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## A CENTRAL LIMIT THEOREM ON THE SPACE OF POSITIVE DEFINITE SYMMETRIC MATRICES

by Piotr GRACZYK

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### 0. Introduction.

Central limit theorems for rotation-invariant random variables on the symmetric space  $\mathcal{P}_n$  of positive definite symmetric  $n \times n$  matrices have been investigated in case  $n = 2$  by Karpelevich, Tutubalin and Shur ([6]), Faraut ([1]) and Terras ([9]), in case  $n = 3$  by Terras ([10]) and for  $n$  arbitrary by Richards ([8]). They find applications in multivariate statistics and in some engineering problems ([9]).

In this paper we prove a central limit theorem of Lindeberg-Feller type on the space  $\mathcal{P}_n$ . It generalizes a theorem obtained by Faraut ([1]) for  $n = 2$ . To state and prove this theorem we introduce on  $\mathcal{P}_n$  some analogs of the mean and dispersion in the real case.

Sections 1 and 2 provide the basic definitions and facts from the harmonic analysis on  $\mathcal{P}_n$  used in the paper. In Section 3 we define and investigate the mean and the dispersions on  $\mathcal{P}_n$ . In Section 4 we derive in a simple way a Taylor expansion of the spherical functions on  $\mathcal{P}_n$ . Section 5 contains the main result of the paper.

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### 1. Preliminaries.

Throughout this paper  $G = GL(n, \mathbb{R})$  will denote the general linear group of  $n \times n$  nonsingular matrices and  $K = O(n)$  the group of  $n \times n$  orthogonal matrices. The symmetric space  $G/K$  is identified with  $\mathcal{P}_n$ , the space of real, positive definite symmetric  $n \times n$  matrices.

The transitive action of  $G$  on  $\mathcal{P}_n$  is defined by

$$X \mapsto X[g] = gXg^t$$

where  $g \in G$ ,  $X \in \mathcal{P}_n$  and  $g^t$  is the transpose of  $g$ . The correspondence of  $G/K$  and  $\mathcal{P}_n$  is given by  $gK \mapsto I[g] = gg^t$ , where  $I \in \mathcal{P}_n$  is the identity matrix. The space  $\mathcal{P}_n$  is a Riemannian manifold with the arc length  $ds^2 = \text{Tr}((X^{-1}dX)^2)$  for  $X = (x_{ij})_{i,j \leq n}$ ,  $dX = (dx_{ij})_{i,j \leq n}$ .

A differential operator  $L$  on  $\mathcal{P}_n$  is said to be  $G$ -invariant if it commutes with the action of  $G$ , that is for every  $g \in G$  and  $f \in C^\infty(\mathcal{P}_n)$

$$(Lf)^g = L(f^g)$$

where  $f^g(X) = f(X[g])$  when  $X \in \mathcal{P}_n$ .

The algebra  $\mathbb{D}(\mathcal{P}_n)$  of all  $G$ -invariant differential operators on  $\mathcal{P}_n$  is commutative and isomorphic with the algebra  $I(\mathfrak{a})$  of symmetric polynomials on the Cartan space  $\mathfrak{a} = \{H | H \text{ diagonal}\} \cong \mathbb{R}^n$ . By Newton's theorem the algebra  $I(\mathfrak{a})$  is generated by the symmetric polynomials :

$$(1) \quad p_j(\mathbf{x}) = \sum_{i=1}^n x_i^j, \quad j = 0, \dots, n.$$

We will denote  $\gamma : \mathbb{D}(\mathcal{P}_n) \rightarrow I(\mathfrak{a})$  the isomorphism of  $\mathbb{D}(\mathcal{P}_n)$  and  $I(\mathfrak{a})$ . Remark that the order of  $L \in \mathbb{D}(\mathcal{P}_n)$  equals to the degree of  $\gamma(L)$  ([4], p. 306).

A function  $h : \mathcal{P}_n \rightarrow \mathbb{C}$  is  $K$ -invariant if  $h^k = h$  for all  $k \in O(n)$ . A  $K$ -invariant function  $h$  is said to be spherical if  $h(I) = 1$  and  $h$  is an eigenfunction of all  $G$ -invariant differential operators on  $\mathcal{P}_n$ . All the spherical functions are given by

$$(2) \quad \Phi_{\mathbf{s}}(X) = \int_K \Delta_1^{s_1 - s_2}(X[k]) \dots \Delta_{n-1}^{s_{n-1} - s_n}(X[k]) \Delta_n^{s_n}(X[k]) dk$$

where  $\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{C}^n$ ,  $\Delta_j(Y)$  is the principal minor of order  $j$  of  $Y$  and  $dk$  denotes the normalised Haar measure on  $K$ . Formula (2)

corresponds to the classical Harish-Chandra integral formula for spherical functions on  $G$  with  $\mathbf{s} = \frac{\lambda + \rho}{2}$ , where  $\rho = (\rho_1, \dots, \rho_n)$ ,  $\rho_j = \frac{1}{2}(2j - n - 1)$ ,  $\lambda \in \mathbb{C}^n$ . We will write  $\varphi_\lambda = \Phi_{\mathbf{s}}$ . By ([4], p. 418) we have then  $L\varphi_\lambda = \gamma(L)(\lambda)\varphi_\lambda$  for  $L \in \mathbb{D}(\mathcal{P}_n)$ .

Terras ([10],[11]) and Richards ([8]) use the coordinates  $\mathbf{r} = (r_1, \dots, r_n)$  with  $\mathbf{r} = \frac{\lambda}{2}$ .

Let  $W$  denote the Weyl group of permutations. Then  $\varphi_\lambda = \varphi_{w\lambda}$  for all  $w \in W$ . One has also  $\Phi_\rho = \varphi_\rho \equiv 1$ . For all the details concerning the spherical functions on  $\mathcal{P}_n$  see e.g.[11].

