\rho_1,\rho_2) = T(\rho_1\rho_2,\ 1/\rho_2).$$

Since the radius $R_2 = 0.\ 97660...$ is close to 1, the torus $(\tau\circ\sigma)T(R_1,R_2) = T(R_1R_2,\ 1/R_2)$ is close to $T(R_1,R_2)$ and, by the invariance of $g(x,y)$ under the action of $\tau\circ\sigma$, we see that each point on $T(R_1,R_2)$ at which $h = \log|g(x,y)|$ has a local maximum is transformed by $\tau\circ\sigma$ into a point close to another local maximum of h on $T(R_1,R_2)$. We are thus led to the choice $\rho_2 = 1$. For every $\rho_1 > 0$, the torus $T(\rho_1,1)$ is now invariant under the action of $\tau\circ\sigma$, and the ten maximal points for h on $T(\rho_1,1)$, say $(x_1,y_1),...,(x_{10},y_{10})$, can be numbered so that $\tau\circ\sigma$ interchanges (x_1,y_1) with (x_2,y_2), (x_3,y_3) with (x_4,y_4), ..., (x_9,y_9) with (x_{10},y_{10}), and the complex conjugation interchanges (x_1,y_1) with (x_3,y_3), (x_2,y_2) with (x_4,y_4), (x_5,y_5) with (x_7,y_7), (x_6,y_6) with (x_8,y_8), and (x_9,y_9) with (x_{10},y_{10}). Thus we have only three distinct values of h at $(x_1,y_1),...,(x_{10},y_{10})$, and it is easy to find that when ρ_1 is approximately 483/307 the maximum b of those three values is minimal. For $\rho_1 = 483/307$ the values of h at (x_k,y_k) $(k = 1,5,9)$ are :

$$x_1 = -1.\ 4234\ 8195... + 0.\ 6168\ 3816...\ i,$$
$$y_1 = -1.\ 5732\ 8795... - 0.\ 0024\ 7301...\ i,$$
$$h = 2.\ 0677\ 1416... = b$$

$$x_5 = -1.\ 5515\ 9378... + 0.\ 0022\ 6745...\ i,$$
$$y_5 = -1.\ 4439\ 8024... - 0.\ 6246\ 2963...\ i,$$
$$h = 2.\ 0676\ 6868...$$

$$x_9 = -1.\ 2660\ 2634... - 0.\ 9835\ 1409...\ i,$$
$$y_9 = -1.\ 4432\ 1610... + 0.\ 6263\ 9315...\ i,$$
$$h = 2.\ 0675\ 1361...$$

Associating b with the value $a = -2.\ 6357\ 2567...$ given in Section 6.2 we find

$$\mu(\zeta(2)) \le (a-b)/(a+2) = 7.\ 398\ 5369...\ .$$

This completes the proof of the theorem.

BIBLIOGRAPHY

[1] K. ALLADI and M.L. ROBINSON, Legendre polynomials and irrationality, J. reine angew. Math., 318 (1980), 137-155.
[2] E.J. ANDERSON and P. NASH, Linear Programming in infinite-dimensional spaces, Wiley-Interscience, 1987.
[3] R. APÉRY, Irrationalité de $\zeta(2)$ et $\zeta(3)$, Astérisque, 61 (1979), 11-13.
[4] F. BEUKERS, A note on the irrationality of $\zeta(2)$ and $\zeta(3)$, Bull. London Math. Soc., 11 (1979), 268-272.
[5] D.V. CHUDNOVSKY and G.V. CHUDNOVSKY, Padé and rational approximations to systems of functions and their arithmetic applications, Lect. Notes in Math., 1052 (1984), 37-84.
[6] D.V. CHUDNOVSKY and G.V. CHUDNOVSKY, Transcendental methods and Theta-functions, Proc. Symp. Pure Math., 49 (1989), part 2, 167-232.
[7] G.V. CHUDNOVSKY, Hermite-Padé approximations to exponential functions and elementary estimates of the measure of irrationality of π, Lect. Notes in Math., 925 (1982), 299-322.
[8] R. DVORNICICH and C. VIOLA, Some remarks on Beukers' integrals, Colloquia Math. Soc. János Bolyai, 51 (1987), 637-657.
[9] M. HATA, Legendre type polynomials and irrationality measures, J. reine angew. Math., 407 (1990), 99-125.
[10] K. MAHLER, On the approximation of π, Proc. K. Ned. Akad. Wet. Amsterdam, A 56 (1953), 30-42.
[11] M. MIGNOTTE, Approximations rationnelles de π et quelques autres nombres, Bull. Soc. Math. France, Mémoire 37 (1974), 121-132.
[12] J.A. NELDER and R. MEAD, A simplex method for function minimization, Computer J., 7 (1965), 308-313.
[13] W.H. PRESS, B.P. FLANNERY, S.A. TEUKOLSKY, W.T. VETTERLING, Numerical Recipes. The art of scientific computing, Cambridge University Press, 1986.
[14] G. RHIN, Approximants de Padé et mesures effectives d'irrationalité, Progr. in Math., 71 (1987), 155-164.
[15] E.A. RUKHADZE, A lower bound for the approximation of $\ln 2$ by rational numbers (Russian), Vestnik Moskov Univ., Ser 1 Math. Mekh., 6 (1987), 25-29.

Manuscrit reçu le 20 septembre 1991,
révisé le 2 juin 1992.

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