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ON EQUIVARIANT HARMONIC MAPS DEFINED ON A LORENTZ MANIFOLD

by MA LI

1. Introduction.

It is interesting to study harmonic maps from a Lorentz manifold into a Riemannian manifold. In this case, the harmonic map equation is a Hyperbolic system of second order. In this paper, we look for equivariant harmonic maps defined on a specific Lorentz manifold; namely, the Lorentz manifold $M = M_0 \times R$ with the space-time metric

$$ds^2 = dt^2 - S^2(t) d\sigma^2$$

where $(M_0, d\sigma^2)$ be the symmetric space for a compact Lie group $G$ with a bi-invariant Riemannian metric $d\sigma^2$ and $S(t)$ is a smooth positive function defined on $R$. The target manifold is a compact Riemannian manifold $(N, h)$ admitting an isometric group action of $G$. This kind of problem is called a $\sigma$-model in Physics literature and one may see [G] and [EL] for further datum. Without loss of generality, we may assume that $N$ is a submanifold of some Euclidean space $R^k$ by Nash's isometrical imbedding theorem, so we may think of $G$ as $\subset SO(k)$ with its Lie algebra $LG \subset so(k)$ the Lie algebra of $SO(k)$ whose elements are skew-matrices.

By definition, a smooth map $u$ from $M$ to $N$ is called a harmonic map if it is a critical point of the following action integral

$$E_f(u) = \int_{M_0 \times I} (Tr_{ds^2} u^* h) S^\alpha(t) d\mu dt$$

$$= \int I \int_{M_0} (|\partial_i u|^2 - S^{-2}(t)|\nabla \theta u|^2) S^\alpha(t) d\mu dt$$

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for every interval \( I = [a, b] \subset \mathbb{R} \) among all maps of its class, here \( n = \dim M_0 \), \( d\mu \) is the invariant measure of \( (M_0, d\sigma) \), \( |\cdot| \) is the usual norm induced by \( R^k \) and \( \nabla_0 \) is the covariant derivative induced by \( d\sigma^2 \) on \( M_0 \).

We will prove the following

**Theorem.** — Let \( M \) and \( N \) be the manifold defined above. Suppose \( S(t) \) is a smooth positive periodic function of period \( 2\pi \), then, there exist infinitely many \( G \)-equivariant harmonic maps which are of period \( 2\pi \) in \( t \) from \( M \) to \( N \).

By equivariant, we mean that the map \( u : M \to N \) satisfies

\[ u(g \cdot m, t) = g \cdot u(m, t) \]

for every \( g \in G \) and \( (m, t) \in M_0 \times \mathbb{R} \). We denote the set of equivariant maps \( \mathcal{M} \) and it is non-empty by our assumptions on \( M \) and \( N \). Select a basis \( \{e_j\}_{j=1}^n \) (note \( n = \dim M_0 = \dim_G G \) of the Lie group \( G \) and let \( \{A_j\}_{j=1}^n \) denote the corresponding basis of its Lie algebra. Fix \( m \in M_0 \) and write \( x(t) = u(m, t) \). Because \( u \) is an equivariant map, \( u(\exp(sA_j)m, t) = \exp(sA_j) u(m, t) \). Differentiating it w.r.t. \( s \) at \( s = 0 \) we get that \( \nabla_0 u(m, t)(A_j) = A_j u(m, t) \) (matrix multiplication in \( R^k \)). From this and the invariance of the metric \( d\sigma^2 \), the action integral \( E_i(u) \) for the \( G \)-equivariant map \( u \) becomes

\[
E_i(u) = \int_{M_0} d\mu \int_I \left( |u_t(m, t)|^2 - S^{-2}(t) \sum_{j=1}^n |A_j(u(m, t))|^2 \right) S^n(t) \, dt \\
= \text{Vol}(M_0) F_i(x),
\]

where the last integral factor \( F_i(x) \) will be written as \( F(x) \) when \( I = S^1 \).

It will be shown by the minimax principle that there exist infinitely many critical points of \( F(\cdot) \) just like closed geodesics in \( N \). But here we should mention that it is conceptually different from the closed geodesic case because the Euler-Lagrange equation for our \( F(\cdot) \) is a non-autonomous one (see Lemma 2 below).

