ANNALES DE L'INSTITUT FOURIER

FRANK MANTLIK

Partial differential operators depending analytically on a parameter

Annales de l'institut Fourier, tome 41, nº 3 (1991), p. 577-599

<http://www.numdam.org/item?id=AIF_1991__41_3_577_0>

Ann. Inst. Fourier, Grenoble
41,3 (1991), 577-599

PARTIAL DIFFERENTIAL OPERATORS DEPENDING ANALYTICALLY ON A PARAMETER

by Frank MANTLIK

0. Introduction.

Consider a linear differential operator in $\mathbf{R}^n$,

$$P(\lambda, D) = \sum_{|\alpha|\leq m} a_\alpha(\lambda) D^\alpha : D = -i\partial \ , \ \partial = \Big(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\Big) \ ,$$

where the coefficients $a_\alpha(\lambda)$ – constant with respect to the variable of differentiation x – may depend analytically on a parameter λ in a complex manifold Λ. We assume that $P(\lambda, D)$ is equally strong for each $\lambda \in \Lambda$.

In [H2], p. 59 L. Hörmander posed the question whether under these conditions there exists a fundamental solution $\mathfrak{f}_\lambda$ of $P(\lambda, D)$ which depends analytically on λ. In 1962 F. Treves [T2] had shown that this is true locally in Λ and that the assumption of constant strength is necessary for this to hold [T1]. Recently the author could construct a global solution in the hypoelliptic case [M]. The proof of this result based on the fact that for each compact subset Λ' of Λ there exists an integration contour in $\mathbf{C}^n$ which yields fundamental solutions of $P(\lambda, D)$ simultaneously for all $\lambda \in \Lambda'$. In a second step we could apply a theorem of J. Leiterer [L] to obtain a global solution $\mathfrak{f}_\lambda$ by means of a Mittag-Leffler procedure.

The aim of the present paper is to eliminate the assumption of hypoellipticity. In section 1 we show that also in the general case one can

Key-words : Linear differential operators – Fundamental solutions – Analytic parameter-dependence.
A.M.S. Classification : 35B30 – 35E05.

always find a uniform integration contour $H_{\Lambda'}$ for all λ in a compact subset Λ' of Λ. As a consequence we obtain an explicit formula for $\mathfrak{f}_\lambda : \lambda \in \Lambda'$. Our proof uses some ideas of Hörmander [H2] concerning asymptotic properties of multivariate polynomials. The rest of this article is essentially an adaptation of the methods of [M] : in section 2 certain distribution spaces are introduced by means of the contours $H_{\Lambda'}$. These spaces constitute the setting for our application of the Leiterer theorem [L]. Section 3 contains the statements and proofs of our main results. We consider the equation $P(\lambda, D)\mathfrak{f}_\lambda = \mathfrak{g}_\lambda$ where $\mathfrak{g}_\lambda$ is a given analytic function of λ with values in some distribution space and prove the existence of a solution $\mathfrak{f}_\lambda$ which also depends analytically on λ. In the special case $\mathfrak{g}_\lambda \equiv \delta$ (the Dirac distribution) we obtain a solution to the problem described above.

1. Construction of a uniform integration contour.

We begin by fixing some notations : for any $n, m \in \mathbb{N}$ let

$$\mathrm{Pol}(n,m) := \{P \in \mathbb{C}[x_1, \ldots, x_n] \mid \deg P \leq m\};$$

$$\mathrm{Pol}'(n,m) := \{P \in \mathrm{Pol}(n,m) \mid \deg P = m\} .$$

If $P, Q \in \mathbb{C}[x_1, \ldots, x_n]$ then we write

$$\delta_P(\xi) := \mathrm{dist}(\xi, \{\zeta \in \mathbb{C}^n \mid P(\zeta) = 0\}) : \quad \xi \in \mathbb{C}^n;$$

$$\widetilde{P}(\xi, t) := \sum_\alpha t^{|\alpha|} |P^{(\alpha)}(\xi)| : \quad \xi \in \mathbb{C}^n,\ t > 0 ,$$

where $|\alpha| := \sum_{j=1}^{n} \alpha_j$ and $P^{(\alpha)} := \partial^\alpha P$;

$$\widetilde{P}(\xi) := \widetilde{P}(\xi, 1);$$

$$P < Q : \Longleftrightarrow \sup\{\widetilde{P}(\xi)/\widetilde{Q}(\xi) \mid \xi \in \mathbb{R}^n\} < \infty;$$

$$P \sim W : \Longleftrightarrow P < Q \wedge Q < P;$$

$$\mathsf{W}(Q) := \{P \in \mathbb{C}[x_1, \ldots, x_n] \mid P < Q\};$$

$$\mathsf{E}(Q) := \{P \in \mathbb{C}[x_1, \ldots, x_n] \mid P \sim Q\} .$$

1.1. *Remarks.*

(i) Note that our definition of $\widetilde{P}(\xi, t)$ differs from that of Hörmander [H2], §10.4, who used the notation $\widetilde{P}(\xi, t) := \left(\sum_\alpha t^{2|\alpha|} |P^{(\alpha)}(\xi)|^2\right)^{1/2}$.

According to [H2], 10.4.3 we have

$$P < Q \iff \sup\{\widetilde{P}(\xi,t)/\widetilde{Q}(\xi,t) \mid \xi \in \mathbf{R}^n,\ t \geq 1\} < \infty .$$

In this case we say that P *is weaker than* Q. If $P \sim Q$ then we say that P *and* Q *are equally strong.*

(ii) $P < Q \Longrightarrow \deg P \leq \deg Q$. This is clear by definition of $\widetilde{P}$. In particular, $\mathbf{W}(Q)$ is a finite-dimensional complex vector space (consequence of [H2], 10.4.1).

(iii) $\mathbf{E}(Q)$ is a linearly convex, open subset of $\mathbf{W}(Q)$ ([H2], 10.4.7). For our purposes it suffices to know that $\mathbf{E}(Q)$ is holomorphically convex (*cf.* [M]).

We assume the integers n, m to be fixed throughout this paper. The letters c, C denote positive constants which only depend on n and m. We use the notations

$$|\xi| := \sum |\xi_j| \ , \ |\xi|_\infty := \max |\xi_j| : \quad \xi = (\xi_1, \dots, \xi_n) \in \mathbf{C}^n .$$

For $\mathbf{K} = \mathbf{R}, \mathbf{C}$ and $\rho \geq 0$ let

$$\mathbf{B}_{\mathbf{K}^n}(\rho) := \{\xi \in \mathbf{K}^n \mid |\xi|_\infty \leq \rho\} .$$

In the case $\rho = 1$ we simply write $\mathbf{B}_{\mathbf{K}^n}$. Further let

$$\mathbf{T}^r := \{z \in \mathbf{C}^r \mid |z_1| = \dots = |z_r| = 1\} \ \text{ if } \ r \in \mathbf{N} .$$

1.2. THEOREM. — *Let* $Q \in \mathrm{Pol}'(n,m)$, $\Pi \subseteq \mathbf{E}(Q)$ *be a compact set and* $\rho \geq 0$. *Then there exists* $A \geq 1$ *and a bounded measurable function* $\eta : \mathbf{R}^n \to \mathbf{R}^n$ *such that*

$$\widetilde{P}(\xi) \leq A|P(\xi + \zeta + z\eta(\xi))| : \quad P \in \Pi, \xi \in \mathbf{R}^n, \zeta \in \mathbf{B}_{\mathbf{C}^n}(\rho),\ z \in \mathbf{T}^1 . \tag{1.1}$$

Our proof of this theorem is long and will occupy the rest of this section. First it requires a detailed study of the function $\widetilde{P}(\xi,t)$:

1.3. LEMMA. — *Let* $Q \in \mathrm{Pol}'(n,m)$ *and* $\Pi \subseteq \mathbf{E}(Q)$ *be compact. Then there exists* $B \geq 1$ *such that*

$$B^{-1} \leq \widetilde{P}(\xi,t)/\widetilde{Q}(\xi,t) \leq B : \quad P \in \Pi,\ \xi \in \mathbf{R}^n,\ t \geq 1 . \tag{1.2}$$

Proof. — By 1.1 (i) the expression $N_Q(P) := \sup\{\widetilde{P}(\xi,t)/\widetilde{Q}(\xi,t) \mid \xi \in \mathbf{R}^n,\ t \geq 1\}$ defines a norm on $\mathbf{W}(Q)$. Now let $R \in \Pi$ be fixed. Since $Q < R$ we have

$$b_R := \inf\{\widetilde{R}(\xi,t)/\widetilde{Q}(\xi,t) \mid \xi \in \mathbf{R}^n,\ t \geq 1\} > 0 .$$

For any $P \in \omega_R := \{P \in \mathbf{W}(Q) \mid N_Q(R-P) < b_R/2\}$ we get

$$\frac{\widetilde{P}(\xi,t)}{\widetilde{Q}(\xi,t)} \geq \frac{\widetilde{R}(\xi,t) - (R-P)^{\sim}(\xi,t)}{\widetilde{Q}(\xi,\tau)} > b_R/2 : \quad \xi \in \mathbf{R}^n, t \geq 1.$$

Since ω_R is an open neighborhood of R it follows from the compactness of Π that there exists $b_0 > 0$ with

$$\widetilde{P}(\xi,t) \geq b_0 \widetilde{Q}(\xi,t) : \quad P \in \Pi,\ \xi \in \mathbf{R}^n,\ t \geq 1 .$$

On the other hand the boundedness of Π implies that

$$B_0 := \sup\{N_Q(P) \mid P \in \Pi\} < \infty ,$$

hence

$$\widetilde{P}(\xi,t) \leq B_0 \widetilde{Q}(\xi,t) : \quad P \in \Pi,\ \xi \in \mathbf{R}^n,\ t \geq 1 .$$

With $B := \max\{1/b_0, B_0\}$ the assertion follows . □

1.4. LEMMA (*cf.* [H2], 11.1.4). — *There exists $C \geq 1$ such that for any $P \in \mathrm{Pol}'(n,m)$ the following holds :*

$$|P^{(\alpha)}(\xi)|\delta_P(\xi)^{|\alpha|} \leq C|P(\xi)| : \quad \xi \in \mathbf{C}^n,\ |\alpha| \leq m . \tag{1.3}$$

$$C^{-1} \leq \delta_P(\xi) \sum_{\alpha \neq 0} |P^{(\alpha)}(\xi)/P(\xi)|^{1/|\alpha|} \leq C : \quad \xi \in \mathbf{C}^n,\ P(\xi) \neq 0 . \tag{1.4}$$

$$|P(\xi)| \leq \widetilde{P}(\xi, \delta_P(\xi)) \leq C|P(\xi)| : \quad \xi \in \mathbf{C}^n . \tag{1.5}$$

