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## REMARKS ON THE LICHNEROWICZ-POISSON COHOMOLOGY

by Izu VAISMAN

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The Lichnerowicz-Poisson (LP) cohomology of a Poisson manifold was defined in [L], and it provides a good framework to express deformation and quantization obstructions [L], [VK], [H], [V2]. The LP cohomology spaces are, generally, very large, and their structure is known only in some particular cases [VK], [X]. The homological algebraic place of these spaces was clarified in [H]. In the present note, we make a number of further remarks on the LP cohomology, most of them related with a certain natural spectral sequence which shows that, in the case of a regular Poisson manifold, the LP cohomology is connected with the cohomology of the sheaves of germs of foliated (i.e., projectable) forms of the symplectic foliation of the manifold (e.g., [V1]).

### 1. General remarks.

Let  $M^m$  be a Poisson manifold with the Poisson bivector  $\Pi$ , and put  $\mathcal{V}^0(M) \stackrel{\text{def}}{=} C^\infty(M)$ ,  $\mathcal{V}(M) = \mathcal{V}^1(M) \stackrel{\text{def}}{=} \mathcal{V}^1(M)$  the space of  $C^\infty$  vector fields of  $M$ ,  $\mathcal{V}^k(M) \stackrel{\text{def}}{=} \mathcal{V}^k(M)$  the space of  $k$ -vector fields (i.e., antisymmetric  $k$ -contravariant tensor fields of  $M$ ),  $\mathcal{V}^*(M) \stackrel{\text{def}}{=} \mathcal{V}^*(M)$  the space of Pfaff forms of  $M$ , and, finally  $\mathcal{L}(M) \stackrel{\text{def}}{=} \bigoplus_{k=0}^m \mathcal{V}^k(M)$  the contravariant Grassmann algebra of  $M$ . The bivector  $\Pi$  has an associated morphism  $\# : T^*M \rightarrow TM$ , defined by  $\beta(\alpha^\#) = \Pi(\alpha, \beta)$ ,  $\forall \alpha, \beta \in T^*M$ , and it yields the Poisson bracket of functions  $\{f, g\} = \Pi(df, dg)$ , as well as Hamiltonian vector fields  $X_f$ ,  $\forall f \in \mathcal{V}^0(M)$ , given by  $X_f g = \{f, g\}$ . These fields define a generalized foliation with symplectic leaves called the *symplectic foliation* of  $(M, \Pi)$  (i.e.,  $\{X_f\}$  generate the tangent spaces of

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the leaves). It is important to remember that the Poisson bracket induces a bracket of Pfaff forms which is the unique natural extension of the formula  $\{df, dg\} = d\{f, g\}$ , and is given by

$$(1.1) \quad \{\alpha, \beta\} = L_{\alpha\#} \beta - L_{\beta\#} \alpha - d(\Pi(\alpha, \beta)).$$

The basic Poisson condition  $[\Pi, \Pi] = 0$ , where  $[ , ]$  denotes the Schouten-Nijenhuis bracket, ensures that  $(\mathcal{V}^0(M), \{ , \})$  and  $(\mathcal{V}^*(M), \{ , \})$  are Lie algebras. The same condition also shows that the operator  $\sigma Q = -[\Pi, Q]$  is a coboundary on  $\mathcal{L}(M)$  (i.e.,  $\sigma^2 = 0$ ), and the cohomology of the cochain complex  $(\mathcal{L}, \sigma)$  is, by definition, the LP cohomology of  $(M, \Pi)$ . Its spaces will be denoted by  $H_{LP}^k(M, \Pi)$ . It is also important to remind that, for  $Q = \mathcal{V}^k(M)$ , one has [BV]

$$(1.2) \quad (\sigma Q)(\alpha_0, \dots, \alpha_k) = \sum_{i=1}^k \alpha_i\#(Q(\alpha_0, \dots, \hat{\alpha}_i, \dots, \alpha_k)) + \sum_{i < j=0}^k (-1)^{i+j} Q(\{\alpha_i, \alpha_j\}, \alpha_0, \dots, \hat{\alpha}_i, \dots, \hat{\alpha}_j, \dots, \alpha_k),$$

where  $\alpha_i \in \mathcal{V}^*(M)$ , and  $\hat{\phantom{x}}$  denotes the absence of an argument.

