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<http://www.numdam.org/item?id=AIF_1990__40_4_965_0>
THE VERSALITY DISCRIMINANT
AND LOCAL TOPOLOGICAL
EQUIVALENCE OF MAPPINGS

by James DAMON (1)

Introduction.

One of the goals of singularity theory going back to the early work of Morse and Whitney and extending through the work of Thom and Mather was to classify germs of mappings between different spaces up to change of coordinates in source and target ($\mathcal{A}$-equivalence). This originally concerned the classification for generic mappings as well as questions involving finite determinacy. However, the presence of moduli (i.e., parameters describing continuous change of smooth or analytic type) yields an uncountable classification unless we consider topological analogues of the above questions. In the case of finitely $\mathcal{A}$-determined germs which are weighted homogeneous, [D1] provided a sufficient condition for a deformation to be topologically trivial allowing one to answer classification questions for topological equivalence of germs of mappings. This was refined in [D2] for non-weighted homogeneous germs $f_0$ which still had finite $\mathcal{A}$-codimension but in a graded sense relative to an algebraic filtration on the tangent space of the infinitesimal deformations of $f_0$. This allowed one to give precise topological determinacy results as well as topological classification results.

Unfortunately, map germs are not finitely determined in general once one leaves the « nice dimensions » [M2]; hence, the above results

(1) Partially supported by a grant from the National Science Foundation.

Key-words : Versality discriminant - Topological triviality - Finite determinacy - Stratified vector fields - Canical neighborhoods.
A.M.S. Classification : 58C27 - 32C40.
are often no longer directly applicable. The alternative is to verify the
Whitney conditions for a deformation of the germ. However, that
involves the checking of Whitney conditions for all pairs of strata,
which is not feasible except for very low dimensions.

Associated with the failure of finite determinacy is a germ of an
analytic set in the target space which we call the versality discriminant.
The *versality discriminant* of a germ \( f_0 \) is the set of points in the target
where \( f_0 \) is not infinitesimally stable, i.e., where \( f_0 \), viewed as an
unfolding of some multi-germ, is not versal. For example, finite
determinacy is equivalent to the versality discriminant being a point (or
empty). Often, the dimension of the versality discriminant can be quite
small even though the dimensions of the source and target spaces are
large.

At points off of the versality discriminant, \( f_0 \) is locally stable and
so small perturbations will not change \( f_0 \) locally near the point. Hence,
we might hope that the knowledge of the behavior of \( f_0 \) near the
versality discriminant will allow us to determine the topological behavior
of \( f_0 \) itself under deformations. The goal of this paper is to prove such
a result.

The general statement will take the form: if an unfolding
\( f: k^{s+q}, 0 \to k^{t+q},0 \) of the germ \( f_0: k^s,0 \to k^t,0 \) is topologically trivial
in a "conical neighborhood" of the versality discriminant of \( f \) then \( f \)
itself is topologically trivial. Here by conical neighborhood we mean a
neighborhood in the punctured target \( k^\setminus\{0\} \).

We will give several versions of this result. As might be expected
in light of the results referred to above, the simplest version of the
theorem (Theorem 1) concerns \( f \) an unfolding of non-negative weight of
a weighted homogeneous germ \( f_0 \). However, the arguments for the proof
in this case are the same that apply to more general situations where
weighted homogeneity can be replaced by conditions stated in terms of
algebraic filtrations (Theorem 2-4). For this reason, we first separately
state Theorem 1 which considers the weighted homogeneous case and
then give the filtered version which states the modifications needed to
deal with the lack of weighted homogeneity.

Also, we shall see that it is not necessary to explicitly find the
versality discriminant to apply the theorems. It is sufficient to apply
the theorems for a germ of a variety which contains the versality
discriminant.
We indicate applications of these results to a variety of examples, including multi-modal singularities, unimodal singularities which are not finitely $\mathcal{A}$-determined, and mappings to $\mathbb{C}^2$. These results have consequences for the versal topological stratification of versal unfoldings as well as the determination of topologically stable map germs for higher modality singularities.

In § 1 we define the versality discriminant and give several examples of its computation. We collect together in § 2 for later use several basic facts about singularity submanifolds and weak stratifications and stratified vector fields. In § 3 we recall certain basic properties of algebraic filtrations which we will need and which follow automatically in the weighted homogeneous case. Then in § 4, we define the local condition which must be satisfied near the versality discriminant and state the topological triviality theorems. We also indicate how to apply these theorems to the earlier examples. In §§ 5 and 6 we prove the theorems by constructing stratified vector fields in § 5 and proving in § 6 that these vector fields are locally integrable, giving the desired topological trivialization. Lastly, in § 7 we give a further refinement of the theorems which basically allows for non-positive weights or filtrations.

1. The versality discriminant.

Let $f: k^{s+q}, 0 \rightarrow k^{s+q}, 0$ be an unfolding of the germ $f_0: k^s, 0 \rightarrow k^s, 0$, so that if $x, y, u$ denote local coordinates for $k^s$, $k^t$, and $k^q$ then $f(x,u) = (f(x,u), u)$ and $f(x,0) = f_0(x)$. Here, if $k = \mathbb{C}$, $f$ is holomorphic and if $k = \mathbb{R}$, then $f$ is real analytic.

We let the algebras of $k$-valued germs (in the appropriate category) on $k^s$, $k^{s+q}$, etc., be denoted by $\mathcal{G}_x$, $\mathcal{G}_{x,u}$, etc. These algebras have maximal ideals of germs vanishing at 0 denoted by $m_x$, $m_{x,u}$, etc. Also a finitely-generated $R$-module generated by $\{\varphi_1, \ldots, \varphi_k\}$ will be denoted by $R(\varphi_1, \ldots, \varphi_k)$ or $R(\varphi)$ if the set of $\varphi_i$ is implicitly understood.

We consider $f_0$ which has finite singularity type. Furthermore, if $k = \mathbb{R}$, we will replace $f_0$ by its complexification, so in either case we have complex germs of finite singularity type, still denoted by $f_0$ and $f$.

There is a neighborhood $U$ of 0 such that $f_0$ has a representative on $U$, again denoted by $f_0$, and a neighborhood $W$ of 0 so that if
\[ \sum(f) \] denotes the singular set of \( f \):

(i) \( f|\sum(f) \cap U \to W \) is proper and finite to one

(ii) \( f^{-1}(0) \cap \sum(f) \cap U = \{0\} \).

We let \( V \subset W \) denote the set of \((y,u) \in W \) such that if \( S = f^{-1}(y,u) \cap \sum(f) \cap U \), then the multi-germ \( f(y,u) : \mathbb{C}^n \to \mathbb{C}^t \), \((y,u)\) fails to be infinitesimally stable. Then, using an argument modeled on that used by Gaffney [Gaf] for giving a geometric characterization of finite \( \mathscr{A} \)-determinacy, we showed [D1, I, II].

**Proposition 1.1.** — *In the above situation*

(i) \( V \) is an analytic subset of \( W \);

(ii) if \( I \) denote that ideal of germs at 0 vanishing on \( V \), then there is a \( k \) so that.

\[
\text{(1.2)} \quad I^k \cdot \theta (\mathcal{F}) \subset \mathscr{G}_{x,u} \left\{ \frac{\partial \mathcal{F}}{\partial x_i} \right\} + C_{x,u} \left\{ \frac{\partial \cdot}{\partial y_j} \right\}
\]

(where \( \theta (\mathcal{F}) = \mathscr{G}_{x,u} \left\{ \frac{\partial \cdot}{\partial y_j} \right\} \), and

(iii) if \( I' \) is another ideal satisfying (1.2), then the germ \( V' = V(I') \) satisfies \( V' \supseteq V \).

The well-known relation between the infinitesimal stability of a germ and its versality when viewed as an unfolding leads us to define.

**Definition 1.3.** — *The versality discriminant of the unfolding \( f \) (respectively \( f_0 \)) is \( V \) (respectively \( V \cap \mathbb{C}^t \)) in the case \( k = \mathbb{C} \) or \( V \cap \mathbb{R}^{1+q} \) (respectively \( V \cap \mathbb{R}^t \)) in the case \( k = \mathbb{R} \).

**Remark 1.4.** — In the theorems, we will allow \( V \) to be an analytic subset of the discriminant which contains the versality discriminant. If \( I \) denotes the ideal of germs at 0, vanishing on \( V \), then (1.2) is still valid. This includes the special case where \( V \) is itself the versality discriminant.

**Examples.**

(1.5) Pham Example [DGai]:

Pham considered the \( \mu \)-constant family

\[
f(x,y) = y^3 + tx^6 + syx^7 + x^9 \quad (4t^3 + 27 \neq 0)
\]
and showed that the versal unfolding

\[ F(x,y,s,t,u,v) = \left( f(x,y) + \sum_{i=0}^{5} u_{i}x^{i}y + \sum_{i=1}^{7} v_{i}x^{i}, s,t,u,v \right) \]

is not topologically a product along the \( t \)-axis near \( t = 0 \). Thus, the topological structure in the versal unfolding depends on the particular values of \( s \) and \( t \). Fixing \( s \) and \( t \) gives an unfolding \( f_{1}: k^{15}, 0 \rightarrow k^{14}, 0 \). In [DGal] the versality discriminant is determined for the case \( s = 0 \) and \( t \) fixed \((\neq 0)\) by considering the family:

\[ y^{3} + t(x-x_{0})^{4}(x+2x_{0})^{2}y + (x-x_{0})^{6}(x+2x_{0})^{3}. \]

Along the curve \( C \) in \( k^{14} \) defined by this deformation, there are \( E_{8} \) and \( D_{4} \) singularities in a fibre at the points \( x = x_{0} \) and \( -2x_{0} \) (and the \( E_{8} \) singularities have a fixed modulus value). However, the dimension of the target space is 14, while the codimensions of \( E_{3} \) and \( E_{4} \) are 10 and 4 respectively. Thus, if the multi-germ in this fibre were multi-transverse, the set of points where it occurred would be isolated and not along a curve. Hence, this curve lies in the versality discriminant for \( f_{1} \). In [DGal] it is proven that this curve is exactly the versality discriminant.

(1.6) Unimodal singularities which are not finitely determined:

In conversations with Terry Wall he pointed out that even in the region around the edge of the nice dimensions, we still have not completely determined the topologically stable map germs because of the presence of unimodal singularities whose negative versal unfolding fails to be finitely determined.

For example, the negative versal unfolding of the germ \( f_{0}: k^{2}, 0 \rightarrow k^{3}, 0 \) defined by \( f_{0}(x,y) = (x^{2}+\varepsilon y^{4}, xy^{3}+ty^{5}, y^{6}) \) is proven in [D1, II], to be finitely \( \mathcal{A} \)-determined as a germ and hence topologically versal (and topologically stable). However, the germ \( f_{1}(x,y) = (x^{2}+\varepsilon y^{4}, xy^{3}+ty^{5}, 0) \) is also unimodal; but as shown in [D1, II], its negative versal unfolding is not finitely \( \mathcal{A} \)-determined as a germ because it fails to be transverse to the \( \mathcal{H} \)-orbits of \( f_{0} \).

Terry Wall suggested that it is possible to modify various inequalities in the proof in [D1, I], for this special case to obtain topological triviality along the \( t \)-axis [Wa].
Alternatively we can determine the versality discriminant and directly apply our results here. If we assign weight 6 to the third coordinate in the target (with $\text{wt}(x,y) = (2, 1)$), then $f_1$ becomes a "bimodal" singularity with weight zero deformation given by

$$f(x,y) = (x^2 + \varepsilon y^4, xy^3 + ty^5, uy^0).$$

For these weights, the proof of prop. 7.2 of [D1, II], shows that the unfolding $F$ of $f_1$, which now includes the parameter $u$ (but fixes $t$), fails only to be transverse to a $\mathcal{H}$-orbit of $f_0$, which is a line parallel to the $u$-axis. Thus, the versality discriminant for this unfolding is exactly this line. We shall then see in § 4 as a consequence of theorem 1 that the versal unfolding of $f_1(x,y) = (x^2 + \varepsilon y^4, xy^3 + ty^5, 0)$ is topologically trivial along the $t$-axis. This same argument will work as well for $(x^2 + \varepsilon y^4, xy^3 + ty^5, 0, 0, \ldots, 0)$ as well as for other finite map germs with similar properties (see [D1, II]).