Proof. — (1.4) is due to Hörmander [H2], 11.1.4. (1.5) is a consequence of (1.3) which follows from (1.4). □

1.5. LEMMA (*cf.* [H2], 11.1.9). — *There exists $c > 0$ such that for any $P, Q \in \mathrm{Pol}'(n,m)$ and $\xi \in \mathbf{C}^n$ we have : if*

$$B^{-1} \leq \widetilde{P}(\xi,t)/\widetilde{Q}(\xi,t) \leq B : \quad t \geq 1 \tag{1.6}$$

holds with some $B \geq 1$ then

$$\frac{c}{1+B^2} \leq \frac{1+\delta_P(\xi)}{1+\delta_Q(\xi)} \leq \frac{1+B^2}{c} . \tag{1.7}$$

Proof. — If $\delta_Q(\xi) \geq 1$ then

$$\sum_\alpha |P^{(\alpha)}(\xi)|\delta_Q(\xi)^{|\alpha|} \overset{(1.6)}{\leq} B \sum_\alpha |Q^{(\alpha)}(\xi)|\delta_Q(\xi)^{|\alpha|}$$

$$\overset{(1.5)}{\leq} C_1 B |Q(\xi)| \overset{(1.6)}{\leq} C_1 B^2 \sum_\alpha |P^{(\alpha)}(\xi)| .$$

When $\delta_Q(\xi) \geq 2C_1B^2 =: D$ (hence $\frac{1}{2}\delta_Q(\xi)^{|\alpha|} \leq \delta_Q(\xi)^{|\alpha|} - \frac{D}{2}$, $\alpha \neq 0$) this yields

$$\sum_\alpha |P^{(\alpha)}(\xi)|\delta_Q(\xi)^{|\alpha|} \leq D|P(\xi)| .$$

In particular then $P(\xi) \neq 0$ and

$$|P^{(\alpha)}(\xi)/P(\xi)|^{1/|\alpha|}\delta_P(\xi) \leq D\delta_P(\xi)/\delta_Q(\xi) : \quad \alpha \neq 0 .$$

Summing up we get

$$C_2B^2\delta_P(\xi)/\delta_Q(\xi) \geq \delta_P(\xi)\sum_{\alpha\neq 0}|P^{(\alpha)}(\xi)/P(\xi)|^{1/|\alpha|} \overset{(1.4)}{\geq} C_3^{-1} ,$$

hence

$$\frac{1+\delta_P(\xi)}{1+\delta_Q(\xi)} \geq \frac{1}{2}\frac{\delta_P(\xi)}{\delta_Q(\xi)} \geq (2C_2C_3B^2)^{-1} \text{ if } \delta_Q(\xi) \geq D .$$

In the case $\delta_Q(\xi) \leq D$ we have

$$\frac{1+\delta_P(\xi)}{1+\delta_Q(\xi)} \geq \frac{1}{1+2C_1B^2} .$$

With suitable $c > 0$ we obtain the lefthand side of (1.7). The second inequality follows from this one by interchanging the roles of P and Q. □

1.6. Lemma (cf. [H2], 10.4.2). — *There exists $C \geq 1$ such that for any $P \in \mathrm{Pol}(n,m)$, $\xi \in \mathbb{C}^n$ and $\tau > 0$:*

(1.8) $$C^{-1}\tilde{P}(\xi,\tau) \leq \max\{|P(\xi+\eta)| \mid \eta \in \mathbf{B}_{\mathbb{K}^n}(\tau)\} \leq C\tilde{P}(\xi,\tau) ;$$

(1.9) $$C^{-1}\tau \leq \max\{\delta_P(\xi+\eta) \mid \eta \in \mathbf{B}_{\mathbb{K}^n}(\tau)\} \text{ if } P \text{ is nonconstant} .$$

This holds for $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$.

Proof. — Assertion (1.8) corresponds to [H2], 10.4.2. (Our use of the ℓ_1-norm in the definition of $\tilde{P}(\xi,t)$ only results in a change of the constants.)

Ad (1.9) : first we note that for $\tau > 0$ and $\eta \in \mathbf{B}_{\mathbb{K}^n}(\tau)$,

$$|P^{(\alpha)}(\xi+\eta)| \leq \sum_\beta |P^{(\alpha+\beta)}(\xi)|\tau^{|\beta|} \leq \tau^{-|\alpha|}\tilde{P}(\xi,\tau)$$

by Taylor's formula. As a consequence we have the estimate

(1.10) $$\tilde{P}(\xi+\eta,\tau) \leq C_1\tilde{P}(\xi,\tau) : \quad P \in \mathrm{Pol}(n,m), \xi \in \mathbb{C}^n, \eta \in \mathbf{B}_{\mathbb{C}^n}(\tau),$$

which will be used later. By (1.8) there exists for fixed $\xi \in \mathbb{C}^n$ and $\tau > 0$ an $\eta \in \mathbf{B}_{\mathbb{K}^n}(\tau)$ such that

$$\tilde{P}(\xi,\tau) \leq C_2|P(\xi+\eta)| .$$

In particular then $P(\xi+\eta)\neq 0$ and

$$\sum_{\alpha\neq 0}|P^{(\alpha)}(\xi+\eta)/P(\xi+\eta)|^{1/|\alpha|}\leq \sum_{1\leq|\alpha|\leq m}(C_2\tau^{-|\alpha|})^{1/|\alpha|}\leq C_3\tau^{-1}\ .$$

From (1.4) it follows that $\delta_P(\xi+\eta)\geq C_4^{-1}\tau$, hence the assertion. □

Now we can already prove a preliminary version of Theorem 1.2 :

1.7. COROLLARY. — *Let $Q\in \mathrm{Pol}'(n,m)$ and $\Pi\subseteq \mathbf{E}(Q)$ compact. Then there exist A, $\mu\geq 1$ such that*

$$(1.11)\qquad \forall\tau\geq\mu,\ \xi\in\mathbf{R}^n\ \exists\eta\in\mathbf{B}_{\mathbf{R}^n}(\tau)\ \forall P\in\Pi\ :\widetilde{P}(\xi,\tau)\leq A|P(\xi+\eta)|\ .$$

Proof. — By Lemma 1.3 there exists $B\geq 1$ such that

$$B^{-1}\leq \widetilde{P}(\xi,t)/\widetilde{Q}(\xi,t)\leq B:\quad P\in\Pi,\ \xi\in\mathbf{R}^n,\ t\geq 1\ .$$

With $A_1:=(1+B^2)/c\geq 1$ we get from (1.7),

$$A_1^{-1}(1+\delta_Q(\xi))\leq 1+\delta_P(\xi):\quad P\in\Pi,\ \xi\in\mathbf{R}^n\ .$$

By (1.9) we have

$$(1.12)\qquad \max\{\delta_Q C\xi+\eta)\mid\eta\in\mathbf{B}_{\mathbf{R}^n}(\tau)\}\geq C_0^{-1}\tau:\quad \xi\in\mathbf{R}^n,\ \tau>0\ .$$

Choose $A_2\geq 1$ with $C_0^{-1}-A_1/A_2>0$ and put

$$\mu:=\max\{1,(A_1-1)/(C_0^{-1}-A_1/A_2)\}\ .$$

If $\tau\geq\mu$ then $(1+C_0^{-1}\tau)/A_1\geq 1+\tau/A_2$. For such a τ and arbitrary $\xi\in\mathbf{R}^n$ we may now choose $\eta\in\mathbf{B}_{\mathbf{R}^n}(\tau)$ with $\delta_Q(\xi+\eta)\geq C_0^{-1}\tau$ according to (1.12). For any $P\in\Pi$ we then obtain

$$1+\delta_P(\xi+\eta)\geq A_1^{-1}(1+\delta_Q(\xi+\eta))\geq A_1^{-1}(1+C_0^{-1}\tau)\geq 1+\tau/A_2\ ,$$

i.e. $\tau\leq A_2\delta_P(\xi+\eta)$. Because of (1.5) this yields

$$\begin{aligned}\widetilde{P}(\xi+\eta,\tau)\leq\widetilde{P}(\xi+\eta,A_2\delta_P(\xi+\eta))&\leq A_2^m\widetilde{P}(\xi+\eta,\delta_P(\xi+\eta))\\ &\leq A_3|P(\xi+\eta)|.\end{aligned}$$

Finally, replacing in (1.10) η by $-\eta$ and ξ by $\xi+\eta$, we obtain

$$\widetilde{P}(\xi,\tau)\leq C_1\widetilde{P}(\xi+\eta,\tau)\leq C_1A_3|P(\xi+\eta)|:\quad P\in\Pi\ .$$ □

For any $R\in\mathbb{C}[x_1,\ldots,x_n]$ and $k\in\mathbb{N}_0$ we put

$$(\Phi_k R)(\xi):=\sum_{|\alpha|=k}R^{(\alpha)}(\xi)\overline{R}^{(\alpha)}(\xi)\ ,$$

where $\overline{R}$ is obtained from R by taking complex conjugates of the coefficients. Note that $\Phi_k R \in \mathbf{R}[x_1, \ldots, x_n]$ and $(\Phi_k R)(\xi) \geq 0$ for $\xi \in \mathbf{R}^n$. With the notation

$$(\Psi_k R)(\xi) := \sum_{|\alpha|=k} |R^{(\alpha)}(\xi)|$$

we have

$$\widetilde{R}(\xi, t) = \sum_{k=0}^{m} t^k (\Psi_k R)(\xi) : \quad R \in \mathrm{Pol}(n, m) .$$

1.8. LEMMA. — *There exists $C \geq 1$ such that for any $P \in \mathrm{Pol}(n,m)$, $k \in \mathbb{N}_0$, $\xi \in \mathbf{R}^n$ and $t > 0$:*

$$C^{-1}(\Phi_k P)^{\sim}(\xi, t) \leq \Big(\sum_{j=k}^{m} t^{j-k} (\Psi_j P)(\xi)\Big)^2 \leq C(\Phi_k P)^{\sim}(\xi, t) . \tag{1.13}$$

Proof. — First we have by (1.8) (note that $\Phi_k P \in \mathrm{Pol}(n, 2m)$),

$$C_1^{-1}(\Phi_k P)^{\sim}(\xi, t) \leq \max_{\eta \in \mathbf{B}_{\mathbf{R}^n}} (\Phi_k P)(\xi + t\eta) \leq C_1 (\Phi_k P)^{\sim}(\xi, t) \tag{1.14}$$

and

$$C_1^{-1} \sum_{|\alpha|=k} (P^{(\alpha)})^{\sim}(\xi, t) \leq \sum_{|\alpha|=k} \max_{\eta \in \mathbf{B}_{\mathbf{R}^n}} |P^{(\alpha)}(\xi + t\eta)| \leq C_1 \sum_{|\alpha|=k} (P^{(\alpha)})^{\sim}(\xi, t) .$$