The following lemma describes the relationship between the spherical functions on  $\mathcal{P}_n$  and the spherical functions on

$$\mathcal{SP}_n = \{X \in \mathcal{P}_n \mid \det X = 1\}.$$

The space  $\mathcal{SP}_n$  may be identified with the symmetric space  $SL(n)/SO(n)$ .

LEMMA 1.

$$(3) \quad \Phi_{\mathbf{s}}(X) = (\det X)^{\frac{1}{n}} \sum_{i=1}^n s_i \Psi_{\mathbf{s}}(X_1)$$

where  $X = (\det X)^{\frac{1}{n}} X_1$  with  $X_1 \in \mathcal{SP}_n$  and

$$\Psi_{\mathbf{s}}(X_1) = \int_{SO(n)} \Delta_1^{s_1 - s_2}(X_1[k]) \dots \Delta_{n-1}^{s_{n-1} - s_n}(X_1[k]) dk_{SO(n)}$$

is a spherical function on  $\mathcal{SP}_n$ .

*Proof.* — By (2) one gets

$$(4) \quad \Phi_{\mathbf{s}}(X) = (\det X)^{\frac{1}{n}} \sum_{i=1}^n s_i \int_K \Delta_1^{s_1 - s_2} \dots \Delta_{n-1}^{s_{n-1} - s_n}(X_1[k]) dk.$$

Using the fact that  $dk|_{SO(n)} = \frac{1}{2} dk_{SO(n)}$  and the invariance of  $dk$  it follows that the integral in (4) equals  $\frac{1}{2}(\Psi_{\mathbf{s}}(X_1) + \Psi_{\mathbf{s}}(X_1[h]))$  for all  $h \in O(n)$  such that  $\det(h) = -1$ . There exist  $k_0 \in SO(n)$  and  $h_0, \det(h_0) = -1$ , such that  $X_1[k_0]$  is diagonal and  $X_1[h_0 k_0] = X_1[k_0]$ . Then  $\det(h_0 k_0) = -1$  and  $\Psi_{\mathbf{s}}(X_1[h_0 k_0]) = \Psi_{\mathbf{s}}(X_1)$ . □

Theorem of Helgason-Johnson ([4], p. 458) and Lemma 1 imply that the spherical function  $\varphi_\lambda$  is bounded on  $\mathcal{P}_n$  if and only if  $\operatorname{Re} \left( \sum_{i=1}^n \lambda_i \right) = 0$

and  $\operatorname{Re} \lambda \in C(\rho)$ , where  $C(\rho)$  denotes the convex envelope of  $\{w\rho | w \in W\}$ . Then  $|\varphi_\lambda| \leq 1$ .

In the sequel we will often use the  $G$ -invariant differential operators on  $\mathcal{P}_n$  of order 1 and 2. Let us state some of their principal properties now.

All the differential operators of order 1 in  $\mathbb{D}(\mathcal{P}_n)$  are given up to a multiplicative constant by the *Euler operator*  $E$  which is defined by

$$(5) \quad Ef(X) = \frac{d}{dt} f(tX)|_{t=1}.$$

Note that  $E$  is homogeneous of degree 0. Lemma 1 shows that a spherical function  $\Phi_{\mathbf{s}}$  is homogeneous of order  $\sum_{i=1}^n s_i$ , so

$$E\Phi_{\mathbf{s}} = \left( \sum_{i=1}^n s_i \right) \Phi_{\mathbf{s}}.$$

We denote the eigenvalue of  $E$  acting on  $\Phi_{\mathbf{s}}$  by  $\gamma_1(\mathbf{s})$ .

If  $f$  is  $K$ -invariant on  $\mathcal{P}_n$  then by the spectral decomposition of  $\mathcal{P}_n$  it suffices to know  $f$  on

$$A = \{a \in \mathcal{P}_n | a \text{ diagonal with } a_{ii} > 0\}.$$

The Lie algebra of  $A$  is given by  $\mathfrak{a}$ . One denotes the diagonal entries of  $H \in \mathfrak{a}$  by  $h_1, \dots, h_n$ . Formula (5) implies

$$(6) \quad Ef(\exp H) = \sum_{i=1}^n \frac{\partial \tilde{f}}{\partial h_i}$$

where  $\tilde{f}(h_1, \dots, h_n) = f(\exp H)$ .

The  $G$ -invariant differential operators of order 2 (without terms of lower order) are given by the linear combinations of  $E^2$  and the *Laplace-Beltrami operator*  $\Delta$  on  $\mathcal{P}_n$ . If  $f$  is  $K$ -invariant then by [2]VI,4.2

$$(7) \quad \Delta f(\exp H) = \sum_j \frac{\partial^2 \tilde{f}}{\partial h_j^2} + \frac{1}{2} \sum_{i < j} \coth \left( \frac{h_i - h_j}{2} \right) \left( \frac{\partial}{\partial h_i} - \frac{\partial}{\partial h_j} \right) \tilde{f}.$$

In order to find the eigenvalues of  $\Delta$  acting on  $\Phi_{\mathbf{s}}$  one may use the horospherical part of  $\Delta$  (see [2]VI,4.4). If  $N$  denotes the nilpotent group of lower triangular matrices with 1 on the diagonal and if  $f$  is  $N$ -invariant on

$\mathcal{P}_n$ , then  $f(X) = F(a_1, \dots, a_n)$  where  $(a_i)_{1 \leq i \leq n}$  are the eigenvalues of  $X$  and

$$\Delta f(X) = \sum_{i=1}^n \left( a_i \frac{\partial}{\partial a_i} \right)^2 F - \sum_{i=1}^n \rho_i a_i \frac{\partial F}{\partial a_i}.$$

Remark that the function under the integral in (2) is  $N$ -invariant. It follows that

$$\Delta \Phi_{\mathbf{s}} = \gamma_2(\mathbf{s}) \Phi_{\mathbf{s}}$$

with  $\gamma_2(\mathbf{s}) = (\mathbf{s} - \rho|\mathbf{s}) = \frac{1}{4}(\|\lambda\|^2 - \|\rho\|^2)$ .