2. Some well-known facts.

Since the \( A_j \) is a skew-symmetric matrix, there exists a non-negative symmetric matrix \( A \) such that

\[
\sum_{j=1}^n A_j^2 = - \sum_{j=1}^n A_j A_j^* = -A^2.
\]
So

\[
F(x) = \int_0^{2\pi} \left( |x'(t)|^2 - S^{-2}(t) |Ax(t)|^2 \right) S^n(t) \, dt.
\]

Think of \( M_0 \times S^1 \) as a Riemannian manifold with the metric \( dt^2 + S^2(t) \, d\sigma^2 \), we may define a Hilbert manifold \( H = W^{1,2}(M_0 \times S^1, N) \) for \( l \) large enough. Now, \( H \) admits an isometric group action \( (u, g) \to g^{-1} \cdot u \cdot g \) of \( G \). Applying the theorem in page 23 of R. S. Palais [P2] to \( F \) on \( H \) and to the fixed point set of the map \( u \to g^{-1} \cdot u \cdot g \), we find

**Lemma 1.** - If \( u \in \mathcal{M} \), then \( u \) is harmonic if and only if \( x(t) = u(m, t) \) is a critical point for \( F_t(x) \) for all intervals \( I \subset \mathbb{R} \).

Let \( \mathcal{O} \) be an open uniform tubular neighborhood of \( N \) in \( \mathbb{R}^k \) such that the \( P: \mathcal{O} \to N \) given by \( P(y) = \) the nearest point in \( N \) to \( y \), is a smooth fibration.

**Lemma 2.** - The Euler-Lagrange equations for an equivariant harmonic map from \( M \) to \( N \) are

\[
S^{-n}(t) (S^n(t) x')' - D^2 P(x', x') + S^{-2}(t) A^2 x = 0,
\]

which is a non-autonomous system except if \( S(t) = \) const.

**Proof.** - Suppose \( x \) is the critical point of \( F(\cdot) \) which corresponds to the equivariant harmonic map we consider. For \( \eta \in W^{1,2}(S^1, \mathbb{R}^k) \), if \( \varepsilon > 0 \) is small enough, we have that \( P(x(\cdot) + s\eta(\cdot)) \) is a smooth curve in \( W^{1,2}(S^1, N) := \{ y \in W^{1,2}(S^1, \mathbb{R}^k); y(t) \in N \} \) passing through \( x \) for \( s \in (-\varepsilon, \varepsilon) \). Hence

\[
0 = 2^{-1} \frac{d}{ds} \bigg|_{s=0} F(P(x + s\eta)) = \frac{1}{2} \frac{d}{ds} \bigg|_{s=0} \int_{S^1} |DP_{x+s\eta} \cdot (x'(t) + s\eta'(t))|^2 S^n(t) \, dt \bigg|_{s=0}
\]

\[
- \int_{S^1} |AP(x(t) + s\eta(t))|^2 S^{-2}(t) \, dt
\]

\[
= \int_{S^1} (D_P x'(t), D^2 P x(x'(t), \eta(t)) + D_P \cdot \eta'(t)) S^n(t) \, dt
\]

\[
- \int_{S^1} \langle A^2 x(t), \eta(t) \rangle S^{-2}(t) \, dt
\]
\[
= \int_{S^1} \langle x'(t), \eta'(t) \rangle S^n(t) \, dt
\]
\[
+ \int_{S^1} \left( \langle D^2 P_x(x'(t), x'(t)) - S^{-2}(t) A^2 x(t), \eta(t) \rangle \right) S^n(t) \, dt.
\]

Since \( P(x) = x \), we have that \( DP_x(x') = x' \). So by integration by part we get (2). Since, for \( S(\cdot) \neq \text{const.} \),
\[
\left( |x'(t)|^2 + S^{-2}(t) |Ax(t)|^2 \right) S^n(t)
\]
is not conserved, (2) is a non-autonomous system.