Furthermore an easy calculation shows that

$$C_2^{-1} \sum_{|\alpha|=k} (P^{(\alpha)})^{\sim}(\xi, t) \leq \sum_{j=k}^{m} t^{j-k} (\Psi_j P)(\xi) \leq C_2 \sum_{|\alpha|=k} (P^{(\alpha)})^{\sim}(\xi, t) ,$$

hence

$$\begin{aligned} C_3^{-1} \sum_{j=k}^{m} t^{j-k} (\Psi_j P)(\xi) &\leq \sum_{|\alpha|=k} \max_{\eta \in \mathbf{B}_{\mathbf{R}^n}} |P^{(\alpha)}(\xi + t\eta)| \\ &\leq C_3 \sum_{j=k}^{m} t^{j-k} (\Psi_j P)(\xi) . \end{aligned} \tag{1.15}$$

Now let $\mathbf{M}(n,k) = \{\alpha \in \mathbb{N}_0^n \mid |\alpha| = k\}$. Obviously the expressions

$$N_1((R_\alpha)_{\alpha \in \mathbf{M}(n,k)}) := \Big(\max_{\eta \in \mathbf{B}_{\mathbf{R}^n}} \sum_{|\alpha|=k} R_\alpha(\eta) \overline{R}_\alpha(\eta)\Big)^{1/2} ,$$

$$N_2((R_\alpha)_{\alpha \in \mathbf{M}(n,k)}) := \sum_{|\alpha|=k} \max_{\eta \in \mathbf{B}_{\mathbf{R}^n}} |R_\alpha(\eta)|$$

define norms on the finite-dimensional vector space $\mathrm{Pol}(n,m)^{\mathbf{M}(n,k)}$, hence they are equivalent. On replacing $R_\alpha(\eta)$ by $P^{(\alpha)}(\xi+t\eta)$ we get

$$C_4^{-1}\sum_{|\alpha|=k}\max_{\eta\in\mathbf{B}_{\mathbf{R}^n}}|P^{(\alpha)}(\xi+t\eta)|\le\Big(\max_{\eta\in\mathbf{B}_{\mathbf{R}^n}}(\Phi_kP)(\xi+t\eta)\Big)^{1/2}$$
$$\le C_4\sum_{|\alpha|=k}\max_{\eta\in\mathbf{B}_{\mathbf{R}^n}}|P^{(\alpha)}(\xi+t\eta)|\ .$$

With (1.14) and (1.15) we obtain the assertion. □

1.9. LEMMA. — *There exist $0<c\le 1\le C$ such that for any $P,Q\in\mathrm{Pol}'(n,m)$ and $\xi\in\mathbf{R}^n$ the following holds : let $0\le k\le m-1$ and $B\ge 1$ with*

$$(1.16)\quad B^{-1}\le\Big(\sum_{j=k}^{m}t^{j-k}(\Psi_jP)(\xi)\Big)/\Big(\sum_{j=k}^{m}t^{j-k}(\Psi_jQ)(\xi)\Big)\le B:\quad t\ge 1.$$

Further let $\nu\ge 1$ such that $\hat\nu:=\big(\frac{c\nu}{1+B^4}-1\big)/C\ge 1$. Then we have with $\check\nu:=C(1+\nu)(1+B^4)$:

$$\text{(i) } (\Psi_kQ)(\xi)\ge\sum_{j=k+1}^{m}\nu^{\,j-k}(\Psi_jQ)(\xi)\Longrightarrow(\Psi_kP)(\xi)\ge\sum_{j=k+1}^{m}\hat\nu^{\,j-k}(\Psi_jP)(\xi),$$

$$\text{(ii) } (\Psi_kQ)(\xi)\le\sum_{j=k+1}^{m}\nu^{j-k}(\Psi_jQ)(\xi)\Longrightarrow(\Psi_kP)(\xi)\le\sum_{j=k+1}^{m}\check\nu^{j-k}(\Psi_jP)(\xi).$$

Proof.

(i) Let $\nu\ge 1$ with $(\Psi_kQ)(\xi)\ge\sum_{j=k+1}^{m}\nu^{j-k}(\Psi_jQ)(\xi)$. Then we have

$$|Q^{(\alpha)}(\xi)|\le\nu^{-(|\alpha|-k)}(\Psi_kQ)(\xi)\le C_1\nu^{-(|\alpha|-k)}\sqrt{(\Phi_kQ)(\xi)}\ :\ |\alpha|\ge k\ .$$

This implies by Leibniz' rule,

$$|(\Phi_kQ)^{(\beta)}(\xi)|=\Big|\sum_{|\alpha|=k}\sum_{\gamma\le\beta}\binom{\beta}{\gamma}Q^{(\alpha+\gamma)}(\xi)\overline{Q}^{(\alpha+\beta-\gamma)}(\xi)\Big|$$
$$\le C_2\nu^{-|\beta|}(\Phi_kQ)(\xi)$$

for any multiindex β ($C_2\ge 1$). In particular then $(\Phi_kQ)(\xi)\ne 0$ and

$$|(\Phi_kQ)^{(\beta)}(\xi)/(\Phi_kQ)(\xi)|^{1/|\beta|}\le C_2\nu^{-1}:\quad\beta\ne 0\ .$$

An application of (1.4) yields

$$C_3^{-1}\le\delta_{\Phi_kQ}(\xi)\sum_{\beta\ne 0}|(\Phi_kQ)^{(\beta)}(\xi)/(\Phi_kQ)(\xi)|^{1/|\beta|}\le C_4\nu^{-1}\delta_{\Phi_kQ}(\xi)\ .$$

By (1.13) and (1.16) we also have

$$(C_5B^2)^{-1} \le (\Phi_k P)^{\sim}(\xi,t)/(\Phi_k Q)^{\sim}(\xi,t) \le C_5B^2 : \quad t \ge 1 .$$

Using (1.7) we obtain

$$\frac{1+\delta_{\Phi_k P}(\xi)}{1+C_3^{-1}C_4^{-1}\nu} \ge \frac{1+\delta_{\Phi_k P}(\xi)}{1+\delta_{\Phi_k Q}(\xi)} \ge \frac{c_1}{1+C_5^2B^4} ,$$

$$\delta_{\Phi_k P}(\xi) \ge \frac{c_1(1+C_3^{-1}C_4^{-1}\nu)}{1+C_5^2B^4} - 1 \ge \frac{c_2\nu}{1+B^4} - 1 =: \tilde{\nu}$$

with $0 < c_2 \le 1$. Let ν be so large that $\tilde{\nu} \ge 1$. Then

$$\begin{aligned}(\Phi_k P)(\xi) &\overset{(1.5)}{\ge} C_6^{-1}(\Phi_k P)^{\sim}(\xi, \delta_{\Phi_k P}(\xi)) \ge C_6^{-1}(\Phi_k P)^{\sim}(\xi,\tilde{\nu}) \\ &\overset{(1.13)}{\ge} C_7^{-1}\Big(\sum_{j=k}^{m} \tilde{\nu}^{\,j-k}(\Psi_j P)(\xi)\Big)^2\end{aligned}$$

with $C_7 \ge 1$, hence

$$\begin{aligned}(\Psi_k P)(\xi) \ge \sqrt{(\Phi_k P)(\xi)} &\ge C_7^{-1/2}\sum_{j=k}^{m} \tilde{\nu}^{\,j-k}(\Psi_j P)(\xi) \\ &\ge \sum_{j=k+1}^{m} (\tilde{\nu}/C_7)^{j-k}(\Psi_j P)(\xi) .\end{aligned}$$

With $c := c_2$, $C \ge C_7$ we obtain the first assertion.

(ii) Now assume that $(\Psi_k Q)(\xi) \le \sum_{j=k+1}^{m} \nu^{\,j-k}(\Psi_j Q)(\xi)$. If then

$$(\Psi_k P)(\xi) \ge \sum_{j=k+1}^{m} \mu^{j-k}(\Psi_j P)(\xi) \text{ and } \tilde{\mu} := \frac{c_2\mu}{1+B^4} - 1 \ge 1$$

with some $\mu \ge 1$ we obtain as above (on interchanging the roles of P and Q) : $(\Psi_k Q)(\xi) \ge \sum_{j=k+1}^{m} (\tilde{\mu}/C_7)^{j-k}(\Psi_j Q)(\xi)$, hence

$$\sum_{j=k+1}^{m} (\tilde{\mu}/C_7)^{j-k}(\Psi_j Q)(\xi) \le \sum_{j=k+1}^{m} \nu^{\,j-k}(\Psi_j Q)(\xi) .$$

This implies $\tilde{\mu}/C_7 \le \nu$, i.e.

$$\mu \le (1+C_7\nu)(1+B^4)/c_2 \le C_7(1+\nu)(1+B^4)/c_2 .$$

Thus, with $C := C_7/c_2$ the second assertion also holds. □

Proof of Theorem 1.2. — The subsequent procedure will yield a decomposition of $\Omega_0 := \mathbf{R}^n$ into $m+1$ disjoint subsets, $\Omega_0 = \Omega'_0 \dot\cup \Omega'_1 \dot\cup \cdots \dot\cup \Omega'_m$, such that the following holds :

$$(1^k) \qquad \begin{gathered} \exists A \geq 1\ \forall k = 0, \ldots, m\ \exists \tau_k \geq 1\ \forall \xi \in \Omega'_k\ \exists \eta_\xi \in \mathbf{B}_{\mathbf{R}^n}(\tau_k) : \\ |P(\xi + z\eta_\xi)| \geq \frac{1}{2A}\widetilde{P}(\xi, \tau_k) : \quad P \in \Pi,\ z \in \mathbf{T}^1 . \end{gathered}$$

Now note that the set

$$\Pi_\rho := \{P(\cdot + \zeta) \mid P \in \Pi,\ \zeta \in \mathbf{B}_{\mathbf{C}^n}(\rho + 1)\}$$

is a compact subset of $\mathbf{E}(Q)$ since for fixed ζ the polynomial $P(\cdot + \zeta)$ is equally strong as P. So we may assume that $(1^0), \ldots, (1^m)$ is already proved for Π_ρ instead of Π. It follows that for any $\vartheta \in \mathbb{Z}^n$ there exists $\eta_\vartheta \in \mathbf{B}_{\mathbf{R}^n}(\tau)$, where $\tau := \max\{\tau_0, \ldots, \tau_m\}$, such that if $|\xi - \vartheta|_\infty \leq 1$ we have for each $P \in \Pi$, $\zeta \in \mathbf{B}_{\mathbf{C}^n}(\rho)$ and $z \in \mathbf{T}^1$:

$$|P(\xi + \zeta + z\eta_\vartheta)| = |P(\vartheta + z\eta_\vartheta + (\xi - \theta + \zeta))| \geq \frac{1}{2A}\widetilde{P}(\vartheta) \overset{(1.10)}{\geq} \frac{1}{2CA}\widetilde{P}(\xi) .$$

