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DAVID B. MASSEY
DIRK SIERSMA

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DEFORMATION OF POLAR METHODS

by D.B. MASSEY and D. SIERSMA

0. Introduction.

In [Ma2], [Ma3], and [Ma4], Massey defines and investigates a collection of analytic invariants which can be attached to a hypersurface singularity, regardless of the dimension of the singular locus. These numbers are the Lê numbers and, in many ways, they appear to be a good generalization of the Milnor number of an isolated singularity. The Lê numbers are defined and investigated in a manner that falls under the general heading of "polar" methods.

Of course, from the topological point of view, it is the Betti numbers of the Milnor fibre that are the interesting invariants. But only in a very few special cases one can explicitly calculate these Betti numbers.

However, in the case that the hypersurface has a one-dimensional singular locus, there are certain other analytic invariants which play an important role in the study of singularities. These invariants are the numbers of certain special types of singularities that occur in generic deformations of the original hypersurface. In some cases, one can determine from this information the Betti numbers, or even the homotopy type, of the Milnor fibre.

Such deformation invariants are studied by de Jong [Jo1], [Jo2], Pellikaan [Pe1], [Pe2], Siersma [Si1],[Si2],[Si3], and de Jong and van Straten

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[JoSt1], [JoSt2], [JoSt3] in the context of deforming the defining equation of the hypersurface, and are studied by Gaffney [Ga1], Mond [Mo1],[Mo2], and Gaffney and Mond [GaMo], in the context of deforming parametrizations of the original hypersurface.

In this paper, the authors show the connection between these two approaches in the case of a one-dimensional critical locus. This connection arises from the fact that the alternating sum of the Lê numbers equals the reduced Euler characteristic of the Milnor fibre [Ma2],[Ma3]. Specifically, we derive a formula involving the Lê numbers, the number of special points in a deformation, and the Euler characteristic of the deformed singular set. In order to state the theorem, we must first fix some terminology.

Let $f_s : (\mathcal{U}, \mathbf{0}) \to (\mathbb{C}, 0)$ be a family of analytic germs, where $\mathcal{U}$ is an open subset of $\mathbb{C}^{n+1}$. Suppose that the dimension of the critical locus, Σf_0, of the germ of f_0 at the origin equals 1 and that the deformation f_s is *equi-transversal* (see 1.10) – this last condition means essentially that the generic tranverse Milnor number is constant in the family f_s. In this situation, we have

THEOREM (1.11 and 2.2). — *For all $\epsilon > 0$ sufficiently small, if B_ϵ is the closed ball of radius ϵ around the origin in $\mathbb{C}^{n+1}$ and a is a small complex number with $0 < |a| \ll \epsilon$, then*

$$\tilde{b}_n(F) - \tilde{b}_{n-1}(F) = \sum_{\mathbf{p}} \tilde{b}_n(F_{\mathbf{p}}) + \sum_{\mathbf{q}} \left(\tilde{b}_n(F_{\mathbf{q}}) - \tilde{b}_{n-1}(F_{\mathbf{q}})\right)$$
$$- \sum_k \overset{\circ}{\mu}_k \Big(\chi(\Sigma_a^k) - \sum_{\mathbf{q} \,\in\, \Sigma_a^k} 1 \Big),$$

where χ denotes the Euler characteristic, $\tilde{b}_i$ denotes the reduced Betti number, F denotes the Milnor fibre of f_0 at the origin, $F_{\mathbf{x}}$ denotes the Milnor fibre of $f_a - f_a(\mathbf{x})$ at the point $\mathbf{x}$, the $\mathbf{p}$'s are summed over all $\mathbf{p} \in B_\epsilon \cap \Sigma f_a$ which are <u>not</u> contained in $V(f_a) := f_a^{-1}(0)$, the $\mathbf{q}$'s are summed over those $\mathbf{q}$ in $B_\epsilon \cap \Sigma f_a$ which are contained in $V(f_a)$ and at which f_a has generic polar curve, the $\{\Sigma_a^k : k = 1, 2, \dots \}$ are the irreducible components of the singular set, $\Sigma V(f_a)$, of $V(f_a)$ in B_ϵ, and $\overset{\circ}{\mu}_k$ is the generic transverse Milnor number of Σ_a^k.

While this formula seems to give a comparison of the two methods, in the proofs we wish to contrast the two methods. Hence, we give two proofs of the formula. In the first author's proof, we use the dynamic properties of intersection numbers coupled with some easy stratified Morse theory to conclude the result. In the second author's proof, one uses the notion of vanishing homology in a deformation, as developed in [Si2] and [Si3], to derive a formula involving the Euler characteristic of the Milnor fibre of the original hypersurface, the number of special points in a deformation, and the Euler characteristic of the deformed singular set.

In section 3, we give a slightly improved version of the formula of Lê and Iomdine [Lê2], [Io2], together with the sharpest possible bound for the validity of the formula. This formula enables one to calculate the Euler characteristic of the Milnor fibre of a one-dimensional singularity in terms of the multiplicity of the Jacobian scheme and the Milnor number of an associated isolated singularity.

In section 4, we give some important special cases and some non-trivial examples. The examples in section 4 include the transverse A_1 case, line singularities, plane curve singularities, homogeneous and quasi-homogeneous singularities, and composed singularities.

In the final remarks, we discuss - among other things - how to effectively calculate these invariants with the aid of a computer.

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1. Lê numbers and deformations.

In this section, we shall define and investigate a collection of analytic cycles - the Lê cycles - which live in the critical set of a analytic map $h : (\mathbb{C}^{n+1}, \mathbf{0}) \to (\mathbb{C}, \mathbf{0})$. Our intention is to generalize the information given by the Milnor number in the case of an isolated singularity. In [Ma2] , we defined the Lê varieties as schemes. However, it appears that only their structure as cycles is important, and this structure is much easier to define and calculate.

Throughout this section, we let $h : (\mathbb{C}^{n+1}, \mathbf{0}) \to (\mathbb{C}, \mathbf{0})$ be an analytic map and $\mathbf{z}$ be a linear choice of coordinate systems for $\mathbb{C}^{n+1}$.

We shall be considering schemes, cycles, and sets; for clarification of what structure we are considering, we shall at times enclose cycles in square brackets, [ ] , and analytic sets in a pair of vertical lines,||. Occasionally, when the notation becomes cumbersome, we shall simply state explicitly whether we are considering V as a scheme, a cycle, or a set.

By the intersection of a collection of closed subschemes, we mean the scheme defined by the sum of the underlying ideal sheaves. By the union of a finite collection of closed subschemes, we mean the scheme defined by the intersection (not the product) of the underlying ideal sheaves. We say that two subschemes, V and W , are equal up to embedded component provided that, in each stalk, the isolated components of the defining ideals (those corresponding to minimal primes) are equal. Our main concern with this last notion is that it implies that the cycles [V] and [W] are equal. We say that two cycles are equal at a point, $\mathbf{p}$, provided that the portions of each cycle which pass through $\mathbf{p}$ are equal.

We will use the notation of [Ma2]. Let W be a scheme and let α be an ideal in A. We wish to consider scheme-theoretically those components of $V(\alpha)$ which are not contained in $|W|$.

Let S be the multiplicatively closed set $A-\bigcup p$ where the union is over all prime ideals $p \in \operatorname{Ass}(A/\alpha)$ with $|V(p)| \not\subseteq |W|$. Then, we define α/W to equal $S^{-1}\alpha \cap A$. Thus, α/W is the ideal consisting of the intersection of those primary components, q , (possibly embedded) , of α such that $|V(q)| \not\subseteq |W|$. If $V = V(\alpha)$, we let V/W denote the scheme $V(\alpha/W)$.

It is important to note that the scheme V/W does not depend on the structure of W as a scheme, but only as an analytic set. This definition coincides with that of a gap-sheaf - a notion which is normally encountered in the analytic context [SiTr]. The gap sheaf notation for V/W is $V[W]$. We shall not use this notation here. In the analytic situation, one does the primary decomposition above on the level of stalks, and must then show that the construction above yields a coherent sheaf.

DEFINITION 1.1. — For $0 \leqslant k \leqslant n$, the *k-th (relative) polar variety*, $\Gamma^k_{h,\mathbf{z}}$, of h with respect to $\mathbf{z}$ is the scheme $V\left(\dfrac{\partial h}{\partial z_k}, \dots, \dfrac{\partial h}{\partial z_n}\right) \Big/ \Sigma h$ (see [Ma2] , [Ma3]). If the choice of the coordinate system is clear, we will sometimes simply write Γ^k_h.

Thus, on the level of defining ideals, $\Gamma^k_{h,\mathbf{z}}$ consists of those components of

$$V\left(\frac{\partial h}{\partial z_k},\ldots,\frac{\partial h}{\partial z_n}\right)$$

which are not contained in $|\Sigma h|$. Note, in particular, that $\Gamma^0_{h,\mathbf{z}}$ is empty and, at a point $\mathbf{p}$ where $\dim_{\mathbf{p}}\Sigma h < k$, we must have $\Gamma^k_{h,\mathbf{z}} = V\left(\frac{\partial h}{\partial z_k},\ldots,\frac{\partial h}{\partial z_n}\right)$.

We naturally refer to the cycle $\left[\Gamma^k_{h,\mathbf{z}}\right]$ as *the k-th polar cycle* of h with respect to $\mathbf{z}$.

The key point of this definition is that the dimension of the singular set of h is allowed to be arbitrary.

Our ideal structure is somewhat non-standard, as we allow for embedded subvarieties. Also, it is important to note that we index by the generic dimension instead of the codimension.

Clearly, as sets, $\varnothing = \Gamma^0_{h,\mathbf{z}} \subseteq \Gamma^1_{h,\mathbf{z}} \subseteq \cdots \subseteq \Gamma^{n+1}_{h,\mathbf{z}} = \mathbb{C}^{n+1}$. In fact, by 0.1. i) of [Ma2] , we have that :

PROPOSITION 1.2. — *$\left(\Gamma^{k+1}_{h,\mathbf{z}} \cap V\left(\frac{\partial h}{\partial z_k}\right)\right) \Big/ \Sigma h = \Gamma^k_{h,\mathbf{z}}$ as schemes, and thus the cycle $\left[\Gamma^{k+1}_{h,\mathbf{z}} \cap V\left(\frac{\partial h}{\partial z_k}\right)\right] - \left[\Gamma^k_{h,\mathbf{z}}\right]$ has only components which are contained in the critical set of the map h.*

As the ideal $\langle\frac{\partial h}{\partial z_k},\ldots,\frac{\partial h}{\partial z_n}\rangle$ is invariant under any linear change of coordinates which leaves $V(z_0,\ldots,z_{k-1})$ invariant, we see that the scheme $\Gamma^k_{h,\mathbf{z}}$ depends only on h and the choice of the first k coordinates. At times, it will be convenient to subscript the k-th polar variety with only the first k coordinates instead of the whole coordinate system.

While it is immediate from the number of defining equations that every component of the analytic set $\left|\Gamma^k_{h,\mathbf{z}}\right|$ has dimension $\geqslant k$, one usually requires that the coordinate system be suitably generic so that $\dim_{\mathbf{p}}\Gamma^k_{h,\mathbf{z}} = k$ at some point, $\mathbf{p}$. When this is the case, we have :

PROPOSITION 1.3. — *If $\dim_{\mathbf{p}}\Gamma^k_{h,\mathbf{z}} = k$, then $\Gamma^k_{h,\mathbf{z}}$ has no embedded subvarieties through $\mathbf{p}$.*

Proof. — This follows from 1.3 of [Ma2]. □

DEFINITION 1.4. – If the intersection of $\Gamma^k_{h,\mathbf{z}}$ and

$$V(z_0 - p_0, \dots, z_{k-1} - p_{k-1})$$

is zero-dimensional, or empty, at a point $\mathbf{p}$, then we say that the *k-th polar number*, $\gamma^k_{h,\mathbf{z}}(\mathbf{p})$, is defined and we set $\gamma^k_{h,\mathbf{z}}(\mathbf{p})$ equal to the intersection number

$$\left(\Gamma^k_{h,\mathbf{z}} \cdot V(z_0 - p_0, \dots, z_{k-1} - p_{k-1})\right)_{\mathbf{p}}.$$

Thus, if $\gamma^k_{h,\mathbf{z}}(\mathbf{p})$ is defined, then $\Gamma^k_{h,\mathbf{z}}$ must be purely k-dimensional, or empty, at $\mathbf{p}$ and so - by 1.3 - $\Gamma^k_{h,\mathbf{z}}$ has no embedded components at $\mathbf{p}$.

We now wish to define the Lê cycles. Unlike the polar varieties and cycles, the Lê cycles are supported on the critical set of h itself. These cycles demonstrate a number of properties which generalize the data given by the Milnor number for an isolated singularity.

