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THE VERSALITY DISCRIMINANT AND LOCAL TOPOLOGICAL EQUIVALENCE OF MAPPINGS

by James DAMON (1)

Introduction.

One of the goals of singularity theory going back to the early work of Morse and Whitney and extending through the work of Thom and Mather was to classify germs of mappings between different spaces up to change of coordinates in source and target ($\mathscr{A}$-equivalence). This originally concerned the classification for generic mappings as well as questions involving finite determinacy. However, the presence of moduli (i.e., parameters describing continuous change of smooth or analytic type) yields an uncountable classification unless we consider topological analogues of the above questions. In the case of finitely $\mathscr{A}$-determined germs which are weighted homogeneous, [D1] provided a sufficient condition for a deformation to be topologically trivial allowing one to answer classification questions for topological equivalence of germs of mappings. This was refined in [D2] for non-weighted homogeneous germs f_0 which still had finite $\mathscr{A}$-codimension but in a graded sense relative to an algebraic filtration on the tangent space of the infinitesimal deformations of f_0. This allowed one to give precise topological determinacy results as well as topological classification results.

Unfortunately, map germs are not finitely determined in general once one leaves the « nice dimensions » [M2]; hence, the above results

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are often no longer directly applicable. The alternative is to verify the Whitney conditions for a deformation of the germ. However, that involves the checking of Whitney conditions for all pairs of strata, which is not feasible except for very low dimensions.

Associated with the failure of finite determinacy is a germ of an analytic set in the target space which we call the versality discriminant. The *versality discriminant* of a germ f_0 is the set of points in the target where f_0 is not infinitesimally stable, i.e., where f_0, viewed as an unfolding of some multi-germ, is not versal. For example, finite determinacy is equivalent to the versality discriminant being a point (or empty). Often, the dimension of the versality discriminant can be quite small even though the dimensions of the source and target spaces are large.

At points off of the versality discriminant, f_0 is locally stable and so small perturbations will not change f_0 locally near the point. Hence, we might hope that the knowledge of the behavior of f_0 near the versality discriminant will allow us to determine the topological behavior of f_0 itself under deformations. The goal of this paper is to prove such a result.

The general statement will take the form : if an unfolding $f: k^{s+q}, 0 \to k^{t+q},0$ of the germ $f_0: k^s,0 \to k^t,0$ is topologically trivial in a « conical neighborhood » of the versality discriminant of f then f itself is topologically trivial. Here by conical neighborhood we mean a neighborhood in the punctured target $k^t \backslash \{0\}$.

We will give several versions of this result. As might be expected in light of the results referred to above, the simplest version of the theorem (Theorem 1) concerns f an unfolding of non-negative weight of a weighted homogeneous germ f_0. However, the arguments for the proof in this case are the same that apply to more general situations where weighted homogeneity can be replaced by conditions stated in terms of algebraic filtrations (Theorem 2-4). For this reason, we first separately state Theorem 1 which considers the weighted homogeneous case and then give the filtered version which states the modifications needed to deal with the lack of weighted homogeneity.

Also, we shall see that it is not necessary to explicitly find the versality discriminant to apply the theorems. It is sufficient to apply the theorems for a germ of a variety which contains the versality discriminant.

We indicate applications of these results to a variety of examples, including multi-modal singularities, unimodal singularities which are not finitely $\mathscr{A}$-determined, and mappings to $\mathbb{C}^2$. These results have consequences for the versal topological stratification of versal unfoldings as well as the determination of topologically stable map germs for higher modality singularities.

In § 1 we define the versality discriminant and give several examples of its computation. We collect together in § 2 for later use several basic facts about singularity submanifolds and weak stratifications and stratified vector fields. In § 3 we recall certain basic properties of algebraic filtrations which we will need and which follow automatically in the weighted homogeneous case. Then in § 4, we define the local condition which must be satisfied near the versality discriminant and state the topological triviality theorems. We also indicate how to apply these theorems to the earlier examples. In §§ 5 and 6 we prove the theorems by constructing stratified vector fields in § 5 and proving in § 6 that these vector fields are locally integrable, giving the desired topological trivialization. Lastly, in § 7 we give a further refinement of the theorems which basically allows for non-positive weights or filtrations.

1. The versality discriminant.

Let f: k^{s+q}, $0 \to k^{t+q}$, 0 be an unfolding of the germ $f_0 : k^s$, $0 \to k^t$, 0, so that if x, y, u denote local coordinates for k^s, k^t, and k^q then $f(x,u) = (\bar{f}(x,u),u)$ and $\bar{f}(x,0) = f_0(x)$. Here, if $k = \mathbb{C}$, f is holomorphic and if $k = \mathbb{R}$, then f is real analytic.

We let the algebras of k-valued germs (in the appropriate category) on k^s, k^{s+q}, etc., be denoted by $\mathscr{C}_x$, $\mathscr{C}_{x,u}$, etc. These algebras have maximal ideals of germs vanishing at 0 denoted by $\mathscr{m}_x$, $\mathscr{m}_{x,u}$, etc. Also a finitely-generated R-module generated by $\{\varphi_1, \dots, \varphi_k\}$ will be denoted by $R\{\varphi_1, \dots, \varphi_k\}$ or $R\{\varphi_i\}$ if the set of φ_i is implicitly understood.

We consider f_0 which has finite singularity type. Furthermore, if $k = \mathbb{R}$, we will replace f_0 by its complexification, so in either case we have complex germs of finite singularity type, still denoted by f_0 and f.

There is a neighborhood U of 0 such that f_0 has a representative on U, again denoted by f_0, and a neighborhood W of 0 so that if

$\sum(f)$ denotes the singular set of f:

(i) $f | \sum(f) \cap U \to W$ is proper and finite to one

(ii) $f^{-1}(0) \cap \sum(f) \cap U = \{0\}$.

We let $V \subset W$ denote the set of $(y,u) \in W$ such that if $S = f^{-1}(y,u) \cap \sum(f) \cap U$, then the multi-germ $f(\ ,u): \mathbb{C}^s, S \to \mathbb{C}^t, (y,u)$ fails to be infinitesimally stable. Then, using an argument modeled on that used by Gaffney [Gaf] for giving a geometric characterization of finite $\mathscr{A}$-determinacy, we showed [D1, I, II].

Proposition 1.1. — *In the above situation*

(i) *V is an analytic subset of W;*

(ii) *if I denote that ideal of germs at 0 vanishing on V, then there is a k so that.*

$$I^k.\theta(\bar{f}) \subset \mathscr{C}_{x,u}\left\{\frac{\partial \bar{f}}{\partial x_i}\right\} + C_{y,u}\left\{\frac{\partial}{\partial y_j}\right\} \tag{1.2}$$

$\left(\text{where } \theta(\bar{f}) = \mathscr{C}_{x,u}\left\{\frac{\partial}{\partial y_j}\right\}\right)$, *and*

(iii) *if I' is another ideal satisfying* (1.2), *then the germ $V' = V(I')$ satisfies $V' \supseteq V$.*

The well-known relation between the infinitesimal stability of a germ and its versality when viewed as an unfolding leads us to define.

Definition 1.3. — *The versality disciminant of the unfolding f (respectively f_0) is V (respectively $V \cap \mathbb{C}^t$) in the case $k = \mathbb{C}$ or $V \cap \mathbb{R}^{t+q}$ (respectively $V \cap \mathbb{R}^t$) in the case $k = \mathbb{R}$.*

Remark 1.4. — In the theorems, we will allow V to be an analytic subset of the discriminant which contains the versality discriminant. If I denotes the ideal of germs at 0, vanishing on V, then (1.2) is still valid. This includes the special case where V is itself the versality discriminant.

Examples.

(1.5) Pham Example [DGal]:

Pham considered the μ-constant family

$$f(x,y) = y^3 + tyx^6 + syx^7 + x^9 \quad (4t^3 + 27 \neq 0)$$

and showed that the versal unfolding

$$F(x,y,s,t,u,v) = \left(f(x,y) + \sum_{i=0}^{5} u_{6\text{-}i}x^i y + \sum_{i=1}^{7} v_{9\text{-}i}x^i,\ s,t,u,v\right)$$

is not topologically a product along the t-axis near $t = 0$. Thus, the topological structure in the versal unfolding depends on the particular values of s and t. Fixing s and t gives an unfolding $f_1: k^{15}, 0 \to k^{14}, 0$. In [DGal] the versality discriminant is determined for the case $s = 0$ and t fixed ($\neq 0$) by considering the family :

$$y^3 + t(x-x_0)^4(x+2x_0)^2\, y + (x-x_0)^6(x+2x_0)^3.$$

Along the curve C in k^{14} defined by this deformation, there are $\tilde{E}_8$ and D_4 singularities in a fibre at the points $x = x_0$ and $-2x_0$ (and the $\tilde{E}_8$ singularities have a fixed modulus value). However, the dimension of the target space is 14, while the codimensions of $\tilde{E}_8$ and D_4 are 10 and 4 respectively. Thus, if the multi-germ in this fibre were multi-transverse, the set of points where it occurred would be isolated and not along a curve. Hence, this curve lies in the versality discriminant for f_1. In [DGal] it is proven that this curve is exactly the versality discriminant.

(1.6) Unimodal singularities which are not finitely determined :

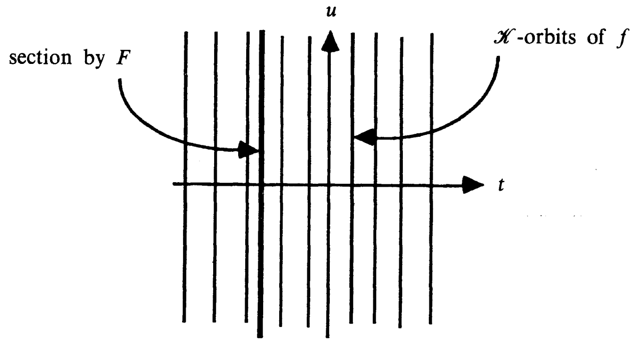
In conversations with Terry Wall he pointed out that even in the region around the edge of the nice dimensions, we still have not completely determined the topologically stable map germs because of the presence of unimodal singularities whose negative versal unfolding fails to be finitely determined.

For example, the negative versal unfolding of the germ $f_0: k^2, 0 \to k^3, 0$ defined by $f_0(x,y) = (x^2+\varepsilon y^4, xy^3+ty^5, y^6)$ is proven in [D1, II], to be finitely $\mathscr{A}$-determined as a germ and hence topologically versal (and topologically stable). However, the germ $f_1(x,y) = (x^2+\varepsilon y^4, xy^3+ty^5, 0)$ is also unimodal; but as shown in [D1, II], its negative versal unfolding is not finitely $\mathscr{A}$-determined as a germ because it fails to be transverse to the $\mathscr{K}$-orbits of f_0.

Terry Wall suggested that it is possible to modify various inequalities in the proof in [D1, I], for this special case to obtain topological triviality along the t-axis [Wa].

Alternatively we can determine the versality discriminant and directly apply our results here. If we assign weight 6 to the third coordinate in the target (with $wt(x,y) = (2, 1)$), then f_1 becomes a « bimodal » singularity with weight zero deformation given by

$$f(x,y) = (x^2 + \varepsilon y^4, xy^3 + ty^5, uy^6).$$



For these weights, the proof of prop. 7.2 of [D1, II], shows that the unfolding F of f_1, which now includes the parameter u (but fixes t), fails only to be transverse to a $\mathscr{K}$-orbit of f_0, which is a line parallel to the u-axis. Thus, the versality discriminant for this unfolding is exactly this line. We shall then see in § 4 as a consequence of theorem 1 that the versal unfolding of $f_1(x,y) = (x^2 + \varepsilon y^4, xy^3 + ty^5, 0)$ is topologically trivial along the t-axis. This same argument will work as well for $(x^2 + \varepsilon y^4, xy^3 + ty^5, 0, 0, \ldots, 0)$ as well as for other finite map germs with similar properties (see [D1, II]).

(1.7) Weighted homogeneous germs f: $\mathbb{C}^n, 0 \to \mathbb{C}^2, 0$:

Suppose that f is weighted homogeneous and has finite singularity type. Let f_t: $\mathbb{C}^n, 0 \to \mathbb{C}^2, 0$ be a deformation of f_0 of nondecreasing weight. By [D2] or [D3] the germ $f_t^{-1}(0)$ is topologically trivial; however, f_0 need not have finite $\mathscr{A}$-codimension and the deformation f_t need not be topologically $\mathscr{A}$-trivial. In fact, in [D3] is given the simple example of a weight zero deformation $f_t(x,y,z) = (xy + tz, x^4 + y^4 + z^2)$ which is not topologically $\mathscr{A}$-trivial. The restriction of f_t as a mapping from the critical set to the discriminant is already not topologically trivial.