(1.7) Weighted homogeneous germs $f: \mathbb{C}^n, 0 \to \mathbb{C}^2, 0$

Suppose that $f$ is weighted homogeneous and has finite singularity type. Let $f_1: \mathbb{C}^n, 0 \to \mathbb{C}^2, 0$ be a deformation of $f_0$ of nondecreasing weight. By [D2] or [D3] the germ $f_1^{-1}(0)$ is topologically trivial; however, $f_0$ need not have finite $\mathcal{A}$-codimension and the deformation $f_1$ need not be topologically $\mathcal{A}$-trivial. In fact, in [D3] is given the simple example of a weight zero deformation $f_1(x,y,z) = (xy + tz, x^4 + y^4 + z^5)$ which is not topologically $\mathcal{A}$-trivial. The restriction of $f_1$ as a mapping from the critical set to the discriminant is already not topologically trivial.

Thus, even the simplest weighted homogeneous germs can be non-finitely $\mathcal{A}$-determined. For such a germ $f_0$, its discriminant $D(f_0)$ is a curve in $\mathbb{C}^2$. The versality discriminant is then an analytic subset of
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$D(f_0)$; if $f_0$ is not finitely $\mathcal{A}$-determined then it is a union of some of the branches of $D(f_0)$. We shall identify in § 4 a class of weighted homogeneous germs which enlarges the classe of finitely $\mathcal{A}$-determined ones. We shall see that for this class it is not even necessary to identify the versality discriminant to apply the theorems here. We shall prove that a weight zero deformation of such a germ its topologically trivial if and only if the deformation restricted to the critical set is topologically trivial.

2. Singularity submanifolds and stratifications.

Next, we introduce the notion of singularity submanifolds which will suffice for our purposes and give conditions which ensure that vector fields are stratified relative to a stratification by such submanifolds.

Let $f: k^n, 0 \rightarrow k^p, 0$ denote a germ of finite singularity type. Let $J'_\ell = J'_\ell(n,p,k)$ denote the $r$-multijets of order $\ell$ and $j'_r f: k^n, 0 \rightarrow J'_\ell$, the jet extension of $f$. Also, we let $\mathcal{A}' = \mathcal{A}'(n,p,k)$ denote the $r$ multijet version of the group of left-right equivalence (see [Ml-V] § 1). Finally, we let $\pi: (k^n)^{(r)} \Delta k^n \rightarrow k^n$ denote the projection $\pi(x_1, \ldots, x_r) = x_1$.

**Definition 2.1.** — A singularity submanifold of $f$ is a submanifold of the form $Tc_0(W)$ (respectively $f^0W$) where $W$ is an $\mathcal{A}$-invariant submanifold of $J'_\ell$ and $f$ denotes a representative on a neighborhood $U$ such that $f|\sum(f) \cap U$ is finite to one.

**Remark 2.2.** — In general, even if $W$ is a submanifold, the associated singularity submanifolds will not necessarily be smooth. However, off the versality discriminant this will be true by [Ml-V, Thm 4.1] when $f|\sum f^{-1}(W)$ is an immersion.

(2.3) We recall [D2, I], that a weak stratification of an open subset $U$ of $k^n$ is a decomposition of $U$ into a finite union of smooth submanifolds $\{V_i\}$, the strata, which satisfy a weak form of the axiom of the frontier, namely,

$$C'(V_i) \subset V_i \cup (\cup V_j) \quad \text{with} \quad \dim V_j < \dim V_i.$$

In [D2, I], we solved questions of topological triviality using stratified vector fields on weak stratifications. Our definition of stratified vector field was especially suited for the algebraic criteria used there. It is
possible to use a weaker notion of stratified vector field and still obtain
one which is locally integrable by Wirthmüller [Wi] (or see §2 of
[D2, I]), and the vector fields constructed in [D2, I], can still be used
as they satisfy this weaker condition.

**Definition 2.4.** — A vector field defined on an open subset $U$ of $\mathbb{k}^n$
is stratified if it is continuous and there is a weak stratification $\{V_i\}$ such
that:

(i) $\xi_i|V_i$ is smooth and tangent to $V_i$ for each $V_i$.

(ii) For each $x \in U$, with say $x \in V_i$, there exists a neighborhood $U_x$ of $x$ and a smooth non-negative function $\rho_x : U_x \to \mathbb{R}$ such that $\rho_x^{-1}(0) = V_i \cap U$ and $|\xi_i(\rho_x)| \leq C \cdot \rho_x$ on $U$.

One problem with stratified vector fields occurs when one wants to
patch them together with a partition of unity. The resulting vector field
need no longer be stratified (nor possibly even locally integrable). We
wish to ensure that the resulting vector field is still stratified to ensure
integrability. For stratifications by singularity submanifolds, we show in
the next proposition that the vector fields will be correctly stratified, at
least where they are smooth.

**Proposition 2.5.** — Given a germ $f : \mathbb{k}^n, 0 \to \mathbb{k}^p, 0$ (in the appropriate
category) and weak stratifications of neighborhoods $U'$ and $U$ of 0 in $\mathbb{k}^n$
and $\mathbb{k}^p$ by singularity submanifolds of $f$. Let $\xi$ and $\eta$ be germs of
vector fields such that $\xi(f) = \eta \circ f$. If $y \in U'$ with $f^{-1}(y) \cap \sum(f) =
S = \{x_1, \ldots, x_m\}$ and $\xi$ and $\eta$ are smooth in neighborhoods of $S$ and $y$,
than $\xi$ and $\eta$ are stratified relative to the given weak stratifications in
the neighborhoods of $S$ and $y$.

**Proof.** — Let the neighborhoods of $S$ and $y$ be denoted by $U'$ and $U$
respectively. Given $y' \in U$ with say $y' \in V$, then there are $x_1, \ldots, x_r \in S$
with $(x_1, \ldots, x_r) \in j'_f f^{-1}(W)$. If $\varphi_t$ and $\psi_t$ denote the local flows
generated by $\xi$ and $\eta$ in smaller neighborhoods of $y'$ and $S' = \{x_1, \ldots, x_r\}$.
Then, $(\varphi_t, \psi_t)$ generate an $\mathscr{A}$-equivalence of germs $f : \mathbb{k}^n, S' \to \mathbb{k}^p, y'$
and $f'_t : \mathbb{k}^n, S'_t \to \mathbb{k}^p, y'_t$ with $S'_t = \varphi_t(S')$ and $y'_t = \psi_t(y')$.

Hence, $S'_t \subset j'_f f^{-1}(W)$ and $y_t \in f(j'_f f^{-1}(W)) = V_t$. Hence $\eta$ is tangent
to $V_t$ at $y'_t$, and hence near $y'$. As $V_t$ is smooth, there exist local coordinates near $y'_t, (y_1, \ldots, y_p)$
so that $V_t$ is given by $y_1, \ldots, y_p = 0$. Let $\rho_y = \sum_{i=1}^t |y_i|^2$. Then, as $\eta$ is
tangent to $V_i$ near $y'$, $\eta = \sum_{i=1}^{p} h_i \cdot \frac{\partial}{\partial y_i}$ where $h_i(0) = 0$ for $i = 1, \ldots, \ell$
so $h_i = \sum_{j=1}^{\ell} g_{ij} \cdot y_j$ for $i = 1, \ldots, \ell$.

Lastly,
$$|\eta(\rho_y)| \leq \sum_{i=1}^{\ell} |\tilde{y}_i| \cdot |\eta(y_i)| \leq \sum_{i,j=1}^{\ell} |g_{ij}| \cdot |\tilde{y}_i| \cdot |y_j| \leq C \cdot \rho_y$$
where $C/\varepsilon^2$ is a bound for all $|g_{ij}|$ in a neighborhood of $y'$. A similar argument works for the strata of $U'_i$.

3. Filtration conditions.

In the weighted homogeneous case, a germ being of finite singularity type or an ideal defining the versal discriminant can be stated in terms of certain algebraic conditions which have immediate analogues for corresponding graded algebraic objects. If in place of weighted homogeneity we use more general algebraic filtrations, then the corresponding graded conditions need not necessarily hold. Instead, we must actually require the stronger graded conditions as part of our criteria.

In this section we briefly recall the properties that we require for the filtration on our rings and modules of vector fields. These properties are described in more detail in §4-6 of [D2, I]. Then, the stronger graded conditions are contained in the filtration properties (F1-F3). These properties lead to certain «jump invariants» which we associate to a germ.

We emphasize that for unfoldings of nondecreasing weight of weighted homogeneous germs the conditions (F1-F3) are always satisfied and if the unfolding $f$ is also weighted homogeneous (allowing nonpositive weights for the unfolding parameters) the jump invariants are always zero. Hence, a reader who wishes to understand the results for the weighted homogeneous case may continue on directly to the statement of theorem 1 in the next section.

Filtrations on rings and modules.

Recall that the algebras of $k$-valued germs (in the appropriate category) on $k^4$, $k^{r+q}$, etc., are denoted by $\mathcal{O}_x$, $\mathcal{O}_{x,u}$, etc. with maximal
ideals $m_x, m_{x,u}$, etc. A filtration on the algebra $\mathcal{E}_x(\text{or } \mathcal{E}_y)$ consists of a sequence of finitely generated ideals of finite codimension $I_1 \supset I_2 \supset \ldots$ such that $I_j \cdot I_i \subseteq I_{i+j}$. Such a filtration will be called convex if there is an $\ell$ and $h_1, \ldots, h_r \in I_\ell$ such that if

$$\rho = \sum_{i=1}^{r} |h_i|^2$$

then given $h \in I_m$, there is a $C > 0$ so that

$$|h| \leq C \cdot \rho^{(m/2\ell)} \text{ on a neighborhood of 0.}$$

Such a $\rho$ will be called a control function of filtration $2\ell$. The functions $h_1, \ldots, h_r$ are called a set of vertices for the filtration at level $\ell$.

A filtration can have many sets of vertices of varying filtration levels, e.g. $\{h_1^1, \ldots, h_r^1\}$ is a set of vertices for level $s \cdot \ell$, as well as many different control functions. However, control functions $\rho_1$ and $\rho_2$ of filtrations $2\ell$ and $2m$ are related by inequalities of the form

$$\rho_2 \leq C \cdot \rho_1^{(m/\ell)} \text{ on a neighborhood of 0.}$$

The filtration extends to smooth functions and any smooth function of filtration $2\ell$ which satisfies inequalities of the form (3.1) for all $h \in I_\ell$ is also referred to as a control function and satisfies (3.1a).

The filtration extends to $\mathcal{E}_{x,u}$ by $\{I_j \cdot \mathcal{E}_{x,u}\}$ (this is no longer of finite codimension). In the holomorphic case, the filtration extends to the ring of complex valued smooth function $\mathcal{E}_{x,u}$ by $(I_j + I) \cdot \mathcal{E}_{x,u}$, where $(\cdot)^c$ denotes complex conjugation.

We assume there are filtrations on both $\mathcal{E}_x$ and $\mathcal{E}_y$ so that $f_0^*$ and $f^*$ both preserve filtrations.

Example: Weight filtration.

If we can assign weights $> 0$ to $x$ and $y$ so that $f_0$ is weighted homogeneous, then we can define weight filtrations on $\mathcal{E}_x$ and $\mathcal{E}_y$ with $I_j$ generated by the monomials of weight $\geq j$. Such filtrations are convex and preserved by $f_0^*$. If $f$ deforms $f_0$ by terms of weight $\geq \text{wt}(y_i)$ in the $i$-th coordinate, then $f$ is an unfolding preserving the weight filtration (we will refer to $f$ more precisely as an unfolding of non-negative weight).
Other important examples of convex filtrations are Newton filtrations ([K] or [DGaf]) or filtrations by integral closures of powers of an ideal [T].

Likewise, we also assume that we have filtrations on the modules

\[ \theta_s = \mathcal{C}_x \left\{ \frac{\partial}{\partial x_i} \right\}, \quad \theta_t = \mathcal{C}_y \left\{ \frac{\partial}{\partial y_j} \right\} \quad \text{and} \quad \theta(f) = \mathcal{C}_x \left\{ \frac{\partial}{\partial y_j} \right\}. \]

For example, for \( \theta_s \) this means there is a decreasing sequence of finitely generated finite codimension \( \mathcal{C}_x \)-modules \( M_r \supset M_{r+1} \supset \ldots \) (beginning with some \( r \in \mathbb{Z} \)) and satisfying:

(i) \( I_j \cdot M_t \subseteq M_{t+j} \)

(ii) if \( \zeta \in M_j \) and \( g \in I_i \), then \( \zeta(g) \in I_{t+j} \)

(\( \zeta(g) \) denotes the directional derivative).