DEFINITION 1.5. — *For* $0 \leqslant k \leqslant n$, *we define the k-th Lê cycle of h with respect to* $\mathbf{z}$, $\left[\Lambda^k_{h,\mathbf{z}}\right]$, *to be*

$$\left[\Gamma^{k+1}_{h,\mathbf{z}} \cap V\left(\frac{\partial h}{\partial z_k}\right)\right] - \left[\Gamma^k_{h,\mathbf{z}}\right].$$

If the choice of coordinate system is clear, we will sometimes simply write $\left[\Lambda^k_h\right]$. Also, as we have given the Lê cycles no structure as schemes, we will sometimes omit the brackets and write $\Lambda^k_{h,\mathbf{z}}$ to denote the Lê cycle - unless we explicitly state that we are considering it as a set only.

Note that as every component of $\left[\Gamma^{k+1}_{h,\mathbf{z}}\right]$ has dimension $\geqslant k+1$, that every component of $\Lambda^k_{h,\mathbf{z}}$ has dimension $\geqslant k$. We say that the cycle, $\left[\Lambda^k_{h,\mathbf{z}}\right]$, or the set, $\left|\Lambda^k_{h,\mathbf{z}}\right|$, has correct dimension at a point $\mathbf{p}$ provided that $\left|\Lambda^k_{h,\mathbf{z}}\right|$ is purely k-dimensional, or empty, at $\mathbf{p}$.

We define the *k-th* Lê *number of h at* $\mathbf{p}$ *with respect to* $\mathbf{z}$,$\lambda^k_{h,\mathbf{z}}(\mathbf{p})$, to equal the intersection number $\left(\Lambda^k_{h,\mathbf{z}} \cdot V(z_0 - p_0, \dots, z_{k-1} - p_{k-1})\right)_{\mathbf{p}}$, provided this intersection is zero-dimensional, or empty, at $\mathbf{p}$. If this intersection has dimension $\geqslant 1$ at $\mathbf{p}$, then we say that the k-th Lê number (of h at $\mathbf{p}$ with respect to $\mathbf{z}$) is *undefined.* Here, when $k = 0$, we mean that

$$\lambda^0_{h,\mathbf{z}}(\mathbf{p}) = \left(\mathbf{\Lambda}^0_{\mathbf{h},\mathbf{z}} \cdot \mathbb{C}^{\mathbf{n}+1}\right)_{\mathbf{p}} = \left[\mathbf{\Gamma}^1_{\mathbf{h},\mathbf{z}} \cap \mathbf{V}\left(\frac{\partial h}{\partial z_0}\right)\right]_{\mathbf{p}} = \left(\left[\mathbf{\Gamma}^1_{\mathbf{h},\mathbf{z}}\right] \cdot \left[\mathbf{V}\left(\frac{\partial h}{\partial z_0}\right)\right]\right)_{\mathbf{p}}$$

(this last equality holds whenever $\lambda^0_{h,\mathbf{z}}(\mathbf{p})$ is defined, for then $\Gamma^1_{h,\mathbf{z}}$ has no embedded components by 1.3).

Note that if $\lambda^k_{h,\mathbf{z}}(\mathbf{p})$ is defined, then $\left|\Lambda^k_{h,\mathbf{z}}\right|$ must have correct dimension at $\mathbf{p}$. Also note that, since $\Gamma^{k+1}_{h,\mathbf{z}}$ and $\Gamma^k_{h,\mathbf{z}}$ depend only on the choice of the coordinates z_0 through z_k, the k-th Lê cycle, $\left[\Lambda^k_{h,\mathbf{z}}\right]$, depends only on the choice of $(z_0, \ldots, z_k)$.

PROPOSITION 1.6. — *The Lê cycles are all non-negative and are contained in the critical set of h. Every component of* $\left|\Lambda^k_{h,\mathbf{z}}\right|$ *has dimension at least k. If* $s = \dim_{\mathbf{p}}\Sigma h$ *then, for all k with* $s < k < n+1$, $\mathbf{p}$ *is not contained in* $\left|\Lambda^k_{h,\mathbf{z}}\right|$, *i.e.* $\left|\Lambda^k_{h,\mathbf{z}}\right|$ *is empty at* $\mathbf{p}$.

Proof. — The first statement follows from 1.2. The second statement follows from the definition of the Lê cycles and the fact that every component of $\Gamma^{k+1}_{h,\mathbf{z}}$ has dimension at least $k+1$. The third statement follows from the first two. □

Remark 1.7. — As we demonstrated in example 1.7 of [Ma2] , in the case of an isolated singularity, $\lambda^0_{h,\mathbf{z}}$ is nothing other the Milnor number.

In the general case, it is tempting to think of $\lambda^0_{h,\mathbf{z}}(\mathbf{p})$ as the local (generic) degree of the Jacobian map of h at $\mathbf{p}$, i.e. the number of points in

$$\overset{\circ}{B}_\varepsilon \cap V\left(\frac{\partial h}{\partial z_0} - a_0, \ldots, \frac{\partial h}{\partial z_n} - a_n\right),$$

where $\overset{\circ}{B}_\varepsilon$ is a small open ball centered at $\mathbf{p}$ and $\mathbf{a}$ is a generic point with length that is small compared to ε; unfortunately, there is no such local degree.

Consider the example $h = z_2^2 + (z_0 - z_1^2)^2$ and let $\mathbf{p}$ be the origin. Then,

$$\overset{\circ}{B}_\varepsilon \cap V\left(\frac{\partial h}{\partial z_0} - a_0,\ \frac{\partial h}{\partial z_1} - a_1,\ \frac{\partial h}{\partial z_2} - a_2\right)$$
$$= \overset{\circ}{B}_\varepsilon \cap V(2(z_0 - z_1^2) - a_0,\ \ 2(z_0 - z_1^2)(-2z_1) - a_1,\ \ 2z_2 - a_2).$$

The solutions to these equations are

$$z_0 = \frac{a_0}{2} + \frac{a_1^2}{4a_0^2}, \qquad z_1 = -\frac{a_1}{2a_0}, \qquad z_2 = \frac{a_2}{2}.$$

The number of solutions of these equations inside any small ball does not just depend on picking small, generic a_0, a_1, and a_2, but also depends on the relative sizes of a_0 and a_1. If a_1 is small relative to a_0, then there will be one solution inside the ball; if a_0 is small relative to a_1, then there will be no solutions inside the ball.

Do either of these numbers actually agree with $\lambda^0_{h,\mathbf{z}}(\mathbf{0})$? Yes, with these coordinates, $\lambda^0_{h,\mathbf{z}}(\mathbf{0}) = 1$. This can be seen from the above calculations together with the discussion below, which shows how "close" $\lambda^0_{h,\mathbf{z}}$ is to being the generic degree of the Jacobian map of h.

We claim that, if $\dim_{\mathbf{p}}\Gamma^1_{h,\mathbf{z}} = 1$, then $\lambda^0_{h,\mathbf{z}}(\mathbf{p})$ exists and equals the number of points in

$$\overset{\circ}{B}_\varepsilon \cap V\left(\frac{\partial h}{\partial z_0} - a_0, \dots, \frac{\partial h}{\partial z_n} - a_n\right),$$

where $\overset{\circ}{B}_\varepsilon$ is a small open ball centered at $\mathbf{p}$, $a_0 \neq 0$ is small compared to ε, and $a_1, \dots, a_n$ are generic, with length that is small compared to that of a_0.

To see this, note that this number of points equals the sum of the intersection numbers given by

$$\sum_{\mathbf{q}} \left(V\left(\frac{\partial h}{\partial z_1}, \dots, \frac{\partial h}{\partial z_n}\right) \cdot V\left(\frac{\partial h}{\partial z_0} - a_0\right)\right)_{\mathbf{q}}$$

where the sum is over all $\mathbf{q}$ in

$$\overset{\circ}{B}_\varepsilon \cap V\left(\frac{\partial h}{\partial z_0} - a_0,\ \frac{\partial h}{\partial z_1}, \dots, \frac{\partial h}{\partial z_n}\right).$$

But, for $a_0 \neq 0$, these points, $\mathbf{q}$, do not occur on the critical locus of h, and so this sum equals

$$\sum_{\mathbf{q}} \left(\Gamma^1_{h,\mathbf{z}} \cdot V(\frac{\partial h}{\partial z_0} - a_0) \right)_{\mathbf{q}} .$$

This last sum is none other than

$$\lambda^0_{h,\mathbf{z}}(\mathbf{p}) = \left(\Gamma^1_{h,\mathbf{z}} \cdot V\left(\frac{\partial h}{\partial z_0}\right) \right)_{\mathbf{p}} .$$

It is also possible to give a more intuitive characterization of $\lambda^s_{h,\mathbf{z}}(\mathbf{p})$ where $s = \dim_{\mathbf{p}} \Sigma h$. Namely, (see Prop. 2.8 of [Ma2])

$$\lambda^s_{h,\mathbf{z}}(\mathbf{p}) = \sum_{\nu} \mathbf{n}_\nu \overset{\circ}{\mu}_\nu,$$

where ν runs over all s-dimensional components of Σh at $\mathbf{p}$, n_ν is the local degree of the map $(z_0, \dots , z_{s-1})$ restricted to ν at $\mathbf{p}$, and $\overset{\circ}{\mu}_\nu$ denotes the generic transverse Milnor number of h along the component ν in a neighborhood of $\mathbf{p}$. In particular, if the coordinate system is generic enough so that n_ν is actually the multiplicity of ν at $\mathbf{p}$ for all ν, then $\lambda^s_{h,\mathbf{z}}(\mathbf{p})$ is merely the multiplicity of the Jacobian scheme of h at $\mathbf{p}$.

For the remainder of the paper, we shall restrict our attention to the case where $\dim_{\mathbf{0}} \Sigma h = 1$. In this case, there are only two (possibly) non-vanishing Lê numbers :

$$\lambda^0_{h,\mathbf{z}}(\mathbf{p}) = \left(\Gamma^1_{h,\mathbf{z}} \cdot V\left(\frac{\partial h}{\partial z_0}\right) \right)_{\mathbf{p}}$$

and

$$\lambda^1_{h,\mathbf{z}}(\mathbf{p}) = \sum_{\nu} n_\nu \overset{\circ}{\mu}_\nu.$$

Correspondingly, there are only two (possibly) non-vanishing reduced Betti numbers of the Milnor fibre, $F_{\mathbf{p}}$, of h at $\mathbf{p}$; namely, $\tilde{b}_n(F_{\mathbf{p}})$ and $\tilde{b}_{n-1}(F_{\mathbf{p}})$ (see [KM]).

In [Ma2],Thm. 2.15, we give a formula for the Euler characteristic of the Milnor fibre in terms of the Lê numbers under the hypothesis that the coordinate system is *polar*. In [Ma3], the dominate generic requirement on the coordinate system is that it be *pre-polar* - a strictly weaker requirement than being polar. However, one has only to go through the proof of 2.15 of [Ma2] to see that all that is used is that the coordinates are prepolar. In the case of a one-dimensional singularity, the result is easy to state :

PROPOSITION 1.8. — *Suppose that $h : (\mathbb{C}^{n+1}, \mathbf{0}) \to (\mathbb{C}, \mathbf{0})$ is an analytic map and that* $\mathbf{z}$ *is a linear choice of coordinates for* $\mathbb{C}^{n+1}$ *such that, at some point* $\mathbf{p}$, *we have* $\dim_{\mathbf{p}}\Sigma(h_{|_{V(z_0-p_0)}}) = 0$. *Then,* $\lambda^0_{h,\mathbf{z}}(\mathbf{p})$ *and* $\lambda^1_{h,\mathbf{z}}(\mathbf{p})$ *are defined and*

$$\lambda^0_{h,\mathbf{z}}(\mathbf{p}) - \lambda^1_{h,\mathbf{z}}(\mathbf{p}) = \tilde{b}_n(F_{\mathbf{p}}) - \tilde{b}_{n-1}(F_{\mathbf{p}}),$$

where $\tilde{b}_i(F_{\mathbf{p}})$ *denotes the i-th reduced Betti number of the Milnor fibre of* h *at* $\mathbf{p}$.

We may use this proposition to calculate $\lambda^0_{h,\mathbf{z}}(\mathbf{p})$ even when the coordinate system has been chosen in a very non-generic way.