Now, the definitions given above have some easy consequences such as

a) [X], [VK].  $H_{LP}^0(M, \Pi) = \{f \in C^\infty(M) / \forall g \in C^\infty(M), X_g f = 0\}$ . (Since  $\sigma f = -X_f$ .)

b) [X], [VK].  $H_{LP}^1(M, \Pi) = \mathcal{V}_\pi(M) / \mathcal{V}_{\mathcal{X}}(M)$ , where

$$\mathcal{V}_\pi(M) \stackrel{\text{def}}{=} \{X \in \mathcal{V}(M) / L_X \Pi = 0\}, \quad \mathcal{V}_{\mathcal{X}}(M) \stackrel{\text{def}}{=} \{X_f / f \in \mathcal{V}^0(M)\}.$$

(Since  $\sigma X = -L_X \Pi$  [L].)

c) [L],  $\sigma \Pi = 0$ , and  $\Pi$  defines a *fundamental class*  $[\Pi] \in H_{LP}^2(M, \Pi)$ .

d) The LP cohomology satisfies the Mayer-Vietoris exact sequence property i.e., if  $U, V$  are open subsets of  $M$ , there is an exact sequence of the form

$$(1.3) \quad \dots \rightarrow H_{LP}^k(U \cup V, \Pi) \rightarrow H_{LP}^k(U, \Pi) \oplus H_{LP}^k(V, \Pi) \rightarrow H_{LP}^k(U \cap V, \Pi) \rightarrow H_{LP}^{k+1}(U \cup V, \Pi) \rightarrow \dots$$

The definition of the arrows and the proof of the exactness are the same as for the de Rham cohomology (e.g., [BT]).

e) [L], [K]. Natural homomorphisms  $\rho : H^k(M, \mathbb{R}) \rightarrow H^k_{LP}(M, \Pi)$ , which are isomorphisms in the symplectic case, exist. Namely,  $\rho$  is defined by the extension of  $\#$  to  $k$ -forms  $\lambda$  by

$$(1.4) \quad \lambda^\#(\alpha_1, \dots, \alpha_k) = (-1)^k \lambda(\alpha_1^\#, \dots, \alpha_k^\#),$$

since (1.2) shows that  $\sigma(\lambda^\#) = (-1)^k(d\lambda)^\#$ .

Because of e), it is natural to ask for a covariant interpretation of the whole LP cohomology via a Riemannian metric, and such an interpretation can be obtained by using Koszul's generating operators of the Schouten-Nijenhuis bracket. If we change signs such as to agree with [L], Koszul's formula for  $[A, B]$  where  $A \in \mathcal{V}^i(M)$ ,  $B \in \mathcal{V}^j(M)$  is [K]

$$(1.5) \quad [A, B] = D_\nabla(A \wedge B) - (D_\nabla A) \wedge B - (-1)^i A \wedge (D_\nabla B),$$

where  $\nabla$  is a torsionless linear connection on  $M$ , and  $D_\nabla$  is defined by the coordinatewise formula

$$(1.6) \quad (D_\nabla A)^{h_2, \dots, h_i} = \nabla_k A^{kh_2, \dots, h_i}.$$

If  $\nabla$  is the Riemannian connection of a metric  $g$ , (1.6) means  $D_\nabla = -\#_g \delta_g \#_g^{-1}$ , where  $\#_g : T^*M \rightarrow TM$  is the well known musical isomorphism, and  $\delta_g$  is the codifferential of  $(M, g)$ . Now, if we denote  $\pi = \#_g^{-1} \Pi$ ,  $B = \#_g \lambda$ , and take  $A = \Pi$  in (1.5), we obtain  $\sigma(\#_g \lambda) = \#_g \delta_\pi$ , where, if  $e(i)$  denotes the exterior (interior) multiplication by a form, one has

$$(1.7) \quad \delta_\pi = \delta_g e(\pi) - e(\pi) \delta_g - e(\delta_g \pi).$$

Hence,  $H^k_{LP}(M, \Pi)$  are isomorphic to the cohomology spaces of the Grassmann complex  $\Lambda M$  endowed with the coboundary  $\delta_\pi$ .