Thus, even the simplest weighted homogeneous germs can be nonfinitely $\mathscr{A}$-determined. For such a germ f_0, its discriminant $D(f_0)$ is a curve in $\mathbb{C}^2$. The versality discriminant is then an analytic subset of

$D(f_0)$; if f_0 is not finitely $\mathscr{A}$-determined then it is a union of some of the branches of $D(f_0)$. We shall identify in § 4 a class of weighted homogeneous germs which enlarges the classe of finitely $\mathscr{A}$-determined ones. We shall see that for this class it is not even necessary to identify the versality discriminant to apply the theorems here. We shall prove that a weight zero deformation of such a germ its topologically trivial if and only if the deformation restricted to the critical set is topologically trivial.

2. Singularity submanifolds and stratifications.

Next, we introduce the notion of singularity submanifolds which will suffice for our purposes and give conditions which ensure that vector fields are stratified relative to a stratification by such submanifolds.

Let $f\colon k^n, 0 \to k^p, 0$ denote a germ of finite singularity type. Let $J^\ell_r = J^\ell_r(n,p,k)$ denote the r-multijets of order ℓ and $j^\ell_r f\colon k^n, 0 \to J^\ell_r$, the jet extension of f. Also, we let $\mathscr{A}^\ell_r = \mathscr{A}^\ell_r(n,p,k)$ denote the r multijet version of the group of left-right equivalence (see [M1-V] § 1). Finally, we let $\pi : (k^n)^{(r)} \backslash \Delta k^n \to k^n$ denote the projection $\pi(x_1, \dots, x_r) = x_1$.

DEFINITION 2.1. — *A singularity submanifold* of f is a submanifold of the form $\pi(j^\ell_r f)^{-1}(W))$ (respectively $f(\pi(j^\ell_r f)^{-1}(W))$ where W is an $\mathscr{A}$-invariant submanifold of J^ℓ_r and f denotes a representative on a neighborhood U such that $f|\sum(f) \cap U$ is finite to one.

Remark 2.2. — In general, even if W is a submanifold, the associated singularity submanifolds will not necessarily be smooth. However, off the versality discriminant this will be true by [M1-V, Thm 4.1] when $f|j^\ell_r f^{-1}(W)$ is an immersion.

(2.3) We recall [D2, I], that a *weak stratification* of an open subset U of k^n is a decomposition of U into a finite union of smooth submanifolds $\{V_i\}$, the strata, which satisfy a weak form of the axiom of the frontier, namely,

$$C\ell(V_i) \subset V_i \cup (\cup V_j) \qquad \text{with} \qquad \dim V_j < \dim V_i .$$

In [D2, I], we solved questions of topological triviality using stratified vector fields on weak stratifications. Our definition of stratified vector field was especially suited for the algebraic criteria used there. It is possible to use a weaker notion of stratified vector field and still obtain one which is locally integrable by Wirthmüller [Wi] (or see § 2 of [D2, I]), and the vector fields constructed in [D2, I], can still be used as they satisfy this weaker condition.

DEFINITION 2.4. — *A vector field defined on an open subset U of k^n is stratified if it is continuous and there is a weak stratification $\{V_i\}$ such that :*

(i) *$\xi \mid V_i$ is smooth and tangent to V_i for each V_i.*

(ii) *For each $x \in U$, with say $x \in V_i$, there exists a neighborhood U_x of x and a smooth non-negative function $\rho_x : U_x \to \mathbb{R}$ such that $\rho_x^{-1}(0) = V_i \cap U$ and $|\xi(\rho_x)| \leqslant C \cdot \rho_x$ on U.*

One problem with stratified vector fields occurs when one wants to patch them together with a partition of unity. The resulting vector field need no longer be stratified (nor possibly even locally integrable). We wish to ensure that the resulting vector field is still stratified to ensure integrability. For stratifications by singularity submanifolds, we show in the next proposition that the vector fields will be correctly stratified, at least where they are smooth.

PROPOSITION 2.5. — *Given a germ $f : k^n, 0 \to k^p, 0$ (in the appropriate category) and weak stratifications of neighborhoods U' and U of 0 in k^n and k^p by singularity submanifolds of f. Let ξ and η be germs of vector fields such that $\xi(f) = \eta \circ f$. If $y \in U'$ with $f^{-1}(y) \cap \sum(f) = S = \{x_1, \ldots, x_m\}$ and ξ and η are smooth in neighborhoods of S and y, then ξ and η are stratified relative to the given weak stratifications in the neighborhoods of S and y.*

Proof. — Let the neighborhoods of S and y be denoted by U' and U respectively. Given $y' \in U$ with say $y' \in V$, then there are $x_1, \ldots, x_r \in S$ with $(x_1, \ldots, x_r) \in j_r^\ell f^{-1}(W)$. If φ_t and ψ_t denote the local flows generated by ξ and η in smaller neighborhoods of y' and $S' = \{x_1, \ldots, x_r\}$. Then, (φ_t, ψ_t) generate an $\mathscr{A}$-equivalence of germs f: k^n, $S' \to k^p$, y' and f_t: k^n, $S'_t \to k^p$, y'_t with $S'_t = \varphi_t(S')$ and $y'_t = \psi_t(y')$.

Hence, $S'_t \subset j_r^\ell f^{-1}(W_i)$ and $y_t \in f(j_r^\ell f^{-1}(W_i)) = V_i$. Hence η is tangent to V_i at y', and hence near y'.

As V_i is smooth, there exist local coordinates near y', $(y_1, \ldots, y_p)$ so that V_i is given by $y_1, \ldots, y_\ell = 0$. Let $\rho_{y'} = \sum_{i=1}^{\ell} |y_i|^2$. Then, as η is tangent to V_i near y', $\eta = \sum_{i=1}^{p} h_i \cdot \frac{\partial}{\partial y_i}$ where $h_i(0) = 0$ for $i = 1, \ldots, \ell$ so $h_i = \sum_{j=1}^{\ell} g_{ij} \cdot y_j$ for $i = 1, \ldots, \ell$.

Lastly,

$$|\eta(\rho_{y'})| \leqslant \sum_{i=1}^{\ell} |\bar{y}_i| \cdot |\eta(y_i)| \leqslant \sum_{i,j=1}^{\ell} |g_{ij}| \cdot |\bar{y}_i| \cdot |y_j| \leqslant C . \rho_{y'}$$

where C/ℓ^2 is a bound for all $|g_{ij}|$ in a neighborhood of y'. A similar argument works for the strata of U'_i. □

3. Filtration conditions.

In the weighted homogeneous case, a germ being of finite singularity type or an ideal defining the versal discriminant can be stated in terms of certain algebraic conditions which have immediate analogues for corresponding graded algebraic objects. If in place of weighted homogeneity we use more general algebraic filtrations, then the corresponding graded conditions need not necessarily hold. Instead, we must actually require the stronger graded conditions as part of our criteria.

In this section we *briefly recall* the properties that we require for the filtration on our rings and modules of vector fields. These properties are described in more detail in § 4-6 of [D2, I]. Then, the stronger graded conditions are contained in the filtration properties (F1-F3). These properties lead to certain « jump invariants » which we associate to a germ.

We emphasize that for unfoldings of nondecreasing weight of weighted homogeneous germs the conditions (F1-F3) are *always satisfied* and if the unfolding f is also weighted homogeneous (allowing nonpositive weights for the unfolding parameters) the jump invariants are always zero. Hence, a reader who wishes to understand the results for the weighted homogeneous case may continue on directly to the statement of theorem 1 in the next section.

Filtrations on rings and modules.

Recall that the algebras of k-valued germs (in the appropriate category) on k^s, k^{s+q}, etc., are denoted by $\mathscr{C}_x$, $\mathscr{C}_{x,u}$, etc. with maximal ideals $\mathscr{m}_x$, $\mathscr{m}_{x,u}$, etc. A *filtration* on the algebra $\mathscr{C}_x$(or $\mathscr{C}_y$) consists of a sequence of finitely generated ideals of finite codimension $I_1 \supset I_2 \supset \ldots$ such that $I_j \cdot I_i \subseteq I_{i+j}$. Such a filtration will be called *convex* if there is an ℓ and $h_1, \ldots, h_r \in I_\ell$ such that if

$$\rho = \sum_{i=1}^{r} |h_i|^2$$

then given $h \in I_m$, there is a $C > 0$ so that

(3.1) $\quad |h| \leqslant C \cdot \rho^{(m/2\ell)}$ on a neighborhood of 0.

Such a ρ will be called a *control function of filtration* 2ℓ. The functions $h_1, \ldots, h_r$ are called a *set of vertices for the filtration at level* ℓ.

A filtration can have many sets of vertices of varying filtration levels, e.g. $\{h_1^s, \ldots, h_r^s\}$ is a set of vertices for level $s \cdot \ell$, as well as many different control functions. However, control functions ρ_1 and ρ_2 of filtrations 2ℓ and $2m$ are related by inequalities of the form.

(3.1a) $\quad \rho_2 \leqslant C \cdot \rho_1{}^{(m/\ell)}$ on a neighborhood of 0.

The filtration extends to smooth functions and any smooth function of filtration 2ℓ which satisfies inequalities of the form (3.1) for all $h \in I_\ell$ is also referred to as a control function and satisfies (3.1a).

The filtration extends to $\mathscr{C}_{x,u}$ by $\{I_j \cdot \mathscr{C}_{x,u}\}$ (this is no longer of finite codimension). In the holomorphic case, the filtration extends to the ring of complex valued smooth function $\mathscr{E}_{x,u}$ by $(I_j + I_j^c) \cdot \mathscr{E}_{x,u}$, where $(\ )^c$ denotes complex conjugation.

We assume there are filtrations on both $\mathscr{C}_x$ and $\mathscr{C}_y$ so that f_0^* and f^* both preserve filtrations.

Example : Weight filtration.

If we can assign weights > 0 to x and y so that f_0 is weighted homogeneous, then we can define weight filtrations on $\mathscr{C}_x$ and $\mathscr{C}_y$ with I_j generated by the monomials of weight $\geqslant j$. Such filtrations are convex and preserved by f_0^*. If $\bar{f}$ deforms f_0 by terms of weight $\geqslant wt(y_i)$ in the *i-th* coordinate, then f is an unfolding preserving the weight filtration (we will refer to f more precisely as an *unfolding of non-negative weight*).

Other important examples of convex filtrations are Newton filtrations ([K] or [DGaf]) or filtrations by integral closures of powers of an ideal [T].

Likewise, we also assume that we have filtrations on the modules

$$\theta_s = \mathscr{C}_x\left\{\frac{\partial}{\partial x_i}\right\}, \quad \theta_t = \mathscr{C}_y\left\{\frac{\partial}{\partial y_j}\right\} \quad \text{and} \quad \theta(f) = \mathscr{C}_x\left\{\frac{\partial}{\partial y_j}\right\}.$$

For example, for θ_s this means there is a decreasing sequence of finitely generated finite codimension $\mathscr{C}_x$-modules $M_r \supset M_{r+1} \supset \ldots$ (beginning with some $r \in \mathbb{Z}$) and satisfying :

(i) $I_j \cdot M_i \subseteq M_{i+j}$

(ii) if $\zeta \in M_j$ and $g \in I_i$, then $\zeta(g) \in I_{i+j}$

($\zeta(g)$ denotes the directional derivative).

(iii) if $\zeta = \sum h_i \frac{\partial}{\partial x_i}$ then $\mathrm{fil}(h_i) \geqslant \mathrm{fil}(\zeta) + \mathrm{fil}(x_i)$.

Also, through § 6 we will assume $\mathrm{fil}(x_i) > 0$ for all i ; we show how to relax this condition in § 7.

Note : here and in what follows if $g \in I_i \backslash I_{i+1}$ then $\mathrm{fil}(g) = i$ and similarly for modules.

An analogous definition holds for θ_t; while for $\theta(f)$ we do not require ii).

These filtrations extend to $\mathscr{C}_{x,u}\left\{\frac{\partial}{\partial x_i}\right\}$, etc., by replacing M_j by the $\mathscr{C}_{x,u}$-submodule generated by the generators of M_j.

If f preserves filtrations and $\mathrm{fil}\left(\frac{\partial \bar{f}}{\partial/u_i}\right) \geqslant \mathrm{fil}(f_0)$ for all i, then f is said to be an *unfolding of non-decreasing filtration.*

Filtration conditions.

We next describe the filtration conditions which will be required for the main theorems. As mentioned above, these conditions will always be satisfied in the (semi-) weighted homogeneous case.