In particular we may choose $\eta(\xi) \equiv \eta_\vartheta$ in any cube $\{\xi \mid \vartheta_j \leq \xi_j < \vartheta_j + 1\}$, where $\vartheta_1, \ldots, \vartheta_n$ are integers, such that (1.1) holds and $\sup_\xi |\eta(\xi)|_\infty \leq \tau$. This completes the proof. The sets Ω'_k will be defined inductively as follows :

$$\Omega'_k := \{\xi \in \Omega_k \mid (\Psi_k Q)(\xi) \geq \sum_{j=k+1}^{m} \nu_k^{j-k}(\Psi_j Q)(\xi)\}\ (0 \leq k \leq m-1)$$

with suitable constants $\nu_k \geq 1$, and

$$\Omega_{k+1} := \Omega_k \setminus \Omega'_k\ ;\ \Omega'_m := \Omega_m .$$

In what follows the statements (2^k) $(0 \leq k \leq m)$ will be needed :

$$(2^k) \qquad \begin{gathered} \exists B_k \geq 1\ \forall P \in \Pi,\ \xi \in \Omega_k,\ t \geq 1 : \\ B_k^{-1} \leq \Big(\sum_{j=k}^{m} t^{j-k}(\Psi_j P)(\xi)\Big) \Big/ \Big(\sum_{j=k}^{m} t^{j-k}(\Psi_j Q)(\xi)\Big) \leq B_k . \end{gathered}$$

With the constants c, C in Lemma 1.9 we set

$$\hat{\nu}_k := \Big(\frac{c\nu_k}{1 + B_k^4} - 1\Big)/C \quad \text{and} \quad \check{\nu}_k := C(1 + \nu_k)(1 + B_k^4) .$$

Then for each $0 \leq k \leq m-1$ we have by (2^k) and Lemma 1.9, if $\hat{\nu}_k \geq 1$,

$$(3^k) \qquad (\Psi_k P)(\xi) \geq \sum_{j=k+1}^{m} \hat{\nu}_k^{j-k}(\Psi_j P)(\xi) : \quad P \in \Pi,\ \xi \in \Omega'_k ,$$

$$(4^k) \qquad (\Psi_k P)(\xi) \le \sum_{j=k+1}^{m} \check{\nu}_k^{\,j-k}(\Psi_j P)(\xi): \quad P \in \Pi,\ \xi \in \Omega_{k+1}\ .$$

Now the proof of (1^k), (2^k) proceeds by induction on k. Recall that by Corollary 1.7 there exist A, $\mu \ge 1$ such that

$$(5) \qquad \forall \tau \ge \mu,\ \xi \in \mathbf{R}^n\ \exists \eta \in \mathbf{B}_{\mathbf{R}_n}(\tau)\ \forall P \in \Pi\ : \widetilde{P}(\xi,\tau) \le A|P(\xi+\eta)|\ .$$

Without loss of generality we may assume that $Q \in \Pi$.

Case $k = 0$. — Lemma 1.3 yields the existence of B_0 satisfying (2^0). Choose $\nu_0 \ge 1$ such that $\hat{\nu}_0 \ge 1$ and define Ω_0', Ω_1 as above. Let $\tau_0 := \hat{\nu}_0$ and for any $\xi \in \Omega_0'$ choose $\eta_\xi := 0 \in \mathbf{B}_{\mathbf{R}_n}(\tau_0)$. We obtain

$$2|P(\xi + z\eta_\xi)| = 2(\Psi_0 P)(\xi) \overset{(3^0)}{\ge} \sum_{j=0}^{m} \hat{\nu}_0^j (\Psi_j P)(\xi) = \widetilde{P}(\xi, \tau_0)$$

for $P \in \Pi$, $z \in \mathbf{T}^1$, i.e. (1^0) is satisfied.

Case $1 \le k \le m$. — The inductive assumption yields (2^{k-1}) and $(4^0), \ldots, (4^{k-1})$. Since $\Omega_k \subseteq \Omega_{k-1}$ this implies for $\xi \in \Omega_k$, $t \ge \check{\nu}_{k-1}$:

$$\begin{aligned}(2B_{k-1})^{-1}\sum_{j=k}^{m} t^{j-k}(\Psi_j Q)(\xi) &\le (2B_{k-1})^{-1}\frac{1}{t}\sum_{j=k-1}^{m} t^{j-(k-1)}(\Psi_j Q)(\xi)\\ &\overset{(2^{k-1})}{\le} \frac{1}{2t}\sum_{j=k-1}^{m} t^{j-(k-1)}(\Psi_j P)(\xi)\\ &\overset{(4^{k-1})}{\le} \sum_{j=k}^{m} t^{j-k}(\Psi_j P)(\xi)\ .\end{aligned}$$

For $1 \le t \le \check{\nu}_{k-1}$ this yields

$$\begin{aligned}(2B_{k-1})^{-1}\sum_{j=k}^{m} t^{j-k}(\Psi_j Q)(\xi) &\le \sum_{j=k}^{m} \check{\nu}_{k-1}^{\,j-k}(\Psi_j P)(\xi)\\ &\le \check{\nu}_{k-1}^{\,m-k}\sum_{j=k}^{m} t^{j-k}(\Psi_j P)(\xi)\ .\end{aligned}$$

Analogous estimates hold with P and Q interchanged. Setting $B_k := 2B_{k-1}\check{\nu}_{k-1}^{\,m-k}$ we obtain (2^k). Now let

$$\mu_k := \max\{\mu, \check{\nu}_0, \ldots, \check{\nu}_{k-1}\} \ (\ge 1)\ .$$

For $P \in \Pi$, $\xi \in \Omega_{j+1}$ $(j = 0, \ldots, k-1)$, $\tau \ge \mu_k$ it follows from (4^j) :

$$(\Psi_j P)(\xi) \le \sum_{i=j+1}^{m} \left(\frac{\mu_k}{\tau}\right)^{i-j} \tau^{i-j}(\Psi_i P)(\xi) \le \frac{\mu_k}{\tau}\sum_{i=j+1}^{m} \tau^{i-j}(\Psi_i P)(\xi)\ .$$

Multiplying by τ^j and summing up this yields (note that $\Omega_k \subseteq \Omega_{j+1}$) :

(6) $$\sum_{j=0}^{k-1} \tau^j (\Psi_j P)(\xi) \leq \frac{k\mu_k}{\tau} \widetilde{P}(\xi, \tau) : \quad P \in \Pi,\ \xi \in \Omega_k,\ \tau \geq \mu_k \ .$$

In the case $k \leq m-1$ we choose $\tau_k, \nu_k \geq 1$ such that

(7) $$\mu_k \leq \tau_k \leq \hat{\nu}_k \ , \ A^{-1} - \frac{2k\mu_k}{\tau_k} - \frac{2\tau_k}{\hat{\nu}_k} \geq \frac{1}{2A}$$

and define Ω'_k, Ω_{k+1} as above. By (3^k) (consequence of (2^k)) we have

(8) $$\sum_{j=k+1}^{m} \tau_k^j (\Psi_j P)(\xi) \leq \frac{\tau_k}{\hat{\nu}_k} \tau_k^k (\Psi_k P)(\xi) \leq \frac{\tau_k}{\hat{\nu}_k} \widetilde{P}(\xi, \tau_k) : \quad P \in \Pi,\ \xi \in \Omega'_k \ .$$

Now let $\xi \in \Omega'_k$ be fixed and choose $\eta_\xi \in \mathbf{B}_{\mathbf{R}^n}(\tau_k)$ such that

(9) $$\widetilde{P}(\xi, \tau_k) \leq A|P(\xi + \eta_\xi)| : \quad P \in \Pi \ \ (cf.\ (5)) \ .$$

An application of Taylor's formula gives for $P \in \Pi$, $z \in \mathbf{T}^1$:

$$\begin{aligned} |P(\xi + z\eta_\xi)| &\geq \Big| \sum_{|\alpha|=k} \frac{P^{(\alpha)}(\xi)}{\alpha!} \eta_\xi^\alpha \Big| - \sum_{j \neq k} \tau_k^j (\Psi_j P)(\xi) \\ &\geq \sum_{j=0}^{m} \Big| \sum_{|\alpha|=j} \frac{P^{(\alpha)}(\xi)}{\alpha!} \eta_\xi^\alpha \Big| - 2 \sum_{j \neq k} \tau_k^j (\Psi_j P)(\xi) \\ &\overset{(6),(8)}{\geq} |P(\xi + \eta_\xi)| - 2\Big\{ \frac{k\mu_k}{\tau_k} + \frac{\tau_k}{\hat{\nu}_k} \Big\} \widetilde{P}(\xi, \tau_k) \\ &\overset{(9)}{\geq} \Big\{ A^{-1} - \frac{2k\mu_k}{\tau_k} - \frac{2\tau_k}{\hat{\nu}_k} \Big\} \widetilde{P}(\xi, \tau_k) \\ &\overset{(7)}{\geq} \frac{1}{2A} \widetilde{P}(\xi, \tau_k) \ . \end{aligned}$$

This yields (1^k).

In the case $k = m$ we choose $\tau_m \geq 1$ such that

(10) $$\mu_m \leq \tau_m,\ A^{-1} - \frac{2m\mu_m}{\tau_m} \geq \frac{1}{2A} \ .$$

Let $\xi \in \Omega'_m := \Omega_m$ be fixed and choose $\eta_\xi \in \mathbf{B}_{\mathbf{R}^n}(\tau_m)$ such that

(11) $$\widetilde{P}(\xi, \tau_m) \leq A|P(\xi + \eta_\xi)| : \quad P \in \Pi \ \ (cf.\ (5)) \ .$$

Using (6), (10) and (11) an analogous computation as above yields (1^m) :

$$|P(\xi + z\eta_\xi)| \geq \Big\{ A^{-1} - \frac{2m\mu_m}{\tau_m} \Big\} \widetilde{P}(\xi, \tau_m) \geq \frac{1}{2A} \widetilde{P}(\xi, \tau_m) : P \in \Pi, z \in \mathbf{T}^1 . \ \square$$

2. Some distribution spaces.

We adopt the standard notations for spaces of test functions and distributions (*cf.* [H1], [H2]) :

$\mathcal{D} = C_c^\infty(\mathbf{R}^n)$ — C^∞-functions with compact support;

$\mathcal{D}' = \mathcal{D}'(\mathbf{R}^n)$ — space of all distributions;

$\mathcal{S} = \mathcal{S}(\mathbf{R}^n)$ — space of rapidly decreasing C^∞-functions;

$\mathcal{S}' = \mathcal{S}'(\mathbf{R}^n)$ — space of tempered distributions.