It is easy to check that  $\Delta$  is elliptic and  $E^2$  is semi-elliptic on  $\mathcal{P}_n$ . All the elliptic second order operators in  $\mathbb{D}(\mathcal{P}_n)$  are given by

$$(8) \quad L = a \left( \Delta - \frac{1}{n} E^2 \right) + bE^2 + cE + d, \quad a, b > 0.$$

Observe that the operator  $\Omega = \Delta - \frac{1}{n} E^2$  restrained to  $\mathcal{SP}_n$  is the Laplace-Beltrami operator on  $\mathcal{SP}_n$ .  $\Omega$  is semi-elliptic on  $\mathcal{P}_n$ .

### 2. $K$ -invariant probability measures on $\mathcal{P}_n$ .

A probability measure  $\mu$  on  $\mathcal{P}_n$  is said to be  $K$ -invariant if for any measurable subset  $B$  of  $\mathcal{P}_n$  and for all  $k \in K$  we have  $\mu(kBk^t) = \mu(B)$ . We shall then write  $\mu \in M^{\natural}(\mathcal{P}_n)$ .

In this paper we consider only  $K$ -invariant measures on  $\mathcal{P}_n$ . We can identify such measures with  $K$ -biinvariant measures on  $G$ . Then the convolution  $\mu_1 * \mu_2$  of two measures in  $M^{\natural}(\mathcal{P}_n)$  is defined by the convolution of the corresponding measures on  $G$  and then projecting on  $\mathcal{P}_n$ . This convolution is commutative.

The spherical Fourier transform of a measure  $\mu \in M^{\natural}(\mathcal{P}_n)$  is defined by

$$\hat{\mu}(\lambda) = \int_{\mathcal{P}_n} \varphi_{\lambda}(X) d\mu(X)$$

for  $\lambda$  such that  $\varphi_{\lambda}$  is bounded. In Section 1 we have formulated sufficient and necessary conditions for such  $\lambda$ . We also write

$$\hat{\mu}(\mathbf{s}) = \int_{\mathcal{P}_n} \Phi_{\mathbf{s}}(X) d\mu(X).$$

The spherical Fourier transform carries the convolution of  $K$ -invariant measures on  $\mathcal{P}_n$  into the usual product

$$\widehat{\mu_1 * \mu_2}(\mathbf{s}) = \hat{\mu}_1(\mathbf{s})\hat{\mu}_2(\mathbf{s}).$$

If  $\mu \in M^{\natural}(\mathcal{P}_n)$  is infinitely divisible, the infinitesimal generator of the continuous semigroup of measures  $(\mu_t)_{t>0}$  with  $\mu_1 = \mu$  is given by the Hunt formula ([5]). It is then natural to call  $\mu$  *Gaussian* if the generator of  $(\mu_t)_{t>0}$  is a second order  $G$ -invariant elliptic differential operator on  $\mathcal{P}_n$  which annihilates constants (cf.[3]). If the generator is semi-elliptic we say that  $\mu$  is *Gaussian degenerate*. By (8) the Fourier transform of a Gaussian measure  $\mu \in M^{\natural}(\mathcal{P}_n)$  has the following form

$$\hat{\mu}(\mathbf{s}) = \exp \left[ a\gamma_2(\mathbf{s}) + \left( b - \frac{1}{n}a \right) \gamma_1^2(\mathbf{s}) + c\gamma_1(\mathbf{s}) \right]$$

with  $a, b > 0, c \in \mathbb{R}$ .

*Examples.* — (i) The Laplace-Beltrami operator  $\Delta$  is the generator of the *heat semigroup*  $(\kappa_t)_{t>0}$  on  $\mathcal{P}_n$  (cf.[11]). We have

$$\hat{\kappa}_t(\mathbf{s}) = \exp[t(\mathbf{s} - \rho|\mathbf{s})].$$

Certainly, the measures  $\kappa_t$  are Gaussian on  $\mathcal{P}_n$ .

(ii) The operator  $\Omega = \Delta - \frac{1}{n}E^2$  is the generator of the semigroup  $(\nu_t)_{t>0}$  on  $\mathcal{P}_n$  which extends naturally the heat semigroup  $(\tilde{\nu}_t)_{t>0}$  on  $\mathcal{SP}_n$ , i.e.  $\nu_t(B) = \tilde{\nu}_t(B \cap \mathcal{SP}_n)$ . We have

$$\hat{\nu}_t(\mathbf{s}) = \exp \left\{ t \left[ (\mathbf{s} - \rho|\mathbf{s}) - \frac{1}{n} \left( \sum_{i=1}^n s_i \right)^2 \right] \right\}$$

and  $\nu_t$  are Gaussian degenerate.

(iii) Let  $\mathfrak{n}_t$  be a random variable on  $\mathbb{R}$  with normal distribution of mean 0 and variance  $t$ . Let  $\eta_t$  be the probability distribution of the random variable  $\exp(\mathfrak{n}_t I)$  on  $\mathcal{P}_n$ . We have then

$$\hat{\eta}_t(\mathbf{s}) = \exp \left[ \frac{t}{2} \left( \sum_{i=1}^n s_i \right)^2 \right].$$

The generator of the semigroup  $(\eta_t)_{t>0}$  equals  $\frac{1}{2}E^2$ . This operator corresponds to the Laplacian in the  $\mathbb{R}^+ I$ -direction of the space  $\mathcal{P}_n$  considered as a product  $\mathcal{SP}_n \times \mathbb{R}^+$ . The measures  $\eta_t$  are Gaussian degenerate on  $\mathcal{P}_n$ .

(iv) If  $\delta_t$  is the Dirac delta in  $\exp(tI)$  then

$$\hat{\delta}_t(\mathbf{s}) = \exp\left(t \sum_{i=1}^n s_i\right)$$

and the generator of  $(\delta_t)_{t>0}$  is  $E$ .