(iii) if \( \zeta = \sum h_i \frac{\partial}{\partial x_i} \) then \( \text{fil}(h_i) \geq \text{fil}(\zeta) + \text{fil}(x_i) \).

Also, through § 6 we will assume \( \text{fil}(x_i) > 0 \) for all \( i \); we show how to relax this condition in § 7.

Note: here and in what follows if \( g \in I_i \backslash I_{i+1} \) then \( \text{fil}(g) = i \) and similarly for modules.

An analogous definition holds for \( \theta_t \); while for \( \theta(f) \) we do not require ii).

These filtrations extend to \( \mathcal{C}_{x,u} \left\{ \frac{\partial}{\partial x_i} \right\}, \text{etc.} \), by replacing \( M_j \) by the \( \mathcal{C}_{x,u} \)-submodule generated by the generators of \( M_j \).

If \( f \) preserves filtrations and \( \text{fil} \left( \frac{\partial f}{\partial u_i} \right) \geq \text{fil}(f_0) \) for all \( i \), then \( f \) is said to be an unfolding of non-decreasing filtration.

**Filtration conditions.**

We next describe the filtration conditions which will be required for the main theorems. As mentioned above, these conditions will always be satisfied in the (semi-) weighted homogeneous case.

As \( f_0 \) has finite singularity type, the Jacobian ideal, generated by \( f_0^* m_y \) and the \( t \times t \) minors of \( df_0 \), has finite codimension. If \( i = (i_1, \ldots, i_t) \),
is a $t$-tuple of $\{1, \ldots, s\}$ and $f_0 = (f_{01}, \ldots, f_{0t})$, let $\Delta_t(f_0)$ denote the $t \times t$ minor $\det\left(\frac{\partial f_{0j}}{\partial x_i}\right)$ and $\Delta(f_0)$ denote the ideal generated by such minors.

**Definition 3.2.** — The filtration conditions are the conditions $F1$-$F3$ to follow.

(F1) there are $g_1, \ldots, g_r \in \Delta(f_0)$ and $g'_1, \ldots, g'_{n,r} \in m_x$ of filtration $m$ such that \{\(g_1, \ldots, g_r, f_0^s(g'_1), \ldots, f_0^s(g'_{n,r})\}\} form a set of vertices for the filtration on $g_x$.

Second, by Cramer's rule, we may write

$$\Delta_t(f_0) \cdot \frac{\partial}{\partial y_i} = \zeta_t(f_0) \quad \text{and} \quad \Delta_t(f) \cdot \frac{\partial}{\partial y_i} = \zeta_t(f).$$

Thus, we may write

$$\Delta_t(f_0) \cdot \frac{\partial}{\partial y_i} = \sum k_{i,j,t}(x) \cdot \Delta_t(f_0) \cdot \frac{\partial}{\partial y_i} = \sum k_{i,j,t} \cdot \zeta_t(f_0).$$

Applying Cramer's rule in exactly the same way to $f$ we obtain

$$\Delta_t(f) \cdot \frac{\partial}{\partial y_i} = \sum k_{i,j,t} \cdot \zeta_t(f) (\text{for } \zeta_t(f))$$

where $G_j(x,0) = g_j(x)$. The second condition is that for each $j$

(F2) \(\text{fil}(G_j) \geq \text{fil}(g_j)\).

Third, if $V' = f^{-1}(V) \cap \sum$, and $I = I(V)$ then the third condition becomes

(F3) \(V' = V(\sum + (G_1, \ldots, G_t) \cdot g_x) \quad (\text{or } \cap \mathbb{R}^{\geq q} \text{ if } k = \mathbb{R})\).

For the general case, we must also allow for «jumps» in filtration to occur in (1.2) and (3.3). Consequently, we give a «graded version» of the definition of versality discriminant as follows.

**Definition 3.4.** — $V$ contains the versality discriminant in the graded sense if there is an integer $d$ and $h_1, \ldots, h_a \in I_m'$, (with say $km' = m$) and $V(h_1, \ldots, h_a) = V$, so that for any $\zeta \in \theta(f)$

$$h^k \cdot \zeta = \zeta_t(f) - \eta_i \circ f, \quad 1 \leq i \leq a$$

with $\text{fil}(\zeta_i), \text{fil}(\eta_i) \geq m + \text{fil}(\zeta) - d$. 

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We define \( \text{jump}_V(f) \) (\( V \) for versality) to be the minimum such integer \( d \geq 0 \) for which this is true for all \( i \). Also, in the construction in (3.3), we define \( \text{jump}_C(f) \) (\( C \) for critical set) to be the smallest integer \( d \geq 0 \) such that

\[
\text{fil}(\xi_i, \ell) \geq m + \text{fil}(y_\ell) - d \quad \text{for all} \ i, \ell.
\]

Remark 3.6. — When \( f \) is weighted homogeneous (where the unfolding parameters have nonpositive weights), \( \text{jump}_V(f) = \text{jump}_C(f) = 0 \). This is because the weighted homogeneity and analyticity of the germs allow us to choose weighted homogeneous \( h_i, G_i, \) and \( \xi_i, \eta_i \) so that (3.3) and (3.5) preserve weights (for weighted homogeneous \( \zeta \)). Even if only \( f_0 \) is weighted homogeneous we may still choose \( \xi_0 \) so that \( \text{jump}_C(f) = 0 \).

We conclude this section by deriving one consequence of the filtration conditions. Let \( \sigma = \sum_{i=1}^r |G_i|^2 \) and \( \rho_0 \) be a control function of filtration \( 2m \) for \( \mathscr{C}_y \). For a germ \( h \) on \( k^{I+q}, 0 \) we let \( \tilde{h} \) denote the pull-back of \( h \) by \( f \).

Then, the assumptions together with the following lemma imply that \( \rho_0^{(1)} = \sigma + \tilde{\rho}_0 \) is a control function of filtration \( 2m \) for the induced filtration on \( \mathscr{C}_{x,u} \).

Lemma 3.7. — Let \( \{f_1, \ldots, f_r\} \) denote a set of vertices of filtration \( m \) for \( \mathscr{C}_x \) and let \( \{F_1(x,u), \ldots, F_r(x,u)\} \) be deformations of non-decreasing filtration. Then \( \tilde{\rho}_0 = \sum_{i=1}^r |F_i|^2 \) is a control function of filtration \( 2m \) for the extension of the filtration to \( \mathscr{C}_{x,u} \).

Proof. — By the assumption that \( F_j(x,u) \) is a deformation of \( f_j(x) \) of non-decreasing filtration, we may write

\[
F_j(x,u) = f_j(x) + \sum_{\alpha} h_{j,\alpha}(x,u) \cdot \varphi_\alpha(x)
\]

summed over a finite set of \( N \) generators \( \{\varphi_\alpha(x)\} \) of \( I_m \). Also, \( h_{j,\alpha}(x,0) = 0 \). Let

\[
\rho_0 = \sum_{i=1}^r |f_i|^2 \quad \text{and} \quad \tilde{\rho}_0 = \sum_{i=1}^r |F_i|^2.
\]
Then, by the triangle inequality
\begin{equation}
\tilde{\rho}_0^{(1/2)} \geq \rho_0^{(1/2)} - \sum_{i,a} |h_{j,a}| \cdot |\varphi_a|.
\end{equation}

By the continuity of $h_{j,a}$, on a small enough neighborhood, $||u|| < \delta_1$, $||x|| < \delta_2$.

$$||h_{j,a}(x,u)|| < \varepsilon/Nr.$$  

Because $\rho_0$ is a control function of filtration $2m$, there is a $C > 0$ so that on a neighborhood of $0$

$$|\varphi_a| \leq C \cdot \rho_0^{(m/2m)} = C \cdot \rho_0^{(1/2)}.$$  

Thus, from (3.8)

\begin{equation}
(3.9) \quad \tilde{\rho}_0^{(1/2)} \geq \rho_0^{(1/2)} - Nr \cdot (\varepsilon/Nr) C \cdot \rho_0^{(1/2)} = (1 - \varepsilon C) \cdot \rho_0^{(1/2)}.
\end{equation}

Choosing $\varepsilon$ so that $\varepsilon C < 1$, we obtain from (3.9)

$$\rho_0 \leq C' \cdot \tilde{\rho}_0 \text{ on a neighborhood of 0}.$$  

If $g \in I_m \cdot \mathcal{C}_{x,u}$ then we may write

$$g = \sum_{i=1}^{n} g_i \cdot \psi_i \quad \text{where} \quad g_i \in I_m \text{ and } \psi_i \in \mathcal{C}_{x,u}.$$  

Thus, on a small enough neighborhood of zero

$$|g| \leq \sum_{i=1}^{n} |g_i| \cdot |\psi_i| \leq \sum_{i=1}^{n} C_i \cdot \rho_0^{(m/2m)} C_i \leq C'' \cdot \tilde{\rho}_0^{(m/2m)}$$

where $C'' = \sum C_i \cdot C_i' \cdot C''^{m/2m}$. This gives the desired conclusion.

4. Stratified topological triviality.

Let $f : k^{s+q}, 0 \to k^{t+q}, 0$ be an unfolding of $f_0$. We let $V$ denote an analytic subset of the discriminant of $f$ which contains the versality discriminant of the unfolding $f$. Also, $V' = f^{-1}(V) \cap \Sigma$, where $\Sigma = \ldots$
\( \sum(f) \) denotes the singular set of \( f \). In this section we shall define what we mean by the unfolding \( f \) being stratified topologically trivial in a conical neighborhood of \( V \). Then, we can state the principal theorems. We shall also indicate how the principal results apply to the examples given in \( \S \, 1 \).

By a control function for an analytic subset \( (W, 0) \subset k^n, 0 \) we mean a germ of a smooth, nonnegative function \( \rho : k^n, 0 \to \mathbb{R} \) such that \( \rho^{-1}(0) = W \). There is a relation between control functions for filtrations and control functions for analytic sets: if \( \rho_0 \) denotes the control function (of filtration \( 2m \)) for \( \mathcal{E}_y \), then \( \rho_0 \) is also a control function for \( V(\mathcal{E}_y, \mathcal{E}_x, u) = V_0 = \{0\} \times k^q \). Also, by the comments preceding lemma 3.7, \( \rho_0^{(1)} = \sigma + \tilde{\rho}_0 \) is a control function for \( V' = \{0\} \times k^q \subset k^{q+s} \). Here again \( \sim \) denotes composition with \( f \).

Let \( \hat{\rho} \) be the control function for \( V \) (of filtration \( 2m \)) defined as follows: if \( I(V) = (h_1, \ldots, h_r) \) then pick \( a_i \) so that \( a_i \cdot \text{fil}(h_i) = m \) and let \( \hat{\rho} = \sum |h_i|^{a_i} \). Then, by (F3), we may use \( \hat{\rho}^{(1)} = \hat{\rho} + \sigma \) as a control function of \( V' \). It is still of filtration \( 2m \).

Next, by a conical neighborhood \( U \) of \( V \) is meant a neighborhood of the form

\[
U = \{(y, u) : \hat{\rho}(y, u) < \varepsilon \cdot \rho_0(y, u)\}
\]

and similarly for a conical neighborhood \( U' \) of \( V' \) using instead \( \hat{\rho}^{(1)} \) and \( \rho_0^{(1)} \).

The notion of stratified topological triviality of \( f \) in a conical neighborhood of \( V \) will be defined in terms of stratified vector fields which trivialize \( f \) on conical neighborhoods of \( V \) and \( V' \). This is described via the conditions (V1)-(V2) to follow.

(V1) There are weak stratifications of neighborhoods of 0 in \( k^{q+s} \) and \( k^{q+s} \) whose strata are singularity submanifolds, include the strata \( V_0 \) and \( V'_0 \), and for which the strata in \( k^{q+s} \setminus \sum(f) \) are of the form \( f^{-1}(V_i) \setminus \sum(f) \) for \( V_i \) strata of \( k^{q+s} \).
(V2) There are conical neighborhoods $U$ of $V$ and $U'$ of $V'$ such that for any smaller conical neighborhoods $U_i$ such that $C\ell(U_i) \subset U$ (closure $C\ell$ is taken in the punctured space) and for any $j$ with $1 \leq j \leq q$, there exist vector fields $\xi_j$ and $\eta_j$ defined on $U'$ and $U$ respectively such that:

1. $\xi_j$ and $\eta_j$ are stratified vector fields relative to the restrictions of the weak stratifications to $U'$ and $U$.