COROLLARY 1.9. — *Suppose that* $h : (\mathbb{C}^{n+1}, \mathbf{0}) \to (\mathbb{C}, \mathbf{0})$ *is an analytic map and that* $\mathbf{z}$ *is a linear choice of coordinates for* $\mathbb{C}^{n+1}$ *such that, at some point* $\mathbf{p}$, $\dim_{\mathbf{p}}\Sigma h = 1$ *and* $\dim_{\mathbf{p}}\Sigma(h_{|_{V(z_0-p_0)}}) = 0$. *In addition, suppose for a generic choice of coordinates that* h *has no polar curve at* $\mathbf{p}$. *Then,* Σh *is itself smooth at* $\mathbf{p}$ *and*

$$\lambda^0_{h,\mathbf{z}}(\mathbf{p}) = \mathring{\mu}\left(\deg_{\mathbf{p}}(z_{0|_{\Sigma h}}) - 1\right),$$

where $\mathring{\mu}$ *denotes the generic transverse Milnor number of* h *near* $\mathbf{p}$ *(see 1.7) and* $\deg_{\mathbf{p}}(z_{0|_{\Sigma h}})$ *is the local degree of the map* z_0 *restricted to* Σh *at* $\mathbf{p}$.

Proof. — That Σh is smooth at $\mathbf{p}$ follows from Lê's non-splitting result [Lê3]. Using [Lê3] again together with the result of [Lê1], we also have that $\tilde{b}_n(F_{\mathbf{p}}) = 0$ and $\tilde{b}_{n-1}(F_{\mathbf{p}}) = \mathring{\mu}$. By remark 1.7, $\lambda^1_{h,\mathbf{z}}(\mathbf{p}) = \mathring{\mu}\left(\deg_{\mathbf{p}}(z_{0|_{\Sigma h}})\right)$. The formula now follows from 1.8. □

We now wish to study **deformations** of one-dimensional singularities. For the remainder of this section, we will let $f_s : \mathbb{C}^{n+1} \to \mathbb{C}$ denote a family of analytic maps in the coordinates $(z_0, \dots, z_n)$ such that $\dim_{\mathbf{0}}\Sigma f_0 = 1$ and $\dim_{\mathbf{0}}\Sigma\left(f_{0|_{V(z_0)}}\right) = 0$. Let $f(\mathbf{z}, s) = f_s(\mathbf{z})$.

PROPOSITION/DEFINITION 1.10. — *The following are equivalent :*

i) *For all* $\mathbf{p} \in \Sigma f_0 - \mathbf{0}$ *near* $\mathbf{0}$, $f_{|_{V(z_0-p_0)}}$ *has a smooth critical locus near* $\mathbf{p}$ *and the family* $f_{s|_{V(z_0-p_0)}}$ *is* μ*-constant (i.e. has constant Milnor number) at* $\mathbf{p}$;

ii) *for all components* ν *of* Σf_0 *through* $\mathbf{0}$, *there exists a unique component* ν' *of* Σf *containing* ν *and, moreover,* $\mathring{\mu}_\nu = \mathring{\mu}_{\nu'}$.

iii) *for all* $\varepsilon > 0$ *sufficiently small, there exist* τ, $\eta > 0$ *such that*

$$\begin{array}{c} \partial B_\varepsilon \times \mathring{\mathbb{D}}_\tau \cap \Psi^{-1}((\mathring{\mathbb{D}}_\eta - 0) \times \mathring{\mathbb{D}}_\tau) \\ \downarrow \ \Psi := (f, s) \\ (\mathring{\mathbb{D}}_\eta - 0) \times \mathring{\mathbb{D}}_\tau \end{array}$$

is a proper, stratified submersion.

We call such a deformation of f_0 an *equi-transversal deformation* at the origin.

Proof. — i) and ii) are both equivalent to :

†) for all $\mathbf{p} \in \Sigma f_0 - \mathbf{0}$ near $\mathbf{0}$, Σf is smooth at $(0, \mathbf{p})$ and

$$\mu(f_{|V(s-a, z_0-q_0)})$$

is independent of the point $(a, \mathbf{q}) \in \Sigma f$ near $(0, \mathbf{p})$.

The proof that these three conditions are equivalent is essentially the argument for the $\mathring{\mu}$-lemma (4.2) of [Ma1]. The point is that as a function of $(a, \mathbf{q})$,

$$\mu(f_{|V(s-a, z_0-q_0)})$$

is upper semi-continuous and so has a generic value on each component of the singular set; namely, $\mathring{\mu}_{\nu'}$. All three conditions are equivalent to saying that $\mathring{\mu}_{\nu'}$ equals the Milnor number of $f_{0|V(z_0-p_0)}$ at $(0, \mathbf{p})$.

By Proposition 4.1 of [Ma1] (or by a double application of Theorem 4.5 of [Ma4]), we see that this implies

††) for all $\mathbf{p} \in \Sigma f_0 - \mathbf{0}$ near $\mathbf{0}$, Σf is smooth at $(0, \mathbf{p})$ and, if $(s_i, \mathbf{p}_i)$ is a sequence points not in Σf such that $T_{(s_i, \mathbf{p}_i)} V(f - f(s_i, \mathbf{p}_i))$ converges to some hyperplane $\mathcal{T}$, then $T_{(0,\mathbf{p})} \Sigma f \subseteq \mathcal{T}$. (In the terminology of [Ma3], this says that $V(s)$ is a pre-polar slice for f at the origin.)

Moreover, ††) certainly implies †), for if $\mu(f_{|V(s-a, z_0-q_0)})$ were not constant near $(0, \mathbf{p})$, then a generic hyperplane slice through $(0, \mathbf{p})$ would have polar curve at $(0, \mathbf{p})$ – and this would give a contradiction to ††).

Thus, our problem now is to show that ††) is equivalent to iii).

That ††) implies iii) is the argument of Lê in Proposition 2.1 of [Lê1] (or, in a more general setting, is Proposition 2.4.i of [Ma3].)

To complete the argument, we shall now show that iii) implies i).

Let B_ε be a Milnor ball for f_0 at the origin (i.e. all spheres contained in B_ε centered at the origin transversely intersect all the strata of a Whitney stratification of $V(f_0)$), and let $\mathbf{p} \in \partial B_\varepsilon \cap \Sigma f_0$. For a particular linear form, L , we will show that the Milnor number in the family $f_{s|_{V(L-L(0,\mathbf{p}))}}$ is constant near $(0,\mathbf{p})$; i) and ii) follow, since they are independent of the linear form, L , so long as $f_{s|_{V(L-L(0,\mathbf{p}))}}$ is a family of isolated singularities.

The linear form, L , that we select is $L(\mathbf{v}) = \langle \mathbf{v}, (0,\mathbf{p})\rangle$, where $\langle , \rangle$ denotes the complex inner-product. We choose this linear form because $\ker(L)$ is contained in $T_{(0,\mathbf{p})}(\mathbb{C} \times \partial B_\varepsilon)$.

Now, suppose that iii) is true, but that $\mu\left(f_{s|_{V(L-L(\mathbf{p}))}}\right)$ is not constant. Then, $f_{|_{V(L-L(0,\mathbf{p}))}}$ possesses polar curve with respect to the linear map s. This polar curve has dimension 2 over the reals and so its intersection with $\mathbb{C} \times \partial B_\varepsilon$ is real one-dimensional. This real curve passes through $(0,\mathbf{p})$ and at each point, $\mathbf{q}$, on this curve, we have that

$$T_{\mathbf{q}}V(f - f(\mathbf{q}), L - L(0,\mathbf{p})) = T_{\mathbf{q}}V(s - s(\mathbf{q}), L - L(0,\mathbf{p}))$$

whence,

$$T_{\mathbf{q}}V(f - f(\mathbf{q}), s - s(\mathbf{q})) = T_{\mathbf{q}}V(s - s(\mathbf{q}), L - L(0,\mathbf{p}))$$

$$\subseteq T_{\mathbf{q}}V(s - s(\mathbf{q})) \cap T_{\mathbf{q}}(\mathbb{C} \times \partial B_\varepsilon).$$

This contradicts iii). □

Note that for an equi-transversal deformation, one must have that

$$\Sigma f \cap V(s) = \Sigma(f_0)$$

as germs of sets at the origin (though this is certainly not sufficient).

We can now state the main theorem in terms of Lê numbers – the translation to Betti numbers is immediate.

THEOREM 1.11. — *Suppose that* $\dim_{\mathbf{0}}\Sigma\left(f_{0|_{V(z_0)}}\right) = 0$ *and suppose that* f_s *is an equi-transversal deformation of* f_0 *at the origin. If* B_ε *is a* *sufficiently small closed ball around the origin in* $\mathbb{C}^{n+1}$ *and* a *is a small complex number with* $0 < |a| \ll \varepsilon$, *then,*

$$\lambda^0_{f_0}(\mathbf{0}) - \lambda^1_{f_0}(\mathbf{0}) = \sum_{\mathbf{p}} \lambda^0_{f_a - f_a(\mathbf{p})}(\mathbf{p}) + \sum_{\mathbf{q}} (\lambda^0_{f_a}(\mathbf{q}) - \lambda^1_{f_a}(\mathbf{q})) \\ - \sum_k \mathring{\mu}_k \big(\chi(\Sigma^k_a) - \sum_{\mathbf{q} \in \Sigma^k_a} 1 \big),$$

where the $\mathbf{p}$*'s are summed over all* $\mathbf{p} \in B_\varepsilon \cap \Sigma f_a$ *which are* <u>*not*</u> *contained in* $V(f_a)$, *the* $\mathbf{q}$*'s are summed over those* $\mathbf{q} \in B_\varepsilon \cap \Sigma f_a$ *which are contained in* $V(f_a)$ *and at which* f_a *has generic polar curve, the* $\{\Sigma^k_a : k = 1, 2, \dots\}$ *are the irreducible components of* $\Sigma V(f_a)$ *in* B_ε, *and* $\mathring{\mu}_k$ *is the generic transverse Milnor number of* Σ^k_a.

Proof. — We use the coordinate system $(s, z_0, \dots, z_n)$ for f.

We will first show that

$$(*) \qquad \lambda^0_{f_0}(\mathbf{0}) = \left(\Gamma^1_f \cdot V(s)\right)_{\mathbf{0}} + \left(\Lambda^1_f \cdot V(s)\right)_{\mathbf{0}},$$

for then - by the dynamic properties of intersection numbers - we can conclude that

$$\lambda^0_{f_0}(\mathbf{0}) = \sum_{\mathbf{p}} \left(\Gamma^1_f \cdot V(s-a)\right)_{\mathbf{p}} + \sum_{\mathbf{q}} \left(\Lambda^1_f \cdot V(s-a)\right)_{\mathbf{q}} + \sum_{\mathbf{r}} \left(\Lambda^1_f \cdot V(s-a)\right)_{\mathbf{r}}$$

where the $\mathbf{p}$'s and $\mathbf{q}$'s are as in the statement of the theorem, and the $\mathbf{r}$'s are summed over all points $\mathbf{r} \in B_\varepsilon \cap \Sigma f_a$ which are contained in $V(f_a)$ and at which f_a has <u>no</u> generic polar curve.

From this, one concludes - from a local application of $(*)$ or see [Ma2] - that

$$(**) \qquad \lambda^0_{f_0}(\mathbf{0}) = \sum_{\mathbf{p}} \lambda^0_{f_a - f_a(\mathbf{p})}(\mathbf{p}) + \sum_{\mathbf{q}} \lambda^0_{f_a}(\mathbf{q}) + \sum_{\mathbf{r}} \lambda^0_{f_a}(\mathbf{r}),$$

and we will replace the r-term to derive the theorem. First though, we show $(*)$.

By definition,

$$\lambda^0_{f_0} = \left(\Gamma^1_{f_0} \cdot V\left(\frac{\partial f_0}{\partial z_0}\right)\right)_{\mathbf{0}} = \left(V\Big(s, \frac{\partial f}{\partial z_1}, \dots \frac{\partial f}{\partial z_n}\Big)/\Sigma f_0 \cdot V\left(\frac{\partial f}{\partial z_0}\right)\right)_{\mathbf{0}}.$$

As f_s is equi-transversal, $\Sigma f \cap V(s) = \Sigma(f_0)$, and so we may use 0.1.i of [Ma2] to conclude that

$$V\left(s, \frac{\partial f}{\partial z_1}, \ldots \frac{\partial f}{\partial z_n}\right) / \Sigma f_0 = \left(V(s) \cap \Gamma_f^2\right) / \Sigma f$$

(see also 5.18 of [Ma3]). Hence, we would like to show that

$$\left(V(s) \cap \Gamma_f^2\right) / \Sigma f = V(s) \cap \Gamma_f^2 = V(s) \cdot \Gamma_f^2$$

as cycles - for then we would have

$$\lambda_{f_0}^0 = \left(\left(V(s) \cdot \Gamma_f^2\right) \cdot V\left(\frac{\partial f}{\partial z_0}\right)\right)_{\mathbf{0}}$$

$$= \left(V(s) \cdot \left(\Gamma_f^2 \cdot V\left(\frac{\partial f}{\partial z_0}\right)\right)\right)_{\mathbf{0}}$$

$$= \left(V(s) \cdot \left(\Gamma_f^1 + \Lambda_f^1\right)\right)_{\mathbf{0}}$$

from which $(*)$ follows. Thus, to prove $(*)$, it remains for us to show that

$$\left(V(s) \cap \Gamma_f^2\right) / \Sigma f = V(s) \cap \Gamma_f^2$$

as cycles. This is where having an equi-transversal deformation is used strongly.