As f_0 has finite singularity type, the Jacobian ideal, generated by $f_0^* m_y$ and the $t \times t$ minors of df_0, has finite codimension. If $i = (i_1, \ldots, i_t)$

is a t-tuple of $\{1, \ldots, s\}$ and $f_0 = (f_{01}, \ldots, f_{0t})$, let $\Delta_i(f_0)$ denote the $t \times t$ minor $\det\left(\dfrac{\partial f_{0j}}{\partial x_{i_\ell}}\right)$ and $\Delta(f_0)$ denote the ideal generated by such minors.

DEFINITION 3.2. — *The filtration conditions are the conditions* F1-F3 *to follow.*

(F1) there are $g_1, \ldots, g_r \in \Delta(f_0)$ and $g'_1, \ldots, g'_{n-r} \in m_y$ of filtration m such that $\{g_1, \ldots, g_r, f_0^*(g'_1), \ldots, f_0^*(g'_{n-r})\}$ form a set of vertices for the filtration on $\mathscr{C}_x$.

Second, by Cramer's rule, we may write

$$\Delta_i(f_0)\cdot\frac{\partial}{\partial_{y_\ell}} = \zeta_i(f_0) \qquad \text{and} \qquad \Delta_i(\bar{f})\cdot\frac{\partial}{\partial_{y_\ell}} = \zeta'_i(\bar{f})\,.$$

Thus, we may write

$$g_j\cdot\frac{\partial}{\partial_{y_\ell}} = \sum k_{i,j,\ell}(x)\cdot\Delta_i(f_0)\cdot\frac{\partial}{\partial_{y_\ell}} = \sum k_{i,j,\ell}\cdot\zeta_i(f_0)\,.$$

Applying Cramer's rule in exactly the same way to $\bar{f}$ we obtain

$$G_j\cdot\frac{\partial}{\partial_{y_\ell}} = \sum k_{i,j,\ell}\cdot\zeta'_i(\bar{f})\ (\stackrel{\text{def}}{=} \zeta_{j,\ell}(\bar{f})) \tag{3.3}$$

where $G_j(x,0) = g_j(x)$. The second condition is that for each j

$$\operatorname{fil}(G_j) \geqslant \operatorname{fil}(g_j)\,. \tag{F2}$$

Third, if $V' = f^{-1}(V) \cap \sum$, and $I = I(V)$ then the third condition becomes

$$V' = V(I + (G_1, \ldots, G_r)\cdot\mathscr{C}_{x,u}) \quad (\text{or} \cap \mathbb{R}^{s+q} \text{ if } k = \mathbb{R})\,. \tag{F3}$$

For the general case, we must also allow for « jumps » in filtration to occur in (1.2) and (3.3). Consequently, we give a « graded version » of the definition of versality discriminant as follows.

DEFINITION 3.4. — *V contains the versality discriminant in the graded sense if there is an integer d and $h_1, \ldots, h_a \in I_{m'}$, (with say $km' = m$) and $V(h_1, \ldots, h_a) = V$, so that for any $\zeta \in \theta(\bar{f})$*

$$h_i^k\cdot\zeta = \xi_i(\bar{f}) - \eta_i \circ f, \qquad 1 \leqslant i \leqslant a \tag{3.5}$$

with

$$\operatorname{fil}(\zeta_i), \operatorname{fil}(\eta_i) \geqslant m + \operatorname{fil}(\zeta) - d\,.$$

We define $\text{jump}_V(f)$ (V for versality) to be the minimum such integer $d \geqslant 0$ for which this is true for all i. Also, in the construction in (3.3), we define $\text{jump}_C(f)$ (C for critical set) to be the smallest integer $d \geqslant 0$ such that

$$\text{fil}(\zeta_{i,\ell}) \geqslant m + \text{fil}(y_\ell) - d \qquad \text{for all } i, \ell .$$

Remark 3.6. – When f is weighted homogeneous (where the unfolding parameters have nonpositive weights), $\text{jump}_V(f) = \text{jump}_C(f) = 0$. This is because the weighted homogeneity and analyticity of the germs allow us to choose weighted homogeneous h_i, G_j, and $\zeta_{i\ell}$, ξ_i, and η_i so that (3.3) and (3.5) preserve weights (for weighted homogeneous ζ). Even if only f_0 is weighted homogeneous we may still choose $\zeta_{i\ell}$ so that $\text{jump}_C(f) = 0$.

We conclude this section by deriving one consequence of the filtration conditions. Let $\sigma = \sum_{i=1}^{r} |G_i|^2$ and ρ_0 be a control function of filtration $2m$ for $\mathscr{C}_y$. For a germ h on $k^{t+q}, 0$ we let $\tilde{h}$ denote the pull-back of h by f.

Then, the assumptions together with the following lemma imply that $\rho_0{}^{(1)} = \sigma + \tilde{\rho}_0$ is a control function of filtration $2m$ for the induced filtration on $\mathscr{C}_{x,u}$.

LEMMA 3.7. – *Let* $\{f_1, \ldots, f_r\}$ *denote a set of vertices of filtration* m *for* $\mathscr{C}_x$ *and* let $\{F_1(x,u), \ldots, F_r(x,u)\}$ *be deformations of non-decreasing filtration. Then* $\bar{\rho}_0 = \sum_{i=1}^{r} |F_i|^2$ *is a control function of filtration* $2m$ *for the extension of the filtration to* $\mathscr{C}_{x,u}$.

Proof. – By the assumption that $F_j(x,u)$ is a deformation of $f_j(x)$ of non-decreasing filtration, we may write

$$F_j(x,u) = f_j(x) + \sum_{\alpha} h_{j,\alpha}(x,u) \cdot \varphi_\alpha(x)$$

summed over a finite set of N generators $\{\varphi_\alpha(x)\}$ of I_m. Also, $h_{j,\alpha}(x,0) = 0$. Let

$$\rho_0 = \sum_{i=1}^{r} |f_i|^2 \qquad \text{and} \qquad \bar{\rho}_0 = \sum_{i=1}^{r} |F_i|^2 .$$

Then, by the triangle inequality

$$(3.8) \qquad \bar{\rho}_0{}^{(1/2)} \geqslant \rho_0{}^{(1/2)} - \sum_{j,\alpha} |h_{j,\alpha}| \cdot |\varphi_\alpha| .$$

By the continuity of $h_{j,\alpha}$, on a small enough neighborhood, $\|u\| < \delta_1$, $\|x\| < \delta_2$.

$$\|h_{j,\alpha}(x,u)\| < \varepsilon/Nr .$$

Because ρ_0 is a control fonction of filtration $2m$, there is a $C > 0$ so that on a neighborhood of 0

$$|\varphi_\alpha| \leqslant C \cdot \rho_0{}^{(m/2m)} = C \cdot \rho_0{}^{(1/2)} .$$

Thus, from (3.8)

$$(3.9) \quad \bar{\rho}_0{}^{(1/2)} \geqslant \rho_0{}^{(1/2)} - Nr \cdot (\varepsilon/Nr) C \cdot \rho_0{}^{(1/2)} = (1 - \varepsilon C) \cdot \rho_0{}^{(1/2)} .$$

Choosing ε so that $\varepsilon C < 1$, we obtain from (3.9)

$$\rho_0 \leqslant C' \cdot \bar{\rho}_0 \quad \text{on a neighborhood of } 0 .$$

If $g \in I_m \cdot \mathscr{C}_{x,u}$ then we may write

$$g = \sum_{i=1}^{n} g_i \cdot \psi_i \qquad \text{where} \qquad g_i \in I_m \quad \text{and} \quad \psi_i \in \mathscr{C}_{x,u} .$$

Thus, on a small enough neighborhood of zero

$$\begin{aligned} |g| &\leqslant \sum_{i=1}^{n} |g_i| \cdot |\psi_i| \leqslant \sum_{i=1}^{n} C_i \cdot \rho_0^{(m/2\ell)} C_i' \\ &\leqslant C''' \cdot \bar{\rho}_0^{(m/2\ell)} \end{aligned}$$

where $C''' = \sum C_i \cdot C_i' \cdot C''^{(m/2\ell)}$. This gives the desired conclusion. □

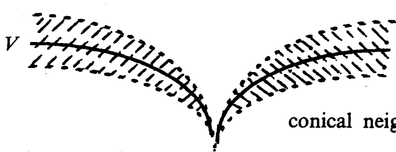
4. Stratified topological triviality.

Let $f : k^{s+q}, 0 \to k^{t+q}, 0$ be an unfolding of f_0. We let V denote an analytic subset of the discriminant of f which contains the versality discriminant of the unfolding f. Also, $V' = f^{-1}(V) \cap \sum$, where $\sum(=$

$\sum(f))$ denotes the singular set of f. In this section we shall define what we mean by the unfolding f being *stratified topologically trivial in a conical neighborhood of V*. Then, we can state the principal theorems. We shall also indicate how the principal results apply to the examples given in § 1.

By a *control function for an analytic subset* $(W,0) \subset k^n, 0$ we mean a germ of a smooth, nonnegative function $\rho : k^n, 0 \to \mathbb{R}$ such that $\rho^{-1}(0) = W$. There is a relation between control functions for filtrations and control functions for analytic sets : if ρ_0 denotes the control function (of filtration $2m$) for $\mathscr{C}_y$, then ρ_0 is also a control function for $V(m_y \cdot \mathscr{C}_{y,u}) = V_0 = \{0\} \times k^q$. Also, by the comments preceding lemma 3.7, $\rho_0^{(1)} = \sigma + \tilde{\rho}_0$ is a control function for $V_0' = \{0\} \times k^q \subset k^{q+s}$. Here again « $\sim$ » denotes composition with f.

Let $\hat{\rho}$ be the control function for V (of filtration $2m$) defined as follows : if $I(V) = (h_1, \ldots, h_r)$ then pick a_i so that $a_i \cdot \mathrm{fil}\,(h_i) = m$ and let $\hat{\rho} = \sum |h_i|^{2a_i}$. Then, by (F3), we may use $\hat{\rho}^{(1)} = \tilde{\hat{\rho}} + \sigma$ as a control function of V'. It is still of filtration $2m$.



Next, by a *conical neighborhood* U of V is meant a neighborhood of the form

$$U = \{(y,u) : \hat{\rho}(y,u) < \varepsilon \cdot \rho_0(y,u)\}$$

and similarly for a conical neighborhood U' of V' using instead $\hat{\rho}^{(1)}$ and $\rho_0{}^{(1)}$.

The notion of stratified topological triviality of f in a conical neighborhood of V will be defined in terms of stratified vector fields which trivialize f on conical neighborhoods of V and V'. This is described via the conditions (V1)-(V2) to follow.

(V1) There are weak stratifications of neighborhoods of 0 in k^{s+q} and k^{t+q} whose strata are singularity submanifolds, include the strata V_0 and V_0', and for which the strata in $k^{s+q}\backslash\sum(f)$ are of the form $f^{-1}(V_i)\backslash\sum(f)$ for V_i strata of k^{t+q}.

(V2) There are conical neighborhoods U of V and U' of V' such that for any smaller conical neighborhoods U_1 such that $C\ell(U_1) \subset U$ (closure $C\ell$ is taken in the punctured space) and for any j with $1 \leqslant j \leqslant q$, there exist vector fields ξ_j and η_j defined on U' and U respectively such that:

(1) ξ_j and η_j are stratified vector fields relative to the restrictions of the weak stratifications to U' and U.

(2) ξ_j and η_j are smooth on $U' \backslash f^{-1}(C\ell(U_1))$ and $U \backslash C\ell(U_1)$.

$$
(3) \qquad \begin{aligned}
\eta_j &= \frac{\partial}{\partial u_j} + \sum n_{ij} \cdot \frac{\partial}{\partial y_i} && \text{with } |n_{ij}| \leqslant C \cdot \rho_0^{a_i}, \\
\xi_j &= \frac{\partial}{\partial u_j} + \xi_j' && \text{with } \|\xi_j'\| \leqslant C \cdot \rho_0^{(1)\,b}
\end{aligned}
$$

and

$$
\begin{aligned}
|\eta_j(\rho_0)| &\leqslant C \cdot \rho_0 && \text{on } U \\
|\xi_j(\rho_0^{(1)})| &\leqslant C \cdot \rho_0^{(1)} && \text{on } U'
\end{aligned}
$$

where C, $b > 0$ and $a_i = -\operatorname{fil}\left(\frac{\partial}{\partial y_i}\right)$.

$$
(4) \qquad \xi_j(f) = \eta_j \circ f .
$$

Remark. – That ξ_j and η_j can be made smooth outside of arbitrarily small neighborhoods of V' and V should not be surprising since V contains the versality discriminant so on its complement $\bar{f}(.,u)$ is stable. In fact, one would hope that the smoothness could be guaranteed off V; however, this may be difficult to achieve while simultaneously satisfying the stratification conditions (see e.g. [D Gal]).