Recall that each of these spaces carries a natural locally convex vector space topology. The scalar product of two vectors $\xi, \zeta \in \mathbf{C}^n$ will be denoted by $[\xi, \zeta] := \sum_{\nu=1}^{n} \xi_\nu \bar{\zeta}_\nu$. If $\varphi \in \mathcal{S}$ then the Fourier transform $\hat{\varphi}$ of φ is the function

$$\hat{\varphi}(\zeta) := \int_{\mathbf{R}^n} \exp(-i[\zeta, x])\varphi(x)dx \ : \ \zeta \in \mathbf{R}^n \ .$$

The Fourier transform $\hat{u}$ of $u \in \mathcal{S}'$ is defined by the formula

$$\langle \hat{u}, \varphi \rangle := \langle u, \hat{\varphi} \rangle : \quad \varphi \in \mathcal{S} \ ,$$

where $\langle \cdot, \cdot \rangle$ denotes the disbribution pairing. The following definitions and results are taken from Hörmander [H2], §10.1.

2.1. Definition.

(a) *A function* $k : \mathbf{R}^n \to (0, \infty)$ *will be called a* temperate weight function *if there exist constants* $a, b > 0$ *such that*

$$k(\xi + \zeta) \leq (1 + a|\xi|)^b k(\zeta) : \quad \xi, \zeta \in \mathbf{R}^n \ .$$

The set of all such functions will be denoted by $\mathcal{K}$.

(b) *If* $k \in \mathcal{K}$ *and* $1 \leq p \leq \infty$ *we denote by* $\mathbf{B}_{p,k}$ *the set of all distributions* $u \in \mathcal{S}'$ *such that* $\hat{u}$ *is a function and*

$$\|u\|_{p,k} := \left((2\pi)^{-n} \int_{\mathbf{R}^n} |k(\xi)\hat{u}(\xi)|^p d\xi\right)^{1/p} < \infty \ .$$

In the case $p = \infty$ *this expression has to be interpreted as* $\operatorname{ess.sup}_{\xi \in \mathbf{R}^n} |k(\xi)\hat{u}(\xi)|$.

By [H2], 10.1.7 we have

$$\mathcal{S} \hookrightarrow \mathbf{B}_{p,k} \hookrightarrow \mathcal{S}' \ ,$$

where $\mathfrak{F} \hookrightarrow \mathfrak{G}$ means a continuous embedding of topological vector spaces $\mathfrak{F}, \mathfrak{G}$. The spaces $\mathbf{B}_{p,k}$ are Banach spaces which for $1 \leq p < \infty$ contain $\mathcal{D}$

as a dense subset. In this case the dual $(\mathbf{B}_{p,k})'$ of $\mathbf{B}_{p,k}$ is (isometrically) isomorphic to $\mathbf{B}_{p',k'}$, where

$$1/p + 1/p' = 1\ ,\quad k'(\xi) := 1/k(-\xi)\ .$$

Any continuous linear form on $\mathbf{B}_{p,k}$ is given by continuous extension of a form $\varphi \mapsto \langle v, \varphi\rangle$, defined for $\varphi \in \mathcal{D}$ with $v \in \mathbf{B}_{p',k'}$. The norm of this functional equals $\|v\|_{p',k'}$ ([H2], 10.1.14). Let

$$\mathbf{B}^{\mathrm{loc}}_{p,k} := \{u \in \mathcal{D}' \mid \psi \cdot u \in \mathbf{B}_{p,k},\ \psi \in \mathcal{D}\}$$

denote the local space associated with $\mathbf{B}_{p,k}$. This is a Fréchet space with the system of seminorms $u \mapsto \|\psi \cdot u\|_{p,k}$, $\psi \in \mathcal{D}$.

In the following we shall consider certain subspaces of $\mathbf{B}^{\mathrm{loc}}_{p,k}$:

2.2. DEFINITION. — *Let $\sigma : [0,\infty) \to \mathbb{R}$ be a C^∞-function satisfying $\lim_{\rho\to+\infty} \sigma(\rho) = +\infty$ and $\sigma^{(j)}$ is bounded for all $j \geq 1$.*

Further let $\tilde{\sigma}(x) := \exp(\sigma([x,x])\cdot\sqrt{1+[x,x]})$, $x \in \mathbb{R}^n$. For $1 \leq p \leq \infty$ and $k \in \mathcal{K}$ we consider the distribution spaces

$$\mathbf{B}^{+\sigma}_{p,k} := \{u/\tilde{\sigma} \mid u \in \mathbf{B}_{p,k}\}\ ;\quad \mathbf{B}^{-\sigma}_{p,k} := \{\tilde{\sigma}\cdot v \mid v \in \mathbf{B}_{p,k}\}\ .$$

Obviously these are Banach spaces with the norms

1) $\|u/\tilde{\sigma}\|^{+\sigma}_{p,k} := \|u\|_{p,k}$

2) $\|\tilde{\sigma}\cdot v\|^{-\sigma}_{p,k} := \|v\|_{p,k}$.

Remarks.

(i) Since $\tilde{\sigma}$, $1/\tilde{\sigma} \in C^\infty(\mathbb{R}^n)$ we have $\mathbf{B}^{\pm\sigma}_{p,k} \subseteq \mathbf{B}^{\mathrm{loc}}_{p,k}$ by [H2], 10.1.23.

(ii) It is our intention to keep the spaces $\mathbf{B}^{-\sigma}_{p,k}$ as small as possible. This can be achieved by letting the function σ tend to $+\infty$ very slowly. For example, choose $\sigma_0 \in C^\infty(\mathbb{R})$ with $\sigma_0(\rho) = \begin{cases} 0, & \rho \leq 0 \\ 1, & \rho \geq 1 \end{cases}$ and put $\sigma(\rho) := \sum_{j=1}^{\infty} \sigma_0(\rho/a_j - a_j)$, where the sequence (a_j) tends to $+\infty$ very fast (e.g. $a_1 := 2$, $a_{j+1} := a_j^{a_j}$).

2.3. LEMMA. — *Let $1 \leq p \leq \infty$, $k \in \mathcal{K}$ and σ as in Definition 2.2. Then we have*

$$\mathbf{B}^{-\sigma}_{p,k} \hookrightarrow \mathbf{B}^{\mathrm{loc}}_{p,k}\ . \tag{2.1}$$

Proof. — Let $\psi \in \mathcal{D}$ and $v \in \mathbf{B}_{p,k}^{-\sigma}$ arbitrary. Since $\psi \cdot \tilde{\sigma} \in \mathcal{D} \subseteq \mathcal{S}$ it follows from [H2], 10.1.15 that

$$\|\psi \cdot v\|_{p,k} = \|\psi \cdot \tilde{\sigma} \cdot v/\tilde{\sigma}\|_{p,k} \le K\|v/\tilde{\sigma}\|_{p,k} = K\|v\|_{p,k}^{-\sigma} ,$$

with $K < \infty$ depending only on $\tilde{\sigma}$, k and ψ. Since the topology of $\mathbf{B}_{p,k}^{\mathrm{loc}}$ is given by the seminorms $v \mapsto \|\psi \cdot v\|_{p,k}$ the proof is complete. □

The same proof shows that if σ_1, σ_2 are such that $\tilde{\sigma}_1/\tilde{\sigma}_2 \in \mathcal{S}$ (e.g. if $\limsup_{\rho\to\infty} \sigma_1(\rho) - \sigma_2(\rho) < 0$) then $\mathbf{B}_{p,k}^{-\sigma_1} \hookrightarrow \mathbf{B}_{p,k}^{-\sigma_2}$.

2.4. *Remark.* — Let $Q \in \mathrm{Pol}'(n,m)$ be fixed and $\Pi \subseteq \mathsf{E}(Q)$ a compact set. By Theorem 1.2 there is a bounded measurable function $\eta : \mathbf{R}^n \to \mathbf{R}^n$ such that

$$\widetilde{P}(-\xi) \le A|P(-\xi - z\eta(\xi))| : \quad P \in \Pi,\ \xi \in \mathbf{R}^n,\ z \in \mathbf{T}^1 .$$

Using this we can for every $P \in \Pi$ define a distribution $\mathfrak{f}_P \in \mathcal{D}'$ through

$$\text{(2.2)} \quad \langle \mathfrak{f}_P, \varphi\rangle := (2\pi)^{-n} \int_{\mathbf{R}^n} \int_{z\in\mathbf{T}^1} \frac{\widehat{\varphi}(\xi + z\eta(\xi))}{P(-\xi - z\eta(\xi))} \frac{dz}{2\pi i z}\, d\xi : \quad \varphi \in \mathcal{D} .$$

This type of formula has been introduced by L. Hörmander. Similarly as in [T2] we could now show that $\mathfrak{f}_P$ is an analytic function of $P \in \Pi$ with values in $\mathbf{B}_{\infty,\widetilde{Q}}^{-\sigma}$ and $\mathfrak{f}_P$ is a fundamental solution of $P(D)$ for each P. (In fact, $\mathfrak{f}_P$ takes its values in the smaller space $\mathbf{B}_{\infty,\widetilde{Q}}^{*H^1}$ defined below, where $H^1 = (\eta)$.) We shall not do so since it is our aim to prove a more general result (Theorem 3.1 below). However, formula (2.2) serves as a motivation for the following

2.5. DEFINITION. — *In order to simplify notations we introduce the measure $|dz| := |dz_1|\cdots|dz_r|$ on the torus $\mathbf{T}^r$ ($r \in \mathbb{N}$). Let $1 \le p \le \infty$, $k \in \mathcal{K}$ and $H^r = (\eta_s)_{s=1}^r : \mathbf{R}^n \longrightarrow (\mathbf{R}^n)^r$ a bounded measurable function. For any $\varphi \in \mathcal{D}$ we set*

$$\|\varphi\|_{p,k}^{H^r} := \Big((2\pi)^{-n-r} \int_{\mathbf{R}^n} \int_{\mathbf{T}^r} |k(\xi)\widehat{\varphi}(\xi + \widetilde{H}^r(\xi,z))|^p\, |dz|\, d\xi\Big)^{1/p} (p < \infty) ,$$

where $\widetilde{H}^r(\xi,z) := \sum_{s=1}^r z_s \cdot \eta_s(\xi)$,

$$\|\varphi\|_{\infty,k}^{H^r} := \sup\{|k(\xi)\widehat{\varphi}(\xi + \widetilde{H}^r(\xi,z))| \mid \xi \in \mathbf{R}^n,\ z \in \mathbf{T}^r\} .$$

The theorem of Paley-Wiener-Schwartz ([H1], §7.3) ensures that $\|\varphi\|_{p,k}^{H^r}$ is finite for each $\varphi \in \mathcal{D}$. Obviously, $(\mathcal{D}, \|\cdot\|_{p,k}^{H^r})$ is a normed space. Its "dual space",

$$\mathbf{B}_{p',k'}^{*H^r} := \{v \in \mathbf{B}_{p',k'}^{\mathrm{loc}} \mid \|v\|_{p',k'}^{*H^r} := \sup\{|\langle v,\varphi\rangle|/\|\varphi\|_{p,k}^{H^r} \mid 0 \ne \varphi \in \mathcal{D}\} < \infty\}$$

will be endowed with the norm $\|\cdot\|^{*H^r}_{p',k'}$. Here $p' := 1$ if $p = \infty$.