2. $\xi_j$ and $\eta_j$ are smooth on $U' \setminus f^{-1}(C\ell(U_i))$ and $U \setminus C\ell(U_i)$.

$$
\eta_j = \frac{\partial}{\partial u_j} + \sum n_{ij} \frac{\partial}{\partial y_i} \quad \text{with } |n_{ij}| \leq C \cdot \rho_0^{a_i},
$$

$$
\xi_j = \frac{\partial}{\partial u_j} + \xi_j' \quad \text{with } ||\xi_j'|| \leq C \cdot \rho_0^{(i)b}
$$

and

$$
|\eta_j(\rho_0)| \leq C \cdot \rho_0 \quad \text{on } U
$$

$$
|\xi_j(\rho_0^{(i)})| \leq C \cdot \rho_0^{(i)} \quad \text{on } U'
$$

where $C, b > 0$ and $a_i = - \text{fil} \left( \frac{\partial}{\partial y_j} \right)$.

$$
(4) \quad \xi_j(f) = \eta_j \circ f.
$$

Remark. — That $\xi_j$ and $\eta_j$ can be made smooth outside of arbitrarily small neighborhoods of $V'$ and $V$ should not be surprising since $V$ contains the versality discriminant so on its complement $\bar{f}(.,u)$ is stable. In fact, one would hope that the smoothness could be guaranteed off $V'$; however, this may be difficult to achieve while simultaneously satisfying the stratification conditions (see e.g. [DGal]).

Definition 4.1. — We say that $f$ is stratified topologically trivial in a conical neighborhood of $V$ if $f$ satisfies the filtration conditions F1-F3 and the conditions V1-V2.

Remark. — The conditions (V1)-(V2) are analogues of the conditions for stratified vector fields in conical neighborhoods in [D2] § 2, so that there exist methods for constructing such vector fields in conical neighborhoods.

The first version of the main theorem is the following.
Theorem 1. — Let \( f \) be an unfolding of non-negative weight of \( f_0 \) so that \( f \) itself is weighted homogeneous (with weights \( \leq 0 \) for the unfolding variables). If \( f \) is stratified topologically trivial in a conical neighborhood of the versality discriminant then \( f \) is a topologically trivial unfolding of \( f_0 \).

More generally, we allow filtrations and varieties \( V \) containing the versality discriminant in the graded sense. We extend definition 4.1:

**Definition 4.1a.** — We say that \( f \) is stratified topologically trivial in a conical neighborhood of the versality discriminant in the graded sense if there is a variety \( V \) which contains the versality discriminant in the graded sense so that \( f \) is stratified topologically trivial in a conical neighborhood of \( V \) and satisfies the stronger condition in \( V_2 \).

\[
(4.2) \quad \text{fil}(n_{ij}) \geq \text{jump}_c(f) + \text{fil}(y_i) \quad \text{all } i, j.
\]

There is the following generalization of Theorem 1.

**Theorem 2.** — Let \( f \) be an unfolding of non-decreasing filtration of \( f_0 \) so that \( f \) is stratified topologically trivial in a conical neighborhood of the versality discriminant in the graded sense. If

\[
(4.3) \quad \text{fil} \left( \frac{\partial f}{\partial u_i} \right) \geq \text{jump}_V(f) + \text{jump}_c(f), \quad 1 \leq i \leq q
\]

then \( f \) is a topologically trivial unfolding of \( f_0 \).

It is immediate from remark 3.6 that Theorem 1 follows from Theorem 2. The proof of Theorem 2 will be given in §§ 5 and 6. Before finishing this section, we indicate how Theorem 1 may be applied to the examples from § 1.

**Examples reconsidered.** — The examples that we consider here are weight zero deformations \( f(x,u) \) of a weighted homogeneous germ \( f_0 \) such that the versality discriminant \( V \) of \( f \) is a topologically trivial family of curves. We may choose sections \( Y_i \) to each branch \( B_i \) of the curve for \( f_0 \), and consider the pull back \( X_i \) via \( \tilde{f} \), with \( S = \tilde{f}^{-1}(y) \cap V' \) (recall \( V' = f^{-1}(V) \cap \sum(f) \)). If \( y \in Y_i \cap V \), let \( Y_y \) denote the isotropy group of \( C^* \) at \( y \). The set \( S \) is invariant under \( G_y \). We can also choose the sections invariant under \( G_y \).
If $\xi_t$ and $\eta_t$ are stratified vector fields (for a weak stratification by singularity submanifolds) which project to $\frac{\partial}{\partial u_t}$ and trivialize the restriction $f|X_t, S \to Y_t, y$ in the direction $u_t$, then we may average them over the finite group $G_y$ and they still have the same properties (since $\text{wt}(u_t) = 0$). We may then use the $\mathbb{C}^*$-action to extend the collection of such vector fields for each branch to conical neighborhoods of $V$ and $V'$ (invariance under $G_y$ ensures that this is well-defined). Provided $\eta_t$ and $\xi_t$ can be chosen to be smooth off an arbitrarily small neighborhood of $y$ and its inverse image, then the $\mathbb{C}^*$-extensions of $\eta_t$ and $\xi_t$ satisfy V1 and 1) and 2), and 4) of V2. Condition 3) of V2 follows from the $\mathbb{C}^*$-invariance of the extensions. The reader can either verify these directly or see for example [DGal] where the details are verified.

Pham example. — In example (1.5) the failure of topological triviality for the Pham example was discussed. The topological triviality of $F$ in the $s$-direction does follow by results of Wirthmüller [Wi]. Thus, it is enough to consider topological triviality along the $t$-direction when $s = 0$. In [DGal], it is then proven that the section to the single branch $C$ of the versality discriminant yields a multi-germ which is stratified topologically trivial in the above sense. Hence, the above line of reasoning together with theorem 1 can be applied to conclude topological triviality along the $t$-axis.

Non-finitely-determined unimodal singularities.

As discussed in § 1, the unfolding $F$ of $f_t(x, y) = (x^2 + t y^4, x y^3 + t y^5, 0)$ which is versal except for the parameter $t$ is not finitely $\mathcal{A}$-determined when viewed as a germ. The versality discriminant of the germ $F$
consists of a line parallel to the \( \mu \)-axis, with \( \mu \) the parameter for the unfolding term \((0,0,\mu^6)\). Hence, the versality discriminant for the unfolding of \( F \) by the parameter \( t \) is the \((\mu,t)\)-plane.

A section to \( V \) in the direction of the other parameters and variables gives the negative versal unfolding to \( f_0(x,y) = (x^2 + \varepsilon y^3, xy^3 + t_0 y^5, u_0 y^6) \). By proposition 8.2 of [Dl, II] and theorem 1 of [Dl, I] this germ is stratified topologically trivial; hence, the above line of reasoning applies, so that theorem 1 allows us to conclude that the versal unfolding of \( f_i \) is topologically a product along the \( t \)-axis. An analogous argument works for the other finite map germs with similar properties (see [D1, II]). Then, the proof of theorem 4 in [D1, I] also shows that the unfolding versal except for the term \( t \) is, in fact, topologically stable.

**Weighted homogeneous germs**

Suppose \( f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^2, 0 \) is weighted homogeneous and of finite singularity type. By the \( C^* \)-action, the germ of \( f_0 \) at a point \( x \) of \( \sum(f_0) \) is equivalent to a product of a hypersurface germ \( f_i : \mathbb{C}^{n-1}, 0 \to \mathbb{C}^1, 0 \) with the identity map on \( \mathbb{C} \). By Lê-Ramanujam [LeR], every equisingular deformation of \( f_i \) is topologically trivial \((n \neq 4)\). However, to see the consequences for the germ \( f_0 \), we consider a special class of hypersurface germs \( f_i \) with the property that every equisingular deformation \( F(x,u) : \mathbb{C}^{n-1+q}, 0 \to \mathbb{C}^1, 0 \) of \( f_i \) is stratified topologically trivial via stratified vector fields \( \xi_i \left( \text{which project to } \frac{\partial}{\partial u_i} \text{, satisfy } \xi_i (f) = 0 \text{ and} \right) \) which are smooth off of \( \{0\} \times \mathbb{C}^q \). This special class includes the simple germs, germs defining plane curve singularities \([T]\), and weighted homogeneous germs (see [DGaf] for deformations of non-decreasing weight which are the equisingular deformations by \([V]\)). Quite possibly all isolated hypersurface singularities belong to this special class.
Then, consider a class of map germs \( f: \mathbb{C}^n, 0 \to \mathbb{C}^2, 0 \) satisfying condition (4.5).

(4.5): For all \( x \in \sum(f_0) \), the associated hypersurface germ \( f: \mathbb{C}^{n-1}, x \to \mathbb{C}^1, f(x) \) belongs to the special class above.

For such germs \( f \), it is not necessary to explicitly determine the versality discriminant.

**Proposition 4.6. —** Suppose that \( f_0: \mathbb{C}^n, 0 \to \mathbb{C}^2, 0 \) is weighted homogeneous, of finite singularity type, and satisfies (4.5). If \( f = (f_t, t) \) is a weight zero unfolding of \( f_0 \), then \( f \) is topologically (\( \mathcal{A} \)-) trivial if and only if the restriction \( f|C(f): C(f) \to D(f) \) is topologically trivial.

The example considered earlier, \( f_0(x, y, z) = (xy + rz, x^4 + y^4 + z^2) \), is a unimodal singularity which only deforms to simple germs; thus, it satisfies (4.5). It is shown in [D3] that \( f|C(f): C(f) \to D(f) \) is topologically trivial if \( t \neq 0, \pm 1/2 \). Thus, proposition 4.6 assures topological triviality off of the points \( t = 0, \pm 1/2 \) (where it does fail).

**Proof (of proposition 4.6). —** If \( f \) is topologically trivial then clearly the restriction to \( C(f) \) is also. Conversely, suppose the restriction \( f|C(f) \) is topologically trivial. By the above discussion, it is sufficient to prove the stratified topological triviality for \( f|X_i, S \to Y_i, y \), where \( Y_i \simeq \mathbb{C} \) is a section through a branch of \( V \) at \( y \). By the topological triviality of \( C(f) \), if \( x_i \in S \), then the germ \( f_i: X_i, x_i \to Y_i, y \) has only one singular point in \( Z_i = C(f) \cap X_i \cap (U_i \times \{u_0\}) \), where \( U_i \) is a sufficiently small neighborhood of \( x_i \) and \( u_0 \in B_{\varepsilon} \) a ball about \( 0 \) of sufficiently small radius \( \varepsilon \). Thus \( f_i|X_i, x_i \to Y_i, y \) is a \( \mu \)-constant deformation and hence is equisingular. Thus, by (4.5) there are \( \zeta_i \) projecting to \( \frac{\partial}{\partial u_i} \), smooth off of \( \{0\} \times \mathbb{C}^n \) and satisfying \( \zeta_i(f) = 0 \). Letting \( \eta_i = \frac{\partial}{\partial u_i} \) and \( \xi_i \) be the disjoint union of the \( \zeta_i \) for each \( x_i \in S \), we obtain from \( (\xi_i, \eta_i) \) the stratified topological trivialization. Hence, by our earlier argument, \( f \) is topologically trivial.

Lastly, consider the general case of weight zero unfoldings \( f(x, u) \) of a weighted homogeneous germ \( f_0: k^n, 0 \to k^p, 0 \). Suppose that the versality discriminant is contained in a weighted homogeneous \( V \subset D(f) \) which is topologically trivial. We may choose sections \( Y \) to each \( k^* \)-orbit \( \subset V \), and consider the pull back \( X \) via \( f \), with \( S = f^{-1}(y) \cap V' \). By arguments analogous to the preceding, we obtain the following.
PROPOSITION 4.7. — Suppose that for all sections $Y, f: X, S \to Y$, $y$ is stratified topologically trivial preserving $V$ and $V'$. Then, $f$ is a topologically $(\mathcal{A}_f)$ trivial unfolding of $f_0$.

5. Construction of the stratified vector fields.

For the unfolding $f$ of $f_0$, we shall find vector fields $\xi_j$ and $\eta_j$, defined on neighborhoods of 0 in $k^{s+q}$ and $k^{t+q}$ respectively and projecting to $\frac{\partial}{\partial u_j}$, which satisfy $\xi_j(f) = \eta_j \circ f$, $1 \leq j \leq q$. We shall show in §6 that these vector fields are stratified (in the sense of §2).

Then, by the local integrability of such vector fields together with standard arguments as in [D1, I], it follows that $f$ is a topologically trivial unfolding of $f_0$. Thus, it is only necessary to carry out the construction for one $\frac{\partial}{\partial u_j}$, which we will henceforth denote by $\frac{\partial}{\partial u}$.