DEFINITION 4.1. – *We say that f is stratified topologically trivial in a conical neighborhood of V if f satisfies the filtration conditions* F1-F3 *and the conditions* V1-V2.

Remark. – The conditions (V1)-(V2) are analogues of the conditions for stratified vector fields in conical neighborhoods in [D2] § 2, so that there exist methods for constructing such vector fields in conical neighborhoods.

The first version of the main theorem is the following.

THEOREM. 1. — Let f *be an unfolding of non-negative weight of* f_0 *so that* f *itself is weighted homogeneous (with weights* $\leqslant 0$ *for the unfolding variables). If* f *is stratified topologically trivial in a conical neighborhood of the versality discriminant then* f *is a topologically trivial unfolding of* f_0.

More generally, we allow filtrations and varieties V containing the versality discriminant in the graded sense. We extend definition 4.1 :

DEFINITION 4.1*a*. — *We say that* f *is stratified topologically trivial in a conical neighborhood of the versality discriminant in the graded sense if there is a variety* V *which contains the versality discriminant in the graded sense so that* f *is stratified topologically trivial in a conical neighborhood of* V *and satisfies the stronger condition in V2.*

$$\text{(4.2)} \qquad \mathrm{fil}(n_{ij}) \geqslant \mathrm{jump}_C(f) + \mathrm{fil}(y_i) \qquad \textit{all } i, j.$$

There is the following generalization of Theorem 1.

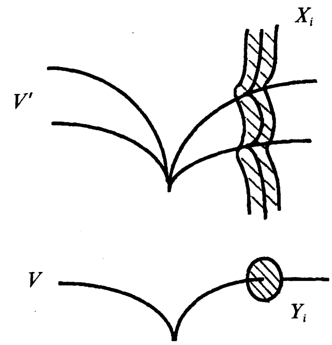
THEOREM 2. — *Let* f *be an unfolding of non-decreasing filtration of* f_0 *so that* f *is stratified topologically trivial in a conical neighborhood of the versality discriminant in the graded sense. If*

$$\text{(4.3)} \qquad \mathrm{fil}\left(\frac{\partial \bar{f}}{\partial u_i}\right) \geqslant \mathrm{jump}_V(f) + \mathrm{jump}_C(f), \qquad 1 \leqslant i \leqslant q$$

then f *is a topologically trivial unfolding of* f_0.

It is immediate from remark 3.6 that Theorem 1 follows from Theorem 2. The proof of Theorem 2 will be given in §§ 5 and 6. Before finishing this section, we indicate how Theorem 1 may be applied to the examples from § 1.

(4.4) *Examples reconsidered.* — The examples that we consider here are weight zero deformations $f(x,u)$ of a weighted homogeneous germ f_0 such that the versality discriminant V of f is a topologically trivial family of curves. We may choose sections Y_i to each branch B_i of the curve for f_0, and consider the pull back X_i via $\bar{f}$, with $S = \bar{f}^{-1}(y) \cap V'$ (recall $V' = f^{-1}(V) \cap \sum(f)$). If $y \in Y_i \cap V$, let Y_y denote the isotropy group of $\mathbb{C}^*$ at y. The set S is invariant under G_y. We can also choose the sections invariant under G_y.



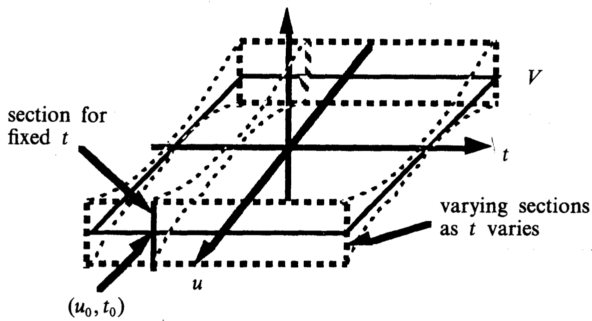
If ξ_i and η_i are stratified vector fields (for a weak stratification by singularity submanifolds) which project to $\frac{\partial}{\partial u_i}$ and trivialize the restriction $f|X_i, S \to Y_i, y$ in the direction u_i, then we may average them over the finite group G_y and they still have the same properties (since $wt(u_i) = 0$). We may then use the $\mathbb{C}^*$-action to extend the collection of such vector fields for each branch to conical neighborhoods of V and V' (invariance under G_y ensures that this is well-defined). Provided η_i and ξ_i can be chosen to be smooth off an arbitrarily small neighborhood of y and its inverse image, then the $\mathbb{C}^*$-extensions of η_i and ξ_i satisfy V1 and 1) and 2), and 4) of V2. Condition 3) of V2 follows from the $\mathbb{C}^*$-invariance of the extensions. The reader can either verify these directly or see for example [DGal] where the details are verified.

Pham example. — In example (1.5) the failure of topological triviality for the Pham example was discussed. The topological triviality of F in the s-direction does follow by results of Wirthmüller [Wi]. Thus, it is enough to consider topological triviality along the t-direction when $s = 0$. In [DGal], it is then proven that the section to the single branch C of the versality discriminant yields a multi-germ which is stratified topologically trivial in the above sense. Hence, the above line of reasoning together with theorem 1 can be applied to conclude topological triviality along the t-axis.

Non-finitely-determined unimodal singularities.

As discussed in § 1, the unfolding F of $f_1(x,y) = (x^2+\varepsilon y^4, xy^3+ty^5, 0)$ which is versal except for the parameter t is not finitely $\mathscr{A}$-determined when viewed as a germ. The versality discriminant of the germ F

consists of a line parallel to the u-axis, with u the parameter for the unfolding term $(0, 0, y^6)$. Hence, the versality discriminant for the unfolding of F by the parameter t is the (u, t)-plane.



A section to V in the direction of the other parameters and variables gives the negative versal unfolding to $f_0(x,y) = (x^2 + \varepsilon y^4, xy^3 + t_0 y^5, u_0 y^6)$. By proposition 8.2 of [D1, II] and theorem 1 of [D1, I] this germ is stratified topologically trivial; hence, the above line of reasoning applies, so that theorem 1 allows us to conclude that the versal unfolding of f_1 is topologically a product along the t-axis. An analogous argument works for the other finite map germs with similar properties (see [D1, II]). Then, the proof of theorem 4 in [D1, I] also shows that the unfolding versal except for the term t is, in fact, topologically stable.

Weighted homogeneous germs $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^2, 0$.

Suppose $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^2, 0$ is weighted homogeneous and of finite singularity type. By the $\mathbb{C}^*$-action, the germ of f_0 at a point x of $\sum(f_0)$ is equivalent to a product of a hypersurface germ $f_1 : \mathbb{C}^{n-1}, 0 \to \mathbb{C}^1, 0$ with the identity map on $\mathbb{C}$. By Lê-Ramanujam [LeR], every equisingular deformation of f_1 is topologically trivial ($n \neq 4$). However, to see the consequences for the germ f_0, we consider a special class of hypersurface germs f_1 with the property that every equisingular deformation $F_1(x,u) : \mathbb{C}^{n-1+q}, 0 \to \mathbb{C}^1, 0$ of f_1 is stratified topologically trivial via stratified vector fields ζ_i $\left(\text{which project to } \frac{\partial}{\partial u_i}, \text{ satisfy } \zeta_i(f) = 0 \text{ and}\right)$ which are smooth off of $\{0\} \times \mathbb{C}^q$. This special class includes the simple germs, germs defining plane curve singularities [T], and weighted homogeneous germs (see [DGaf] for deformations of non-decreasing weight which are the equisingular deformations by [V]). Quite possibly all isolated hypersurface singularities belong to this special class.

Then, consider a class of map germs $f:\mathbb{C}^n, 0 \to \mathbb{C}^2, 0$ satisfying condition (4.5).

(4.5): For all $x \in \sum(f_0)$, the associated hypersurface germ f: $\mathbb{C}^{n-1}, x \to \mathbb{C}^1, f(x)$ belongs to the special class above.

For such germs f, it is not necessary to explicitly determine the versality discriminant.

PROPOSITION 4.6. — *Suppose that f_0: $\mathbb{C}^n$, $0 \to \mathbb{C}^2$, 0 is weighted homogeneous, of finite singularity type, and satisfies* (4.5). *If $f = (f_t, t)$ is a weight zero unfolding of f_0, then f is topologically ($\mathscr{A}$-) trivial if and only if the restriction $f|C(f): C(f) \to \mathrm{D}(f)$* is topologically trivial.

The example considered earlier, $f_t(x,y,z) = (xy+tz, x^4+y^4+z^2)$, is a unimodal singularity which only deforms to simple germs; thus, it satisfies (4.5). It is shown in [D3] that $f|C(f): C(f) \to D(f)$ is topologically trivial if $t \neq 0, \pm 1/2$. Thus, proposition 4.6 assures topological tiviality off of the points $t = 0, \pm 1/2$ (where it does fail).

Proof (*of proposition* 4.6). — If f is topologically trivial then clearly the restriction to $C(f)$ is also. Conversely, suppose the restriction $f|C(f)$ is topologically trivial. By the above discussion, it is sufficient to prove the stratified topological triviality for $\bar{f}|X_i$, $S \to Y_i$, y, where $Y_i \simeq \mathbb{C}$ is a section through a branch of V at y. By the topological triviality of $C(f)$, if $x_i \in S$, then the germ $\bar{f}|: X_i, x_i \to Y_i, y$ has only one singular point in $Z_i = C(f) \cap \mathrm{X}_i \cap (\mathrm{U}_i \times \{\mathrm{u}_0\})$, where U_i is a sufficiently small neighborhood of x_i and $u_0 \in B_\varepsilon$ a ball about 0 of sufficiently small radius ε. Thus $\bar{f}|: \mathrm{X}_i, x_i \to Y_i, y$ is a μ-constant deformation and hence is equisingular. Thus, by (4.5) there are ζ_i projecting to $\dfrac{\partial}{\partial u_i}$, smooth off of $\{0\} \times \mathbb{C}^q$ and satisfying $\zeta_i(f) = 0$. Letting $\eta_i = \dfrac{\partial}{\partial u_i}$ and ξ_i be the disjoint union of the ζ_i for each $x_i \in S$, we obtain from (ξ_i, η_i) the stratified topological trivialization. Hence, by our earlier argument, f is topologically trivial. □

Lastly, consider the general case of weight zero unfoldings $f(x,u)$ of a weighted homogeneous germ $f_0: k^n, 0 \to k^p, 0$. Suppose that the versality discriminant is contained in a weighted homogeneous $V \subset D(f)$ which is topologically trivial. We may choose sections Y to each k^*-orbit $\subset V$, and consider the pull back X via $\bar{f}$, with $S = \bar{f}^{-1}(y) \cap V'$. By arguments analogous to the preceding, we obtain the following.

PROPOSITION 4.7. — *Suppose that for all sections Y, $f: X, S \to Y, y$ is stratified topologically trivial preserving V and V'. Then, f is a topologically ($\mathscr{A}$-) trivial unfolding of f_0.*

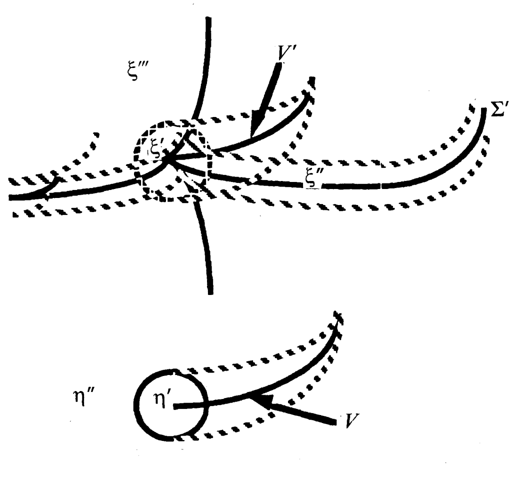
5. Construction of the stratified vector fields.

For the unfolding f of f_0, we shall find vector fields ξ_j and η_j, defined on neighborhoods of 0 in k^{s+q} and k^{t+q} respectively and projecting to $\frac{\partial}{\partial u_j}$, which satisfy $\xi_j(f) = \eta_j \circ f$, $1 \leqslant j \leqslant q$. We shall show in § 6 that these vector fields are stratified (in the sense of § 2).

Then, by the local integrability of such vector fields together with standard arguments as in [D1, I], it follows that f is a topologically trivial unfolding of f_0. Thus, it is only necessary to carry out the construction for one $\frac{\partial}{\partial u_j}$, which we will henceforth denote by $\frac{\partial}{\partial u}$.

We will construct the vector fields ξ and η from three pieces of vector fields which will be patched together using partitions of unity.