The reason why we have introduced the space $\mathbf{B}^{-\sigma}_{q,k}$ is that it contains each $\mathbf{B}^{*H^r}_{q,k}$, yet it is small enough to give quite precise information on the growth at infinity of solutions of the equation $P(D)\mathfrak{f}_P = \delta$ when P runs through $\mathsf{E}(Q)$ and $\mathfrak{f}_P$ depends analytically on P (*cf.* the remark at the end of [M]).

2.6. Lemma. — *Let $H^{r+1} = (\eta_s)^{r+1}_{s=1}$ as in Definition 2.5. With $H^r := (\eta_s)^r_{s=1}$ we then have*

$$\|\varphi\|_{p,k} \le \|\varphi\|^{H^r}_{p,k} \le \|\varphi\|^{H^{r+1}}_{p,k} : \quad \varphi \in \mathcal{D}\,, \tag{2.3}$$

hence

$$\mathbf{B}_{p',k'} \hookrightarrow \mathbf{B}^{*H^r}_{p',k'} \hookrightarrow \mathbf{B}^{*H^{r+1}}_{p',k'}\,. \tag{2.4}$$

Proof. — By Cauchy's formula and the Hölder inequality we have, if $p < \infty$,

$$|\widehat{\varphi}(\xi + \widetilde{H}^r(\xi, z'))|^p \le \int_{z_{r+1}\in\mathbf{T}^1} |\widehat{\varphi}(\xi + \widetilde{H}^{r+1}(\xi, z))|^p \frac{|dz_{r+1}|}{2\pi}\,,$$

where $z = (z', z_{r+1})$. Inserting this in the definition of $\|\varphi\|^{H^{r+1}}_{p,k}$ yields the second inequality in (2.3). In the case $p = \infty$ we can argue similarly using the maximum principle. Choosing $H^0 \equiv 0$ we also get $\|\varphi\|_{p,k} = \|\varphi\|^{H^0}_{p,k} \le \|\varphi\|^{H^r}_{p,k}$. The embedding (2.4) is a direct consequence of these estimates. □

2.7. Lemma. — *Let σ as in Definition 2.2 and H^r as in Definition 2.5. Then there exists a constant $K < \infty$ such that*

$$\|\varphi\|^{H^r}_{p,k} \le K\|\varphi\|^{+\sigma}_{p,k} : \quad \varphi \in \mathcal{D}\,. \tag{2.5}$$

Proof. — Let $\rho := 1 + \sup\{|\widetilde{H}^r(\xi, z)|_\infty \mid \xi \in \mathbf{R}^n,\ z \in \mathbf{T}^r\}$. For any $\varphi \in \mathcal{D}$ and fixed $\xi \in \mathbf{R}^n$, $z \in \mathbf{T}^r$ we have

$$|\widehat{\varphi}(\xi + \widetilde{H}^r(\xi, z))|^p \le \left(\frac{\rho^p}{2\pi}\right)^n \int_{\mathbf{T}^n} |\widehat{\varphi}(\xi + \rho\zeta)|^p |d\zeta| \text{ if } p < \infty\,.$$

This implies

$$\begin{aligned}(\|\varphi\|^{H^r}_{p,k})^p &\le \frac{\rho^{np}}{(2\pi)^{2n}} \int_{\mathbf{R}^n}\int_{\mathbf{T}^n} |k(\xi)\cdot\widehat{\varphi}(\xi + \rho\zeta)|^p |d\zeta|\, d\xi \\ &= \left(\frac{\rho^p}{2\pi}\right)^n \int_{\mathbf{T}^n} (2\pi)^{-n} \int_{\mathbf{R}^n} |k(\xi)\cdot \exp(-i[\rho\zeta,\cdot])\varphi)^\wedge(\xi)|^p d\xi\, |d\zeta| \qquad (2.6) \\ &= \left(\frac{\rho^p}{2\pi}\right)^n \int_{\mathbf{T}^n} (\|\exp(-i[\rho\zeta,\cdot])\varphi\|_{p,k})^p\, |d\zeta|\,.\end{aligned}$$

Now consider the functions

$$\Phi_\zeta(x) := \exp(-i[\rho\zeta, x])/\tilde{\sigma}(x) : \quad \zeta \in \mathbf{T}^n .$$

It is not hard to check that $\{\Phi_\zeta\}$ is a bounded subset of $\mathcal{S}$. With the weight function $M_k \in \mathcal{K}$ (cf. [H2], §10.1),

$$M_k(\xi) := \sup_{\xi' \in \mathbf{R}^n} k(\xi + \xi')/k(\xi') : \quad \xi \in \mathbf{R}^n ,$$

we have $\mathcal{S} \hookrightarrow \mathbf{B}_{1,M_k}$ ([H2], 10.1.7), hence

$$\sup\{\|\Phi_\zeta\|_{1,M_k} \mid \zeta \in \mathbf{T}^n\} =: K < \infty .$$

It follows from [H2], 10.1.15 that

$$\sup\{\|\Phi_\zeta \cdot \psi\|_{p,k} \mid \zeta \in \mathbf{T}^n\} \leq K\|\psi\|_{p,k} : \quad \psi \in \mathcal{D} .$$

From (2.6) we thus obtain with $\psi = \tilde{\sigma} \cdot \varphi$:

$$\|\varphi\|_{p,k}^{H^r} \leq \left(\left(\frac{\rho^p}{2\pi}\right)^n \int_{\mathbf{T}^n} (\|\Phi_\zeta \cdot \tilde{\sigma} \cdot \varphi\|_{p,k})^p |d\zeta|\right)^{1/p} \leq K\rho^n \|\tilde{\sigma} \cdot \varphi\|_{p,k} = K'\|\varphi\|_{p,k}^{+\sigma} .$$

The case $p = \infty$ can be treated analoguously. □

2.8. Corollary. — *Under the assumptions of Lemma 2.7 the mapping $v \mapsto \langle v, \cdot\rangle$ identifies $\mathbf{B}_{p',k'}^{*H^r}$ isometrically with the dual of the normed space $(\mathcal{D}, \|\cdot\|_{p,k}^{H^r})$. In particular, $\mathbf{B}_{p',k'}^{*H^r}$ is complete. Furthermore we have*

$$\mathbf{B}_{p',k'}^{*H^r} \hookrightarrow \mathbf{B}_{p',k'}^{-\sigma} . \tag{2.7}$$

Proof. — Clearly, $v \mapsto \langle v, \cdot\rangle$ defines an isometric embedding of $\mathbf{B}_{p',k'}^{*H^r}$ into $(\mathcal{D}, \|\cdot\|_{p,k}^{H^r})'$. We have to show that it is onto. So let ℓ be a continuous linear form on $(\mathcal{D}, \|\cdot\|_{p,k}^{H^r})$. By Lemma 2.7 we have

$$|\langle \ell/\tilde{\sigma}, \varphi\rangle| \leq \|\ell\| \; \|\varphi/\tilde{\sigma}\|_{p,k}^{H^r} \leq K\|\ell\| \; \|\varphi\|_{p,k} : \quad \varphi \in \mathcal{D} . \tag{2.8}$$

If $p < \infty$ then $\mathbf{B}_{p',k'}$ is the dual space of $\mathbf{B}_{p,k}$, so $\ell \in \mathbf{B}_{p',k'}^{-\sigma} \subseteq \mathbf{B}_{p',k'}^{\text{loc}}$. Hence $\ell \in \mathbf{B}_{p',k'}^{*H^r}$ and $\|\ell\|_{p',k'}^{-\sigma} = \|\ell/\tilde{\sigma}\|_{p',k'} \leq K\|\ell\|_{p',k'}^{*H^r}$ by (2.8).

In the case $p = \infty$ we can analoguously derive (2.8) with σ replaced by $\sigma_1(\rho) := \sigma(\rho) - 1$. Since $\mathcal{S} \hookrightarrow \mathbf{B}_{\infty,k}$ the functional $\ell_1 := \ell/\tilde{\sigma}_1$ can be extended such that $|\langle \ell_1, \varphi\rangle| \leq K\|\ell\| \; \|\varphi\|_{\infty,k}$ holds for all $\varphi \in \mathcal{S}$. Hence $\ell_1 \in \mathcal{S}'$ and the Fourier transform of ℓ_1 is a continuous linear form on $\mathcal{S}$ equipped with the norm $\sup_\xi |k(-\xi)\varphi(\xi)|$. But then $\langle \hat{\ell}_1, \varphi\rangle = \int \varphi(\xi) d\mu(\xi)$ with a measure $d\mu$ in $\mathbf{R}^n$ of total mass $\int |d\mu(\xi)|/k(-\xi) < \infty$. Noting that $\tau := \tilde{\sigma}_1/\tilde{\sigma} \in \mathcal{S}$ we obtain $\ell/\tilde{\sigma} = \tau \cdot \ell_1 \in \mathcal{S}'$ and $(\ell/\tilde{\sigma})^\wedge = (2\pi)^{-n}\hat{\tau} * d\mu$ which is a $\mathcal{C}^\infty$-function satisfying $\int |(\ell/\tilde{\sigma})^\wedge(\xi)|/k(-\xi)\, d\xi < \infty$, *i.e.* $(\ell/\tilde{\sigma}) \in \mathbf{B}_{1,k'}$. As in the case $p < \infty$ we conclude that $\ell \in \mathbf{B}^{*H^r}_{1,k'}$ and $\|\ell\|^{-\sigma}_{1,k'} \leq K'\|\ell\|^{*H^r}_{1,k'}$ by the closed graph theorem. □

Now we shall investigate how a differential operator with constant coefficients acts in the spaces $\mathbf{B}^{*H^r}_{q,k}$ $(1 \leq q \leq \infty, k \in \mathcal{K})$. If $P(x) = \sum_{|\alpha|\leq m} a_\alpha x^\alpha$ is a polynomial in $x \in \mathbf{R}^n$ we consider the differential expression

$$P(D) := \sum_{|\alpha|\leq m} a_\alpha D^\alpha \text{ where } D := -i\Big(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\Big).$$

2.9. Proposition. — *Let $P, Q \in \mathrm{Pol}'(n,m)$ with $P < Q$ and $H^r = (\eta_s)^r_{s=1}$ as in Definition 2.5. Then the operator $P(D)$ maps $\mathbf{B}^{*H^r}_{q,k\widetilde{Q}}$ continuously into $\mathbf{B}^{*H^r}_{q,k}$.*