We must show that

$$\left(V(s) \cap \Gamma_f^2\right) / \Sigma f = V(s) \cap \Gamma_f^2$$

up to embedded subvariety - that is, we must show that $V(s) \cap \Gamma_f^2$ has no (isolated) components contained in Σf. But, if $V(s) \cap \Gamma_f^2$ had a component contained in Σf, then $f_{s|_{V(z_0 - p_0)}}$ would have polar curve (see 7.5 of [Ma3]) and, hence, would not have constant Milnor number; a contradiction of condition i) for an equi-transversal deformation. Therefore, we have shown $(*)$, from which $(**)$ follows.

We have finished now with the intersection theoretic portion of the proof. The remainder consists of replacing the term

$$\sum_{\mathbf{r}} \lambda_{f_a}^0(\mathbf{r})$$

in $(**)$ with some more topological information. Using 1.9 together with condition ii) for an equi-transversal deformation, we find that

$$(\dagger) \qquad \sum_{\mathbf{r}} \lambda^0_{f_a}(\mathbf{r}) = \sum_k \overset{\circ}{\mu}_k \sum_{\mathbf{r} \,\in\, \Sigma^k_a} \left(\deg_{\mathbf{r}}(z_0|_{\Sigma^k_a}) - 1\right),$$

where each $\mathbf{r}$ is counted only once since $\Sigma V(f_a)$ is itself smooth at the $\mathbf{r}$'s, as there is no generic polar curve at these points.

We now use some stratified Morse theory [GoMac] to rewrite

$$\sum_{\mathbf{r} \,\in\, \Sigma^k_a} \left(\deg_{\mathbf{r}}(z_0|_{\Sigma^k_a}) - 1\right).$$

For b small $\geqslant 0$, we consider the real valued function $\psi = \mathrm{Re}(z_0 + b)$ on Σ^k_a, and analyze what happens as the value of ψ increases. By the equi-transversal condition, for $\psi \leqslant 0$, our space has the homotopy type of

$$\sum_{\nu} \deg_{\mathbf{0}}(z_0|_{\nu})$$

points, where the sum is over all components, ν , of $\Sigma V(f_0)$ which are contained in the unique component ν' of $\Sigma V(f)$ such that

$$\Sigma^k_a = B_\varepsilon \cap V(s - a) \cap \nu'.$$

The critical points of ψ occur precisely at the points $\mathbf{q}$ and $\mathbf{r}$ which are contained in Σ^k_a. Moreover, as we pass through each of these critical points, we attach (on the level of homology) $\deg\left(z_0|_{\Sigma^k_a} - 1\right)$ one-cells (this is because, locally, one starts with something which is homotopic to $\deg\left(z_0|_{\Sigma^k_a}\right)$ points and end up with something that is contractible). Thus, we arrive at the following equality of Hurewicz type

$$\chi\left(\Sigma^k_a\right) = \sum_{\nu} \deg_{\mathbf{0}}(z_0|_{\nu}) - \sum_{\mathbf{q} \,\in\, \Sigma^k_a} \left(\deg_{\mathbf{q}}(z_0|_{\Sigma^k_a}) - 1\right) - \sum_{\mathbf{r} \,\in\, \Sigma^k_a} \left(\deg_{\mathbf{r}}(z_0|_{\Sigma^k_a}) - 1\right)$$

and, by combining this with $\dagger$), we obtain

$$(\dagger\dagger) \quad \sum_{\mathbf{r}} \lambda^0_{f_a}(\mathbf{r}) = \sum_k \overset{\circ}{\mu}_k \left(\sum_{\nu} \deg_{\mathbf{0}}(z_0|_{\nu}) - \sum_{q \,\in\, \Sigma^k_a} \left(\deg_{\mathbf{q}}(z_0|_{\Sigma^k_a}) - 1\right) - \chi\left(\Sigma^k_a\right)\right).$$

But, by definition of λ^1 and using that $\mathring{\mu}_k = \mathring{\mu}_\nu$, we have

$$\lambda^1_{f_0}(\mathbf{0}) = \sum_k \mathring{\mu}_k \sum_\nu \deg_{\mathbf{0}}(z_{0|_\nu}),$$

and

$$\sum_{\mathbf{q}} \lambda^1_{f_a}(\mathbf{q}) = \sum_k \mathring{\mu}_k \sum_{q\,\in\,\Sigma^k_a} \deg_{\mathbf{q}}(z_{0|_{\Sigma^k_a}}).$$

Combining these two formulas with ††) and (**) yields the theorem. □

COROLLARY 1.12. – *Suppose that* $\dim_{\mathbf{0}}\Sigma\left(f_{0|_{V(z_0)}}\right) = 0$ *and suppose that* f_s *is an equi-transversal deformation of* f_0 *at the origin. Further, suppose that the value of* $\mathring{\mu}_\nu$ *is independent of the component* ν *of* Σf_0 *through the origin - denote this common value by* $\mathring{\mu}$. *Then, in the notation of the theorem, we have*

$$\lambda^0_{f_0}(\mathbf{0}) - \lambda^1_{f_0}(\mathbf{0}) = \sum_{\mathbf{p}} \lambda^0_{f_a - f_a(\mathbf{p})}(\mathbf{p}) + \sum_{\mathbf{q}} \left(\lambda^0_{f_a}(\mathbf{q}) - \lambda^1_{f_a}(\mathbf{q}) + \mathring{\mu}\right)$$
$$- \mathring{\mu} \cdot \chi(B_\varepsilon \cap \Sigma V(f_a)).$$

Proof. — A quick Euler characteristic calculation (using, say, a simplicial decomposition in which all $\mathbf{q}$'s are vertices) gives

$$\chi(B_\varepsilon \cap \Sigma V(f_a)) - \sum_{\mathbf{q}} 1 = \sum_k \left(\chi(\Sigma^k_a) - \sum_{\mathbf{q} \in \Sigma^k_a} 1\right).$$

Combining this with the theorem yields the result. □

2. Vanishing homology and the main theorem.

In this section, we will re-prove the main theorem as stated in theorem 1.11, but now – instead of polar methods – we shall use purely topological methods.

As in section 1, we let $f_0 : (\mathbb{C}^{n+1}, \mathbf{0}) \to (\mathbb{C}, \mathbf{0})$ be a germ of a holomorphic function such that $\dim_0\Sigma(f_0) = 1$. Choose a representative $f_0 : X \to \mathbb{D}$ satisfying the conditions of the Milnor fibration. More precisely, $\mathbb{D} = \mathbb{D}_\eta$ and $X = f_0^{-1}(\mathbb{D}_\eta) \cap B_\varepsilon$ for η and ε small enough, $f_0^{-1}(0)$

transversely intersects ∂B_ε as a stratified set, and $f_0^{-1}(t)$ transversely intersects ∂B_ε for $0 < |t| \leqslant \eta$. The Milnor construction implies that $f_0 : X \to \mathbb{D}$ is a locally trivial fibre bundle over $\mathbb{D}^* = \mathbb{D} - \{0\}$.

We next consider a holomorphic deformation of f_0 :

$$f : X \times S \to \mathbb{D}$$

where $S = B_\rho \subset \mathbb{C}^r$ (mostly we shall take $r = 1$) and we require $f(x, 0) = f_0(x)$. We define $F : X \times S \to \mathbb{D} \times S$ by $F(x, s) = (f(x, s), s)$.

Recall the equivalent characterizations of an *equi-transversal deformation* as given in 1.10.

LEMMA 2.1. — *Let f be an equi-transversal deformation of f_0. Let ΣF be the critical locus of $F : X \times S \to \mathbb{D} \times S$ and let $\Delta(F)$ be the image $F(\Sigma F)$.*

Then,

$$F|_{F^{-1}(\mathbb{D} \times S \setminus \Delta(F))} : F^{-1}(\mathbb{D} \times S \setminus \Delta(F)) \to \mathbb{D} \times S \setminus \Delta(F)$$

is a locally trivial fibration.

Proof. — By 1.10.iii, F is a proper stratified submersion on ∂B_ε. Thus, it suffices to show that F is submersion at all interior points. However, this is the case, as we have explicitly removed the interior critical points. □

We will also need :

LEMMA 2.2. — *Let f be an equi-transversal deformation of f_0. Then, for all a sufficiently small,*

i) *$f_a^{-1}(\mathbb{D}_\eta) \cap B_\varepsilon$ is homeomorphic to $f_0^{-1}(\mathbb{D}_\eta) \cap B_\varepsilon$ and is therefore contractible,*

ii) *$f_a^{-1}(\xi)$ transversely intersects ∂B_ε (in a stratified sense) for all small ξ.*

Proof. — i) : We will show that

$$\begin{array}{c} B_\varepsilon \times \mathbb{C} \cap f^{-1}(\mathbb{D}_\eta) \\ \downarrow s \\ \mathbb{C} \end{array}$$

is a proper, stratified submersion for all s small.

Certainly, in a neighborhood of the origin, we may Whitney stratify $f^{-1}(\mathbb{D}_\eta)$ by $f^{-1}(\partial\mathbb{D}_\eta)$ and $f^{-1}(\mathring{\mathbb{D}}_\eta)$.

That s is a stratified submersion on $B_\varepsilon \times \mathbb{C} \cap f^{-1}(\mathring{\mathbb{D}}_\eta)$ requires nothing.

That s is a submersion on $\mathring{B}_\varepsilon \times \mathbb{C} \cap f^{-1}(\partial\mathbb{D}_\eta)$ follows from the fact that the intersection of the polar curve, $\Gamma^1_{f,s}$, with $\mathring{B}_\varepsilon \times 0$ is just $\{\mathbf{0}\}$.

Finally, that s is a submersion on $\partial B_\varepsilon \times \mathbb{C} \cap f^{-1}(\partial\mathbb{D}_\eta)$ follows from condition iii) of being equi-transversal.

This proofs i).

ii) Except where $f = 0$, this follows from condition iii) for being equi-transversal. We will now show that it is also true along $V(f)$.

Equi-transversality implies that $V(s)$ transversely intersects $V(f) - \Sigma V(f)$ in a neighborhood of the origin and that $\partial B_\varepsilon \times 0$ transversely intersects $\Sigma V(f)$, where $\Sigma V(f)$ is smooth along this intersection by equi-transversality.

Using nothing, it follows then that $\partial B_\varepsilon \times \{s\}$ transversely intersects $\Sigma V(f)$ for all s small.

We would like to see that $V(f_{s_0}) - \Sigma V(f_{s_0})$ transversely intersects $\partial B_\varepsilon \times \{s_0\}$ for all s_0 small.

Now, $\partial B_\varepsilon \times \{s_0\}$ misses $\Gamma^1_{f,s}$ for s_0 small. Hence, along the intersection in which we are interested, $V(s - s_0)$ transversely intersects $V(f) - \Sigma V(f)$ and

$$V(f_{s_0}) - \Sigma V(f_{s_0}) = V(s - s_0) \cap (V(f) - \Sigma V(f)).$$

Thus, what we want to show is that

$$\begin{array}{c} \partial B_\varepsilon \times \mathbb{C} \cap (V(f) - \Sigma V(f)) \\ \downarrow s \\ \mathbb{C} \end{array}$$

is a submersion for all s small.

Suppose not. We would have a sequence of points

$$(\mathbf{p}_i, s_i) \to (\mathbf{q}, 0) \in \partial B_\varepsilon \times 0,$$

where $\mathbf{p}_i \in \partial B_\varepsilon, f(\mathbf{p}_i, s_i) = 0,\ T_{(\mathbf{p}_i, s_i)} V(f) \to \mathcal{T}$, and

$$(*) \quad T_{(\mathbf{p}_i, s_i)} V(f) \cap (\mathbb{C}^{n+1} \times 0) \subseteq T_{\mathbf{p}_i} \partial B_\varepsilon \times 0.$$

If $(\mathbf{q}, 0)$ is a smooth point of $V(f)$, then $\mathcal{T} = T_{(\mathbf{q},0)} V(f)$ and $(*)$ would imply that $T_{\mathbf{q}} V(f_0) \subseteq T_{\mathbf{q}} \partial B_\varepsilon$. However, we could have initially chosen B_ε so as to make this impossible.