Note that  $\kappa_t = \nu_t * \eta_{2tn^{-1}}$ . All the Gaussian measures on  $\mathcal{P}_n$  have the form  $\nu_t * \eta_u * \delta_w$  for some  $t, u$  positive and  $w \in \mathbb{R}$ .

### 3. The mean and the dispersions on $\mathcal{P}_n$ .

In order to analyse the asymptotic behavior of  $K$ -invariant measures on  $\mathcal{P}_n$  we need some analogs of the mean and the covariance of a measure on  $\mathbb{R}^n$ . In this section we introduce in a natural way such analogs and prove some properties of them.

#### 3.1. Choice of dispersions on $\mathcal{P}_n$ .

To have an analog of the covariance on  $\mathcal{P}_n$  one seeks an application

$$D : M^{\natural}(\mathcal{P}_n) \rightarrow [0, \infty]$$

satisfying for all  $\mu_1, \mu_2 \in M^{\natural}(\mathcal{P}_n)$  the condition

$$(9) \quad D(\mu_1 * \mu_2) = D(\mu_1) + D(\mu_2).$$

One also assumes that there exists an analytic,  $K$ -invariant function  $Q$  on  $\mathcal{P}_n$  such that

$$(10) \quad D(\mu) = \int Q(X) d\mu(X)$$

for all  $\mu \in M^{\natural}(\mathcal{P}_n)$ . The function  $D$  will be called a *dispersion* on  $\mathcal{P}_n$ . Observe that the condition (9) is equivalent to

$$(11) \quad \int_K Q(Y[xk]) dk = Q(I[x]) + Q(Y)$$

for all  $x \in G$  and  $Y \in \mathcal{P}_n$ .

In the real case the covariance of a centralised measure may be represented by a second order derivative of its Fourier transform in 0. In



the case of  $\mathcal{P}_n$  the spherical function  $\varphi_\lambda \equiv 1$  if  $\lambda = \rho$ . It turns out that on  $\mathcal{P}_n$  we have the following analogous property.

**THEOREM 1.** — *If  $Q$  is an analytic,  $K$ -invariant function on  $\mathcal{P}_n$  verifying (11) then there exists a vector  $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$  such that*

$$(12) \quad Q(X) = \frac{\partial \varphi_\lambda(X)}{\partial \mathbf{v}} \Big|_{\lambda=\rho}, \quad X \in \mathcal{P}_n.$$

*Proof.* — First observe that if  $Q$  satisfies (11) then for all  $L \in \mathbb{D}(\mathcal{P}_n)$  annihilating constants  $LQ = \text{const}$ . Indeed, let us apply  $L$  to (11) for  $x$  fixed and then put  $Y = I$ . One gets  $LQ(xx^t) = LQ(I)$  for all  $x \in G$ , so  $LQ$  is constant on  $\mathcal{P}_n$ .

Next remark that if  $Q_1$  and  $Q_2$  verify (11) and  $LQ_1 = LQ_2$  for all  $L \in \mathbb{D}(\mathcal{P}_n)$  annihilating constants then  $Q_1 = Q_2$ . This property comes out from the analyticity of  $Q_1$  and  $Q_2$  and the fact that (11) implies  $Q_1(I) = Q_2(I) = 0$ .

If  $Q = \frac{\partial \varphi_\lambda}{\partial \mathbf{v}} \Big|_{\lambda=\rho}$  for any vector  $\mathbf{v}$  then by the convolution property of the spherical Fourier transform on  $\mathcal{P}_n$  the condition (9) holds for  $D$  corresponding to such  $Q$ .

Let now  $Q$  be arbitrary satisfying (11). To prove the theorem it suffices to show that there exists a vector  $\mathbf{v} = (v_1, \dots, v_n)$  such that

$$L_j Q = L_j \left( \frac{\partial \varphi_\lambda}{\partial \mathbf{v}} \Big|_{\lambda=\rho} \right) = \mathbf{v} \cdot \text{grad}(L_j \varphi_\lambda) \Big|_{\lambda=\rho}$$

for  $L_j = \gamma^{-1}(p_j)$  with  $p_j$  as in (1),  $j = 1, \dots, n$ . We have  $\mathbf{l}_j = \text{grad}(L_j \varphi_\lambda) \Big|_{\lambda=\rho} = \text{grad} \gamma(L_j) \Big|_{\lambda=\rho} = \text{grad} p_j \Big|_{\lambda=\rho} = j(\rho_1^{j-1}, \dots, \rho_n^{j-1})$ . It follows that  $\mathbf{l}_1, \dots, \mathbf{l}_n$  are independent and the system of equations

$$L_j Q = \mathbf{v} \cdot \mathbf{l}_j, \quad j = 1, \dots, n$$

has a solution. □

Now we want to choose the directions  $\mathbf{v}$  of derivation in (12) so as the function  $Q$  was nonnegative. The decomposition (3) shows that this is possible only for  $\sum_{i=1}^n v_i = 0$ .

By reasons of convexity (or more generally by the Helgason-Johnson theorem) the spherical functions verify  $0 < \varphi_\lambda \leq 1$  for  $\lambda \in C(\rho)$ . Thus, if

$\rho + t\mathbf{v} \in C(\rho)$  for  $t$  positive sufficiently small then  $-\frac{\partial\varphi_\lambda}{\partial\mathbf{v}}|_{\lambda=\rho}$  is nonnegative. One will differentiate in the directions of neighbour vertices of  $\rho$  in  $C(\rho)$ . They are given by the permutations of neighbour entries of  $\rho$  :

$$\begin{aligned} \beta_1 &= (\rho_2, \rho_1, \rho_3, \dots, \rho_n) \\ &\dots \\ \beta_{n-1} &= (\rho_1, \rho_2, \dots, \rho_n, \rho_{n-1}). \end{aligned}$$

Then the vectors  $\mathbf{v}_j = \beta_j - \rho = (0, \dots, 1, -1, \dots, 0)$  lie on the edges of  $C(\rho)$  beginning in  $\rho$ . Observe that  $\mathbf{v}_j = -\alpha_j$ ,  $j = 1, \dots, n - 1$ , where  $\alpha_j$  are the simple positive roots corresponding to the Iwasawa decomposition  $G = NAK$  with  $N$  lower triangular. The vectors  $\mathbf{v}_j$ ,  $j = 1, \dots, n - 1$ , are independent and span all the hyperplane  $\{\lambda | \sum \lambda_i = 0\}$ . By reasons of normalisation we will differentiate with respect to the vectors  $2\mathbf{v}_j$ .