We will construct the vector fields $\xi$ and $\eta$ from three pieces of vector fields which will be patched together using partitions of unity.

First, there are vector fields $\xi'$ and $\eta'$ defined on conical neighborhoods $U'$ of $V'$ and $U$ of $V$, respectively. These are defined by assumption. Secondly, we shall use (3.5) to construct vector fields $\xi''$ and $\eta''$ defined off of $V$, respectively $f^{-1}(V)$. Thirdly, we shall use (3.3) to construct a vector field $\xi'''$ defined off of the set $\Sigma'$ where $\sigma = 0$ (this set is essentially $\Sigma(f)$).
Using partitions of unity we will patch together these pieces and extend them to be \( \frac{\partial}{\partial u} \) on \( V_0 \) and \( V'_0 \) to give what will be the desired \( \xi \) and \( \eta \).

**Construction of \( \eta'' \) and \( \xi'' \).**

If we multiply (3.5) with \( \zeta = \frac{\partial f}{\partial u} \) by \( - (h_i^*)^k \), where \( h_i^* \) denotes the complex conjugate of \( h_i \), and we sum over \( i \) then we obtain

\[
(5.1) \quad - \bar{\rho} \cdot \frac{\partial f}{\partial u} = \xi_0(f) - \eta_0 \circ f.
\]

If \( k = \mathbb{R} \), then the left hand side of (5.1) is real, so we replace \( \xi_0 \) and \( \eta_0 \) by the real parts of these vector fields and the result still holds.

Define

\[
\eta'' = \frac{\partial}{\partial u} + \bar{\rho}^{-1} \cdot \eta_0 \quad \text{and} \quad \xi'' = \frac{\partial}{\partial u} + \bar{\rho}^{-1} \cdot \xi_0.
\]

To define \( \xi'' \) will take considerably more work. To begin, we multiply (3.3) by \( - G_j^* \frac{\partial f}{\partial u} \) (where \( f \) denotes the \( \ell \)-th coordinate function of \( f \) and \( G_j^* \) denotes the complex conjugate of \( G_j \) if \( k = \mathbb{C} \) or \( G_j \) if \( k = \mathbb{R} \)). Summing over \( j \) and \( \ell \), we obtain

\[
(5.2) \quad - \sigma_j \frac{\partial f}{\partial u} = \zeta'(f).
\]

Similarly, by multiplying (3.3) by \( - G_j^* \) and summing over \( j \) we obtain

\[
(5.3) \quad - \sigma \frac{\partial}{\partial y} = \zeta' \circ f.
\]

Before we construct \( \xi'' \) from \( \zeta' \) and \( \zeta' \), it will first be necessary to completely define \( \eta \). Furthermore, to do this we must single out certain conical neighborhoods with special properties via the following lemmas.

**Lemma 5.4.** - If \( U^{(1)} \) is a conical neighborhood of \( V \) there is a conical neighborhood \( U^{(1)}' \) of \( V' \) such that \( f(U^{(1)}) \subset U^{(1)} \).

**Proof.** - For the proof recall: \( \rho_0 \) and \( \rho_0^{(1)} = \bar{\rho}_0 + \sigma \) are control functions for \( V_0 \) and \( V'_0 \) (respectively), while \( \bar{\rho} \) and \( \bar{\rho}^{(1)} = \sigma + \bar{\rho} \) are control functions for \( V \) and \( V' \).
Suppose $U^{(1)}$ is defined by

$$U^{(1)} = \{(y,u) : \hat{p}(y,u) < \varepsilon p_0(y,u)\}.$$ 

Consider

$$U^{(1)'} = \{(x,u) : \hat{p}^{(1)}(x,u) < \varepsilon' p_0^{(1)}(x,u)\}$$

with $\varepsilon'$ to be determined. Since $\hat{p}^{(1)} = \sigma + \hat{p}$, if $x' = (x,u) \in U^{(1)'}$, then

$$\hat{p}(x') \leq \hat{p}^{(1)}(x') < \varepsilon' p_0^{(1)}(x').$$

Let $W$ denote a conical neighborhood of $f^{-1}(V_0)$ defined by

$$W = \{x' : \hat{p}_0(x') < \varepsilon'' p_0^{(1)}(x')\}.$$ 

Consider $x' \in U^{(1)'} \cap W$. We have

$$p_0^{(1)}(x') = (\hat{p}_0 + \sigma)(x') \leq (\hat{p}_0 + \hat{p}^{(1)})(x') < (\varepsilon' + \varepsilon'') p_0^{(1)}(x')$$

or $U^{(1)'} \cap W = \emptyset$ if $\varepsilon' + \varepsilon'' < 1$.

Thus, choosing $\varepsilon'$, $\varepsilon''$ this small we have for $x' \in U^{(1)'}$ (and $x' \notin W$)

$$\hat{p}(x') < \varepsilon' p_0^{(1)}(x') \leq \varepsilon' \cdot \varepsilon''^{-1} \hat{p}_0(x').$$

Further choosing $\varepsilon'$, $\varepsilon''$ so that $\varepsilon' \cdot \varepsilon''^{-1} < \varepsilon$ we conclude

$$f(U^{(1)'}) \subset U^{(1)}.$$ 

Next, we let $\sum' = \{(x,u) : \sigma(x,u) = 0\}$. Although $\sum'$ may be slightly larger than the critical set $\sum(f)$ it is still true that $\sum' \cap f^{-1}(V) = V'$ by (F3).

**Lemma 5.5.** Given a conical neighborhood $U'$ of $V'$, there is a conical neighborhood $U''$ of $\sum'$ such that $U'' \cap f^{-1}(V) \subset U'$ and $U'' \cap f^{-1}(V_0) = \emptyset$. Moreover, there is a conical neighborhood $U^{(1)}$ of $V$ so that $f(U'' \setminus U') \cap U^{(1)} = \emptyset$.

**Proof.** We let $x' = (x,u)$, then

$$U' = \{x' : \hat{p}^{(1)}(x') < \varepsilon \cdot p_0^{(1)}(x')\}.$$ 

Pick $0 < \varepsilon' < \varepsilon$ and let

$$U'' = \{x' : p(x') < \varepsilon' \cdot p_0^{(1)}(x')\}.$$
Then, if \( x' \in U'' \cap f^{-1}(V) \) then \( \tilde{\rho}(x') = \tilde{\rho}(f(x')) = 0 \); thus,

\[
\tilde{\rho}^{(1)}(x') = \tilde{\rho}(x') + \sigma(x') < \varepsilon' \cdot \rho_0^{(1)}(x'),
\]

and so \( x' \in U' \). Furthermore, by repeating the argument in lemma 5.4 we can assure that by shrinking \( U'' \) if necessary that \( U'' \cap W = \emptyset \) for a sufficiently small conical neighborhood \( W \) of \( f^{-1}(V_0) \). In particular, \( U'' \cap f^{-1}(V_0) = \emptyset \).

For the second claim, with \( 0 < \varepsilon' < \varepsilon \) as in the first part, let \( \varepsilon'' = \varepsilon - \varepsilon' > 0 \). Also for \( y' = (y,u) \), define

\[
U^{(1)} = \{ y' : \tilde{\rho}(y') < \varepsilon'' \cdot \rho_0(y') \}.
\]

If \( x' \in U'' \setminus U' \), then

\[
\tilde{\rho}(x') + \sigma(x') \geq \varepsilon' \cdot \rho_0^{(1)}(x') \quad \text{and} \quad \sigma(x') < \varepsilon' \cdot \rho_0^{(1)}(x').
\]

Hence, as \( \rho_0^{(1)} = \tilde{\rho}_0 + \sigma \)

\begin{equation}
(5.6) \quad \tilde{\rho}(x') \geq (\varepsilon - \varepsilon')\tilde{\rho}_0(x') = \varepsilon'' \tilde{\rho}_0(x').
\end{equation}

If \( y' = f(x') \), (5.6) implies \( y' \notin U^{(1)} \). \( \square \)
Completing the construction of \( \eta \).

By lemma 5.4, we may assume \( f(U') \subset U \). Then, by lemma 5.5, there are conical neighborhoods \( U^{(1)} \subset U, U^{(1)'} \subset U' \), and \( U'' \) of \( \Sigma' \) such that

\[
U'' \cap f^{-1}(V) \subset U^{(1)'}, \quad U'' \cap f^{-1}(V_0) = \emptyset,
\]

and \( f(U'' \setminus U^{(1)'}) \cap U^{(1)} = \emptyset \).

Let \( U^{(2)} \) and \( U^{(3)} \) be conical neighborhoods of \( V \) with \( \text{Cl}(U^{(3)}) \subset U^{(2)} \subset \text{Cl}(U^{(2)}) \subset U^{(1)} \) (\( \text{Cl} \) denotes closure in \( k^{t+q} \setminus V_0 \)). Then, by (V2), there are stratified vector fields \( \eta' \) and \( \xi' \) on \( U \) and \( U' \), which are smooth off of \( U^{(3)} \) and \( f^{-1}(U^{(3)}) \). There is a partition of unity \( \{ \chi_1, \chi_2 \} \) on \( k^{t+q} \setminus V_0 \) subordinate to \( \{ U^{(1)}, (k^{t+q} \setminus V_0) \setminus \text{Cl}(U^{(2)}) \} \). We define

\[
\eta = \begin{cases} 
\chi_1 \eta' + \chi_2 \eta'' & \text{off of } V_0 \\
\frac{\partial}{\partial u} & \text{on } V_0
\end{cases}
\]

The construction of \( \xi'' \).

We may write

\[
\eta = \frac{\partial}{\partial u} + \sum_{i=1}^t n_i \frac{\partial}{\partial y_i}.
\]

Then, we define

\[
(5.7) \quad \xi'' = \frac{\partial}{\partial u} + \sigma^{-1} \xi' + \sum_{i=1}^t \sigma^{-1} \eta_i \cdot \xi_i.
\]

Then, \( \xi'' \) is defined off of \( \Sigma' \).

Completing the construction of \( \xi \).

Next, for \( \xi \), we first define \( \xi \) on \( U'' \). Let

\[
\xi_1 = \tilde{\chi}_1 \xi' + \tilde{\chi}_2 \xi'' \quad \text{on } U''.
\]

To see that \( \xi_1 \) is defined on \( U'' \), we first observe that \( \tilde{\chi}_1 \) and \( \tilde{\chi}_2 \) are defined and smooth on \( U'' \) since \( U'' \cap f^{-1}(V_0) = \emptyset \). Then, \( \tilde{\chi}_2 \xi'' \) is defined and smooth on \( U'' \) provided that \( \tilde{\chi}_2 = 0 \) on an open neighborhood of \( f^{-1}(V) \cap U'' \). However, by assumption \( f^{-1}(V) \cap U'' \subset U^{(1)'} \) and \( f(U^{(1)'}) \subset U \). Since \( \chi_2 = 0 \) on an open neighborhood of \( V \), \( \tilde{\chi}_2 = 0 \) on an open neighborhood of \( f^{-1}(V) \) in \( U^{(1)'} \), which gives an open neighborhood of \( f^{-1}(V) \cap U'' \) in \( U'' \).
Secondly, \( \tilde{\chi}_1 \xi' \) is defined on \( U'' \) (and smooth on \( U'' \setminus \mathcal{U}^{(3)} \)) provided \( \tilde{\chi}_1 \equiv 0 \) on an open neighborhood of \( U'' \setminus \mathcal{U}^{(3)} \). However, \( \chi_1 \equiv 0 \) on a neighborhood of \( (k^{t+q} \setminus V'_0) \setminus U^{(1)} \) and \( f(U'' \setminus U^{(3)}) \subset (k^{t+q} \setminus V'_0) \setminus U^{(1)} \); hence, \( \tilde{\chi}_1 \equiv 0 \) on a neighborhood of \( U'' \setminus U^{(1)} \). Thus, \( \xi \) is defined.

Let \( U''' \) be a conical neighborhood of \( \Sigma' \) with \( C^\infty(U''' \setminus \mathcal{U}) \subset U'' \) (in \( k^{t+q} \setminus V'_0 \)). Let \( \{ \chi_3, \chi_4 \} \) be a partition of unity subordinate to \( \{ U'' \setminus (k^{t+q} \setminus V'_0), C^\infty(U''' \setminus \mathcal{U}) \} \). Then define

\[
\xi = \left\{ \begin{array}{ll}
\frac{\partial}{\partial u} + \chi_3 \xi'' & \text{off of } V'_0 \\
0 & \text{on } V'_0
\end{array} \right.
\]

We claim that \( \eta \) and \( \xi \) are our desired stratified vector fields. We will prove this claim in the next section.