Proof. — Let $\rho := \sup\{|\widetilde{H}^r(\xi,z)|_\infty \mid \xi \in \mathbf{R}^n, z \in \mathbf{T}^r\}$ and $\xi \in \mathbf{R}^n$, $z \in \mathbf{T}^r$ fixed. With $\zeta := \widetilde{H}^r(\xi, z)$ we have for any $\varphi \in \mathcal{D}$:

$$\begin{aligned}|(k\widetilde{Q})'(\xi)\cdot(P(-D)\varphi)^\wedge(\xi+\zeta)| &= |(k\widetilde{Q})'(\xi)\cdot P(-\xi-\zeta)\cdot\hat{\varphi}(\xi+\zeta)|\\ &\leq |(k\widetilde{Q})'(\xi)\cdot\widetilde{P}(-\xi,\rho)\cdot\hat{\varphi}(\xi+\zeta)|\\ &\leq (1+\rho)^m\frac{\widetilde{P}(-\xi)}{\widetilde{Q}(-\xi)}|k'(\xi)\cdot\hat{\varphi}(\xi+\zeta)|\,.\end{aligned}$$

Since $\sup_{\xi\in\mathbf{R}^n}\dfrac{\widetilde{P}(-\xi)}{\widetilde{Q}(-\xi)} < \infty$ we obtain

$$\|P(-D)\varphi\|^{H^r}_{q',(k\widetilde{Q})'} \leq K\|\varphi\|^{H^r}_{q',k'} : \quad \varphi \in \mathcal{D}\,. \tag{2.9}$$

Now, if $v \in \mathbf{B}^{*H^r}_{q,k\widetilde{Q}} \subseteq \mathbf{B}^{\mathrm{loc}}_{q,k\widetilde{Q}}$ it follows from [H2], 10.1.22 that $P(D)v \in \mathbf{B}^{\mathrm{loc}}_{q,k}$. Furthermore, (2.9) implies that

$$\begin{aligned}|\langle P(D)v,\varphi\rangle| = |\langle v, P(-D)\varphi\rangle| &\leq \|v\|^{*H^r}_{q,k\widetilde{Q}}\|P(-D)\varphi\|^{H^r}_{q',(k\widetilde{Q})'}\\ &\leq K\|v\|^{*H^r}_{q,k\widetilde{Q}}\|\varphi\|^{H^r}_{q',k'}\end{aligned}$$

for any $\varphi \in \mathcal{D}$. In particular this means that $P(D)v \in \mathbf{B}^{*H^r}_{q,k}$ and

$$\|P(D)v\|^{*H^r}_{q,k} \leq K\|v\|^{*H^r}_{q,k\widetilde{Q}}\,.$$ □

2.10. Proposition. — *Let P, $Q \in \mathrm{Pol}'(n,m)$ with $P \sim Q$, $H^r = (\eta_s)^r_{s=1}$ as in Definition 2.5 and $\rho := \sup\{|\widetilde{H}^{r-1}(\xi,z')|_\infty \mid \xi \in \mathbf{R}^n$,*

$z' \in \mathbf{T}^{r-1}\}$ ($\rho := 0$ *if* $r = 1$). *Assume that with some constant* $A > 0$ *we have*

$$\widetilde{P}(-\xi) \le A|P(-\xi - \zeta - z_r\eta_r(\xi))| : \quad \xi \in \mathbf{R}^n, \zeta \in \mathbf{B}_{\mathbf{C}^n}(\rho),\ z_r \in \mathbf{T}^1 .$$

Then the operator $P(D) : \mathbf{B}^{*H^r}_{q,k\widetilde{Q}} \longrightarrow \mathbf{B}^{*H^r}_{q,k}$ *is surjective.*

Proof. — Since $\widetilde{Q}(-\xi) \le B\widetilde{P}(-\xi)$ the assumption implies that

$$\|P(-D)\varphi\|^{H^r}_{q',(k\widetilde{Q})'} \ge (AB)^{-1}\|\varphi\|^{H^r}_{q',k'} : \quad \varphi \in \mathcal{D} . \tag{2.10}$$

Now let $w \in \mathbf{B}^{*H^r}_{q,k}$ be given. Then by (2.10) the mapping

$$P(-D)\varphi \longmapsto \langle w, \varphi\rangle$$

is a well-defined continuous linear form on the subspace $P(-D)\mathcal{D}$ of $E := (\mathcal{D}, \|\cdot\|^{H^r}_{q',(k\widetilde{Q})'})$. By the Hahn-Banach theorem there exists a continuous extension v of this form to the whole of E and Corollary 2.8 implies that $v \in \mathbf{B}^{*H^r}_{q,k\widetilde{Q}}$. Finally it is clear that

$$\langle P(D)v, \varphi\rangle = \langle v, P(-D)\varphi\rangle = \langle w, \varphi\rangle : \quad \varphi \in \mathcal{D} ,$$

i.e. $P(D)v = w$. □

3. Parameter depending differential operators.

We come back to the main topic of this article. Let $Q \in \mathrm{Pol}'(n, m)$ be fixed. Consider a family of differential operators

$$P(\lambda, D) = \sum_{|\alpha| \le m} a_\alpha(\lambda) D^\alpha , \tag{3.1}$$

where the coefficients a_α (constant with respect to x) are analytic functions of a parameter λ varying in a complex manifold Λ. The only assumption we make is that for each value of λ the polynomial $P(\lambda, \cdot)$ is equally strong as Q. Denoting by $\{R_1, \dots, R_\nu\}$ any fixed basis of the vector space $\mathbf{W}(Q)$ we can write

$$P(\lambda, D) = \sum_{\mu=1}^{\nu} b_\mu(\lambda) R_\mu(D) \tag{3.2}$$

with analytic functions $b_\mu : \Lambda \to \mathbf{C}$. Recall (1.1 (iii)) that the set $\mathbf{E}(Q)$ is a holomorphically convex open submanifold of $\mathbf{W}(Q)$. Hence we may take in (3.2) $\Lambda = \mathbf{E}(Q)$ and $\{b_\mu\}$ as the coordinate functions of P with respect to the basis $\{R_\mu\}$.

It $\mathcal{E}$ is a locally convex vector space we denote by $\mathcal{H}(\Lambda, \mathcal{E})$ the set of all analytic functions $e : \Lambda \to \mathcal{E}$. Further let $\sigma \in \mathcal{C}^\infty[0, \infty)$ be any fixed weight function as in Definition 2.2. Recall that $\mathbf{B}^{-\sigma}_{q,k} \hookrightarrow \mathbf{B}^{\text{loc}}_{q,k}$ for $1 \leq q \leq \infty$, $k \in \mathcal{K}$.

3.1. Theorem. — *Let $1 \leq q \leq \infty$ and $k \in \mathcal{K}$. Assume that Λ is a Stein manifold. Then for any $\mathfrak{g} \in \mathcal{H}(\Lambda, \mathbf{B}_{q,k})$ there exists $\mathfrak{f} \in \mathcal{H}(\Lambda, \mathbf{B}^{-\sigma}_{q,k\tilde{Q}})$ such that*

(i) $P(\lambda, D)\mathfrak{f}(\lambda) = \mathfrak{g}(\lambda), \ \lambda \in \Lambda;$

(ii) $R(D)\mathfrak{f} \in \mathcal{H}(\Lambda, \mathbf{B}^{-\sigma}_{q,k})$ *for any* $R \in \mathbf{W}(Q)$.

In the following corollaries we do not make any assumptions concerning Λ :

3.2 Corollary. — *Let $1 \leq q \leq \infty$ and $k \in \mathcal{K}$. Then for any $\mathfrak{g}_0 \in \mathbf{B}_{q,k}$ there exists $\mathfrak{f} \in \mathcal{H}(\Lambda, \mathbf{B}^{-\sigma}_{q,k\tilde{Q}})$ such that $P(\lambda, D)\mathfrak{f}(\lambda) \equiv \mathfrak{g}_0$, and 3.1* (ii) *holds.*

Proof. — By our above remark we may take P itself as a parameter varying in the Stein manifold $\mathbf{E}(Q)$. Theorem 3.1 yields a function $\tilde{\mathfrak{f}} \in \mathcal{H}(\mathbf{E}(Q), \mathbf{B}^{-\sigma}_{q,k\tilde{Q}})$ such that $P(D)\tilde{\mathfrak{f}}(P) = \mathfrak{g}_0$, $P \in \mathbf{E}(Q)$. Since the mapping $\lambda \mapsto \mathfrak{p}(\lambda) := P(\lambda, \cdot)$ is analytic with values in $\mathbf{E}(Q)$ we have $\mathfrak{f} := \tilde{\mathfrak{f}} \circ \mathfrak{p} \in \mathcal{H}(\Lambda, \mathbf{B}^{-\sigma}_{q,k\tilde{Q}})$ and $P(\lambda, D)\mathfrak{f}(\lambda) \equiv \mathfrak{g}_0$. □

By δ we denote the Dirac distribution at 0, $\langle \delta, \varphi \rangle := \varphi(0)$. The next corollary answers a question of L. Hörmander ([H2], p. 59) :

3.3. Corollary. — *There exists $\mathfrak{f} \in \mathcal{H}(\Lambda, \mathbf{B}^{-\sigma}_{\infty,\tilde{Q}})$ such that $P(\lambda, D)\mathfrak{f}(\lambda) \equiv \delta$, and 3.1* (ii) *holds with $q = \infty$, $k \equiv 1$.*

Proof. — This is a special case of Corollary 3.2 since with $k \equiv 1$ we have $\delta = \mathfrak{g}_0 \in \mathbf{B}_{\infty,k}$. □

3.4. *Remark.* — If Λ is an open subset of $\mathbf{R}^d$ (or a real analytic manifold) then the analogues of Theorem 3.1 and its corollaries hold with "analytic" replaced by "real analytic".

Proof. — By a result of Grauert [G] there exists a neighborhood basis of Λ in $\mathbf{C}^d$ consisting of holomorphically convex open sets. Using this the real analytic case can be reduced to the analytic one (*cf.* [M]). □

It remains to prove Theorem 3.1. If $\mathfrak{F}$, $\mathfrak{G}$ are Banach spaces we denote by $\mathcal{L}(\mathfrak{F},\mathfrak{G})$ the space of all bounded linear operators from $\mathfrak{F}$ to $\mathfrak{G}$ equipped with the operator norm topology. In the proof of 3.1 we shall make use of the following result of J. Leiterer [L].