If $(\mathbf{q}, 0) \in \Sigma V(f)$, then by †† of 1.10, $T_{(\mathbf{q},0)} \Sigma V(f) \subseteq \mathcal{T}$. Hence, as $\partial B_\varepsilon \times 0$ transversely intersects $\Sigma V(f)$, we are finished. □

We now give our second proof of the main theorem.

Theorem 2.3. — *Let B_ε be a sufficiently small closed ball around the origin in $\mathbb{C}^{n+1}$ and let a be a small value of the deformation parameter. Then,*

$$\tilde{b}_n(F) - \tilde{b}_{n-1}(F) = \sum_{\mathbf{p}} \tilde{b}_n(F_{\mathbf{p}}) + \sum_{\mathbf{q}} \left(\tilde{b}_n(F_{\mathbf{q}}) - \tilde{b}_{n-1}(F_{\mathbf{q}}) \right)$$
$$- \sum_k \mathring{\mu}_k \Big(\chi(\Sigma_a^k) - \sum_{\mathbf{q} \in \Sigma_a^k} 1 \Big),$$

where F denotes the Milnor fibre of f_0 at the origin, $F_{\mathbf{x}}$ denotes the Milnor fibre of $f_a - f_a(\mathbf{x})$ at the point $\mathbf{x}$, the $\mathbf{p}$'s are summed over all $\mathbf{p} \in B_\varepsilon \cap \Sigma f_a$ which are <u>not</u> contained in $V(f_a)$, the $\mathbf{q}$'s are summed over those $\mathbf{q}$ in $B_\varepsilon \cap \Sigma f_a$ which are contained in $V(f_a)$ and at which f_a has generic polar curve, the $\{\Sigma_a^k : k = 1, 2, \dots\}$ are the irreducible components of $\Sigma V(f_a)$ in B_ε, and $\mathring{\mu}_k$ is the generic transverse Milnor number of Σ_a^k.

Proof. — We consider the critical locus $\Sigma(f_a)$ of f_a. Let $\Sigma(f_a) = \Sigma_a \cup P$ where Σ_a is the 1-dimensional piece of $\Sigma(f_a)$ and $P = \{\mathbf{p}_1, \dots, \mathbf{p}_m\}$ is the set of isolated points of $\Sigma(f_a)$.

First define for every $\mathbf{p} \in P$ pairs $(E_{\mathbf{p}}, F_{\mathbf{p}})$ consisting of a local Milnor ball, $E_{\mathbf{p}}$, and a local Milnor fibre, $F_{\mathbf{p}}$. This construction is also used in [Si3].

We next consider a neighborhood of the 1-dimensional piece

$$\Sigma_a = \Sigma_a^1 \cup \dots \cup \Sigma_a^r.$$

Let $\{\mathbf{q}_1, \dots, \mathbf{q}_d\}$ be the following collection of special points on Σ_a :

- all points where Σ_a is non-smooth
- the smooth points of Σ_a where there is a non-void generic polar in curve.

Let $Q = Q_1 \cup \dots \cup Q_d$ be the union of well-chosen distinct local Milnor balls around each $\mathbf{q}$. Then, there exists a neighborhood, E^0 , of Σ_a together with $F^0 = f_a^{-1}(t_0) \cap E^0$, where t_0 is close to $f_a(\Sigma_a)$, a (piece of a) Milnor fibre, such that

$$(E^0 \backslash Q, F^0 \backslash Q) \rightarrow \Sigma_a \backslash Q$$

is a locally trivial fibre bundle pair, where the fibre pair is $(\overline{E}_k, \overline{F}_k)$ – the Milnor pair of the transversal singularity Y_k on $\Sigma_a^k - 0$.

Let $E = f_0^{-1}(\mathbb{D}_\eta) \cap B_\varepsilon$. According to lemma 2.2.i, E is homeomorphic to $f_a^{-1}(\mathbb{D}_\eta) \cap B_\varepsilon$ and, according to lemma 2.1, F is homeomorphic to $f_a^{-1}(t_0) \cap B_\varepsilon$.

Now we use the direct sum formula for the vanishing homology

$$H_*(E, F) = \bigoplus_{\mathbf{p}} H_*(E_{\mathbf{p}}, F_{\mathbf{p}}) \oplus H_*(E^0, F^0).$$

This formula already occurs in [Si3] and is stated there for the transversal type A_1. Under the transversality conditions of lemma 2.2.ii, the formula holds in general and the proof remains unchanged.

As a corollary, we have

$$\chi(E, F) = \sum_{\mathbf{p}} \chi(E_{\mathbf{p}}, F_{\mathbf{p}}) + \chi(E^0, F^0),$$

where

$$\chi(E, F) = \sum_j (-1)^j \dim H_j(E, F) = \chi(E) - \chi(F) = 1 - \chi(F).$$

At the isolated singularities, $\mathbf{p}$, we have :

$$\chi(E_{\mathbf{p}}, F_{\mathbf{p}}) = \chi(E_{\mathbf{p}}) - \chi(F_{\mathbf{p}}) = (-1)^n b_n(F_{\mathbf{p}}).$$

The remaining part is $\chi(E^0, F^0)$. We assume that the neighborhood E^0 of Σ_a is chosen small enough so that it contains Σ_a as a deformation retract. Then, we have :

$$\chi(E^0, F^0) = \chi(E^0) - \chi(F^0) = \chi(\Sigma_a) - \chi(F^0).$$

Next, cut F^0 into pieces, according to the components of Q and of $\Sigma_a^k \backslash Q$. So,

$$\chi(F^0) = \sum_k \chi(\Sigma_a^k \backslash Q) \times \ \chi(Y_k) \ + \ \sum_{\mathbf{q}} \chi(F_{\mathbf{q}}),$$

where :

$F_{\mathbf{q}} =$ the Milnor fibre of the local singularity at $\mathbf{q}$

$Y_k =$ the transversal Milnor fibre of $\Sigma_a^k - \{0\}$.

Remark that :

$$\chi(\Sigma_a^k \backslash Q) = \chi(\Sigma_a^k) - \sum_{\mathbf{q}} n_{k,\mathbf{q}},$$

where

$$n_{k,\mathbf{q}} = \begin{matrix} 0 & \text{if} & \mathbf{q} \notin \Sigma_a^k \\ 1 & \text{if} & \mathbf{q} \in \Sigma_a^k. \end{matrix}$$

We conclude :

$$\chi(F^0) = \sum_k \chi(\Sigma_a^k) \cdot \chi(Y_k) - \sum_k \sum_{\mathbf{q}} n_{k,\mathbf{q}} \chi(Y_k) + \sum \chi(F_{\mathbf{q}}).$$

So :

$$\chi(E,F) = \chi(\Sigma_a) - \sum_k \chi(\Sigma_a^k) \cdot \chi(Y_k) + \sum_k \sum_{\mathbf{q}} n_{k,\mathbf{q}} \chi(Y_k) \\ - \sum_{\mathbf{q}} \chi(F_{\mathbf{q}}) + \sum_{\mathbf{p}} \chi(E_{\mathbf{p}}, F_{\mathbf{p}}).$$

One concludes the theorem from this formula by substituting the following easy identities :

1. $\chi(E,F) = (-1)^{n+1}\{\tilde{b}_n(F) - \tilde{b}_{n-1}(F)\}$
2. $\chi(E_{\mathbf{p}}, F_{\mathbf{p}}) = (-1)^{n+1}\tilde{b}_n(F_{\mathbf{p}})$
3. $\chi(Y_k) = 1 + (-1)^n \mathring{\mu}_k$
4. $\chi(F_{\mathbf{q}}) = (-1)^{n+1}\{\tilde{b}_n(F_{\mathbf{q}}) - \tilde{b}_{n-1}(F_{\mathbf{q}}) + 1\}$
5. $\sum_k \chi(\Sigma_a^k) = \chi(\Sigma_a) + \sum_k \sum_{\mathbf{q}} (n_{k,\mathbf{q}} - 1).$ $\square$

As in section 1, we immediately have the following corollary, which we now state solely in terms of Betti numbers.

COROLLARY 2.4. — *Suppose that* $\dim_{\mathbf{0}}\Sigma\left(f_{0|_{V(z_0)}}\right) = 0$ *and suppose that* f_s *is an equi-transversal deformation of* f_0 *at the origin. Further, suppose that the value of* $\overset{\circ}{\mu}_\nu$ *is independent of the component* ν *of* Σf_0 *through the origin - denote this common value by* $\overset{\circ}{\mu}$. *Then, in the notation of the theorem, we have*

$$\tilde{b}_n(F) - \tilde{b}_{n-1}(F) = \sum_{\mathbf{p}} \tilde{b}_n(F_{\mathbf{p}}) + \sum_{\mathbf{q}} \left(\tilde{b}_n(F_{\mathbf{q}}) - \tilde{b}_{n-1}(F_{\mathbf{q}}) + \overset{\circ}{\mu}\right) - \overset{\circ}{\mu} \cdot \chi(B_\varepsilon \cap \Sigma V(f_a)) .$$

3. The Lê-Iomdine formula.

We now wish to give a slightly improved version of the formula of Iomdine and Lê as found in [Lê2] and [Io2]. See also [Ma2], [Ma3], [Ma4], [Pe3], and [Si5]. The improvement is a more precise bound on how large the exponent in the formula must be chosen. This new bound is especially useful in the homogeneous and quasi-homogeneous cases.

PROPOSITION 3.1. — *Let* $h : (\mathbb{C}^{n+1}, \mathbf{0}) \to (\mathbb{C}, \mathbf{0})$ *be analytic and let* $\mathbf{z}$ *be a linear choice of coordinates for* $\mathbb{C}^{n+1}$ *such that* $\dim_{\mathbf{0}}\Sigma(h_{|_{V(z_0)}}) = 0$. *Let* j *be an integer* $\geqslant 2$.

Then, $h + \varepsilon z_0^j$ *has an isolated singularity at the origin for all but finitely many* ε. *Moreover, if* j *is not a polar ratio (for the definition of polar ratio, see [Si5] or the proof below), then* $h + \varepsilon z_0^j$ *has an isolated singularity at the origin for all* $\varepsilon \neq 0$.

Finally, if j *is greater than or equal to the maximum of the polar ratios and* $h + \varepsilon z_0^j$ *has an isolated singularity at the origin, then*

$$\lambda^0_{h+\varepsilon z_0^j}(\mathbf{0}) = \lambda^0_h(\mathbf{0}) + (j-1)\lambda^1_h(\mathbf{0}) = \lambda^0_h(\mathbf{0}) - \lambda^1_h(\mathbf{0}) + j\lambda^1_h(\mathbf{0})$$

$$= \tilde{b}_n(F_{\mathbf{0}}) - \tilde{b}_{n-1}(F_{\mathbf{0}}) + j\lambda^1_h(\mathbf{0}) ,$$

where $\lambda^0_{h+\varepsilon z_0^j}(\mathbf{0})$ *is equal to the Milnor number of* $h + \varepsilon z_0^j$.

Proof. — The proof is the same as that of 3.1.ii and 3.2 of [Ma3] but, as our statement here is slightly different, we shall recall the proof.

Since $\Gamma^1_{h,\mathbf{z}}$ is contained in $V\left(\dfrac{\partial h}{\partial z_1}, \dots, \dfrac{\partial h}{\partial z_n}\right)$,

$$\Gamma^1_{h,\mathbf{z}} \cap V(z_0) \subseteq \Sigma(h_{|V(z_0)}).$$

Thus, it follows from our hypothesis that $\dim_0 \Gamma^1_{h,\mathbf{z}} \cap V(z_0) = 0$. It is immediate then that $\dim_0 \Gamma^1_{h,\mathbf{z}} \leqslant 1$ and $\dim_0 \Gamma^1_{h,\mathbf{z}} \cap V\left(\dfrac{\partial h}{\partial z_0}\right) = 0$.