Of course,  $\pi$  must satisfy the condition  $\delta_\pi \pi = 0$ , which is equivalent to  $[\Pi, \Pi] = 0$  i.e., we must have

$$(1.8) \quad \delta_g(\pi \wedge \pi) = 2\pi \wedge (\delta_g \pi),$$

and this is a new characterization of a Poisson structure which may have some usefulness. For instance, it shows that the parallel 2-forms of a Riemannian manifold (if any) and the harmonic 2-forms of a compact Riemannian symmetric space (where the exterior product of two harmonic forms is again a harmonic form) define Poisson structures. Formulas (1.7), (1.8) may also be used if we are looking for compatible

Poisson structures on a given symplectic manifold  $M$  with symplectic form  $\omega$  i.e., Poisson bivectors  $\Pi$  such that  $[\omega^{-1}, \Pi] = 0$  (e.g., [G]). After the choice of a metric  $g$  on  $M$ , this problem amounts to solving the equations

$$(1.9) \quad \delta_{\#_g^{-1}(\omega^{-1})}\pi = 0, \quad \delta_g(\pi \wedge \pi) = 2\pi \wedge \delta_g\pi,$$

where also, if we ask  $g$  to be almost Hermitian  $\omega$ -compatible, then  $\#_g^{-1}(\omega^{-1}) = \omega$ . For instance, (1.9) shows that, if  $M$  is a compact Hermitian symmetric space, and  $\omega$  is its Kähler form, then any harmonic form of  $M$  defines an  $\omega$ -compatible Poisson structure. On the other hand, we shall notice that, in case  $M$  is compact and oriented,  $\delta_\pi$  has the formal adjoint

$$(1.10) \quad d_\pi = i(\pi)d - di(\pi) - i(\delta_g\pi),$$

and we may expect to be able to apply the abstract Hodge decomposition theorem of [LT]. (From the expression of the Schouten-Nijenhuis bracket [L], it follows easily that the complex

$$\dots \rightarrow \mathcal{V}^k(M) \xrightarrow{\sigma} \mathcal{V}^{k+1}(M) \rightarrow \dots$$

is elliptic along the leaves of the symplectic foliation of  $(M, \Pi)$ .)

Finally, we make a remark which will be important for the next sections of this paper. Namely, that there is a Serre-Hochschild spectral sequence associated with the LP cohomology. Let  $\mathcal{V}_0^*(M) \stackrel{\text{def}}{=} \ker \# =$  the space of *conormal 1-forms* of the symplectic foliation of  $(M, \Pi)$ . Since the bracket (1.1) satisfies  $\{\alpha, \beta\}^\# = [\alpha^\#, \beta^\#]$  [BV],  $\mathcal{V}_0^*(M)$  is an abelian ideal of  $(\mathcal{V}^*(M), \{, \})$ , and we may define the *filtration degree* of  $Q \in \mathcal{V}^k(M)$  to be  $h$  if  $Q(\alpha_1, \dots, \alpha_k) = 0$  as soon as  $\geq k - h + 1$  of the arguments are conormal. This yields a differential filtration of the LP complex  $\mathcal{L}(M)$ , where  $S_h^k(M) \stackrel{\text{def}}{=} \text{the space of } k\text{-vector fields of filtration degree } h \text{ is equal to the locally finite span of } \{f_0 X_{f_1} \wedge \dots \wedge X_{f_h} \wedge Y_1 \wedge \dots \wedge Y_{k-h} / f_i \in \mathcal{V}^0(M), Y_j \in \mathcal{V}^1(M)\}$ . Now, the spectral sequence which we have in mind, and which we shall denote by  $E_r^{p,q}(M, \Pi)$ , is the one associated with this filtration i.e., the Serre-Hochschild sequence of the pair of Lie algebras  $(\mathcal{V}^*(M), \mathcal{V}_0^*(M), \{, \})$ . This sequence converges to  $H_{LP}^*(M, \Pi)$ , and one has (e.g., [F])

$$(1.11) \quad E_2^{p,q}(M, \Pi) = H^p(V^*(M) / \mathcal{V}_0^*(M); H^q(\mathcal{V}_0^*(M); C^\infty(M))).$$