We complete this section by showing that \( \xi(f) = \eta \circ f \) on an open dense subset of \( k^{t+q} \). The continuity of \( \eta \) and \( \xi \) to be established in the next section will then imply this equation holds on all of \( k^{t+q} \) (in a neighborhood of 0).

Off of \( V'_0 \),

\[
(5.8) \quad \xi(f) = \chi_3 \xi_1(f) + \chi_4 \xi''(f).
\]

Furthermore,

\[
\xi_1(f) = \tilde{\chi}_1 \xi'(f) + \tilde{\chi}_2 \xi''(f)
\]

\[
= \tilde{\chi}_1 (\eta' \circ f) + \tilde{\chi}_2 (\eta'' \circ f)
\]

\[
= (\tilde{\chi}_1 \cdot \eta' + \tilde{\chi}_2 \cdot \eta'') \circ f
\]

\[
= \eta \circ f \quad \text{off of } f^{-1}(V'_0).
\]

Next, off of \( \Sigma' \),

\[
\xi''(f) = \frac{\partial}{\partial u} + \frac{\partial f}{\partial u} + \sigma^{-1} \xi'(f) + \sum_{i=1}^t \sigma^{-1} \tilde{n}_i \cdot \zeta_i(f)
\]

\[
= \frac{\partial}{\partial u} + 0 + \sum_{i=1}^t \tilde{n}_i \cdot \frac{\partial}{\partial y_i}
\]

by (5.2) and (5.3); or

\[
\xi''(f) = \eta \circ f \quad \text{off of } \Sigma'.
\]
Thus, from (5.8),
\[ \xi(f) = \chi_3 \eta \circ f + \chi_2 \eta \circ f \]
\[ = \eta \circ f \quad \text{off of } f^{-1}(V_0) \cup \Sigma. \]

Thus, it remains to prove \( \xi \) and \( \eta \) are stratified.

6. Verifying that the vector fields are stratified.

We claim that the vector fields \( \xi \) and \( \eta \) defined in the previous section are stratified relative to the weak stratifications given by (V1). We shall establish the stratification conditions in four steps:

- a) \( \eta \) satisfies the condition for being a stratified vector field except for \( \eta \) being continuous and satisfying the control conditions at points of \( V_0 \);
  - a') \( \eta \) is continuous and satisfies the control conditions at points of \( V_0 \);

- b) given a) and a'), \( \xi \) satisfies the condition for being a stratified vector field except for \( \xi \) being continuous and satisfying the control conditions at points of \( V_0 \); also, \( \xi(f) = \eta \circ f \) in a neighborhood of 0;
  - b') \( \xi \) is continuous and satisfies the control conditions at points of \( V_0 \).

If all of a), a'), b) and b') are satisfied then \( \eta \) and \( \xi \) are stratified vector fields, which by b) yield a topological trivialization of \( f \). This would prove theorem 2 (and hence theorem 1). We shall see that a) and b) follow easily from the construction of \( \eta \) and \( \xi \) using (V2) and proposition 2.5. Then a') and b') are more subtle requiring certain estimates based on the filtration properties of the germs.

In verifying the stratification conditions in a) and b), we shall continually make use of the simple observation that to verify that a vector field \( \xi \) defined on an open set \( W \) is stratified (relative to a stratification \( \mathcal{V} \)) it is sufficient to verify that \( \xi \) is stratified (for \( \mathcal{V} \)) on each open set \( W_i \) in a covering \( \{W_i\} \) of \( W \).

**Verification of a) :**

For \( \eta \), we know that \( \eta|_{U^{(2)}} = \chi_1 \eta' \). Let \( \check{V} \) denote the union of the positive codimension strata in \( C'(U^{(3)}) \). Hence, since \( \chi_1 \) is smooth off
of $V_0$ and $\eta'$ is stratified and is smooth off of $\tilde{V}$, so is $\eta|U^{(2)} = \chi_1 \eta'$. Secondly, off of $C\ell(U^{(3)})$, $\chi_2 \eta''$ is smooth, as is $\chi_1 \eta'$. Hence, $\eta$ is smooth off of $C\ell(U^{(3)})$ (in $k^{s+q}\backslash V_0$). Suppose we can show that $\xi$ is smooth off of $f^{-1}(C\ell(U^{(3)}))$. Since $\xi(f) = \eta \circ f$ wherever they are continuous (by the previous section), it follows by proposition 2.5 that $\eta$ is stratified relative to a stratification by singularity submanifolds off of $C\ell(U^{(3)})$. Then, it would follow that $\eta$ is stratified on $U^{(2)} \cup (k^{s+q}\backslash C\ell(U^{(3)})) = k^{s+q}\backslash V_0$. Since $\eta|V_0 = \frac{\partial}{\partial u}$, which is smooth and tangent to $V_0$, to show that $\eta$ is stratified it only remains to show $a')$ that $\eta$ is continuous at points of $V_0$ and satisfies the local control condition for $V_0$.

**Smoothness of $\xi$ off $f^{-1}(\tilde{V})$.**

We actually show something stronger than used in $a)$, namely, that $\xi$ is smooth off $f^{-1}(\tilde{V})$. We first begin with $\xi_1$ on $U''$. By construction, $\tilde{\chi}_1 \xi'$ is smooth off $f^{-1}(\tilde{V}) \cap U''$ and $\tilde{\chi}_2 \xi''$ is smooth on $U''$; hence, $\xi_1$ is smooth on $U''$ off $f^{-1}(\tilde{V}) \cap U''$. As $\eta$ is smooth off $\tilde{V}$, it follows from the definition of $\xi''$ in (5.7) that it is smooth off of $f^{-1}(\tilde{V}) \cup \sum\xi$. Thus, by (5.8) $\xi$ is smooth off of $f^{-1}(\tilde{V})$.

**Verification of $b)$:**

The stratification assertions in $b)$ state that $\xi$ is stratified on $k^{s+q}\backslash V_0$. By the smoothness of $\xi$ off $f^{-1}(\tilde{V})$ we know that $\xi$ is stratified on $W_1 = k^{s+q}\backslash (V_0 \cup f^{-1}(\tilde{V}))$. By lemma 5.5, $f^{-1}(\tilde{V})$ is covered in $k^{s+q}\backslash V_0$ by the open sets

$$W_2 = f^{-1}(U^{(2)}) \cap U^{(1)'}, U'' \quad \text{and} \quad W_3 = k^{s+q}\backslash \sum\xi'. $$

**Second claim :** $\xi_1$ is stratified on $U''$.

$\xi'$ is stratified on $U'$ and $\tilde{\chi}_1$ is smooth on $U''$, so $\tilde{\chi}_1 \xi'$ is stratified on $U'' \cap U''$. By construction, $\tilde{\chi}_1 = 0$ on a neighborhood of $U'' \backslash U'$; thus, $\tilde{\chi}_1 \xi'$ is stratified on $U''$. Also, $\xi''$ is smooth on $k^{s+q}\backslash f^{-1}(V)$. Since $\tilde{\chi}_2 = 0$ on $f^{-1}(U^{(2)}) \supset f^{-1}(V)$, $\tilde{\chi}_2 \xi''$ is smooth on $U''$. Thus, $\xi_1$ is stratified on $U''$.

**Third claim :** $\xi$ is stratified on $W_2$.

By claim 2, $\xi_1$ is stratified on $U'' \supset W_2$. Since $\chi_3$ is smooth on $W_2$ and $\chi_4 = 0$ on $U'' \supset W_2$, $\xi|W_2 = \chi_3 \xi_1$ is also stratified on $W_2$. 


Fourth claim: $\xi$ is stratified on $W_3$.

By the second claim, $\xi_1$ is stratified on $U''$. Since $\chi_3 = 0$ off $U''$ (in $k^{*+q}\setminus V'_0$), we conclude that $\chi_3\xi_1$ is stratified on $k^{*+q}\setminus V'_0$. Thus, to conclude that $\xi$ is stratified on $W_3$, it is sufficient show that $\xi''$ is stratified on $k^{*+q}\setminus \Sigma'$. Since $\chi_3$ is smooth on $k^{*+q}\setminus V'_0$, so will $\chi_3\xi''$ be stratified; and hence, also $\xi$.

Provided we know from $a')$ that $\eta$ is continuous, it follows from the definition of $\xi''$ in (5.7) that it is continuous off of $\Sigma'$ (hence, we know that $\xi$ is continuous off $V'_0$). It remains to show that $\xi''$ satisfies the stratification conditions on the strata in $f^{-1}(V)\setminus \Sigma'$.

From $\xi''(f) = \eta \circ f$ off $\Sigma'$ it follows that for $x'$ in a stratum $f^{-1}(V_i)\setminus \Sigma'$,

$$df_x(\xi''(x')) = \eta(f(x'))$$

is tangent to $V_i$.

Hence, $\xi''$ is tangent to $f^{-1}(V_i)\setminus \Sigma'$ at $x'$.

In the definition of $\xi''$ in (5.7), $n_j|V_i$ are smooth hence so are their pullbacks to $f^{-1}(V_i)\setminus \Sigma'$. Thus, $\xi''$ restricted to $f^{-1}(V_i)\setminus \Sigma'$ is smooth and tangent.

Lastly, from the assumption of $a')$ we see that the local control conditions are satisfied for $\xi''$ on strata in $f^{-1}(V)\setminus \Sigma'$. Suppose $\eta$ is shown to satisfy the local control conditions for $y \in V_i$ using the local control function $p_y$. We use the local control function $\tilde{p}_y = p_y \circ f$ for $x \in f^{-1}(V_i)\setminus \Sigma'$, with $y = f(x)$. Then,

$$|\xi(\tilde{p}_y)| = |\xi(f)(p_y)| = |(\eta \circ f)(p_y)|$$

$$= |\eta(p_y)| \circ f \leq (C \cdot p_y) \circ f$$

$$\leq C \cdot \tilde{p}_y.$$

Hence, the local control conditions are satisfied and $\xi''$ is stratified.

We can also conclude by continuity that $\xi(f) = \eta \circ f$ off of $V'_0$, while on $V'_0$ it is trivially true; hence the equation holds in a neighborhood of 0.
Filtration properties and estimates.

Before establishing the remaining conditions for \( \eta \) and \( \xi \), we first make a few basic observations concerning the filtration properties of vector fields. We consider for example

\[
\zeta \in \mathfrak{g}_{x,u} \left\{ \frac{\partial}{\partial x_i} \right\} \quad \text{with} \quad \fil(\zeta) = r.
\]

\[
\|\zeta\| \leq C \cdot \rho_0^{(1+r+1/2m)}.
\]

In fact, if \( \zeta = \sum_i \frac{\partial}{\partial x_i} \) then by the properties of the filtrations,

\[
\fil(\zeta) \geq \fil(x_i) + \fil(\zeta)
\]

or since \( \fil(x_i) > 0 \),

\[
\fil(\zeta) \geq r + 1.
\]

Thus, (6.1) follows from \( \|\zeta\| \leq \sum_i |\zeta_i| \). A similar remark applies to the other modules of vector fields.

Then, in (5.1) \( \xi_0 \) is a sum of terms \( - (h_i)\xi_i \); and (with \( m = k \cdot m', m' = \fil(h_i) \))

\[
\fil(- (h_i)\xi_i) \geq m + \fil(\xi_i)
\]

\[
\geq m + \fil \left( \frac{\partial f}{\partial u} \right) - \text{jump}_c(f)
\]

\[
\geq 2m + \text{jump}_c(f)
\]

by (4.3). Hence, summing over \( i \) gives

\[
\fil(\xi_0) \geq 2m + \text{jump}_c(f).
\]

Thus, by (6.1)

\[
\|\xi_0\| \leq C \cdot \rho_0^{(1+1/2m)}.
\]

A similar argument implies

\[
\fil(\eta_0) \geq 2m + \text{jump}_c(f).
\]

Hence,

\[
\|\eta_0\| \leq C'' \cdot \rho_0^{(1+1/2m)}.
\]
Next, we consider $\zeta'$ defined in (5.2). To estimate $\|\zeta'\|$, we argue in a similar fashion to the preceding case with one exception. We observe by the properties of the filtration
\[
\text{fil} \left( \frac{\partial f}{\partial u} \right) \geq \text{fil}(y_r) + \text{fil} \left( \frac{\partial f}{\partial u} \right).
\]
We also know
\[
\text{fil}(\zeta_{\omega}) \geq m - \text{fil}(y_r) - \text{jump}_c(f).
\]
Hence
\[
\text{fil} \left( -G_j^\ell \cdot \frac{\partial f}{\partial u} \cdot \zeta_{\omega} \right) \geq m + \text{fil} \left( \frac{\partial f}{\partial u} \right) + \text{fil}(\zeta_{\omega}) \geq 2m + \text{fil} \left( \frac{\partial f}{\partial u} \right) - \text{jump}_c(f) \geq 2m + \text{jump}_v(f).
\]
Summing over $j, \ell$ yields $\text{fil}(\zeta') \geq 2m \pm \text{jump}_v(f)$, thus, we conclude as in the earlier case
\[
(6.6) \quad \|\zeta'\| \leq C_6 \cdot \rho_0^{(1 + (1/2m))}.
\]
Similarly, arguing as above for $\zeta_\nu$ defined by (5.3)
\[
(6.7) \quad \text{fil}(\zeta_\nu) \geq 2m - b_\nu
\]
and
\[
(6.8) \quad \|\zeta_\nu\| \leq C_6 \cdot \rho_0^{(1 - (b_\nu - 1)/2m)}
\]
where $b_\nu = \text{fil}(y_\nu) + \text{jump}_c(f)$. We are now ready to establish the remaining conditions for $\xi$ and $\eta$.