Now, write the cycle $\Gamma^1_{h,\mathbf{z}}$ as $\sum_\eta k_\eta[\eta]$, where the η are the irreducible components of $\Gamma^1_{h,\mathbf{z}}$. Consider the intersection number

$$\left(\Gamma^1_{h,\mathbf{z}} \cdot V\left(\frac{\partial h}{\partial z_0} + j\varepsilon z_0^{j-1}\right)\right)_{\mathbf{0}} = \sum k_\eta \left(\eta \cdot V\left(\frac{\partial h}{\partial z_0} + j\varepsilon z_0^{j-1}\right)\right)_{\mathbf{0}}.$$

We would like to impose conditions on j and ε so that each

$$\left(\eta \cdot V\left(\frac{\partial h}{\partial z_0} + j\varepsilon z_0^{j-1}\right)\right)_{\mathbf{0}}$$

actually equals

$$\left(\eta \cdot V\left(\frac{\partial h}{\partial z_0}\right)\right)_{\mathbf{0}}$$

for then we would have

$$(\dagger) \qquad \left(\Gamma^1_{h,\mathbf{z}} \cdot V\left(\frac{\partial h}{\partial z_0} + j\varepsilon z_0^{j-1}\right)\right)_{\mathbf{0}} = \left(\Gamma^1_{h,\mathbf{z}} \cdot V\left(\frac{\partial h}{\partial z_0}\right)\right)_{\mathbf{0}} = \lambda^0_h(\mathbf{0}).$$

By [Fu], if we let $\alpha_\eta(t)$ be a parametrization of η, we may calculate the intersection number

$$\left(\eta \cdot V\left(\frac{\partial h}{\partial z_0} + j\varepsilon z_0^{j-1}\right)\right)_{\mathbf{0}}$$

by taking the t-multiplicity of $\left(\dfrac{\partial h}{\partial z_0} + j\varepsilon z_0^{j-1}\right)_{|\alpha_\eta(t)}$.

Using this same method of calculating the intersection number twice more, we conclude that

$$\left(\eta \cdot V\left(\frac{\partial h}{\partial z_0} + j\varepsilon z_0^{j-1}\right)\right)_{\mathbf{0}}$$
$$= \min\left\{\left(\eta \cdot V\left(\frac{\partial h}{\partial z_0}\right)\right)_{\mathbf{0}}, \left(\eta \cdot V\left(z_0^{j-1}\right)\right)_{\mathbf{0}}\right\}$$

with the exception of the single value of ε which makes the lowest degree terms of $\left(\frac{\partial h}{\partial z_0}\right)|_{\alpha_\eta(t)}$ and $\left(j\varepsilon z_0^{j-1}\right)|_{\alpha_\eta(t)}$ add up to zero.

Thus, using that

$$\left(\eta \cdot V\left(z_0^{j-1}\right)\right)_{\mathbf{0}} = (j-1)\left(\eta \cdot V(z_0)\right)_{\mathbf{0}},$$

we find that if we have

$$(*) \qquad j \geqslant \frac{\left(\eta \cdot V\left(\frac{\partial h}{\partial z_0}\right)\right)_{\mathbf{0}}}{\left(\eta \cdot V(z_0)\right)_{\mathbf{0}}} + 1$$

for all η, then (†) holds for all but a finite number of ε. In addition, if we choose strict inequalities in $(*)$, then (†) holds for every value of ε.

Now, the Milnor number of $h + \varepsilon z_0^j$ at the origin equals

$$\left(V\left(\frac{\partial h}{\partial z_0} + j\varepsilon z_0^{j-1}, \frac{\partial h}{\partial z_1}, \ldots, \frac{\partial h}{\partial z_n}\right)\right)_{\mathbf{0}}$$
$$= \left(V\left(\frac{\partial h}{\partial z_0} + j\varepsilon z_0^{j-1}\right) \cdot V\left(\frac{\partial h}{\partial z_1}, \ldots, \frac{\partial h}{\partial z_n}\right)\right)_{\mathbf{0}}$$
$$= \left(V\left(\frac{\partial h}{\partial z_0} + j\varepsilon z_0^{j-1}\right) \cdot \left(\Gamma_h^1 + \Lambda_h^1\right)\right)_{\mathbf{0}}$$
$$= \left(V\left(\frac{\partial h}{\partial z_0} + j\varepsilon z_0^{j-1}\right) \cdot \Gamma_h^1\right)_{\mathbf{0}} + \left(V\left(\frac{\partial h}{\partial z_0} + j\varepsilon z_0^{j-1}\right) \cdot \Lambda_h^1\right)_{\mathbf{0}}.$$

But, as Λ_h^1 is contained in $V\left(\frac{\partial h}{\partial z_0}\right)$, if $\varepsilon \neq 0$ then we have

$$\left(V\left(\frac{\partial h}{\partial z_0} + j\varepsilon z_0^{j-1}\right) \cdot \Lambda_h^1\right)_{\mathbf{0}} = (j-1)\left(\Lambda_h^1 \cdot V(z_0)\right)_{\mathbf{0}}.$$

Combining this with the above and (†), one obtains the result - with the condition given in $(*)$.

Finally, we need to show that $(*)$ is equivalent to the more usual requirement on the polar ratios. As was shown in [Lê1] and [Ma3], a quick application of the chain rule yields that, for every component η of Γ^1_h,

$$(\eta \cdot V(h))_{\mathbf{0}} = \left(\eta \cdot V\left(\frac{\partial h}{\partial z_0}\right)\right)_{\mathbf{0}} + (\eta \cdot V(z_0))_{\mathbf{0}}.$$

Therefore $(*)$ can be re-written as

$$j \geqslant \frac{(\eta \cdot V(h))_{\mathbf{0}}}{(\eta \cdot V(z_0))_{\mathbf{0}}},$$

where the right-hand side gives precisely the polar ratios (or, as is frequently done, one may push everything down into the Cerf diagram). □

The maximum of the polar ratios as a bound already occurs (in a slightly hidden form) in section 3 of [Si5]. In that paper, the Lê-Iomdine formula is generalized to the eigenvalues of the monodromy. For the same kind of statements for the spectrum we refer to Steenbrink[St] and M. Saito [Sa].

Remark 3.2. — Examining the above proof or section 3 of [Si5] slightly closer, it is possible to show that :

$$\lambda^0_{h+\varepsilon z_0^j}(\mathbf{0}) < \lambda^0_h(\mathbf{0}) - \lambda^1_h(\mathbf{0}) + j\lambda^1_h(\mathbf{0}),$$

provided that j is smaller than the maximum of the polar ratios (see, also, the closing remarks). This shows that our bound is sharp.

4. Examples and special cases.

4.A. Transverse A_1 singularities.

In this sub-section, we will restrict ourselves to the case where the generic transverse singularity of f_0 is of type A_1.

COROLLARY 4.A.1. — *Suppose that f_s is a family of analytic maps such that, as germs of sets at the origin, $\Sigma f \cap V(s) = \Sigma(f_0)$. Suppose* *also that* $\dim_{\mathbf{0}}\Sigma\left(f_{0|_{V(z_0)}}\right) = 0$ *and* $\overset{\circ}{\mu}_\nu = 1$ *for every component* ν *of* Σf_0 *through the origin. Then,* f_s *is an equi-transversal deformation of* f_0 *and*

$$\lambda^0_{f_0}(\mathbf{0}) - \lambda^1_{f_0}(\mathbf{0}) = \sum_{\mathbf{p}} \lambda^0_{f_a - f_a(\mathbf{p})}(\mathbf{p}) + \sum_{\mathbf{q}} \left(\lambda^0_{f_a}(\mathbf{q}) - \lambda^1_{f_a}(\mathbf{q}) + 1\right) \\ -\chi\left(B_\varepsilon \cap \Sigma V(f_a)\right).$$

In terms of Betti numbers, this says

$$\tilde{b}_n(F) - \tilde{b}_{n-1}(F) = \sum_{\mathbf{p}} \tilde{b}_n(F_{\mathbf{p}}) + \sum_{\mathbf{q}} \left(\tilde{b}_n(F_{\mathbf{q}}) - \tilde{b}_{n-1}(F_{\mathbf{q}}) + 1\right) \\ - \chi\left(B_\varepsilon \cap \Sigma V(f_a)\right),$$

where the notation is the same as that of theorems 1.11 and 2.3.

Proof. — That f_s must be an equi-transversal deformation follows from the upper-semicontinuity of the Milnor number - if the generic transverse Milnor number of f_0 is 1, then for $s \neq 0$ small the generic transverse Milnor number must be 0 or 1. But it can not be 0 since $\Sigma f \cap V(s) = \Sigma(f_0)$. Thus, the deformation is equi-transversal.

Now, the equations follow immediately from 1.11 and 2.3. □

Example 4.A.2. — Suppose that $\Sigma = \Sigma(f_0)$ is a 1-dimensional, isolated complete intersection singularity. This situation is studied in detail in [Si3]. In this case, there exist equi-transversal deformations with Σ_a smooth (equal to the Milnor fibre of the singular curve Σ) and with f_a having only A_∞ and D_∞ singularities on Σ_a, and the isolated singularities of f_a are all of type A_1.

The main theorem of [Si3] is :

In this case, the homotopy type of the Milnor fibre, F, of f_0 is a bouquet of spheres; there are two cases :

if $\#D_\infty > 0$, then $F \simeq S^n \vee \ldots \vee S^n$;

if $\#D_\infty = 0$, then $F \simeq S^{n-1} \vee S^n \vee \ldots \vee S^n$.

Moreover,

$$\tilde{b}_n(F) - \tilde{b}_{n-1}(F) = \mu(\Sigma) - 1 + 2\#D_\infty + \#A_1,$$

where $\mu(\Sigma)$ is the Milnor number of Σ.

Note that this formula agrees with that of our main theorem.

The same statement on the homotopy type of the Milnor fibre is true in all cases where f allows a deformation with only A_∞, D_∞, and A_1 - singularities. The example $T_{\infty,\infty,\infty}$, given by $f = xyz$, shows that such deformations do not always exist.

There exist the following formulas [Pe1],[Pe2] relating $\#A_1$ and $\#D_\infty$ to the dimension of a certain local ring :

$$j(f) = \dim \frac{I}{J(f)} = \#A_1 + \#D_\infty,$$

where I is the reduced ideal defining Σ and $J(f)$ is the Jacobian ideal of f; and (still under the assumption that Σ is a complete intersection)

$$\delta(f) = \dim \frac{O_\Sigma}{\det(h_{ij})} = \#D_\infty,$$

where $f = \Sigma h_{ij} g_i g_j$ and $\{g_1, \dots, g_n\}$ define Σ as a reduced 1-dimensional icis.

In the next two examples, we remain in the transverse A_1-case, but consider the case where there exists a deformation of f_0 with only A_∞ , D_∞ , $T_{\infty\infty\infty}$, and A_1 singularities. In this case, we have

COROLLARY 4.A.3. — *Suppose that f_s is a family of analytic maps such that, as germs of sets at the origin, $\Sigma f \cap V(s) = \Sigma(f_0)$. Suppose also that $\dim_{\mathbf{0}} \Sigma \left(f_{0|_{V(z_0)}} \right) = 0$ and $\mathring{\mu}_\nu = 1$ for every component ν of Σf_0 through the origin. In addition, suppose that the deformation f_s has only A_1 singularities off $V(f_t)$ and only A_∞, D_∞, and $T_{\infty,\infty,\infty}$ singularities on $V(f_t)$, then*

$$\lambda^0_{f_0}(\mathbf{0}) - \lambda^1_{f_0}(\mathbf{0}) = \tilde{b}_n(F) - \tilde{b}_{n-1}(F) = \#A_1 + 2\#D_\infty - \chi\left(B_\varepsilon \cap \Sigma V(f_a)\right).$$

Proof. — This follows from 4.A.1 and some quick calculations. $\lambda^0 = 1$ at a quadractic singularity, $\lambda^0_{f_a} = 2$ and $\lambda^1_{f_a} = 1$ at a D_∞ point, and $\lambda^0_{f_a} = 2$ and $\lambda^1_{f_a} = 3$ at a $T_{\infty,\infty,\infty}$ point. □

Example 4.A.4. — In his papers [Mo1],[Mo2], D. Mond considers finitely $\mathcal{A}$-determined map germs $F : (\mathbb{C}^2, \mathbf{0}) \to (\mathbb{C}^3, \mathbf{0})$. The image $F(\mathbb{C}^2)$ is a hypersurface germ at the origin, given by some $f = 0$, and has a 1-dimensional singular locus Σ with transversal type A_1 on $\Sigma - \{0\}$.

One can consider the versal unfolding

$$\tilde{G} : \mathbb{C}^2 \times \mathbb{C}^d \to \mathbb{C}^3 \times \mathbb{C}^d.$$

Let $\tilde{G}(x,s) = (G_s(x), s)$ and let the image $G_s(\mathbb{C}^2)$ be the hypersurface germ with defining equation $g_s = 0$. According to Mond, near the origin, the map $g_s : \mathbb{C}^3 \to \mathbb{C}$ has for all $s \in \mathbb{C}^d$ only one fibre with non-isolated singularities and for generic $s \in \mathbb{C}^d$ the only singularities are of types A_∞, D_∞, $T_{\infty\infty\infty}$, and A_1.