DEFINITION. — The dispersions  $D_j$  on  $\mathcal{P}_n$ ,  $j = 1, \dots, n - 1$ , are defined by

$$D_j(\mu) = \int Q_j(X) d\mu(X)$$

where

$$Q_j(X) = -2 \frac{\partial\varphi_\lambda(X)}{\partial\mathbf{v}_j} \Big|_{\lambda=\rho}.$$

Then one has

$$(13) \quad Q_j(X) = 2 \left( \frac{\partial}{\partial\lambda_{j+1}} - \frac{\partial}{\partial\lambda_j} \right) \varphi_\lambda(X) \Big|_{\lambda=\rho} = \left( \frac{\partial}{\partial s_{j+1}} - \frac{\partial}{\partial s_j} \right) \Phi_{\mathbf{s}}(X) \Big|_{\mathbf{s}=\rho}$$

and for  $\mu \in M^{\natural}(\mathcal{P}_n)$

$$(14) \quad D_j(\mu) = 2 \left( \frac{\partial}{\partial\lambda_{j+1}} - \frac{\partial}{\partial\lambda_j} \right) \hat{\mu}(\lambda) \Big|_{\lambda=\rho} = \left( \frac{\partial}{\partial s_{j+1}} - \frac{\partial}{\partial s_j} \right) \hat{\mu}(\mathbf{s}) \Big|_{\mathbf{s}=\rho}.$$

Example. — A direct calculation using (14) allows to find the dispersions of the measures considered in Section 2 :

$$\begin{aligned} D_j(\kappa_t) &= D_j(\nu_t) = t ; \\ D_j(\eta_t) &= 0 ; \\ D_j(\delta_t) &= 0. \end{aligned}$$

We shall give now some properties of the functions  $Q_j$ .

THEOREM 2.

- (i)  $Q_j(X) = Q_j(tX)$ ,  $j = 1, \dots, n - 1$ , for all  $X \in \mathcal{P}_n$  and  $t > 0$ .
- (ii)  $Q_j(I) = 0$ ,  $j = 1, \dots, n - 1$ .
- (iii)  $Q_1(X) + \dots + Q_{n-1}(X) > 0$  for all  $X \neq tI$ .

*Proof.* — (i) follows obviously from (3) and (13). (ii) follows from  $\Phi_{\mathbf{s}}(I) = 1$  for all  $\mathbf{s}$ . To prove (iii) it suffices to consider  $X \in \mathcal{SP}_n$ ,  $X \neq I$ . Suppose that  $Q_1(X) = \dots = Q_{n-1}(X) = 0$ . Then  $\frac{\partial \varphi_\lambda}{\partial \mathbf{u}}|_{\lambda=\rho} = 0$  for all the directions  $\mathbf{u}$  such that  $\sum u_i = 0$ . If  $\sum \lambda_i = 0$ , by (3)  $\varphi_\lambda(X) = \psi_\lambda(X)$  where  $\psi_\lambda$  denotes the spherical function on  $\mathcal{SP}_n$  in the Harish-Chandra notation. Thus the application  $\lambda \mapsto \psi_\lambda(X)$ ,  $\sum \lambda_i = 0$ , has a critical point in  $\lambda = \rho$ , and by  $W$ -invariance also in  $\lambda = w\rho$  for  $w \in W$ .

On the other hand  $\psi_\lambda$  is given by the formula of Harish-Chandra :

$$(15) \quad \psi_\lambda(X) = \int_{SO(n)} e^{(\lambda - \rho|\mathcal{H}(ak))} dk_{SO(n)}$$

where  $a \in A \cap SL(n)$  is such that  $X = a^2[k_0]$  for some  $k_0 \in SO(n)$  and  $g = k \exp \mathcal{H}(g).n$  is the Iwasawa decomposition of  $SL(n)$ . Denote by  $\mu_X$  the image of  $dk_{SO(n)}$  by the mapping  $k \mapsto \mathcal{H}(ak)$ . Then  $\mu_X$  is a probability measure on  $\tilde{a} = \{H|H \text{ diagonal and } \text{Tr}H = 0\}$ . By (15) we have then for any  $\mathbf{u} \in \mathbb{R}^n$

$$\begin{aligned} \psi_\lambda(X) &= \int_{\tilde{a}} e^{(\lambda - \rho|H)} d\mu_X(H) \\ \frac{\partial \psi_\lambda}{\partial \mathbf{u}}(X) &= \int_{\tilde{a}} (\mathbf{u}|H) e^{(\lambda - \rho|H)} d\mu_X(H) \\ (16) \quad \frac{\partial^2 \psi_\lambda}{\partial \mathbf{u}^2}(X) &= \int_{\tilde{a}} (\mathbf{u}|H)^2 e^{(\lambda - \rho|H)} d\mu_X(H). \end{aligned}$$

By a theorem of Kostant ([7])  $\text{supp } \mu_X = C(\log a)$ . Since  $X \neq I$  we have  $a \neq I$  and  $\log a \neq 0$ . The space  $\mathcal{SP}_n$  is irreducible so by [4]IV,10.11  $\dim C(\log a) = \dim \tilde{a}$  and (16) implies  $\frac{\partial^2 \psi_\lambda}{\partial \mathbf{u}^2}(X) > 0$  for all  $\lambda$  and  $\mathbf{u} \neq 0$ . It means that  $\psi_\lambda(X)$  is strictly convex and in particular it has at most one critical point on  $\tilde{a}$ . For  $w \in W$  different from identity  $w\rho \neq \rho$ . That gives a contradiction. □