*Verification of a') :

First, to establish the local control conditions for $\eta$ and $V_0$, consider
\[
(6.9) \quad |\eta(\rho_0)| \leq \chi_1 |\eta'(\rho_0)| + \chi_2 |\eta''(\rho_0)| \leq C_1 \rho_0 + \chi_2 \left( \frac{\partial \rho_0}{\partial u} + \rho^{-1}|\eta'(\rho_0)| \right)
\]
By our choice of $\rho_0$, $\frac{\partial \rho_0}{\partial u} = 0$ and by (6.4)
\[
(6.10) \quad |\eta_0(\rho_0)| \leq C_3 \cdot (\rho_0)^{(2m+r)/2m}
\]
where $r = \text{fil}(\eta_0) \geq 2m$. 


If $\chi_2 = 0$, the control condition holds by (6.9), while if $\chi_2(y') \neq 0$, then $y' \notin U^{(2)}$, hence $\hat{\rho}(y') \geq \epsilon'\rho_0(y')$. Combining this with (6.10) implies for (6.9)

\begin{equation}
|\eta_0(\rho_0)| \leq C_1 \cdot \rho_0 + \epsilon'^{-1} \cdot \rho_0^{-1} C_3 \cdot \rho_0^5 \quad |\eta_0(\rho_0)| \leq C_4 \cdot \rho_0 \quad \text{off of } U^{(2)}.
\end{equation}

Secondly, to verify continuity on $V_0$, we may write

\begin{equation}
\chi_1 \eta' + \chi_2 \eta'' = \left( \frac{\partial}{\partial u} \right) + \chi_1 \left( \eta' - \frac{\partial}{\partial u} \right) + \chi_2 \left( \eta'' - \frac{\partial}{\partial u} \right).
\end{equation}

Also, by assumption (V2) of the definition of stratified topological triviality (and (4.2))

\begin{equation}
\left\| \eta' - \frac{\partial}{\partial u} \right\| \leq C \cdot \rho_0^8 \quad \text{for some } C, \delta > 0
\end{equation}

while if $\chi_2(y') \neq 0$, $y' \notin U^{(2)}$ and as above by (6.5)

\begin{equation}
\chi_2 \|\eta'' - \frac{\partial}{\partial u}\| = \chi_2 \cdot \hat{\rho}^{-1}\|\eta_0\| \leq \epsilon'^{-1} \cdot \rho_0^{-1} C'' \cdot \rho_0^8 + \delta' \leq C \cdot \rho_0^8.
\end{equation}

Thus, $\eta$ decomposes in a neighborhood of 0 as $\eta = \frac{\sigma}{\partial u} + \eta_2$ and from 6.13 and 6.14

\begin{equation}
\eta_2 = \begin{cases} 
\chi_1 \left( \eta' - \frac{\partial}{\partial u} \right) + \chi_2 \left( \eta'' - \frac{\partial}{\partial u} \right) & \text{off } V_0 \\
0 & \text{on } V_0 
\end{cases}
\end{equation}

with

\begin{equation}
\|\eta_2\| \leq C_1 \cdot \rho_0^8 \quad \text{with } C = \max\{C, C''\}, \quad \delta'' = \min\{\delta, \delta'\}.
\end{equation}

This shows that $\eta$ is continuous on $V_0$ as well. Thus, $\eta$ is stratified.

\textit{Verification of b') :}

Next, we establish the local control conditions for $\xi$ and $V_0$.

\begin{equation}
|\xi(\rho_0^{(1)})| \leq \chi_3 |\xi_1(\rho_0^{(1)})| + \chi_4 |\xi''(\rho_0^{(1)})|.
\end{equation}

Now when $\chi_3 \neq 0$, i.e., in $U''$,

\begin{equation}
|\xi_1(\rho_0^{(1)})| \leq \tilde{\chi}_1 |\xi'(\rho_0^{(1)})| + \tilde{\chi}_2 |\xi''(\rho_0^{(1)})| \leq C_1 \cdot \rho_0^{(1)} + \tilde{\chi}_2 \left( \left| \frac{\partial \rho_0^{(1)}}{\partial u} \right| + \hat{\rho}^{-1} |\xi_0(\rho_0^{(1)})| \right).
\end{equation}
The argument proceeds as in (6.9) for \( \eta \). If \( \tilde{\chi}_4(x') \neq 0 \), \( f(x') \notin U^{(4)} \). However, by lemma 5.4, there is a conical neighborhood \( U^{(4)} \) of \( V' \) such that \( f(U^{(4)}) \subset U^{(4)} \), thus, \( x' \notin U^{(4)} \). Since \( f \) is an unfolding of non-decreasing filtration and satisfies F2, we obtain

\[
(6.18) \quad \left| \frac{\partial \rho^{(1)}_0}{\partial u} \right| = \left| \frac{\partial}{\partial u} (\sigma + \tilde{\rho}_0) \right| \leq \left| \frac{\partial \sigma}{\partial u} \right| + \left| \frac{\partial f}{\partial u} (\rho_0) \right| \leq C_1 \cdot \rho^{(1)}_0 + C_2 \cdot \rho^{(1)}_0 = C_3 \cdot \rho^{(1)}_0.
\]

Hence, arguing exactly as for \( \eta \),

\[
(6.19) \quad |\xi_1(\rho^{(1)}_0)| \leq C \cdot \rho^{(1)}_0
\]

(note that \( x' \in U'' \) implies \( \sigma(x') < \varepsilon' \rho^{(1)}_0(x') \) which implies \( \sigma(x') < \varepsilon'' \tilde{\rho}_0(x') \) where \( \varepsilon'' = \varepsilon'/(1-\varepsilon') \), or \( \rho^{(1)}_0(x') < (1 + \varepsilon'') \tilde{\rho}_0(x') \). Since \( f(x') \notin U^{(3)} \), \( \tilde{\rho}(x') > \varepsilon' \tilde{\rho}_0(x') \) giving \( \tilde{\rho}(x') > C \rho^{(1)}_0(x') \)).

Similarly, if \( \chi_4(x') \neq 0 \), \( x' \notin U'' \) so that \( \sigma \geq \varepsilon'' \rho^{(1)}_0 \). Then, by (5.7)

\[
(6.20) \quad \chi_4 |\hat{\xi}''(\rho^{(1)}_0)| = \left| \frac{\partial \rho^{(1)}_0}{\partial u} \right| + \sigma^{-1} |\xi'_1(\rho^{(1)}_0)| + \sum_{i=1}^{t} \sigma^{-1} |\tilde{n}_i| \cdot |\xi_i(\rho^{(1)}_0)|.
\]

Also, from §5, \( \text{fil}(\zeta') \geq 2m \), hence

\[
|\xi'(\rho^{(1)}_0)| \leq C \cdot \rho^{(1)}_0^2.
\]

For the third term, we must more accurately estimate \( |n_j| \). If

\[
\eta' = \frac{\partial}{\partial u} + \sum n_i \frac{\partial}{\partial y_i} \quad \text{and} \quad \eta_0 = \sum n_{0i} \frac{\partial}{\partial y_i},
\]

then by the assumption (4.2) on \( \eta' \),

\[
|n_j| \leq C \cdot \rho^{(1)}_0 \quad \text{where} \quad b_j = \text{jump}_c(f) + \text{fil}(y_i).
\]

Also, as \( \text{fil}(\eta_0) \geq 2m + \text{jump}_c(f) \),

\[
\text{fil}(n_0) \geq 2m + \text{fil}(y_i) + \text{jump}_c(f).
\]

Then,

\[
|n_j| = \chi_1 |n'_j| + \chi_2 |\hat{\rho}^{-1} n_{0j}|.
\]

Arguing exactly as we did in obtaining (6.11),

\[
|n_j| \leq C_j \cdot \rho^{(1)}_0.
\]

Also, by (6.7)

\[
\text{fil}(\zeta_j) \geq 2m - b_j
\]

or

\[
|\xi'(\rho^{(1)}_0)| \leq C \cdot \rho^{(1)}_0(2 - (b_j/2m)).
\]
Thus (6.20) becomes
\[
\chi_4|\xi'''(\rho_0^{(1)})| \leq C_1 \cdot \rho_0^{(1)} + C_2 \sigma^{-1} \cdot \rho_0^{(1)2} + C_3 \sigma^{-1} \cdot \tilde{\rho}_0^{(b/2m)} \cdot \rho_0^{(1)(b-(b/2m))}.
\]
Thus, since \(\tilde{\rho}_0 \leq \rho_0^{(1)}\) and \(\sigma^{-1} \leq \varepsilon''^{-1} \rho_0^{(1)-1}\), (6.20) becomes
\[
(6.21) \quad \chi_4|\xi'''(\rho_0^{(1)})| \leq C_1 \cdot \rho_0^{(1)}.
\]
Combining (6.16) with (6.19) and (6.21) we conclude
\[
|\xi(\rho_0^{(1)})| \leq C''' \cdot \rho_0^{(1)}.
\]
Lastly, we must establish that \(\xi\) is continuous at points of \(V'_0\). We claim that
\[
\begin{align*}
|\xi - \frac{\partial}{\partial u}| &\leq C \cdot \rho_0^{(1)\delta'} \quad \text{for some } C, \delta' > 0.
\end{align*}
\]
Consider
\[
(6.23) \quad \xi - \frac{\partial}{\partial u} = \chi_3 \left(\xi' - \frac{\partial}{\partial u}\right) + \chi_4 \left(\xi'' - \frac{\partial}{\partial u}\right)
\]
where
\[
\xi' - \frac{\partial}{\partial u} = \tilde{\chi}_1 \left(\xi' - \frac{\partial}{\partial u}\right) + \tilde{\chi}_2 \left(\xi'' - \frac{\partial}{\partial u}\right).
\]
By assumption,
\[
\begin{align*}
|\xi - \frac{\partial}{\partial u}| &\leq C \cdot \rho_0^{(1)\delta} \quad \text{for some } C, \delta > 0
\end{align*}
\]
in a neighborhood of 0 (wherever \(\xi'\) is defined).