First, there are vector fields ξ' and η' defined on conical neighborhoods U' of V' and U of V, respectively. These are defined by assumption. Secondly, we shall use (3.5) to construct vector fields ξ'' and η'' defined off of V, respectively $f^{-1}(V)$. Thirdly, we shall use (3.3) to construct a vector field ξ''' defined off of the set $\sum'$ where $\sigma = 0$ (this set is essentially $\sum(f)$).



Using partitions of unity we will patch together these pieces and extend them to be $\frac{\partial}{\partial u}$ on V_0 and V'_0 to give what will be the desired ξ and η.

Construction of η'' *and* ξ''.

If we multiply (3.5) with $\zeta = \frac{\partial \bar{f}}{\partial u}$ by $-(h_i^c)^k$, where h_i^c denotes the complex conjugate of h_i, and we sum over i then we obtain

$$(5.1) \qquad - \hat{\rho}\cdot\frac{\partial \bar{f}}{\partial u} = \xi_0(f) - \eta_0 \circ f.$$

If $k = \mathbb{R}$, then the left hand side of (5.1) is real, so we replace ξ_0 and η_0 by the real parts of these vector fields and the result still holds.

Define

$$\eta'' = \frac{\partial}{\partial u} + \hat{\rho}^{-1}\cdot\eta_0 \qquad \text{and} \qquad \xi'' = \frac{\partial}{\partial u} + \tilde{\rho}^{-1}\cdot\xi_0 .$$

To define ξ''' will take considerably more work. To begin, we multiply (3.3) by $-G_j^c\cdot\frac{\partial \bar{f}_\ell}{\partial u}$ (where $\bar{f}_\ell$ denotes the ℓ-th coordinate function of $\bar{f}$ and G_j^c denotes the complex conjugate of G_j if $k = \mathbb{C}$ or G_j if $k = \mathbb{R}$). Summing over j and ℓ, we obtain

$$(5.2) \qquad - \sigma\cdot\frac{\partial \bar{f}}{\partial u} = \zeta'(f).$$

Similarly, by multiplying (3.3) by $-G_j^c$ and summing over j we obtain

$$(5.3) \qquad - \sigma\cdot\frac{\partial}{\partial y} = \zeta_\ell(f).$$

Before we construct ξ''' from ζ' and ζ_ℓ, it will first be necessary to completely define η. Furthermore, to do this we must single out certain conical neighborhoods with special properties via the following lemmas.

LEMMA 5.4. — *If $U^{(1)}$ is a conical neighborhood of V there is a conical neighborhood $U^{(1)'}$ of V' such that $f(U^{(1)'}) \subset U^{(1)}$.*

Proof. — For the proof recall: ρ_0 and $\rho_0^{(1)} = \tilde{\rho}_0 + \sigma$ are control functions for V_0 and V'_0 (respectively), while $\hat{\rho}$ and $\hat{\rho}^{(1)} = \sigma + \tilde{\hat{\rho}}$ are control functions for V and V'.

Suppose $U^{(1)}$ is defined by

$$U^{(1)} = \{(y,u) : \hat{\rho}(y,u) < \varepsilon\rho_0(y,u)\}.$$

Consider

$$U^{(1)'} = \{(x,u) : \hat{\rho}^{(1)}(x,u) < \varepsilon'\rho_0^{(1)}(x,u)\}$$

with ε' to be determined. Since $\hat{\rho}^{(1)} = \sigma + \tilde{\hat{\rho}}$, if $x' = (x,u) \in U^{(1)'}$, then

$$\tilde{\hat{\rho}}(x') \leqslant \hat{\rho}^{(1)}(x') < \varepsilon'\rho_0^{(1)}(x').$$

Let W denote a conical neighborhood of $f^{-1}(V_0)$ defined by

$$W = \{x' : \tilde{\rho}_0(x') < \varepsilon''\rho_0^{(1)}(x')\}.$$

Consider $x' \in U^{(1)'} \cap W$. We have

$$\rho_0^{(1)}(x') = (\tilde{\rho}_0+\sigma)(x') \leqslant (\tilde{\rho}_0+\hat{\rho}^{(1)})(x') < (\varepsilon'+\varepsilon'')\rho_0^{(1)}(x')$$

or $U^{(1)'} \cap W = \varnothing$ if $\varepsilon' + \varepsilon'' < 1$.

Thus, choosing ε', ε'' this small we have for $x' \in U^{(1)'}$(and $x' \notin W$)

$$\tilde{\hat{\rho}}(x') < \varepsilon'\rho_0^{(1)}(x') \leqslant \varepsilon' \cdot \varepsilon''^{-1}\tilde{\rho}_0(x').$$

Further choosing ε', ε'' so that $\varepsilon' \cdot \varepsilon''^{-1} < \varepsilon$ we conclude $f(U^{(1)'}) \subset U^{(1)}$.

Next, we let $\sum' = \{(x,u) : \sigma(x,u) = 0\}$. Although $\sum'$ may be slightly larger than the critical set $\sum(f)$ it is still true that $\sum' \cap f^{-1}(V) = V'$ by (F3).

LEMMA 5.5. – *Given a conical neighborhood U' of V', there is a conical neighborhood U'' of $\sum'$ such that $U'' \cap f^{-1}(V) \subset U'$ and $U'' \cap f^{-1}(V_0) = \varnothing$. Moreover, there is a conical neighborhood $U^{(1)}$ of V so that $f(U''\backslash U') \cap U^{(1)} = \varnothing$.*

Proof. – We let $x' = (x,u)$, then

$$U' = \{x' : \hat{\rho}^{(1)}(x') < \varepsilon \cdot \rho_0^{(1)}(x')\}.$$

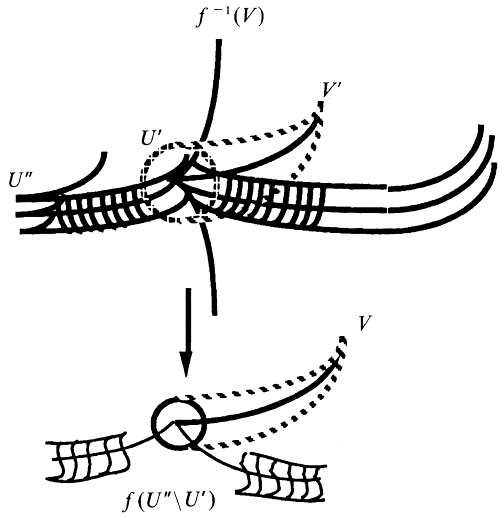
Pick $0 < \varepsilon' < \varepsilon$ and let

$$U'' = \{x' : \rho(x') < \varepsilon' \cdot \rho_0^{(1)}(x')\}.$$

Then, if $x' \in U'' \cap f^{-1}(V)$ then $\tilde{\hat{\rho}}(x') = \hat{\rho}(f(x')) = 0$; thus,

$$\hat{\rho}^{(1)}(x') = \tilde{\hat{\rho}}(x') + \sigma(x') < \varepsilon' \cdot \rho_0^{(1)}(x'),$$

and so $x' \in U'$. Furthermore, by repeating the argument in lemma 5.4 we can assure that by shrinking U'' if necessary that $U'' \cap W = \varnothing$ for a sufficiently small conical neighborhood W of $f^{-1}(V_0)$. In particular, $U'' \cap f^{-1}(V_0) = \varnothing$.



For the second claim, with $0 < \varepsilon' < \varepsilon$ as in the first part, let $\varepsilon'' = \varepsilon - \varepsilon' > 0$. Also for $y' = (y,u)$, define

$$U^{(1)} = \{y' : \hat{\rho}(y') < \varepsilon'' \cdot \rho_0(y')\}.$$

If $x' \in U''\backslash U'$, then

$$\tilde{\hat{\rho}}(x') + \sigma(x') \geqslant \varepsilon' \cdot \rho_0^{(1)}(x') \qquad \text{and} \qquad \sigma(x') < \varepsilon' \cdot \rho_0^{(1)}(x').$$

Hence, as $\rho_0^{(1)} = \tilde{\rho}_0 + \sigma$

$$\tilde{\hat{\rho}}(x') \geqslant (\varepsilon - \varepsilon')\tilde{\rho}_0(x') = \varepsilon''\tilde{\rho}_0(x'). \tag{5.6}$$

If $y' = f(x')$, (5.6) implies $y' \notin U^{(1)}$.

□

Completing the construction of η.

By lemma 5.4, we may assume $f(U') \subset U$. Then, by lemma 5.5, there are conical neighborhoods $U^{(1)} \subset U$, $U^{(1)'} \subset U'$, and U'' of $\sum'$ such that

$$U'' \cap f^{-1}(V) \subset U^{(1)'}, \qquad U'' \cap f^{-1}(V_0) = \varnothing,$$
$$\text{and} \qquad f(U''\backslash U^{(1)'}) \cap U^{(1)} = \varnothing.$$

Let $U^{(2)}$ and $U^{(3)}$ be conical neighborhoods of V with $C\ell(U^{(3)}) \subset U^{(2)} \subset C\ell(U^{(2)}) \subset U^{(1)}$ ($C\ell$ denotes closure in $k^{t+q}\backslash V_0$). Then, by (V2), there are stratified vector fields η' and ξ' on U and U', which are smooth off of $U^{(3)}$ and $f^{-1}(U^{(3)})$. There is a partition of unity $\{\chi_1, \chi_2\}$ on $k^{t+q}\backslash V_0$ subordinate to $\{U^{(1)}, (k^{t+q}\backslash V_0)\backslash C\ell(U^{(2)})\}$. We define

$$\eta = \begin{cases} \chi_1\eta' + \chi_2\eta'' & \text{off of } V_0 \\ \dfrac{\partial}{\partial u} & \text{on } V_0 \end{cases}$$

The construction of ξ'''.

We may write

$$\eta = \frac{\partial}{\partial u} + \sum_{i=1}^{t} n_i \cdot \frac{\partial}{\partial y_i}.$$

Then, we define

$$\xi''' = \frac{\partial}{\partial u} + \sigma^{-1}\zeta' + \sum_{i=1}^{t} \sigma^{-1}\tilde{n}_i \cdot \zeta_i. \tag{5.7}$$

Then, ξ''' is defined off of $\sum'$.

Completing the construction of ξ.

Next, for ξ, we first define ξ on U''. Let

$$\xi_1 = \tilde{\chi}_1\xi' + \tilde{\chi}_2\xi'' \qquad \text{on } U''.$$

To see that ξ_1 is defined on U'', we first observe that $\tilde{\chi}_1$ and $\tilde{\chi}_2$ are defined and smooth on U'' since $U'' \cap f^{-1}(V_0) = \varnothing$. Then, $\tilde{\chi}_2\xi''$ is defined and smooth on U'' provided that $\tilde{\chi}_2 = 0$ on an open neighborhood of $f^{-1}(V) \cap U''$. However, by assumption $f^{-1}(V) \cap U'' \subset U^{(1)'}$ and $f(U^{(1)'}) \subset U$. Since $\chi_2 = 0$ on an open neighborhood of V, $\tilde{\chi}_2 = 0$ on an open neighborhood of $f^{-1}(V)$ in $U^{(1)'}$, which gives an open neighborhood of $f^{-1}(V) \cap U''$ in U''.

Secondly, $\tilde{\chi}_1\xi'$ is defined on U'' (and smooth on $U''\backslash f^{-1}(U^{(3)})$) provided $\tilde{\chi}_1 \equiv 0$ on an open neighborhood of $U''\backslash U^{(1)'}$. However, $\chi_1 \equiv 0$ on a neighborhood of $(k^{t+q}\backslash V_0)\backslash U^{(1)}$ and $f(U''\backslash U^{(1)'}) \subset (k^{t+q}\backslash V_0)\backslash U^{(1)}$; hence, $\tilde{\chi}_1 \equiv 0$ on a neighborhood of $U''\backslash U^{(1)'}$. Thus, ξ_1 is defined.

Let U''' be a conical neighborhood of $\sum'$ with $C\ell(U''') \subset U''$ (in $k^{t+q}\backslash V_0'$). Let $\{\chi_3, \chi_4\}$ be a partition of unity subordinate to $\{U'', (k^{s+q}\backslash V'_0)\backslash C\ell(U''')\}$. Then define

$$\xi = \begin{cases} \chi_3\xi_1 + \chi_4\xi''' & \text{off of } V_0' \\ \dfrac{\partial}{\partial u} & \text{on } V_0' \end{cases}$$

We claim that η and ξ are our desired stratified vector fields. We will prove this claim in the next section.

We complete this section by showing that $\xi(f) = \eta \circ f$ on an open dense subset of k^{s+q}. The continuity of η and ξ to be established in the next section will then imply this equation holds on all of k^{s+q} (in a neighborhood of 0).