It is shown in [Mo2] and [Si7] that the special (non-isolated) fibre is homotopy equivalent to a bouquet of 2-spheres

$$g_s^{-1}(0) \simeq S^2 \vee \ldots \vee S^2,$$

where the number of spheres is equal to $\#A_1$.

Moreover, the Milnor fibre of g_s is, in this case, also a bouquet of spheres :

$$F \simeq S^2 \vee \ldots \vee S^2.$$

The number of spheres is, according to [Si7], equal to :

$$b_n(F) = 2\#D_\infty - 1 + 2\#T_{\infty\infty\infty} - \chi(\tilde{\Sigma}_s) + \#A_1,$$

where $\tilde{\Sigma}_s$ is the normalization of Σ_s.

This formula agrees with the formula of our main theorem since

$$\chi(\tilde{\Sigma}_s) = \chi(\Sigma_s) + 2\#T_{\infty\infty\infty}.$$

A list of these very interesting examples appears in [Mo1].

Example 4.A.5. — In [Pe1] and [Pe2], Pellikaan considers the polynomial $f_0 = x^2y^2 + y^2z^2 + 4z^2x^2$. There exist two totally different equi-transversal deformations of f_0.

- $f_s = x^2y^2 + y^2z^2 + 4z^2x^2 + sxyz,$

  in which Σ_s consists of the three coordinate axes, and

$$\#A_1 = 4 \quad \#D_\infty = 6 \quad \#T_{\infty\infty\infty} = 1 \quad \chi(\Sigma_s) = 1.$$

Therefore, $b_2(F) - b_1(F) = -1 + 12 + 4 = 15$. We remark that, by using a computer, this Euler characteristic calculation can be produced quickly via the Iomdine-Lê formula – see the closing remarks.

- $f_a = (xy - a_2x - a_1y)^2 + (yz + a_3y - a_2z)^2 + (2xz + a_3x - a_1z)^2.$

This deformation is induced by the miniversal deformation of $(\Sigma\ ,0)$, which can be defined by the vanishing of the $2 \times\ 2$ minors of the matrix

$$\begin{pmatrix} x & y & z \\ x+a_1 & 2y+a_2 & 3z+a_3 \end{pmatrix}.$$

Here, Σ_a is a Milnor fibre of Σ, and $\mu(\Sigma) = 2$.

$$\#A_1 = 6 \qquad \#D_\infty\ = 4 \qquad \#T_{\infty\infty\infty} = 0 \qquad \chi(\Sigma_s) = -1.$$

Therefore, $b_2(F) - b_1(F) = 6 + 8 + 1 = 15$.

Since the deformation has only A_∞, D_∞, and A_1 -points, we conclude that

$$F \simeq\ S^2 \vee\ \ldots\ \vee\ S^2,$$

where the number of spheres is exactly 15.

Note that, despite the ease with which they may be calculated, the Lê numbers and the formula of Iomdine and Lê provide no indication that the homology is trivial in dimension 1.

Remark 4.A.6. — We should mention here one other formula for the Euler characteristic of the Milnor fibre in the transversal A_1 case. In [Jo2], de Jong gives the following formula :

$$b_n(F) - b_{n-1}(F) = j(f) + VD_\infty\ + \mu(\Sigma) - 1,$$

where F, μ , and Σ are as before, VD_∞ is the virtual number of D_∞ points, and $j(f) = \dim_{\mathbb{C}} \dfrac{I}{J(f)}$, where $J(f)$ is the Jacobian ideal and I is its radical.

While it seems that this formula should follow directly from ours, we have yet to show that this is, in fact, the case.

4.B. Line singularities.

The topology of line singularities is studied in [Si1] for transversal type A_1 and in [Jo1] for the transversal types

$$S \in\ \{A_1, A_2, A_3, D_4, E_6, E_7(n = 2), E_8(n = 2)\}.$$

De Jong produced a list of elementary non-trivial line singularities, F_iS , of type S; a list which is complete up to stable equivalence.

Next, he constructed an equi-transversal deformation with only F_iS points and A_1 points as special points. His main theorem states :

Let $f : (\mathbb{C}^{n+1}, \mathbf{0}) \to (\mathbb{C}, 0)$ be a line singularity of type S. Then, the Milnor fibre F of f has the homotopy type of a bouquet of spheres :

$$\bigvee_{\varepsilon} S^{n-1} \vee \bigvee_{\mu+\varepsilon} S^n$$

with

$$\mu = b_n(F) - b_{n-1}(F) = \sum \alpha_i h_i + \#A_1 - \overset{\circ}{\mu},$$

where h_i is the number of F_iS points in the above-mentioned equi - transversal deformation and α_i and ε can be computed explicitly. Only in exceptional cases is $\varepsilon \neq 0$ and in these cases ε is small, $\varepsilon = 0, 1,$ or $\overset{\circ}{\mu}$.

The formula for μ above is connected with those in 4.A.1. In fact, if $\mathbf{q}$ is a point of F_iS, then

$$\alpha_i = b_n(F_{\mathbf{q}}) - b_{n-1}(F_{\mathbf{q}}) + \overset{\circ}{\mu}$$

in our notation. Thus, the formulas are all equivalent in the cases considered by de Jong.

4.C. Plane curve singularities.

Example 4.C.1. — Let h be an analytic map in the variables x and y, and suppose that $h = P \prod Q_i^{\alpha_i}$, where P and $\prod Q_i^{\alpha_i}$ are relatively prime and $\alpha_i \geqslant 2$, i.e. h gives a non-reduced curve singularity. We wish to calculate the Lê numbers of h at the origin.

Let $z_0 = ax + by$, where $a \neq 0$, and let $z_1 = y$. Then,

$$\left[V\left(\frac{\partial h}{\partial z_1}\right)\right] = \left[V\left(\frac{\partial h}{\partial x}\left(\frac{-b}{a}\right) + \frac{\partial h}{\partial y}\right)\right]$$

$$= \sum \left[V\left(Q_i^{\alpha_i - 1}\right)\right] + \left[V\left(\frac{\frac{\partial h}{\partial x}\left(\frac{-b}{a}\right) + \frac{\partial h}{\partial y}}{\prod Q_i^{\alpha_i - 1}}\right)\right] = \Lambda^1_{h,\mathbf{z}} + \Gamma^1_{h,\mathbf{z}}.$$

Thus, whenever

$$\left[V\left(\frac{\dfrac{\partial h}{\partial x}\left(\dfrac{-b}{a}\right)+\dfrac{\partial h}{\partial y}}{\prod Q_i^{\alpha_i-1}}\right)\right]$$

has no components contained in the critical locus of h (an easy argument shows that this is the case for a generic choice of (a,b)), we have that

$$\lambda^1_{h,\mathbf{z}}=\sum(\alpha_i-1)\left(V(Q_i)\cdot V(ax+by)\right)_{\mathbf{0}}$$

and

$$\lambda^0_{h,\mathbf{z}}=\left(V\left(\frac{\dfrac{\partial h}{\partial x}\left(\dfrac{-b}{a}\right)+\dfrac{\partial h}{\partial y}}{\prod Q_i^{\alpha_i-1}}\right)\cdot V\left(\frac{\partial h}{\partial x}\right)\right)_{\mathbf{0}},$$

where we have used that $V\left(\dfrac{\partial h}{\partial z_0}\right)=V\left(\dfrac{\partial h}{\partial x}\right)$.

Note that the formula

$$\lambda^1_{h,\mathbf{z}}=\sum(\alpha_i-1)\left(V(Q_i)\cdot V(ax+by)\right)_{\mathbf{0}}$$

agrees with our earlier formula

$$\lambda^1_{h,\mathbf{z}}=\sum_{\nu}n_\nu\mathring{\mu}_\nu,$$

since we clearly have $n_\nu=\left(V(Q_i)\cdot V(ax+by)\right)_{\mathbf{0}}$ and $\mathring{\mu}_\nu=\alpha_i-1$.

Example 4.C.2. — Deformation of non-reduced plane curve singularities are studied by Schrauwen in [Sc]. Among others, he considers especially deformations where the singularity splits up into $D[p,q]$ singularities only, having local equation $x^py^q=0$.

Let Σ^p be the curve consisting of all branches with multiplicity p. We give Σ^p the reduced structure. Then, Schrauwen gives

FORMULA. — *Let f_a be an equi-transversal deformation which makes each Σ^p smooth. Then,*

$$\begin{aligned}&b_1(F)-b_0(F)\\&\quad=\sum_{p<q}(p+q-1)\#D[p,q]+\#D[1,1]+\sum_k(k-1)\left(\mu(\Sigma^k)-1\right)-1,\end{aligned}$$

where the first summation is over the $D[p,q]$-points with $p < q$, and the second is over all multiplicities k.

This formula is now a direct consequence of our main theorem, once you have noticed that the zero Betti numbers, b_0 , have to be replaced by the reduced Betti numbers, $\tilde{b}_0$, and use that $b_1(F_q) - \tilde{b}_0(F_q) = 1$ for each $D[p,q]$-point, and $\chi(\Sigma_a^k) = 1 - \mu(\Sigma^k)$.

Schrauwen considers in his paper also a second kind of deformation, called a *network map deformation.* Such deformations arise as follows. Deform first the reduced singularity f_R in such a way that one gets the maximal number of normal crossings δ, the (virtual) number of double points.

The branches of f_R and $(f_R)_a$ are in one-to-one correspondence. We get the network map deformation by giving the branches of $(f_R)_a$ the correct multiplicity of f. Then, we have :

FORMULA FOR NETWORK MAP DEFORMATIONS

$$b_1(F) - b_0(F) = \sum (p+q)\#D^0[p,q] - S,$$

where the first sum runs over all $D[p,q]$-points on $f_a^{-1}(0)$ with $p \leqslant q$ and $S = \sum m_i$ = the number of all branches counted with multiplicities.

This formula is similar to $\mu = 2\delta - r + 1$ in the case of an isolated singularity. It has a very easy topological proof, which has no direct relation with our main formula.

4.D. Homogeneous and quasi-homogeneous singularities.

Example 4.D.1. — We now turn to the case where $h : (\mathbb{C}^{n+1}, \mathbf{0}) \to (\mathbb{C}, \mathbf{0})$ is a homogeneous polynomial of degree d with $\dim_{\mathbf{0}} \Sigma(h_{|_{V(z_0)}}) = 0$.

As h is homogeneous, Σh is a collection of lines which are transversely intersected by $V(z_0)$ (since $\dim_{\mathbf{0}} \Sigma(h_{|_{V(z_0)}}) = 0$). Hence,

$$\lambda_h^1 = \sum_\nu \overset{\circ}{\mu}_\nu .$$

To calculate λ_h^0 we proceed in a manner similar to [MiOr]. We consider the partial derivatives of h as defining a collection of hypersurfaces in $\mathbb{P}^{n+1}$

with homogeneous coordinates $(z_0 : \ldots : z_n : w)$. Then, if a is not zero, we have that the hypersurfaces (in $\mathbb{P}^{n+1}$)

$$V\left(\frac{\partial h}{\partial z_0} + aw^{d-1}\right), V\left(\frac{\partial h}{\partial z_1}\right), \ldots, V\left(\frac{\partial h}{\partial z_n}\right)$$

intersect in a finite number of points with total intersection multiplicity $(d-1)^{n+1}$.

Now, on the patch $w \neq 0$, one immediately sees that the number of intersection points (counted with multiplicity) is precisely λ^0_h. It remains for us to count the number of intersection points where $w = 0$.

But, the intersection points where $w = 0$ correspond exactly to the lines making up the singular locus. Thus, all of these points occur on the patch $z_0 \neq 0$, and the contribution of these points to the total intersection multiplicity was

$$\deg\left(\frac{\partial h}{\partial z_0}\right) \times \sum_{\mathbf{p}} \left(V\left(\frac{\partial h_{|z_0=1}}{\partial z_1}\right) \cdot \ \ldots \ \cdot V\left(\frac{\partial h_{|z_0=1}}{\partial z_n}\right)\right)_{\mathbf{p}},$$

where the sum is over all $\mathbf{p}$ in $V(z_0 - 1) \cap \Sigma h$. But, this equals

$$(d-1)\sum_{\nu} \mathring{\mu}_\nu = (d-1)\lambda^1_h.$$

Therefore, we find that $\lambda^0_h = (d-1)^{n+1} - (d-1)\lambda^1_h$, and so

$$\tilde{b}_n(F) - \tilde{b}_{n-1}(F) = \lambda^0_h - \lambda^1_h = (d-1)^{n+1} - d\lambda^1_h.$$

Remark 4.D.2. — As was shown in [Si5], this agrees with the formula that one would attain by applying the Iomdine-Lê formula in 3.1, using that the polar ratios are each exactly d and that the Milnor number of a homogeneous degree d polynomial in $n+1$ variables is $(d-1)^{n+1}$ [MiOr]. (That the polar ratios are all d follows from the fact that the polar curve is homogeneous and is, hence, a collection of lines.)