Next,
\[
|\xi'' - \frac{\partial}{\partial u}| = |\tilde{\rho}^{-1} \xi_0| \leq C \cdot \tilde{\rho}^{-1} \rho_0^{(1)(1+(1/2m))}.
\]
However, if \(\tilde{\chi}_3(x') \neq 0\), and \(x' \in U''\), then we can argue as for (6.19) to conclude
\[
|\xi'' - \frac{\partial}{\partial u}| \leq C' \rho_0^{(1/2m)}.
\]
Hence, within \(U''\)
\[
(6.24) \quad |\xi' - \frac{\partial}{\partial u}| \leq C'' \cdot \rho_0^{(1)\delta'}
\]
for \(\delta' = \min\{\delta, 1/2m\}, C'' = C + C'.
\]
It remains to check $\xi'' - \frac{\partial}{\partial u}$ when $\chi_4 \neq 0$.

$$\left\| \xi'' - \frac{\partial}{\partial u} \right\| \leq \sigma^{-1}\|\xi \| + \sum_{i=1}^{t} \sigma^{-1}\|\tilde{\eta}_i \|. $$

If $\chi_4(x') \neq 0$, then $x' \notin U''$, hence $\sigma(x') \geq \epsilon'' \cdot \rho_0^{(1)}(x')$. Then, since $\text{fil}(\zeta') \geq 2m$,

$$\sigma^{-1}\|\zeta \| \leq C\sigma^{-1}\rho_0^{(1)(1+\frac{1}{2m})} \leq C\epsilon''^{-1}\rho_0^{(1)(1/2m)}. $$

From the earlier computation of (6.20), and using (6.1)

$$\sigma^{-1}\|\tilde{\eta}_j \| \leq C_j\sigma^{-1}\cdot \rho_0^{(b_j/2m)} \cdot \rho_0^{(1-\frac{1}{b_j-1}/2m)} \leq C\epsilon''^{-1}\rho_0^{(1)(1/2m)}. $$

Thus, for (6.25), if $\chiug_a \neq 0$,

$$\left\| \xi'' - \frac{\partial}{\partial u} \right\| \leq C\epsilon''^{(1/2m)}. $$

Finally, from (6.24) and (6.26)

$$\left\| \xi - \frac{\partial}{\partial u} \right\| \leq C_1 \cdot \rho_0^{(1/2)}. $$

This completes the proof of $b'\rangle$ showing that $\xi$ and $\eta$ are stratified.

\[\Box\]

7. A Version of the theorem for germs with negative filtration.

Consider the case where $f_0 : k^s, 0 \to k^t, 0$ is itself an unfolding of a germ $f_1 : k^s, 0 \to k^t, 0$. Let $x = (x+, x')$ and $y = (y+, y')$ with $x+$ local coordinates for $k^s$, $x'$ for $k'(r=s-s')$ and similarly for $y$. We write $f_0(x+, x') = (f_0(x+, x'), x')$ with $(f_0(x+, 0)) = f_1(x+)$. This time we suppose that there are convex filtrations for $\mathcal{C}_{x+}$ and $\mathcal{C}_{y+}$ (preserved by $f_0^g$) and filtrations on $\theta_{x+}, \theta_{y+}$, and $\theta(f_1)$ as $\mathcal{C}_{x+}$ and $\mathcal{C}_{y+}$-modules, so that $f_0$ is an unfolding of non-decreasing filtration of $f_1$. In essence, this says that the $x_i$ have non-positive filtration, e.g., if $f_0$ is weighted homogeneous then the $x_i$ must have non-positive weight.
Now, let $f$ be an unfolding of $f_0$ of the form

$$f(x_+, x', u) = \tilde{f}(x_+, x', u, x', u)$$

so that $f$ is also an unfolding of $f_1$ of non-decreasing filtration (again all germs are real analytic or holomorphic). For example, in the weighted homogeneous case this situation occurs if $f_0$ is the unfolding of $f_1$ versal in weight $\leq m$ for some $m \geq 0$ (as defined in [D1, I]). We describe how the statements of Theorems 1 and 2 must be modified to allow for this situation.

We suppose:

(F1') There are $g_1, \ldots, g_r \in \Delta(f_1)$ and $g'_1, \ldots, g'_{n-r} \in m_{\gamma_+}$ of filtration $\pi$ such that \{g_1, \ldots, g_r, f'_*(g'_1), \ldots, f'_*(g'_{n-r})\} form a set of vertices for the filtration on $\mathcal{G}_{\gamma_+}$, and that if

$$g_j \left( \frac{\partial}{\partial y_{++}} \right) = \sum_i h_{i,j,\ell} \cdot \Delta_i(f) \cdot \left( \frac{\partial}{\partial y_{+\ell}} \right)$$

then $G_j$ defined as in (3.3) still satisfies

(F2') $\text{fil}(G_j) \geq \text{fil}(g_j)$.

Now, let $\rho_0$ denote a control function of filtration $2m$ for the filtration on $\mathcal{G}_{\gamma_+}$, and let $V_0 = \{0\} \times k^{r+q}$. We define $\sigma = \sum |G_j|^2$ so that using lemma 3.7, we see that $\rho_0^{(1)} = \sigma + \tilde{\rho}_0$ vanishes on $V'_0 = \{0\} \times k^{r+q}$. Let $\rho^{(1)} = \sigma + \tilde{\rho}$.

We now assume that $f$ is stratified topologically trivial in conical neighborhoods $U$ of $V$ and $U'$ of $V'$ (with $V' = \sum(f) \cap f^{-1}(V)$) in that it satisfies the conditions (F1') (F2'), and (F3), and the conditions (V1) - (V2) of definition 4.1. The version for weighted homogeneous $f$ becomes:

THEOREM 3. – Suppose that $f$, $f_0$ (and $f_1$) are as in the above situation (for a weight filtration) with $f$ weighted homogeneous. If $f$ is stratified topologically trivial in a conical neighborhood of the versality discriminant and

$$\text{wt}(u_i) > \max \{\text{wt}(x_j)\}, \quad 1 \leq i \leq q$$

then $f$ is a topologically trivial unfolding of $f_0$. 
VERSALITY DISCRIMINANT AND TOPOLOGICAL EQUIVALENCE

This is a special case of the general result for the filtered case. Just as in Theorem 2, to state the general result we must assume the versality discriminant is graded and use a modification analogous to (4.2).

First we modify the definition of \( \text{jump}_\nu(f) \). In our case, (2.2) can be written in the equivalent form

\[
(I^k \cdot \theta(f)) \subseteq \Phi_{x_+} \left\{ \frac{\partial f}{\partial x_+} \right\} + \Phi_{y_+} \left\{ \frac{\partial}{\partial y_+} \right\} + \Phi_{x_+} \left\{ \frac{\partial f}{\partial x_+} \right\}.
\]

For \( \psi \in \Phi_{x,u} \left\{ \frac{\partial}{\partial x_i} \right\} \), if \( \psi = \sum r_i(y,u) \cdot \frac{\partial}{\partial x_i} \), then we define

\[
\text{fil}(\psi) = \min_i \left\{ \text{fil}(r_i) + \text{fil} \left( \frac{\partial f}{\partial x_i} \right) \right\}.
\]

We then suppose that \( V \) contains the versality discriminant in a graded sense by modifying (3.5) to require that there exist \( h_1, \ldots, h_r \in I \) of filtration \( m' \) with \( km' = m \), and whose vanishing still defines \( V \), and an integer \( d \geq 0 \) so that for \( \zeta \in \theta(f) \)

\[
(7.2) \quad h_i \cdot \zeta = \xi_i(f) + \psi_i(f) - \eta_i \circ f, \quad 1 \leq i \leq \ell
\]

with

\[
\text{fil}(\xi_i), \text{fil}(\psi_i), \text{fil}(\eta_i) \geq m + \text{fil}(\zeta) - d.
\]

Again the minimum such integer \( d \geq 0 \) will be denoted \( \text{jump}_\nu(f) \). Also, \( \text{jump}_c(f) \) is defined exactly as before. If now, \( \eta \) in the condition for stratified topological triviality is given by

\[
\eta_j = \frac{\partial}{\partial u_j} + \sum_{i=1}^r n_{ij} \cdot \frac{\partial}{\partial y_{i+}} + \sum_{i=1}^r n'_{ij} \cdot \frac{\partial}{\partial y_{i}}
\]

then (4.2) is replaced by

\[
(7.3a) \quad \text{fil}(n_{ij}) \geq \text{jump}_c(f) + \text{fil}(y_{i+})
\]

\[
(7.3b) \quad \text{fil}(n'_{ij}) \geq \text{jump}_c(f).
\]

Remark. - If \( f \) is weighted homogeneous then again \( \text{jump}_\nu(f) = \text{jump}_c(f) = 0 \) and \( \text{fil} \left( \frac{\partial f}{\partial x_i} \right) = -\text{wt}(y_i) \geq 0 \) so that (7.3) reduces to the original condition for (V2) for stratified topological triviality.
Theorem 4. — Let \( f \), as above, be an unfolding of non-decreasing filtration of \( f_0 \) (and \( f_1 \)) so that \( f \) is stratified topologically trivial in a conical neighborhood of the versality discriminant in the graded sense. If

\[
(7.4) \quad \text{fil}
\left( \frac{\partial f}{\partial u_i} \right)
\geq \text{jump}_y(f) + \text{jump}_c(f)
\]

\[
+ \max_j \left\{ \text{fil}
\left( \frac{\partial f}{\partial x'_j} \right) \right\}, \quad 1 \leq i \leq q
\]

then \( f \) is a topologically trivial unfolding of \( f_0 \).

The only way that the proofs of Theorems 3 and 4 differ from those of Theorems 1 and 2 is in the construction of \( \xi_0, \eta_0 \) and \( \xi'' \). Instead of (5.1), we use (7.2) to solve

\[
(7.5) \quad -\hat{\rho} \cdot \frac{\partial \bar{f}}{\partial u} = \xi'_0(\bar{f}) + \psi_0(\bar{f}) - \eta'_0 \circ f
\]

where if \( \psi_0 = \sum_{i=1}^r r_i(y, u) \frac{\partial}{\partial y'_i} \) then by (7.4)

\[
(7.6) \quad \text{fil}(\bar{r}_i) \geq 2m + \text{fil}
\left( \frac{\partial \bar{f}}{\partial u} \right) - \text{jump}_y(f) + \text{fil}
\left( \frac{\partial f}{\partial x'_i} \right)
\]

\[
\geq 2m + \text{jump}_c(f).
\]

If we also let \( \psi'_0 = \sum_{i=1}^r r_i(y, u) \cdot \frac{\partial}{\partial y'_i} \), then we define

\[
\xi'_0 = \xi'_0 + \psi'_0 \quad \text{and} \quad \eta'_0 = \eta'_0 + \psi'_0.
\]

Then, (7.6) together with analogous calculations for \( \xi'_0 \) and \( \eta'_0 \) imply that for these \( \xi'_0 \) and \( \eta'_0 \), (6.3) and (6.5) as well as (6.10) and (6.19) hold. The construction for \( \eta \) proceeds as before.

To define \( \xi'' \) let

\[
\eta'' \circ f = \eta'_0 \circ f - \psi_0(\bar{f}).
\]

As

\[
\psi_0(\bar{f}) = \sum_{i=1}^r \bar{r}_i \cdot \frac{\partial \bar{f}}{\partial x'_i} = \sum_{i=1}^r \bar{n}_{0i} \cdot \frac{\partial}{\partial y'_{+i}},
\]

\[
\eta'' = \sum_{i=1}^r (n_{0i} - \bar{n}_{0i}) \cdot \frac{\partial}{\partial y'_{+i}}
\]
where \( \eta'_0 = \sum n_{0i} \frac{\partial}{\partial y_{+i}} \). By (7.6)

\[
\text{fil} (n'_{0i}) \geq \min \{ \text{fil} (r_j) \} + \text{fil} \left( \frac{\partial \tilde{f}}{\partial x'_i} \right) + \text{fil} (y_+) \geq 2m + \text{jump}_c (f) + \text{fil} (y_+).
\]

By an earlier calculation, a similar inequality applies to \( n_{0i} \). Similarly, if we write

\[
\eta' = \frac{\partial}{\partial u} + \sum n_i \frac{\partial}{\partial y_{+i}} + \sum n_i' \frac{\partial}{\partial y'_i}
\]

we let

\[
\psi''_0 = \sum_{i=1}^{r} n'_i \frac{\partial}{\partial x'_i} \quad \text{and} \quad \eta'_1 \circ f = \eta' \circ f - \psi''_0 (\tilde{f}).
\]

Then

\[
\eta'_1 \circ f = \eta' \circ f - \sum_{i=1}^{r} n'_i \frac{\partial \tilde{f}}{\partial x'_i} = \sum_{i=1}^{s'} (\tilde{n}_i - \tilde{n}'_i) \frac{\partial}{\partial y_{+i}}
\]

with

\[
\text{fil} (n''_i) \geq \min \{ \text{fil} (n'_j) \} + \text{fil} \left( \frac{\partial \tilde{f}}{\partial x'_j} \right) + \text{fil} (y_+) \geq \text{jump}_c (f) + \text{fil} (y_+)
\]

and similarly for \( n_i \) by assumption. We let

\[
\xi''' = \frac{\partial}{\partial u} + \psi_0 + \psi''_0 + \sigma^{-1} \xi + \sigma^{-1} \sum_{i=1}^{s'} \tilde{n}'_i \xi_i
\]

with

\[
n_i^{(1)} = (n_{0i} - n'_{0i}) + (n_i - n''_i).
\]

The assumptions (7.3) together with (7.6) and the same arguments used in §6 allow us to deduce that \( \xi''' \) and hence \( \xi \) have the desired properties.
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Manuscrit reçu le 28 février 1990.

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