3.5. THEOREM. — *Let $\mathfrak{F}$, $\mathfrak{G}$ be Banach spaces and Λ a complex Stein manifold. Let $\mathfrak{T} \in \mathcal{H}(\Lambda, \mathcal{L}(\mathfrak{F},\mathfrak{G}))$ such that $\mathfrak{T}(\lambda)\mathfrak{F} = \mathfrak{G}$ for each $\lambda \in \Lambda$. Then*

(a) *There exists for each function $\mathfrak{g} \in \mathcal{H}(\lambda, \mathfrak{G})$ a function $\mathfrak{f} \in \mathcal{H}(\Lambda, \mathfrak{F})$ such that $\mathfrak{T}(\lambda)\mathfrak{f}(\lambda) = \mathfrak{g}(\lambda)$, $\lambda \in \Lambda$.*

(b) *For any open subset Λ' of Λ let $\mathcal{N}(\Lambda') := \{\mathfrak{f} \in \mathcal{H}(\Lambda',\mathfrak{F}) \mid \mathfrak{I}(\lambda)\mathfrak{f}(\lambda) \equiv 0\}$. If Λ' is holomorphically convex then the set $\mathcal{N}(\Lambda)_{|\Lambda'}$ of restrictions to Λ' of functions in $\mathcal{N}(\Lambda)$ is dense in $\mathcal{N}(\Lambda')$.*

Proof of Theorem 3.1. — Let $\{\Lambda_r\}_{r\in\mathbf{N}}$ be an exhausting sequence of open submanifolds of Λ such that each Λ_r is holomorphically convex, $\overline{\Lambda}_r$ is compact and $\overline{\Lambda}_r \subseteq \Lambda_{r+1}$. For each $r \in \mathbf{N}$ we inductively choose a bounded measurable function $H^r = (\eta_s)_{s=1}^r : \mathbf{R}^n \longrightarrow (\mathbf{R}^n)^r$ in the following way : set $\rho_r := \sup\{|\widetilde{H}^{r-1}(\xi, z')|_\infty \mid \xi \in \mathbf{R}^n, z' \in \mathbf{T}^{r-1}\}$ ($\rho_1 := 0$). Then by Theorem 1.2 there exist $A_r \geq 1$ and a bounded measurable function $\eta_r : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ such that for all $\lambda \in \overline{\Lambda}_r$, $\xi \in \mathbf{R}^n$, $\zeta \in \mathbf{B}_{\mathbf{C}^n}(\rho_r)$, $z_r \in \mathbf{T}^1$ we have

$$\widetilde{P}(\lambda, -\xi) \leq A_r |P(\lambda, -\xi - \zeta - z_r\eta_r(\xi))| \,. \tag{3.3}$$

Thus, H^r is defined for each $r \in \mathbf{N}$. Now consider the spaces

$$\mathfrak{F}_r := \mathbf{B}^{*H^r}_{q,k\widetilde{Q}} \,,\; \mathfrak{G}_r := \mathbf{B}^{*H^r}_{q,k} : \quad r \in \mathbf{N} \,.$$

By (2.1), (2.4) and (2.7) we have the embeddings

$$\mathfrak{F}_r \hookrightarrow \mathfrak{F}_{r+1} \hookrightarrow \mathfrak{F} := \mathbf{B}^{-\sigma}_{q,k\widetilde{Q}} \hookrightarrow \mathbf{B}^{\text{loc}}_{q,k\widetilde{Q}} \,, \tag{3.4}$$

$$\mathbf{B}_{q,k} \hookrightarrow \mathfrak{G}_r \hookrightarrow \mathfrak{G}_{r+1} \hookrightarrow \mathfrak{G} := \mathbf{B}^{-\sigma}_{q,k} \hookrightarrow \mathbf{B}^{\text{loc}}_{q,k} \,. \tag{3.5}$$

Consider the representation (3.2) of $P(\lambda, D)$. From Proposition 2.9 we know that each $R_\mu(D)$ induces a bounded linear operator from $\mathfrak{F}_r$ into $\mathfrak{G}_r$. Hence the mapping $\lambda \mapsto P(\lambda, D)$ is analytic with values in $\mathcal{L}(\mathfrak{F}_r, \mathfrak{G}_r)$. From (3.3) and Proposition 2.10 we conclude that $P(\lambda, D)\mathfrak{F}_r = \mathfrak{G}_r$ for each $\lambda \in \overline{\Lambda}_r$. Furthermore, $\mathfrak{g} \in \mathcal{H}(\Lambda, \mathfrak{G}_r)$ by (3.5). It follows from part (a) of Theorem 3.5 that there exists for each $r \in \mathbf{N}$ a function $\tilde{\mathfrak{f}}_r \in \mathcal{H}(\Lambda_r, \mathfrak{F}_r)$ such that

$$P(\lambda, D)\tilde{\mathfrak{f}}_r(\lambda) = \mathfrak{g}(\lambda) : \quad \lambda \in \Lambda_r \,.$$

We construct a sequence of functions $\mathfrak{f}_r \in \mathcal{H}(\Lambda_r, \mathfrak{F}_r)$ as follows. Put $\mathfrak{f}_1 := \tilde{\mathfrak{f}}_1$ and assume that $\mathfrak{f}_1, \ldots, \mathfrak{f}_r$ are already defined. Consider then

$$\delta_{r+1}(\lambda) := \tilde{\mathfrak{f}}_{r+1}(\lambda) - \mathfrak{f}_r(\lambda) : \quad \lambda \in \Lambda_r .$$

By (3.4) we have $\delta_{r+1} \in \mathcal{H}(\Lambda_r, \mathfrak{F}_{r+1})$ and we may assume inductively that

$$P(\lambda, D)\delta_{r+1}(\lambda) = 0 : \quad \lambda \in \Lambda_r .$$

By part (b) of Theorem 3.5 there exists for arbitrary $\varepsilon_{r+1} > 0$ a function $\mathfrak{c}_{r+1} \in \mathcal{H}(\Lambda_{r+1}, \mathfrak{F}_{r+1})$ with the properties

$$P(\lambda, D)\mathfrak{c}_{r+1}(\lambda) = 0 : \lambda \in \Lambda_{r+1} \ ; \ \sup_{\lambda\in\Lambda_{r-1}} \|\delta_{r+1}(\lambda) - \mathfrak{c}_{r+1}(\lambda)\|_{\mathfrak{F}_{r+1}} \le \varepsilon_{r+1} \ ,$$

where for convenience we put $\Lambda_0 := \emptyset$. Since $\mathfrak{F}_{r+1} \hookrightarrow \mathfrak{F}$, $\mathfrak{G}_{r+1} \hookrightarrow \mathfrak{G}$ and the operators $R_\mu(D) : \mathfrak{F}_{r+1} \longrightarrow \mathfrak{G}_{r+1}$ $(\mu = 1, \ldots, \nu)$ are continuous (Proposition 2.9) one can choose ε_{r+1} so small that

$$\sup_{\lambda\in\Lambda_{r-1}} \|\delta_{r+1}(\lambda) - \mathfrak{c}_{r+1}(\lambda)\|_{\mathfrak{F}} \le 2^{-r} \ ,$$

$$\sup_{\lambda\in\Lambda_{r-1}} \|R_\mu(D)(\delta_{r+1}(\lambda) - \mathfrak{c}_{r+1}(\lambda))\|_{\mathfrak{G}} \le 2^{-r} : \quad \mu = 1, \ldots, \nu .$$

With this choice of $\mathfrak{c}_{r+1}$ we set

$$\mathfrak{f}_{r+1}(\lambda) := \tilde{\mathfrak{f}}_{r+1}(\lambda) - \mathfrak{c}_{r+1}(\lambda) : \quad \lambda \in \Lambda_{r+1} .$$

We obtain a sequence of functions $\mathfrak{f}_r \in \mathcal{H}(\Lambda_r, \mathfrak{F}_r) \subseteq \mathcal{H}(\Lambda_r, \mathfrak{F})$ with the properties

(3.6) $$P(\lambda, D)\mathfrak{f}_r(\lambda) = \mathfrak{g}(\lambda) : \quad \lambda \in \Lambda_r \ ,$$

(3.7) $$\sup_{\lambda\in\Lambda_{r-1}} \|\mathfrak{f}_{r+1}(\lambda) - \mathfrak{f}_r(\lambda)\|_{\mathfrak{F}} \le 2^{-r} \ ,$$

(3.8) $$\sup_{\lambda\in\Lambda_{r-1}} \|R_\mu(D)(\mathfrak{f}_{r+1}(\lambda) - \mathfrak{f}_r(\lambda))\|_{\mathfrak{G}} \le 2^{-r} : \quad \mu = 1, \ldots, \nu .$$

By (3.7) the limit

$$\mathfrak{f}(\lambda) := \lim_{r\to\infty} \mathfrak{f}_r(\lambda)$$

exists in $\mathfrak{F}$ for each $\lambda \in \Lambda$, and $\mathfrak{f} \in \mathcal{H}(\Lambda, \mathfrak{F})$. Since $\{R_\mu\}$ is a basis of $\mathbf{W}(Q)$ we conclude from (3.8) that $R(D)\mathfrak{f} \in \mathcal{H}(\Lambda, \mathfrak{G})$ for any $R \in \mathbf{W}(Q)$. Finally it is clear by (3.6) that $P(\lambda, D)\mathfrak{f}(\lambda) \equiv \mathfrak{g}(\lambda)$ since for fixed $\lambda \in \Lambda$ the sequence $\{\mathfrak{f}_r(\lambda)\}$ converges in $\mathbf{B}^{\mathrm{loc}}_{q,k\tilde{Q}}$ and the operator $P(\lambda, D) : \mathbf{B}^{\mathrm{loc}}_{q,k\tilde{Q}} \longrightarrow \mathbf{B}^{\mathrm{loc}}_{q,k}$ is continuous ([H2], 10.1.22). The proof is complete. □

Acknowledgement. — I wish to thank Prof. L. Hörmander for a hint which has made it possible to include the case $q = 1$ in Theorem 3.1 and its corollaries.

BIBLIOGRAPHY

[G] H. GRAUERT, On Levi's problem and the embedding of real-analytic manifolds, Ann. Math., 68, 2 (1958), 460–472.

[H1] L. HÖRMANDER, The analysis of linear partial differential operators I, Grundlehren d. mathem. Wissensch., 256, Springer (1983).

[H2] L. HÖRMANDER, The analysis of linear partial differential operators II, Grundlehren d. mathem. Wissensch., 257, Springer (1983).

[L] J. LEITERER, Banach coherent analytic Fréchet sheaves, Math. Nachr., 85 (1978), 91–109.

[M] F. MANTLIK, Fundamental solutions for hypoelliptic differential operators depending analytically on a parameter, to appear.

[T1] F. TRÈVES, Un théorème sur les équations aux dérivées partielles à coefficients constants dépendant de paramètres, Bull. Soc. Math. France, 90 (1962), 473–486.

[T2] F. TRÈVES, Fundamental solutions of linear partial differential equations with constant coefficients depending on parameters, Am. J. Math., 84 (1962), 561–577.

Manuscrit reçu le 24 janvier 1991,
révisé le 29 mai 1991.

Frank MANTLIK,
Universität Dortmund
Fachbereich Mathematik
Postfach 50 05 00
4600 Dortmund 50 (Germany).