Define
\[
\Lambda^1 = \Lambda^1(N) = W^{1,2}(S^1, N).
\]

It is well-known that \( \Lambda^1(N) \) is a Hilbert manifold [P1]. Since \( N \) is compact, there exist constants \( c_i > 0 \) \( (i = 1, 2, 3) \) such that
\[
c_1 D(y) - c_3 \leq F(y) \leq c_2 D(y) - c_3,
\]
here \( D(y) := |y|^2 = \int_{S^1} |y'|^2 \) for every \( y \in \Lambda^1 \). We will also need the following inequality
\[
|y|_\infty \leq |y(0)| + c_4 |y|_1,
\]
for every \( y \in \Lambda^1 \) and the Sobolev imbedding \( W^{1,2}(S^1, R^k) \rightarrow C^0(S^1, R^k) \) is compact.

**Lemma 3.** — i) \( F(\cdot) \) satisfies Palais and Smale condition C;

ii) For every \( c > 0 \), there exists an integer \( n = \bar{n}(c) \) such that
\[
H^n(I_c) = 0
\]
for \( n > \bar{n} \), where \( I_c = D^{-1}(-\infty, c] \).

**Proof.** — i) Suppose \( \{x_m\} \subset \Lambda^1 \) is a sequence such that
\[
F(x_m) \rightarrow c
\]
and
\[
dF(x_m) \rightarrow 0, \quad \text{in} \ H^{-1}.
\]
Since $N$ is compact, we may assume that $x_m(0) \to p$. By inequality (3), we get that, there exist a constant $C > 0$ such that

\[(7) \quad D(x_m) \leq C.\]

So we may assume that $x_m \to x$ in $C^0(S^1, N)$. Now,

\[
\langle dF(x_m), \eta \rangle = 2 \int_{S^1} \left( \langle x', \eta' \rangle + \langle D^2 P(x', x'), \eta \rangle - S^{-2}(t) \langle A^2 x, \eta \rangle \right) S^n(t).
\]

Take $\eta = x_m - x_n$ and $x = x_m, x_n$ in (6), we get by (7) and (4) that

\[
o(1) = 2^{-1} \langle dF(x_m) - dF(x_n), x_m - x_n \rangle \\
\geq c_1 D(x_m - x_n) - 2C|x_m - x_n|_\infty \\
- \int_{S^1} S^{n-2}(t) \langle A^2(x_m - x_n), x_m - x_n \rangle \\
\geq c_1 D(x_m - x_n) - 2C_5 o(1).
\]

Here we implicitly used boundness of the positive function $S(t)$. Hence, $D(x_m - x_n) = o(1)$.

ii) This is borrowed from Milnor's book (see theorem 16.2 in [M]). Since $I_c$ is a strong deformation retract of a finite dimensional manifold, whose dimension $n$ depends on $c$, then, we get the conclusion if we let $\bar{n}(c) = n$.

Now, let us recall a result of M. Vigue-Poirrier and D. Sullivan [V-PS] about the topology of $\Lambda^1$.

**Proposition 4.** If $N$ is compact and simply connected, then there exists an infinite set of positive integers $\mathbb{M} \subset \mathbb{N}$ such that

\[H^q(\Lambda^1) \neq 0\]

for $q \in \mathbb{M}$.

**3. Final argument.**

Consider a non-trivial $\alpha \in H^*(\Lambda^1)$ and set

\[(8) \quad \bar{\alpha} = \{B \in \Lambda^1; i^*_H(\alpha) \neq 0\},\]

where

\[i^*_H : H^*(\Lambda^1) \to H^*(B)\]
is the homomorphism induced by the inclusion

\[ i_B : B \to \Lambda^1. \]

**Remark 5.** — \( \bar{\alpha} \) defined in (8) is non-empty and contains the compact support of a \( k \)-chain \( a \in \alpha \), \( k = \deg \alpha \), which is not homologous to constant by the nontrivial property of \( \alpha \).

**Lemma 6.** — Let \( \alpha \in H^*(\Lambda^1) \), \( \alpha \neq 0 \) and define

\[ c_\alpha = \inf_{B \in \bar{\alpha}} \sup_{B} F(B). \]

Then, \( c_\alpha \) is a critical value of \( F \) on \( \Lambda^1 \); moreover, if we assume that \( H^q(\Lambda^1) \neq 0 \) for infinitely many \( q \), there exists a sequence \( \{ c_\alpha \} \) of critical values of \( F \) defined as in (9) which satisfies that

\[ c_\alpha \to + \infty, \quad \text{as} \quad \deg \alpha \to + \infty. \]

**Proof.** — By our Remark 5 we have

\[ c_\alpha < + \infty. \]