Off of V_0',

$$\xi(f) = \chi_3\xi_1(f) + \chi_4\xi'''(f). \tag{5.8}$$

Furthermore,

$$\begin{aligned} \xi_1(f) &= \tilde{\chi}_1\xi'(f) + \tilde{\chi}_2\xi''(f) \\ &= \tilde{\chi}_1(\eta'\circ f) + \tilde{\chi}_2(\eta''\circ f) \\ &= (\tilde{\chi}_1\cdot\eta' + \tilde{\chi}_2\cdot\eta'')\circ f \\ &= \eta\circ f \qquad \text{off of } f^{-1}(V_0). \end{aligned}$$

Next, off of $\sum'$,

$$\begin{aligned} \xi'''(f) &= \frac{\partial}{\partial u} + \frac{\partial \bar{f}}{\partial u} + \sigma^{-1}\zeta'(f) + \sum_{i=1}^{t} \sigma^{-1}\tilde{n}_i\cdot\zeta_i(f) \\ &= \frac{\partial}{\partial u} + 0 + \sum_{i=1}^{t} \tilde{n}_i\cdot\frac{\partial}{\partial y_i} \end{aligned}$$

by (5.2) and (5.3); or

$$\xi'''(f) = \eta\circ f \qquad \text{off of } \textstyle\sum'.$$

Thus, from (5.8),

$$\begin{aligned}\xi(f) &= \chi_3\eta\circ f + \chi_4\eta\circ f \\ &= \eta\circ f \qquad \text{off of } f^{-1}(V_0)\cup\textstyle\sum'.\end{aligned}$$

Thus, it remains to prove ξ and η are stratified.

6. Verifying that the vector fields are stratified.

We claim that the vector fields ξ and η defined in the previous section are stratified relative to the weak stratifications given by (V1). We shall establish the stratification conditions in four steps:

a) η satisfies the condition for being a stratified vector field except for η being continuous and satisfying the control conditions at points of V_0;

a') η is continuous and satisfies the control conditions at points of V_0;

b) given *a)* and *a')*, ξ satisfies the condition for being a stratified vector field except for ξ being continuous and satisfying the control conditions at points of V_0'; also, $\xi(f) = \eta\circ f$ in a neighborhood of 0;

b') ξ is continuous and satisfies the control conditions at points of V_0'.

If all of *a)*, *a')*, *b)* and *b')* are satisfied then η and ξ are stratified vector fields, which by *b)* yield a topological trivialization of f. This would prove theorem 2 (and hence theorem 1). We shall see that *a)* and *b)* follow easily from the construction of η and ξ using (V2) and proposition 2.5. Then *a')* and *b')* are more subtle requiring certain estimates based on the filtration properties of the germs.

In verifying the stratification conditions in *a)* and *b)*, we shall continually make use of the simple observation that to verify that a vector field ζ defined on an open set W is stratified (relative to a stratification $\mathscr{V}$) it is sufficient to verify that ζ is stratified (for $\mathscr{V}$) on each open set W_i in a covering $\{W_i\}$ of W.

Verification of a) :

For η, we know that $\eta\,|\,U^{(2)} = \chi_1\eta'$. Let $\tilde{V}$ denote the union of the positive codimension strata in $C\ell(U^{(3)})$. Hence, since χ_1 is smooth off of V_0 and η' is stratified and is smooth off of $\tilde{V}$, so is $\eta|U^{(2)'} = \chi_1\eta'$. Secondly, off of $C\ell(U^{(3)})$, $\chi_2\eta''$ is smooth, as is $\chi_1\eta'$. Hence, η is smooth off of $C\ell(U^{(3)})$ (in $k^{t+q}\backslash V_0$). Suppose we can show that ξ is smooth off of $f^{-1}(C\ell(U^{(3)})$. Since $\xi(f) = \eta \circ f$ wherever they are continuous (by the previous section), it follows by proposition 2.5 that η is stratified relative to a stratification by singularity submanifolds off of $C\ell(U^{(3)})$. Then, it would follow that η is stratified on $U^{(2)} \cup (k^{t+q}\backslash C\ell(U^{(3)}) = k^{t+q}\backslash V_0$. Since $\eta | V_0 = \dfrac{\partial}{\partial u}$, which is smooth and tangent to V_0, to show that η is stratified it only remains to show a') that η is continuous at points of V_0 and satisfies the local control condition for V_0.

Smoothness of ξ off $f^{-1}(\tilde{V})$.

We actually show something stronger than used in a), namely, that ξ is smooth off $f^{-1}(\tilde{V})$. We first begin with ξ_1 on U''. By construction, $\tilde{\chi}_1\xi'$ is smooth off $f^{-1}(\tilde{V}) \cap U''$ and $\tilde{\chi}_2\xi''$ is smooth on U''; hence, ξ_1 is smooth on U'' off $f^{-1}(\tilde{V}) \cap U''$. As η is smooth off $\tilde{V}$, it follows from the definition of ξ''' in (5.7) that it is smooth off of $f^{-1}(\tilde{V}) \cup \sum'$. Thus, by (5.8) ξ is smooth off of $f^{-1}(\tilde{V})$.

Verification of b) :

The stratification assertions in b) state that ξ is stratified on $k^{s+q}\backslash V_0'$. By the smoothness of ξ off $f^{-1}(\tilde{V})$ we know that ξ is stratified on $W_1 = k^{s+q}\backslash(V_0' \cup f^{-1}(\tilde{V}))$. By lemma 5.5, $f^{-1}(\tilde{V})$ is covered in $k^{s+q}\backslash V_0'$ by the open sets

$$W_2 = f^{-1}(U^{(2)}) \cap U^{(1)'} \cap U''' \qquad \text{and} \qquad W_3 = k^{s+q}\backslash \textstyle\sum'.$$

Second claim : ξ_1 is stratified on U''.

ξ' is stratified on U' and $\tilde{\chi}_1$ is smooth on U'', so $\tilde{\chi}_1\xi'$ is stratified on $U' \cap U''$. By construction, $\tilde{\chi}_1 = 0$ on a neighborhood of $U''\backslash U'$; thus, $\tilde{\chi}_1\xi'$ is stratified on U''. Also, ξ'' is smooth on $k^{s+q}\backslash f^{-1}(V)$. Since $\tilde{\chi}_2 = 0$ on $f^{-1}(U^{(2)}) \supset f^{-1}(V)$, $\tilde{\chi}_2\xi''$ is smooth on U''. Thus, ξ_1 is stratified on U''.

Third claim : ξ is stratified on W_2.

By claim 2, ξ_1 is stratified on $U'' \supset W_2$. Since χ_3 is smooth on W_2 and $\chi_4 = 0$ on $U''' \supset W_2$, $\xi | W_2 = \chi_3\xi_1$ is also stratified on W_2.

Fourth claim : ξ is stratified on W_3.

By the second claim, ξ_1 is stratified on U''. Since $\chi_3 = 0$ off U'' (in $k^{s+q}\backslash V_0'$), we conclude that $\chi_3\xi_1$ is stratified on $k^{s+q}\backslash V_0'$. Thus, to conclude that ξ is stratified on W_3, it is sufficient show that ξ''' is stratified on $k^{s+q}\backslash\sum'$. Since χ_4 is smooth on $k^{s+q}\backslash V_0'$, so will $\chi_4\xi'''$ be stratified; and hence, also ξ.

Provided we know from a') that η is continuous, it follows from the definition of ξ''' in (5.7) that it is continuous off of $\sum'$ (hence, we know that ξ is continuous off V_0'). It remains to show that ξ''' satisfies the stratification conditions on the strata in $f^{-1}(\tilde{V})\backslash\sum'$.

From $\xi'''(f) = \eta \circ f$ off $\sum'$ it follows that for x' in a stratum $f^{-1}(V_i)\backslash\sum'$

$$df_{x'}(\xi'''(x')) = \eta(f(x')) \text{ is tangent to } V_i.$$

Hence, ξ''' is tangent to $f^{-1}(V_i)\backslash\sum'$ at x'.

In the definition of ξ''' in (5.7), $n_j | V_i$ are smooth hence so are their pullbacks to $f^{-1}(V_i)\backslash\sum'$. Thus, ξ''' restricted to $f^{-1}(V_i)\backslash\sum'$ is smooth and tangent.

Lastly, from the assumption of a') we see that the local control conditions are satisfied for ξ''' on strata in $f^{-1}(\tilde{V})\backslash\sum'$. Suppose η is shown to satisfy the local control conditions for $y \in V_i$ using the local control function ρ_y. We use the local control function $\tilde{\rho}_y = \rho_y \circ f$ for $x \in f^{-1}(V_i)\backslash\sum'$, with $y = f(x)$. Then,

$$\begin{aligned} |\xi(\tilde{\rho}_y)| &= |\xi(f)(\rho_y)| = |(\eta\circ f)(\rho_y)| \\ &= |\eta(\rho_y)| \circ f \leqslant (C \cdot \rho_y) \circ f \\ &\leqslant C \cdot \tilde{\rho}_y . \end{aligned}$$

Hence, the local control conditions are satisfied and ξ''' is stratified.

We can also conclude by continuity that $\xi(f) = \eta \circ f$ off of V_0', while on V_0' it is trivially true; hence the equation holds in a neighborhood of 0.

Filtration properties and estimates.

Before establishing the remaining conditions for η and ξ, we first make a few basic observations concerning the filtration properties of vector fields. We consider for example

$$\zeta \in \mathscr{C}_{x,u}\left\{\frac{\partial}{\partial x_i}\right\} \qquad \text{with } \operatorname{fil}(\zeta) = r.$$

$$\|\zeta\| \leqslant C \cdot \rho_0^{(1)((r+1)/2m)}. \tag{6.1}$$

In fact, if $\zeta = \sum \ell_i \dfrac{\partial}{\partial x_i}$ then by the properties of the filtrations,

$$\operatorname{fil}(\ell_i) \geqslant \operatorname{fil}(x_i) + \operatorname{fil}(\zeta)$$

or since $\operatorname{fil}(x_i) > 0$,

$$\operatorname{fil}(\ell_i) \geqslant r + 1.$$

Thus, (6.1) follows from $\|\zeta\| \leqslant \sum_{i=1}^{s} |\ell_i|$. A similar remark applies to the other modules of vector fields.

Then, in (5.1) ξ_0 is a sum of terms $-(h_i^c)^k \xi_i$; and (with $m = k \cdot m'$, $m' = \operatorname{fil}(h_i)$)

$$\begin{aligned} \operatorname{fil}(-(h_i^c)^k \xi_i) &\geqslant m + \operatorname{fil}(\xi_i) \\ &\geqslant m + m + \operatorname{fil}\left(\frac{\partial f}{\partial u}\right) - \operatorname{jump}_V(f) \\ &\geqslant 2m + \operatorname{jump}_C(f) \end{aligned}$$

by (4.3). Hence, summing over i gives

$$\operatorname{fil}(\xi_0) \geqslant 2m + \operatorname{jump}_C(f). \tag{6.2}$$

Thus, by (6.1)

$$\|\xi_0\| \leqslant C \cdot \rho_0^{(1)(1+(1/2m))}. \tag{6.3}$$

A similar argument implies

$$\operatorname{fil}(\eta_0) \geqslant 2m + \operatorname{jump}_C(f). \tag{6.4}$$

Hence,

$$\|\eta_0\| \leqslant C'' \cdot \rho_0^{(1+(1/2m))}. \tag{6.5}$$

Next, we consider ζ' defined in (5.2). To estimate $\|\zeta'\|$, we argue in a similar fashion to the preceding case with one exception. We observe by the properties of the filtration

$$\operatorname{fil}\left(\frac{\partial \bar{f}_\ell}{\partial u}\right) \geqslant \operatorname{fil}(y_\ell) + \operatorname{fil}\left(\frac{\partial \bar{f}}{\partial u}\right).$$

We also know

$$\operatorname{fil}(\zeta_{i\ell}) \geqslant m - \operatorname{fil}(y_\ell) - \operatorname{jump}_C(f).$$

Hence

$$\begin{aligned}\operatorname{fil}\left(-G_j^c \cdot \frac{\partial \bar{f}_\ell}{\partial u} \cdot \zeta_{i\ell}\right) &\geqslant m + \operatorname{fil}\left(\frac{\partial \bar{f}_\ell}{\partial u}\right) + \operatorname{fil}(\zeta_{i\ell}) \\ &\geqslant 2m + \operatorname{fil}\left(\frac{\partial \bar{f}}{\partial u}\right) - \operatorname{jump}_C(f) \\ &\geqslant 2m + \operatorname{jump}_V(f).\end{aligned}$$

Summing over j, ℓ yields $\operatorname{fil}(\zeta') \geqslant 2m \pm \operatorname{jump}_V(f)$, thus, we conclude as in the earlier case

$$\|\zeta'\| \leqslant C_0 \cdot \rho_0^{(1)(1+(1/2m))}. \tag{6.6}$$

Similarly, arguing as above for ζ_ℓ defined by (5.3)

$$\operatorname{fil}(\zeta_\ell) \geqslant 2m - b_\ell \tag{6.7}$$

and

$$\|\zeta_\ell\| \leqslant C_\ell \cdot \rho_0^{(1)(1-(b_\ell - 1)/2m)}, \tag{6.8}$$

where $b_\ell = \operatorname{fil}(y_\ell) + \operatorname{jump}_C(f)$. We are now ready to establish the remaining conditions for ξ and η.