COROLLARY 1. — Let  $\mu \in M^{\natural}(\mathcal{P}_n)$ . Then

$$D_1(\mu) = \dots = D_{n-1}(\mu) = 0$$

if and only if  $\mu$  is concentrated on  $\{tI | t > 0\}$ . □

*Example.* — In the case  $n = 2$  the explicit form of the dispersion density is known ([1]) :

$$Q(a_r) = 2 \log \left( \operatorname{ch} \frac{r}{2} \right)$$

where

$$a_r = \begin{pmatrix} e^r & 0 \\ 0 & e^{-r} \end{pmatrix}.$$

For  $n \geq 3$  one may give the following integral formula

$$Q_j(X) = \int_K \{ \log \Delta_{j-1}(X[k]) - 2 \log \Delta_j(X[k]) + \log \Delta_{j+1}(X[k]) \} dk$$

but the explicit form of  $Q_j(X)$  is not known.

### 3.2. The mean and the variance in $\mathbb{R}^+I$ -direction.

Theorem 2 and Corollary 1 show that the dispersions  $D_j$  do not control the behaviour of measures in  $M^{\natural}(\mathcal{P}_n)$  in the direction  $\mathbb{R}^+I$ . In fact, via (3) one may say that  $D_j$  are the dispersions in the direction of  $\mathcal{SP}_n$ . We will introduce now some complementary characteristics of  $\mu \in M^{\natural}(\mathcal{P}_n)$ .

Having in mind the decomposition (3) of a spherical function on  $\mathcal{P}_n$  and denoting  $w = \sum_{i=1}^n s_i$  it is natural to have the following definition.

DEFINITION. — Let  $\mu \in M^{\natural}(\mathcal{P}_n)$ . We define :

the mean of  $\mu$  by

$$M(\mu) = \frac{\partial}{\partial w} \hat{\mu} |_{s=\rho} = \frac{1}{n} \int \log(\det X) d\mu(X)$$

the second moment of  $\mu$  by

$$M_2(\mu) = \frac{\partial^2}{\partial w^2} \hat{\mu} |_{s=\rho} = \frac{1}{n^2} \int \log^2(\det X) d\mu(X)$$

the variance of  $\mu$  by  $d^2(\mu) = M_2(\mu) - M^2(\mu)$ .

Note that any measure  $\mu \in M^1(\mathcal{P}_n)$  may be centralised by putting

$$\tilde{\mu}(B) = \mu(e^{M(\mu)}B)$$

for  $B$  measurable. Then  $M(\tilde{\mu}) = 0$  and  $d^2(\mu) = M_2(\tilde{\mu})$ . Observe that

$$\begin{aligned} M(\mu_1 * \mu_2) &= M(\mu_1) + M(\mu_2) \\ d^2(\mu_1 * \mu_2) &= d^2(\mu_1) + d^2(\mu_2) \end{aligned}$$

but  $d^2(\mu) = \int q(X)d\mu(X)$  with  $q(X) = \frac{1}{n^2} \log^2(\det X)$  only for centralised measures  $\mu$ .

The derivative  $\frac{\partial}{\partial w}$  in the definition of  $M$  and  $d^2$  equals in the  $(s_i)$ -coordinates

$$\frac{\partial}{\partial w} = \frac{1}{n} \sum_{i=1}^n \frac{\partial}{\partial s_i}.$$

This makes possible to calculate  $M(\mu)$  and  $d^2(\mu)$  if one knows the spherical Fourier transform of  $\mu$ .

*Example.* — For the measures considered in Section 2 we have :

$$\begin{aligned} M(\kappa_t) &= 0, & d^2(\kappa_t) &= 2tn^{-1}; \\ M(\nu_t) &= 0, & d^2(\nu_t) &= 0; \\ M(\eta_t) &= 0, & d^2(\eta_t) &= t; \\ M(\delta_t) &= t, & d^2(\delta_t) &= 0. \end{aligned}$$

**COROLLARY 2.** — If  $\mu \in M^1(\mathcal{P}_n)$  and  $M(\mu) = d^2(\mu) = 0$ ,  $D_j(\mu) = 0$  for  $j = 1, \dots, n-1$ , then  $\mu = \delta_I$ .  $\square$

Note that a Gaussian measure  $\mu$  on  $\mathcal{P}_n$  is fully characterized by its mean  $M(\mu)$ , variance  $d^2(\mu)$  and dispersion  $D_1(\mu)$ .

#### 4. Taylor expansion of spherical functions.

In this section we will derive a useful Taylor expansion of spherical functions on  $\mathcal{P}_n$ . This expansion will be more detailed than that of Richards ([8]) and we will prove it in a simpler way.

A spherical function  $\Phi_{\mathfrak{s}}$  is  $K$ -invariant so it is enough to consider  $\Phi_{\mathfrak{s}}(\exp H)$ ,  $H \in \mathfrak{a}$ . One may treat  $\Phi_{\mathfrak{s}}(\exp H)$  as a function of  $h_1, \dots, h_n$ . It is then symmetric in  $h_1, \dots, h_n$ . Remark that  $\Phi_{\mathfrak{s}}(\exp H)$  is real analytic

since  $\Phi_{\mathbf{s}}$  is a solution of an elliptic differential equation with analytic coefficients :  $\Delta\Phi_{\mathbf{s}} = \gamma_2(\mathbf{s})\Phi_{\mathbf{s}}$ . Making use of the symmetry and analyticity of the function in the  $h_j$  we get the following Taylor expansion at  $H = 0$  :

$$(17) \quad \Phi_{\mathbf{s}}(\exp H) = 1 + a(\mathbf{s}) \sum h_i + b(\mathbf{s}) \sum h_i^2 + c(\mathbf{s})(\sum h_i)^2 + R_{\mathbf{s}}(H)$$

with

$$(18) \quad R_{\mathbf{s}}(H) = \sum f_{\alpha}(\mathbf{s})P_{\alpha}(H)$$

where  $P_{\alpha}(H)$  are symmetric polynomials in  $h_1, \dots, h_n$  homogeneous of order greater or equal to 3.