Suppose some \( c_\alpha \) is not a critical value of \( F \), then by lemma 3 i) and a well-known deformation lemma in page 125 of R. S. Palais [P1], we know that there exists a positive number \( \varepsilon \) and a homeomorphism \( \eta \) on \( \Lambda^1 \) such that

\[ \eta(1_{c_\alpha^{-1} + \varepsilon}) = 1_{c_\alpha^{-1} - \varepsilon}. \]

Since

\[ \eta^* : H^q(\eta(\Lambda^1)) \to H^q(\Lambda^1) \]

is an isomorphism, we have that

\[ i_{\eta(B)}^*(x) = (\eta^*)^{-1} i_B(x) \neq 0 \]

for all \( B \in \bar{\alpha} \). Hence \( \eta \) leaves \( \bar{\alpha} \) invariant. But, by the definition of \( c_\alpha \), there exists \( B \in \bar{\alpha} \) such that

\[ \sup F(B) < c_\alpha + \varepsilon. \]

So by (10) and \( \eta(B) \in \bar{\alpha} \) we have

\[ \sup F(\eta(B)) < c_\alpha - \varepsilon. \]

It is absurd.
To get (9'), we take $k \in \mathbb{N}$. By lemma 3 ii), there exists $\tilde{n} = \tilde{n}(k) \in \mathbb{N}$ such that $H^q(I_k) = 0$ for $q > \tilde{n}$. By our assumption on $H^*(\Lambda^1)$ we may take $q_k > \tilde{n}$ with $H^{q_k}(\Lambda^1) \neq 0$ and consider $\alpha \in H^{q_k}(\Lambda^1)$, $\alpha \neq 0$. Denote

$$I^k = \{x \in \Lambda^1; D(x) > k\},$$

we claim that

$$\forall B \in \bar{\alpha}, B \cap I^k \neq 0. \tag{11}$$

Suppose it is not true, then, there exists $B \in \bar{\alpha}$ such that

$$B \subset \Lambda^1 \setminus I^k := I_k,$$

then

$$H^{q_k}(\Lambda') \xrightarrow{i^*_2} H^{q_k}(I_k) \xrightarrow{i^*_1} H^{q_k}(B), \tag{12}$$

where $i^*_2$, $i^*_1$ are the homomorphisms induced by the inclusion maps

$$i_2: I_k \to \Lambda^1, \quad i_1: B \to I_k.$$

Then, by $B \in \bar{\alpha}$ we have that

$$i^*_2 \cdot i^*_1(\alpha) = i^*_1(\alpha) \neq 0. \tag{13}$$

From (12) and (13) we obtain that $H^{q_k}(I_k) \neq 0$, a contradiction to our assumption on $q_k$. So (11) is true.

By (11) and our choices of $c_\alpha$ we have that

$$c_\alpha \geq c_1 k - C$$

which implies our conclusion.

Proof of Theorem. 
1) If $N$ is simply-connected, then the result follows from Proposition 4 and Lemma 6.

2) If $\pi_1(N) \neq 0$ and finite. Then the universal covering $(\bar{N}, \Pi)$ is compact. By 1) we have infinitely many critical points $\tilde{x}_n: S^1 \to \bar{N}$ of $F$ such that

$$F(\tilde{x}_n) \to +\infty, \quad \text{as } n \to \infty.$$

Therefore, set $x_n = \Pi(\tilde{x}_n)$, we obtain the existence of infinitely many critical points of $F$, and infinitely distinct harmonic maps of periodic $2\pi$ in $t$ from $M$ to $N$ by Lemma 1.
3) If $\pi_1(N) = \infty$. We may get a minimizer of $F$ in each homotopy class by the Palais-Smale condition in lemma 3 i).

Remark 7. — (1) Suppose $S(t)$ is not periodic in $t$. Take $I = [0,1]$, $x(0)$ and $x(1)$ two point in $N$, we can prove as in our theorem that there are infinitely many geometrical distinct critical points of $F$. It is interesting to consider the behavior of the orbit of some critical point of $F$ just like that of the geodesic in $N$.

(2) It is an open question to obtain our theorem when $S(t) = 1 - \cos(t)$. In this case, the Lorentz manifold $M$ is called Friedman-Robertson-Walker space-time in general relativity.

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