Verification of a′) :

First, to establish the local control conditions for η and V_0, consider

$$\begin{aligned}|\eta(\rho_0)| &\leqslant \chi_1 |\eta'(\rho_0)| + \chi_2 |\eta''(\rho_0)| \\ &\leqslant C_1\rho_0 + \chi_2\left(\left|\frac{\partial \hat{\rho}_0}{\partial u}\right| + \hat{\rho}^{-1}|\eta_0(\hat{\rho}_0)|\right).\end{aligned} \tag{6.9}$$

By our choice of ρ_0, $\dfrac{\partial \rho_0}{\partial u} = 0$ and by (6.4)

$$|\eta_0(\rho_0)| \leqslant C_3 \cdot (\rho_0)^{(2m+r)/2m} \tag{6.10}$$

where $r = \operatorname{fil}(\eta_0) \geqslant 2m$.

If $\chi_2 = 0$, the control condition holds by (6.9), while if $\chi_2(y') \neq 0$, then $y' \notin U^{(2)}$, hence $\hat{\rho}(y') \geqslant \varepsilon'\rho_0(y')$. Combining this with (6.10) implies for (6.9)

$$(6.11) \qquad \begin{aligned} |\eta_0(\rho_0)| &\leqslant C_1 \cdot \rho_0 + \varepsilon'^{-1} \cdot \rho_0^{-1} C_3 \cdot \rho_0^2 \\ |\eta_0(\rho_0)| &\leqslant C_4 \cdot \rho_0 \qquad \text{off of } U^{(2)}. \end{aligned}$$

Secondly, to verify continuity on V_0, we may write

$$(6.12) \quad \chi_1\eta' + \chi_2\eta'' = \left(\frac{\partial}{\partial u}\right) + \chi_1\left(\eta' - \frac{\partial}{\partial u}\right) + \chi_2\left(\eta'' - \frac{\partial}{\partial u}\right).$$

Also, by assumption (V2) of the definition of stratified topological triviality (and (4.2))

$$(6.13) \qquad \left\| \eta' - \frac{\partial}{\partial u} \right\| \leqslant C \cdot \rho_0^\delta \qquad \text{for some } C,\ \delta > 0$$

while if $\chi_2(y') \neq 0$, $y' \notin U^{(2)}$ and as above by (6.5)

$$(6.14) \quad \chi_2 \left\|\eta'' - \frac{\partial}{\partial u}\right\| = \chi_2 \cdot \hat{\rho}^{-1} \|\eta_0\| \leqslant \varepsilon'^{-1} \cdot \rho_0^{-1} C''' \cdot \rho_0^{1+\delta'} \leqslant C \cdot \rho_0^{\delta'}.$$

Thus, η decomposes in a neighborhood of 0 as $\eta = \dfrac{\partial}{\partial u} + \eta_2$ and from 6.13 and 6.14

$$\eta_2 = \begin{cases} \chi_1\left(\eta' - \dfrac{\partial}{\partial u}\right) + \chi_2\left(\eta'' - \dfrac{\partial}{\partial u}\right) & \text{off } V_0 \\ 0 & \text{on } V_0 \end{cases}$$

with

$$(6.15) \quad \|\eta_2\| \leqslant C_1 \cdot \rho_0^{\delta''} \quad \text{with } C = \max\{C, C'''\}, \quad \delta'' = \min\{\delta, \delta'\}.$$

This shows that η is continuous on V_0 as well. Thus, η is stratified.

Verification of b') :

Next, we establish the local control conditions for ξ and V_0.

$$(6.16) \qquad |\xi(\rho_0^{(1)})| \leqslant \chi_3 |\xi_1(\rho_0^{(1)})| + \chi_4 |\xi'''(\rho_0^{(1)})|.$$

Now when $\chi_3 \neq 0$, i.e., in U'',

$$(6.17) \quad \begin{aligned} |\xi_1(\rho_0^{(1)})| &\leqslant \tilde{\chi}_1 |(\xi'(\rho_0^{(1)})| + \tilde{\chi}_2 |\xi''(\rho_0^{(1)})| \\ &\leqslant C_1 \cdot \rho_0^{(1)} + \tilde{\chi}_2\left(\left| \frac{\partial \rho_0^{(1)}}{\partial u} \right| + \tilde{\hat{\rho}}^{-1} |\xi_0(\rho_0^{(1)})| \right). \end{aligned}$$

The argument proceeds as in (6.9) for η. If $\tilde{\chi}_2(x') \neq 0$, $f(x') \notin U^{(2)}$. However, by lemma 5.4, there is a conical neighborhood $U^{(4)}$ of V' such that $f(U^{(4)}) \subset U^{(2)}$, thus, $x' \notin U^{(4)}$. Since f is an unfolding of non-decreasing filtration and satisfies F2, we obtain

$$(6.18) \qquad \left|\frac{\partial \rho_0^{(1)}}{\partial u}\right| = \left|\frac{\partial}{\partial u}(\sigma + \tilde{\rho}_0)\right| \leqslant \left|\frac{\partial \sigma}{\partial u}\right| + \left|\left(\frac{\partial f}{\partial u}\right)(\rho_0)\right| \leqslant C_1 \cdot \rho_0^{(1)} + C_2 \cdot \rho_0^{(1)} = C_3 \cdot \rho_0^{(1)}.$$

Hence, arguing exactly as for η,

$$(6.19) \qquad |\xi_1(\rho_0^{(1)})| \leqslant C \cdot \rho_0^{(1)}$$

(note that $x' \in U''$ implies $\sigma(x') < \varepsilon' \rho_0^{(1)}(x')$ which implies $\sigma(x') < \varepsilon''' \tilde{\rho}_0(x')$ where $\varepsilon''' = \varepsilon'/(1-\varepsilon')$, or $\rho_0^{(1)}(x') < (1 + \varepsilon''')\, \tilde{\rho}_0(x')$. Since $f(x') \notin U^{(2)}$, $\tilde{\tilde{\rho}}(x') > \varepsilon' \tilde{\rho}_0(x')$ giving $\tilde{\tilde{\rho}}(x') > C\rho_0^{(1)}(x'))$.

Similarly, if $\chi_4(x') \neq 0$, $x' \notin U'''$ so that $\sigma \geqslant \varepsilon'' \rho_0^{(1)}$. Then, by (5.7)

$$(6.20) \quad \chi_4 |\xi'''(\rho_0^{(1)})| = \left|\frac{\partial \rho_0^{(1)}}{\partial u}\right| + \sigma^{-1} |\zeta'(\rho_0^{(1)})| + \sum_{i=1}^{t} \sigma^{-1} |\tilde{n}_i| \cdot |\zeta_i(\rho_0^{(1)})| .$$

Also, from § 5, $\mathrm{fil}(\zeta') \geqslant 2m$, hence

$$|\zeta'(\rho_0^{(1)})| \leqslant C \cdot \rho_0^{(1)^2}.$$

For the third term, we must more accurately estimate $|n_j|$. If $\eta' = \dfrac{\partial}{\partial u} + \sum n_i' \dfrac{\partial}{\partial y_i}$ and $\eta_0 = \sum n_{0i} \dfrac{\partial}{\partial y_i}$, then by the assumption (4.2) on η',

$$|n_j'| \leqslant C \cdot \rho_0^{b_j} \qquad \text{where } b_j = \mathrm{jump}_C(f) + \mathrm{fil}\,(y_j).$$

Also, as $\mathrm{fil}(\eta_0) \geqslant 2m + \mathrm{jump}_C(f)$,

$$\mathrm{fil}\,(n_{0j}) \geqslant 2m + \mathrm{fil}\,(y_j) + \mathrm{jump}_C(f).$$

Then,

$$|n_j| = \chi_1 |n_j'| + \chi_2 |\hat{\rho}^{-1} n_{0j}| .$$

Arguing exactly as we did in obtaining (6.11),

$$|n_j| \leqslant C_j' \cdot \rho_0^{b_j}.$$

Also, by (6.7)

$$\mathrm{fil}(\zeta_j) \geqslant 2m - b_j$$

or

$$|\zeta'(\rho_0^{(1)})| \leqslant C \cdot \rho_0^{(1)(2-(b_j/2m))}.$$

Thus (6.20) becomes

$$\chi_4|\xi'''(\rho_0^{(1)})| \leqslant C_1\cdot\rho_0^{(1)} + C_2\sigma^{-1}\cdot\rho_0^{(1)^2} + C_3\sigma^{-1}\cdot\tilde{\rho}_0^{(b_j/2m)}\cdot\rho_0^{(1)^{(2-(b_j/2m))}}.$$

Thus, since $\tilde{\rho}_0 \leqslant \rho_0^{(1)}$ and $\sigma^{-1} \leqslant \varepsilon''^{-1}\rho_0^{(1)^{-1}}$, (6.20) becomes

$$\chi_4|\xi'''(\rho_0^{(1)})| \leqslant C_1\cdot\rho_0^{(1)}. \tag{6.21}$$

Combining (6.16) with (6.19) and (6.21) we conclude

$$|\xi(\rho_0^{(1)})| \leqslant C'''\cdot\rho_0^{(1)}.$$

Lastly, we must establish that ξ is continuous at points of V_0'. We claim that

$$\left\|\xi-\frac{\partial}{\partial u}\right\| \leqslant C\cdot\rho_0^{(1)^{\delta'}} \quad \text{for some} \quad C,\ \delta'>0.$$

Consider

$$\xi - \frac{\partial}{\partial u} = \chi_3\left(\xi_1-\frac{\partial}{\partial u}\right) + \chi_4\left(\xi'''-\frac{\partial}{\partial u}\right) \tag{6.23}$$

where

$$\xi_1 - \frac{\partial}{\partial u} = \tilde{\chi}_1\left(\xi'-\frac{\partial}{\partial u}\right) + \tilde{\chi}_2\left(\xi''-\frac{\partial}{\partial u}\right).$$

By assumption,

$$\left\|\xi'-\frac{\partial}{\partial u}\right\| \leqslant C\cdot\rho_0^{(1)^{\delta}} \quad \text{for some } C,\ \delta>0$$

in a neighborhood of 0 (wherever ξ' is defined).

Next,

$$\left\|\xi''-\frac{\partial}{\partial u}\right\| = \|\tilde{\rho}^{-1}\xi_0\| \leqslant C\cdot\tilde{\rho}^{-1}\rho_0^{(1)^{(1+(1/2m))}}.$$

However, if $\tilde{\chi}_2(x')\neq 0$, and $x'\in U''$, then we can argue as for (6.19) to conclude

$$\left\|\xi''-\frac{\partial}{\partial u}\right\| \leqslant C'\rho_0^{(1)^{(1/2m)}}.$$

Hence, within U''

$$\left\|\xi_1-\frac{\partial}{\partial u}\right\| \leqslant C''\cdot\rho_0^{(1)^{\delta'}} \tag{6.24}$$

for $\delta' = \min\{\delta, 1/2m\}$, $C'' = C + C'$.