In order to find the functions  $a, b, c$  in (17) let us apply the operators  $E, E^2$  and  $\Delta$  to (17) at  $H = 0$ .

For the operators  $E$  and  $E^2$  one uses (6).  $E$  considered as a differential operator on functions of  $h_1, \dots, h_n$  is homogeneous of order 1 while the polynomials  $P_{\alpha}$  are homogeneous of order at least 3. That implies  $ER_{\mathbf{s}}(0) = E^2R_{\mathbf{s}}(0) = 0$  and

$$\begin{aligned} \gamma_1(\mathbf{s}) &= na(\mathbf{s}) \\ \gamma_1^2(\mathbf{s}) &= 2nb(\mathbf{s}) + 2n^2c(\mathbf{s}). \end{aligned}$$

For the operator  $\Delta$  one applies (7). By an argument of homogeneity one obtains  $\Delta R_{\mathbf{s}}(0) = 0$ . We have then

$$\gamma_2(\mathbf{s}) = (n^2 + n)b(\mathbf{s}) + 2nc(\mathbf{s}).$$

Solving these equations we get

THEOREM 3.

$$\Phi_{\mathbf{s}}(\exp H) = 1 + a(\mathbf{s}) \sum h_i + b(\mathbf{s}) \sum h_i^2 + c(\mathbf{s})(\sum h_i)^2 + R_{\mathbf{s}}(H)$$

with

$$(19) \quad \begin{aligned} a(\mathbf{s}) &= \frac{1}{n}\gamma_1(\mathbf{s}) \\ b(\mathbf{s}) &= \frac{n\gamma_2(\mathbf{s}) - \gamma_1^2(\mathbf{s})}{n(n-1)(n+2)} \\ c(\mathbf{s}) &= \frac{(n+1)\gamma_1^2(\mathbf{s}) - 2\gamma_2(\mathbf{s})}{2n(n-1)(n+2)} \end{aligned}$$

where  $\gamma_1(\mathbf{s}) = \sum s_i$ ,  $\gamma_2(\mathbf{s}) = (\mathbf{s} - \rho|\mathbf{s})$  and  $R_{\mathbf{s}}(H)$  is as in (18). □

By (13) and by differentiating of (19) one obtains the following expansion of the functions  $Q_j$  at  $H = 0$  :

COROLLARY 3.

$$(20) \quad Q_j(\exp H) = \frac{1}{(n-1)(n+2)} \sum h_i^2 - \frac{1}{n(n-1)(n+2)} (\sum h_i)^2 + R'_j(H)$$

where  $R'_j(H) = \sum c_{\alpha j} P_\alpha(H)$  with  $P_\alpha$  as in (18),  $j = 1, \dots, n-1$ . □

Writing as in Section 3  $q(X) = \frac{1}{n^2} \log^2(\det X)$  we have  $(\sum h_i)^2 = n^2 q(X)$ . Replacing  $\sum h_i^2$  in (19) by the expression for  $\sum h_i^2$  obtained from (20) we get :

$$(21) \quad \begin{aligned} \Phi_s(\exp H) = 1 + \frac{1}{n} \gamma_1(\mathbf{s}) \sum h_i + \left( \gamma_2 - \frac{1}{n} \gamma_1^2 \right) Q_j(\exp H) \\ + \frac{1}{2} \gamma_1^2 q(\exp H) + R_{j,\mathbf{s}}(H) \end{aligned}$$

where

$$(22) \quad R_{j,\mathbf{s}}(H) = \sum f_{j\alpha}(\mathbf{s}) P_\alpha(H)$$

with  $P_\alpha(H)$  as in (18).

For  $H = (h_1, \dots, h_n)$  we put  $\|H\| = \sum |h_i|$ . Then we have

$$(23) \quad R_{j,\mathbf{s}}(H) = \mathcal{O}(\|H\|^3) \quad \text{if } H \rightarrow 0.$$

In order to estimate  $R_{j,\mathbf{s}}(H)$  when  $\|H\| \rightarrow \infty$  and  $\Phi_s$  is bounded one has to estimate  $Q_j$  in infinity.

LEMMA 2. —  $Q_j(\exp H) \leq \|H\|$  for all  $H \in \mathfrak{a}$  and  $j = 1, \dots, n-1$ .

*Proof.* — By Theorem 2(i)  $Q_j(\exp H) = Q_j(\exp H')$  with  $H' = (h_1 - \frac{1}{n} \sum h_i, \dots, h_n - \frac{1}{n} \sum h_i) \in \tilde{\mathfrak{a}}$ . Then  $Q_j(\exp H') = -2 \frac{\partial \psi_\lambda}{\partial \mathbf{v}_j} |_{\lambda=\rho}$  where  $\psi_\lambda$  is spherical on  $SP_n$ . By the Harish-Chandra formula

$$\begin{aligned} Q_j(\exp H') &= 2 \int_{SO(n)} \left( -\mathbf{v}_j | \mathcal{H} \left( \exp \frac{1}{2} H' \cdot k \right) \right) dk_{SO(n)} \\ &= 2 \int_{SO(n)} \left( \alpha_j | \mathcal{H} \left( \exp \frac{1}{2} H' \cdot k \right) \right) dk_{SO(n)}. \end{aligned}$$

By  $K$ -invariance of  $Q_j$  one may assume that  $H' \in \tilde{a}^+$ , i.e.  $h'_1 \leq \dots \leq h'_n$ .  
 By [4]IV,6.5  $\mathcal{H}(ak) \leq \mathcal{H}(a)$  for  $a \in \exp(\tilde{a}^+), k \in SO(n)$ , so

$$Q_j(\exp H') \leq 2 \left( \alpha_j |\mathcal{H} \left( \exp \frac{1}{2} H' \right) \right) = h'_{j+1} - h'_j = h_{j+1} - h_j.$$

Finally

$$Q_j(\exp H) \leq |h_j| + |h_{j+1}| \leq \|H\|.$$

□

COROLLARY 4. — For every  $j = 1, \dots, n - 1$  and  $\mathbf{s}$  such that  $\Phi_{\mathbf{s}}$  is bounded

$$R_{j,\mathbf{s}}(H) = \mathcal{O}(\|H\|) + \mathcal{O}((\sum h_i)^2)$$

when  $\|H\| \rightarrow \infty$ .