In 4.D.1, we saw that the Euler characteristic of the Milnor fibre of a homogeneous polynomial depends only on the degree, the number of variables, and λ^1. However, the homotopy type of the Milnor fibre is more sensitive and depends on more data. Famous in this context are the Zariski-examples of curves of degree 6 with 6 cusps on a conic or not on a conic. The homology of the complement of the space is related to the eigenspaces of the monodromy with eigenvalue 1 (cf Dimca [Di1]).

Example 4.D.3. — Continuation of Pellikaan example (example 4.A.5).

Let $f = xyz$ then $d = 3$ and transversal type is 3 times A_1 and so $b_2(F) - b_1(F) = 8 - 9 = -1$.

Let $f = x^2y^2 + y^2z^2 + 4z^2x^2$ then $d = 4$ and the transversal type is again 3 times A_1 and now : $b_2(F) - b_1(F) = 27 - 12 = 15$.

Example 4.D.4. — Zariski example with 6 cusps on a conic. The degree is 6, the transversal type is 6 times A_2. So $b_2(F) - b_1(F) = 125 - 72 = 53$.

Example 4.D.5. — In case f is a *quasi-homogeneous polynomial* with weights $w_0, \cdots, w_n$ and of degree d it is sometimes possible to compute $\tilde{b}_n(F) - \tilde{b}_{n-1}(F)$ by using the Lê-Iomdine formula. This is especially the case if x can be embedded in a quasi-homogeneous coordinate system : $x = x_0$ with $d/w_0 \in \mathbb{N}$ and if moreover $f_{d/w_0} = f + \varepsilon x_0^{d/w_0}$ defines an isolated singularity. Similar cases are considered by Dimca [Di2] with the help of differential forms.

We find the following formula :

$$\tilde{b}_n(F) - \tilde{b}_{n-1}(F) = \prod \frac{d - w_i}{w_i} - d \sum_{\nu} \frac{\mathring{\mu}_\nu}{k_\nu}$$

with $k_\nu = gcd(w_i | x_i \neq 0 \text{ on } \Sigma_\nu)$.

Note that all terms in this formula are independent from x.

4.E. Composed singularities.

Example 4.E.1. — Let $f : (\mathbb{C}^{n+1}, \mathbf{0}) \to (\mathbb{C}, \mathbf{0})$ be a holomorphic germ, which can be written as the composition

$$f = P(g_1, g_2)$$

where

$$g = (g_1, g_2) : (\mathbb{C}^{n+1}, \mathbf{0}) \to (\mathbb{C}^2, \mathbf{0})$$

defines an isolated complete intersection singularity and

$$P : (\mathbb{C}^2, \mathbf{0}) \to (\mathbb{C}, \mathbf{0})$$

is a holomorphic germ.

This situation is studied by A. Némethi in his paper [Ne2]. This paper contains information about the homotopy type of the Milnor fibre and the monodromy of these composed singularities. If P has an isolated singularity then the singular locus of f is $(n-1)$-dimensional. In case $n=2$ we have a 1-dimensional singular locus and we can try to apply the methods of our paper.

Consider the deformation

$$f_a = P(g_1 - a_1, g_2 - a_2)$$

for generic values $a = (a_1, a_2)$ of g. The singular locus of f_a consists of the 1-dimensional set

$$\Sigma_a = g^{-1}(a)$$

and some isolated singularities. Moreover the deformation is equi-transversal. Consider for this purpose a generic linear function z_0, which is at the point $\mathbf{p} \in \Sigma f_0 - \mathbf{0}$ transversal to $g^{-1}(\mathbf{0})$. Near $\mathbf{p}$ we can use z_0, g_1, g_2 as a system of coordinates. The functions g_1, g_2 are coordinates on the slice $V(z_0 - p_0)$. On this set we have near $\mathbf{0}$:

$$f_{a|V(z_0-p_0)} = P(g_1 - a_1, g_2 - a_2).$$

This shows that the family $f_{a|V(z_0-p_0)}$ is μ-constant at $\mathbf{p}$. The transversal type is given by the plane curve singularity P. Since there are no special points on Σ_a, our main formula is in this case :

$$b_2(F) - b_1(F) = \sum_{\mathbf{p}} b_2(F_{\mathbf{p}}) - \overset{\circ}{\mu}\chi(\Sigma_a).$$

Nemethi gives the following formula :

$$\chi(F) = (1 - \overset{\circ}{\mu})\chi(\Sigma_a) + (\gamma_f(\Sigma), f)_0.$$

The last expression is an intersection number of the 1-dimensional components of the critical set Σ (with a non-reduced structure) and $f^{-1}(\mathbf{0})$. For more details cf. [Ne2].

Example 4.E.2. (Generalized Zariski examples).

$$f(x, y, z) = (y^q + z^q)^p + (x^p + z^p)^q.$$

This is a composed singularity with

$$g_1(x, y, z) = y^q + z^q$$

$$g_2(x,y,z) = x^p + z^p$$

$$P(u,v) = u^p + v^q.$$

Since f is homogeneous of degree pq the equation $f(x,y,z) = 0$ defines a curve $C_{p,q}$ in the projective plane $\mathbb{P}^2(\mathbb{C})$.

The curve has exactely pq critical points, each with local equation $P(u,v) = u^p + v^q$. These curves are studied by Oka [Ok], who computed the fundamental group of the complement, see also Némethi [Ne1]. If $p = 2$ and $q = 3$ the curve is just Zariski's example with six cusps on a conic.

Following the above recipe we get the following equi-transversal deformation :

$$f_a(x,y,z) = (y^q + z^q - a_1)^p + (x^p + z^p - a_2)^q.$$

In this case, it is not difficult to compute the critical set of f_a, including the multiplicities of the isolated critical points :

$pq - 1$ points with multiplicity $(p-1)(q-1)$,

p points with multiplicity $(q-1)^3$,

q points with multiplicity $(p-1)^3$.

This implies :

$$\sum_{\mathbf{p}} b_2(F_{\mathbf{p}}) = (pq-1)(p-1)(q-1) + p(q-1)^3 + q(p-1)^3.$$

The singular locus of f_a has $\Sigma_a = g^{-1}(a)$ as 1-dimensional component. Σ_a is the Milnor fibre of $\Sigma = g^{-1}(\mathbf{0})$. The map g is (quasi)-homogeneous with weights $w_1 = w_2 = w_3 = 1$ and degrees $a_1 = p$ and $a_2 = q$. We can apply Giusti's formula [Gi1], [Gi2], [Si4] :

$$\mu(\Sigma) = 1 + (\sum a_j - \sum w_i)\frac{\prod a_j}{\prod w_j} = 1 + (p+q-3)pq.$$

The last ingredient we need is

$$\overset{\circ}{\mu} = (p-1)(q-1).$$

Therefore,

$$b_2(F) - b_1(F) = (pq-1)(p-1)(q-1) + p(q-1)^3 + q(p-1)^3 \\ +(p-1)(q-1)(p+q-3)pq.$$

There is also an other way to derive the same answer, using the formula for homogeneous singularities from 4.D.2 :

$$b_2(F) - b_1(F) = (d-1)^3 - d\sum_{\nu} \overset{\circ}{\mu}_\nu = (pq-1)^3 - p^2q^2(p-1)(q-1).$$

After expansion you find out that the two answers coincide.

5. Remarks and questions.

It is reasonable to ask just how effectively calculable our numerical invariants are. In particular, one would like to know if a computer can be of any assistance in calculations for specific examples.

For the Lê numbers the answer is definitely : yes. Any computer program which can calculate the multiplicities of ideals in a polynomial ring, given a set of generators, can calculate the Lê numbers of a polynomial. (A number of programs have this capability, but by far the most efficient that we know of is *Macaulay* – a public domain program written by Michael Stillman and Dave Bayer.)

Given such a program and a polynomial, f , with a one-dimensional singular set, one proceeds as follows to calculate the Lê numbers, λ^0 and λ^1 , at the origin with respect to a generic set of coordinates.

As we saw in 1.7, λ^1 is nothing other than the multiplicity of the Jacobian scheme of f. So, one can have the program calculate it.

Now, we need a hyperplane that is generic enough so that its intersection number (at the origin) with the (reduced) singular set is, in fact, equal to the multiplicity of the singular set. Usually, one knows the singular set (as a set) well enough to know such a hyperplane. (Alternatively, there are programs which can find the singular set for you – though how they present the answer is not always helpful.) We shall assume now, in addition to having λ^1 , that we also have such a hyperplane, $V(L)$, for some linear form, L.

By the work of Iomdine [Io2] and Lê [Lê2] (or our generalization in 3.1), we have that : for all k sufficiently large, $f + L^k$ has an isolated singularity at the origin and the Milnor number $\mu(f+L^k)$ equals $\lambda^0 + (k-1)\lambda^1$. But, the Milnor number is again nothing other than the multiplicity of the Jacobian scheme, and so we may use our program to calculate it. Thus, we can find λ^0 – provided that we have an effective method for knowing when we have chosen k large enough so that the formula of Iomdine and Lê holds.

However, we have such a method. If $f+L^k$ has an isolated singularity, let μ_k denote its Milnor number. (Given a particular k , one must either check by hand whether $f+L^k$ has an isolated singularity or have a program do it. *Macaulay* will tell you the dimension of the singular set in the course of calculating the multiplicity of the Jacobian scheme.) A quick look at the proof of the Iomdine-Lê formula in 3.1 shows that the formula holds provided that

$$\mu_k - (k-1)\lambda^1 \leqslant k-2.$$

Therefore, to find λ^0 , one starts with a relatively small k and checks whether $\mu_k \leqslant k-2+(k-1)\lambda^1$. If the inequality is false, pick a larger k. Eventually, the inequality will hold and then

$$\lambda^0 = \mu_k - (k-1)\lambda^1.$$

There is a similar alternative method for calculating not only the Lê numbers but also the maximum polar ratio of f. Extending remark 3.2 slightly more, it is not difficult to see that, if μ_i is as above, then $\mu_{k+1} - \mu_k = \lambda^1$ if and only if $k \geqslant$ the maximum polar ratio.

Hence, to find the maximum polar ratio, one calculates μ_k for successive values of k – looking for a difference of λ^1. Once this occurs, $k \geqslant$ the maximum polar ratio and, as before, we conclude that

$$\lambda^0 = \mu_k - (k-1)\lambda^1.$$

While this method requires one to calculate at least two Milnor numbers, μ_k, it will still be a more efficient way of calculating λ^0 – provided that the maximum polar ratio is significantly smaller than λ^0 itself. This would be the case, for instance, if the polar curve had a large number of components.

As an example of using this last method to calculate the maximum polar ratio and the Lê numbers, consider the polynomial $f = xy^3 + x^3y^2$. Using the notation above, we find

$k =$	2	3	4	5	6	7	8
$\mu_k =$	3	6	9	11	13	15	16

As one easily checks, $\lambda^1 = 1$ and so, from the table, we see that the maximum polar ratio is 7 and $\lambda^0 = 15-(7-1)1 = 9$. One can verify directly that, in fact, the polar curve has two components, with polar ratios 4 and 7. To apply the first method above, one would have to use a value for k that is $\geqslant 2+\lambda^0 = 11$.

As we saw in example 4.A.5, the Lê numbers are less sensitive invariants than are the number of special points in generic deformations. However, this means that requiring the Lê numbers to be constant in a family is a less stringent condition. Despite the apparent weakness of this assumption, the main result of [Ma3] – stated in the case of one-dimensional singularities – is that, for families of n-dimensional hypersurfaces with one-dimensional singularities, the constancy of the Lê numbers implies the constancy of the fibre-homotopy type of the Milnor fibrations if $n \geqslant 3$, and implies the constancy of the diffeomorphism type of the Milnor fibrations if $n \geqslant 4$.

Related to the result discussed in the last paragraph, T. Gaffney has shown the following [Ga2] :

PROPOSITION. — *Suppose that* $f_t : (\mathbb{C}^2, \mathbf{0}) \to (\mathbb{C}^3, \mathbf{0})$ *is a family of finitely determined germs with rank* 1 *at the origin, and suppose that the Lê numbers, at the origin, of the images are independent of* t. *Then, the family* f_t *is Whitney equisingular.*

In general, however, the relationship between the Lê numbers and the Whitney conditions is very unclear.

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D.B. MASSEY,
Dept. of Mathematics
Northeastern University
Boston, MA, 02115 (USA)

&

D. SIERSMA,
Mathematisch Instituut
Rijksuniversiteit Utrecht
P.O. Box 80.010
3508 TA Utrecht (The Netherlands).