It remains to check $\xi''' - \dfrac{\partial}{\partial u}$ when $\chi_4 \neq 0$.

$$\left\| \xi''' - \frac{\partial}{\partial u} \right\| \leqslant \sigma^{-1} \|\zeta'\| + \sum_{i=1}^{t} \sigma^{-1} |\tilde{n}_i| \cdot \|\zeta_i\|. \tag{6.25}$$

If $\chi_4(x') \neq 0$, then $x' \notin U'''$, hence $\sigma(x') \geqslant \varepsilon'' \cdot \rho_0^{(1)}(x')$. Then, since fil $(\zeta') \geqslant 2m$,

$$\sigma^{-1} \|\zeta'\| \leqslant C\sigma^{-1} \rho_0^{(1)(1+(1/2m))} \leqslant C\varepsilon''^{-1} \rho_0^{(1)(1/2m)}.$$

From the earlier computation of (6.20), and using (6.1)

$$\begin{aligned}\sigma^{-1} |\tilde{n}_j| \cdot \|\zeta_j\| &\leqslant C_j \sigma^{-1} \cdot \tilde{\rho}_0^{(b_j/2m)} \cdot \rho_0^{(1)(1-(b_j-1)/2m)} \\ &\leqslant C_j \sigma^{-1} \cdot \rho_0^{(1)(1+(1/2m))} \leqslant C_j \varepsilon''^{-1} \rho_0^{(1)(1/2m)}.\end{aligned}$$

Thus, for (6.25), if $\chi_4 \neq 0$,

$$\left\| \xi''' - \frac{\partial}{\partial u} \right\| \leqslant C''' \cdot \rho_0^{(1)(1/2m)}. \tag{6.26}$$

Finally, from (6.24) and (6.26)

$$\left\| \xi - \frac{\partial}{\partial u} \right\| \leqslant C_1 \cdot \rho_0^{(1)^{\delta'}}.$$

This completes the proof of b') showing that ξ and η are stratified. □

7. A Version of the theorem for germs with negative filtration.

Consider the case where $f_0 : k^s, 0 \to k^t, 0$ is itself an unfolding of a germ $f_1 : k^{s'}, 0 \to k^{t'}, 0$. Let $x = (x_+, x')$ and $y = (y_+, y')$ with x_+ local coordinates for $k^{s'}$, x' for $k^r (r = s - s')$ and similarly for y. We write $f_0(x_+, x') = (\bar{f}_0(x_+, x'), x')$ with $(\bar{f}_0(x_+, 0) = f_1(x_+)$. This time we suppose that there are convex filtrations for $\mathscr{C}_{x_+}$ and $\mathscr{C}_{y_+}$ (preserved by f_1^*) and filtrations on $\theta_{s'}$, $\theta_{t'}$, and $\theta(f_1)$ as $\mathscr{C}_{x_+}$ and $\mathscr{C}_{y_+}$-modules, so that f_0 is an unfolding of non-decreasing filtration of f_1. In essence, this says that the x_i' have non-positive filtration, e.g., if f_0 is weighted homogeneous then the x_i' must have non-positive weight.

Now, let f be an unfolding of f_0 of the form

$$f(x_+, x', u) = \bar{f}(x_+, x', u), x', u)$$

so that f is also an unfolding of f_1 of non-decreasing filtration (again all germs are real analytic or holomorphic). For example, in the weighted homogeneous case this situation occurs if f_0 is the unfolding of f_1 versal in weight $\leqslant m$ for some $m \geqslant 0$ (as defined in [D1, I]). We describe how the statements of Theorems 1 and 2 must be modified to allow for this situation.

We suppose:

(F 1′) There are $g_1, \ldots, g_r \in \Delta(f_1)$ and $g'_1, \ldots, g'_{n-r} \in m_{y_+}$ of filtration m such that $\{g_1, \ldots, g_r, f_1^*(g'_1), \ldots, f_1^*(g'_{n-r})\}$ form a set of vertices for the filtration on $\mathscr{C}_{x_+}$, and that if

$$g_j \cdot \left(\frac{\partial}{\partial y_{+_\ell}}\right) = \sum_i h_{i,j,\ell} \cdot \Delta_i(f) \cdot \left(\frac{\partial}{\partial y_{+_\ell}}\right)$$

then G_j defined as in (3.3) still satisfies

(F 2′) $$\operatorname{fil}(G_j) \geqslant \operatorname{fil}(g_j).$$

Now, let ρ_0 denote a control function of filtration $2m$ for the filtration on $\mathscr{C}_{y_+}$, and let $V_0 = \{0\} \times k^{r+q}$. We define $\sigma = \sum |G_j|^2$ so that using lemma 3.7, we see that $\rho_0^{(1)} = \sigma + \tilde{\rho}_0$ vanishes on $V'_0 = \{0\} \times k^{r+q}$. Let $\rho^{(1)} = \sigma + \tilde{\tilde{\rho}}$.

We now assume that f is stratified topologically trivial in conical neighborhoods U of V and U' of V' (with $V' = \sum(f) \cap f^{-1}(V)$) in that it satisfies the conditions (F 1′) (F 2′), and (F 3), and the conditions (V 1) – (V 2) of definition 4.1. The version for weighted homogeneous f becomes:

THEOREM 3. — *Suppose that f, f_0 (and f_1) are as in the above situation (for a weight filtration) with f weighted homogeneous. If f is stratified topologically trivial in a conical neighborhood of the versality discriminant and*

$$wt(u_i) > \max_j \{wt(x_j)\}, \qquad 1 \leqslant i \leqslant q$$

then f is a topologically trivial unfolding of f_0.

This is a special case of the general result for the filtered case. Just as in Theorem 2, to state the general result we must assume the versality discriminant is graded and use a modification analogous to (4.2).

First we modify the definition of $\mathrm{jump}_V(f)$. In our case, (2.2) can be written in the equivalent form

$$(7.1) \qquad I^k \cdot \theta(\bar{f}) \subseteq \mathscr{C}_{x,u}\left\{\frac{\partial \bar{f}}{\partial x_{+i}}\right\} + \mathscr{C}_{y,u}\left\{\frac{\partial}{\partial y_{+j}}\right\} + \mathscr{C}_{y,u}\left\{\frac{\partial \bar{f}}{\partial x'_i}\right\}.$$

For $\psi \in \mathscr{C}_{y,u}\left\{\frac{\partial}{\partial x'_i}\right\}$, if $\psi = \sum r_i(y,u)\cdot\frac{\partial}{\partial x'_i}$ then we define

$$\mathrm{fil}\,(\psi) = \min_i \left\{\mathrm{fil}\,(r_i) + \mathrm{fil}\left(\frac{\partial \bar{f}}{\partial x'_i}\right)\right\}.$$

We then suppose that V contains *the versality discriminant in a graded sense* by modifying (3.5) to require that there exist $h_1, \ldots, h_\ell \in I$ of filtration m' with $km' = m$, and whose vanishing still defines V, and an integer $d \geqslant 0$ so that for $\zeta \in \theta(f)$

$$(7.2) \qquad h_i^k \cdot \zeta = \xi_i(f) + \psi_i(f) - \eta_i \circ f, \qquad 1 \leqslant i \leqslant \ell$$

with

$$\mathrm{fil}\,(\xi_i),\ \mathrm{fil}\,(\psi_i),\ \mathrm{fil}\,(\eta_i) \geqslant m + \mathrm{fil}\,(\zeta) - d.$$

Again the minimum such integer $d \geqslant 0$ will be denoted $\mathrm{jump}_V(f)$. Also, $\mathrm{jump}_C(f)$ is defined exactly as before. If now, η in the condition for stratified topological triviality is given by

$$\eta_j = \frac{\partial}{\partial u_j} + \sum_{i=1}^{t'} n_{ij}\cdot\frac{\partial}{\partial y_{+i}} + \sum_{i=1}^{r} n'_{ij}\cdot\frac{\partial}{\partial y'_i}$$

then (4.2) is replaced by

$$(7.3a) \qquad \mathrm{fil}\,(n_{ij}) \geqslant \mathrm{jump}_C(f) + \mathrm{fil}\,(y_{+i})$$

$$(7.3b) \qquad \mathrm{fil}\,(n'_{ij}) \geqslant \mathrm{jump}_C(f).$$

Remark. — If f is weighted homogeneous then again $\mathrm{jump}_V(f) = \mathrm{jump}_C(f) = 0$ and $\mathrm{fil}\left(\frac{\partial \bar{f}}{\partial x'_i}\right) = -\,wt(y'_i) \geqslant 0$ so that (7.3) reduces to the original condition for (V2) for stratified topological triviality.

THEOREM 4. — *Let f, as above, be an unfolding of non-decreasing filtration of f_0 (and f_1) so that f is stratified topologically trivial in a conical neighborhood of the versality discriminant in the graded sense. If*

$$\text{(7.4)}\quad \operatorname{fil}\left(\frac{\partial \bar{f}}{\partial u_i}\right) > \operatorname{jump}_V(f) + \operatorname{jump}_C(f) + \max_j\left\{\operatorname{fil}\left(\frac{\partial \bar{f}}{\partial x'_j}\right)\right\}, \qquad 1 \leqslant i \leqslant q$$

then f is a topologically trivial unfolding of f_0.

The only way that the proofs of Theorems 3 and 4 differ from those of Theorems 1 and 2 is in the construction of ξ_0, η_0 and ξ'''. Instead of (5.1), we use (7.2) to solve

$$\text{(7.5)}\qquad -\hat{\rho}\cdot\frac{\partial \bar{f}}{\partial u} = \xi'_0(\bar{f}) + \psi_0(\bar{f}) - \eta'_0\circ f$$

where if $\psi_0 = \sum_{i=1}^{r} \tilde{r}_i(y,u)\dfrac{\partial}{\partial x'_i}$ then by (7.4)

$$\text{(7.6)}\qquad \begin{aligned} \operatorname{fil}(\tilde{r}_i) &\geqslant 2m + \operatorname{fil}\left(\frac{\partial \bar{f}}{\partial u}\right) - \operatorname{jump}_V(f) + \operatorname{fil}\left(\frac{\partial \bar{f}}{\partial x'_i}\right) \\ &> 2m + \operatorname{jump}_C(f). \end{aligned}$$

If we also let $\psi'_0 = \sum_{i=1}^{r} r_i(y,u)\cdot\dfrac{\partial}{\partial y'_i}$, then we define

$$\xi_0 = \xi'_0 + \psi_0 \qquad \text{and} \qquad \eta_0 = \eta'_0 + \psi'_0.$$

Then, (7.6) together with analogous calculations for ξ'_0 and η'_0 imply that for these ξ_0 and η_0, (6.3) and (6.5) as well as (6.10) and (6.19) hold. The construction for η proceeds as before.

To define ξ''' let

$$\eta''_0\circ f = \eta'_0\circ f - \psi_0(\bar{f}).$$

As

$$\psi_0(\bar{f}) = \sum_{i=1}^{r} \tilde{r}_i\cdot\frac{\partial \bar{f}}{\partial x'_i} = \sum_{i=1}^{t'} \tilde{n}'_{0i}\cdot\frac{\partial}{\partial y_{+i}},$$

$$\eta''_0 = \sum_{i=1}^{t'} (n_{0i} - n'_{0i})\cdot\frac{\partial}{\partial y_{+i}}$$

where $\eta_0' = \sum n_{0i} \cdot \frac{\partial}{\partial y_{+_i}} \cdot$ By (7.6)

$$\begin{aligned}\operatorname{fil}(n_{0i}') &\geqslant \min_j \{\operatorname{fil}(r_j)\} + \operatorname{fil}\left(\frac{\partial \bar{f}}{\partial x_i'}\right) + \operatorname{fil}(y_{+_i}) \\ &\geqslant 2m + \operatorname{jump}_C(f) + \operatorname{fil}(y_{+_i}).\end{aligned}$$

By an earlier calculation, a similar inequality applies to n_{0i}. Similarly, if we write

$$\eta' = \frac{\partial}{\partial u} + \sum n_i \cdot \frac{\partial}{\partial y_{+_i}} + \sum n_i' \cdot \frac{\partial}{\partial y_i'}$$

we let

$$\psi_0'' = \sum_{i=1}^{r} \tilde{n}_i' \cdot \frac{\partial}{\partial x_i'} \qquad \text{and} \qquad \eta_1' \circ f = \eta' \circ f - \psi_0''(\bar{f}).$$

Then

$$\eta_1' \circ f = \eta' \circ f - \sum_{i=1}^{r} n_i' \cdot \frac{\partial \bar{f}}{\partial x_i'} = \sum_{i=1}^{s'} (\tilde{n}_i - \tilde{n}_i'') \cdot \frac{\partial}{\partial y_{+_i}}$$

with

$$\begin{aligned}\operatorname{fil}(n_i'') &\geqslant \min_j \{\operatorname{fil}(n_j')\} + \operatorname{fil}\left(\frac{\partial \bar{f}}{\partial x_j'}\right) + \operatorname{fil}(y_{+_i}) \\ &\geqslant \operatorname{jump}_C(f) + \operatorname{fil}(y_{+_i})\end{aligned}$$

and similarly for n_i by assumption. We let

$$\xi''' = \frac{\partial}{\partial u} + \psi_0 + \psi_0'' + \sigma^{-1}\zeta + \sigma^{-1} \sum_{i=1}^{s'} \tilde{n}_i^{(1)} \zeta_i$$

with

$$n_i^{(1)} = (n_{0i} - n_{0i}') + (n_i - n_i'').$$

The assumptions (7.3) together with (7.6) and the same arguments used in § 6 allow us to deduce that ξ''' and hence ξ have the desired properties. □

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