□

### 5. Central limit theorem.

Let  $\{\mu_{m_j}\}, m \in \mathbb{N}, 1 \leq j \leq k_m$  be a family of  $K$ -invariant probability measures on  $\mathcal{P}_n$ . Put

$$\mu_m = \mu_{m_1} * \mu_{m_2} * \dots * \mu_{m_{k_m}}.$$

Denote by  $H(X)$  the diagonal matrix of logarithms of eigenvalues of  $X \in \mathcal{P}_n$ . Then we have the following central limit theorem :

THEOREM 4. — Suppose that the measures  $\{\mu_{m_j}\}_{m \in \mathbb{N}, 1 \leq j \leq k_m}$  satisfy the following conditions :

$$(24) \quad \begin{aligned} M(\mu_{m_j}) &= 0 \\ \lim_{m \rightarrow \infty} D_1(\mu_m) &= t \end{aligned}$$

$$(25) \quad \lim_{m \rightarrow \infty} d^2(\mu_m) = u$$

$$(26) \quad \lim_{m \rightarrow \infty} \sum_{j=1}^{k_m} \int \frac{\|H\|^3}{1 + \|H\|^2} d\mu_{m_j} = 0$$

$$(27) \quad \lim_{m \rightarrow \infty} \sum_{j=1}^{k_m} \int_{\{\|H\| > 1\}} (\text{Tr} H)^2 d\mu_{m_j} = 0.$$

Then the measures  $\mu_m$  converge weakly to the Gaussian measure  $\nu_t * \eta_u$ .



*Proof.* — First observe that for any  $\varepsilon > 0$

$$\begin{aligned} \int \|H\| d\mu_{mj} &= \int_{\{\|H\| \leq \varepsilon\}} \|H\| d\mu_{mj} + \int_{\{\|H\| > \varepsilon\}} \|H\| d\mu_{mj} \\ &\leq \varepsilon + \frac{1 + \varepsilon^2}{\varepsilon^2} \int \frac{\|H\|^3}{1 + \|H\|^2} d\mu_{mj}. \end{aligned}$$

Then Lemma 2 and (26) imply that  $\lim_m D_1(\mu_{mj}) = 0$  uniformly in  $j$ . Similarly,

$$\int (\sum h_i)^2 d\mu_{mj} \leq \int \|H\| d\mu_{mj} + \int_{\{\|H\| > 1\}} (\text{Tr}H)^2 d\mu_{mj}$$

and by (27)  $\lim_m d^2(\mu_{mj}) = 0$  uniformly in  $j$ .

Fix  $\mathbf{s}$  such that  $\Phi_{\mathbf{s}}$  is bounded. By Corollary 4 and (23) the conditions (26) and (27) imply

$$(28) \quad \lim_m \sum_{j=1}^{k_m} \int |R_{1,\mathbf{s}}(H)| d\mu_{mj} = 0.$$

In particular  $\int |R_{1,\mathbf{s}}(H)| d\mu_{mj}$  tends to 0 uniformly with respect to  $j$ . Then (21) implies

$$(29) \quad \lim_m \sup_{1 \leq j \leq k_m} |\hat{\mu}_{mj}(\mathbf{s}) - 1| = 0.$$

By (21)

$$\sum_j [1 - \hat{\mu}_{mj}(\mathbf{s})] = - \left( \gamma_2 - \frac{1}{n} \gamma_1^2 \right) D_1(\mu_m) - \frac{1}{2} \gamma_1^2 d^2(\mu_m) - \sum_j \int R_{1,\mathbf{s}}(H) d\mu_{mj}.$$

By (24),(25),(28) and (29) we have then

$$\begin{aligned} \lim_m \sum_j [1 - \hat{\mu}_{mj}(\mathbf{s})] &= - \left( \gamma_2 - \frac{1}{n} \gamma_1^2 \right) t - \frac{1}{2} \gamma_1^2 u; \\ \lim_m \sum_j [1 - \hat{\mu}_{mj}(\mathbf{s})]^2 &= 0. \end{aligned}$$

Using again (29) we get

$$\begin{aligned} \lim_m \hat{\mu}_m(\mathbf{s}) &= \exp \left[ \lim_m \sum_j \log \hat{\mu}_{mj}(\mathbf{s}) \right] \\ &= \exp \left[ \left( \gamma_2 - \frac{1}{n} \gamma_1^2 \right) t + \frac{1}{2} \gamma_1^2 u \right] = \nu_t * \widehat{\eta}_u(\mathbf{s}). \end{aligned}$$

By the Lévy continuity theorem on  $\mathcal{P}_n$  (look [5]Thm.4.2 in the case of  $G$  semisimple; the proof of Gangolli works on  $\mathcal{P}_n$ ) we get  $\mu_m \Rightarrow \nu_t * \eta_u$ .  $\square$

*Remark.* — Under the hypotheses of Theorem 4  $\lim_m D_l(\mu_m) = t$  for  $l = 2, \dots, n - 1$  (see Corollary 3).

**COROLLARY 5.** — *Let  $(\kappa_t)_{t>0}$  be the heat semigroup on  $\mathcal{P}_n$ . Let the family of measures  $\{\mu_{m_j}\}$  be centralised and verify (26) and (27). If*

$$\begin{aligned} \lim_m D_1(\mu_m) &= D_1(\kappa_t) = t \\ \lim_m d^2(\mu_m) &= d^2(\kappa_t) = 2tn^{-1} \end{aligned}$$

then

$$\mu_m \Rightarrow \kappa_t. \quad \square$$

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