2. The regular case.

In the remaining part of this paper we assume that  $\Pi$  is of the constant rank  $2n$ , and  $m = 2n + s$ . This is the *regularity condition*, and then the symplectic foliation of  $(M, \Pi)$ , hereafter to be denoted by  $\mathcal{S}$ , is regular. Hence, we can and shall define a transversal distribution  $\mathcal{S}'$ , and  $TM = \mathcal{S}' \oplus T\mathcal{S}$ ,  $T^*M = \mathcal{S}'^* \oplus T^*\mathcal{S}$  induce a bigrading of the covariant and contravariant tensors of  $M$ . A tensor whose transversal degree is  $p$  and whose leafwise degree is  $q$  is said to be of the type  $(p, q)$ . We shall denote by  $\mathcal{V}^{p,q}(M)$  and  $\Lambda^{p,q}(M)$  the spaces of  $k$ -vector fields and  $k$ -forms ( $k = p + q$ ) of the type  $(p, q)$  of  $M$ , respectively. For instance, it is easy to understand that  $\ker \#$  (i.e.,  $\mathcal{V}_0^*(M)$ ) is just  $\mathcal{S}'^* =$  the space of the 1-forms of type  $(1, 0)$ , and that type  $\Pi = (0, 2)$ . (E.g., see [V1] for details on the bigrading of differential forms.)

Now, if  $Q \in \mathcal{V}^k(M)$  is of type  $(p, q)(p + q = k)$ , and if we use bihomogeneous arguments  $\alpha_i$  in (1.2), we see that  $\sigma = \sigma' + \sigma''$  where type  $\sigma' = (-1, 2)$ , type  $\sigma'' = (0, 1)$ , and, for arguments  $\alpha$  of type  $(1, 0)$  and  $\beta$  of type  $(0, 1)$ , one has

$$(2.1) \quad (\sigma' Q)(\alpha_0, \dots, \alpha_{p-2}, \beta_0, \dots, \beta_{q+1}) = \sum_{i < j=0}^{q+1} (-1)^{i+j} Q(\{\beta_i, \beta_j\}, \alpha_0, \dots, \alpha_{p-2}, \beta_0, \dots, \hat{\beta}_i, \dots, \hat{\beta}_j, \dots, \beta_{q+1}),$$

$$(2.2) \quad (\sigma'' Q)(\alpha_0, \dots, \alpha_{p-1}, \beta_0, \dots, \beta_q) = \sum_{i=0}^q (-1)^{p+i} \beta_i^\# (Q(\alpha_0, \dots, \alpha_{p-1}, \beta_0, \dots, \hat{\beta}_i, \dots, \beta_q) + \sum_{i=0}^{p-1} \sum_{j=0}^q (-1)^{p+i+j} Q(\{\alpha_i, \beta_j\}, \alpha_0, \dots, \hat{\alpha}_i, \dots, \alpha_{p-1}, \beta_0, \dots, \hat{\beta}_j, \dots, \beta_q) + \sum_{i < j=0}^q (-1)^{p+i+j} Q(\alpha_0, \dots, \alpha_{p-1}, \{\beta_i, \beta_j\}'' , \beta_0, \dots, \hat{\beta}_i, \dots, \hat{\beta}_j, \dots, \beta_q)).$$

Remember that type  $\alpha = (1, 0)$  means  $\alpha \in \mathcal{V}_0^*(M)$ , and that the latter is an ideal of  $\mathcal{V}^*(M)$ . On the other hand, we denoted by  $\{ , \}'$ ,  $\{ , \}''$  the type  $(1, 0)$  and  $(0, 1)$  components of  $\{ , \}$ . Particularly, if type  $X = (1, 0)$ , we get easily

$$(2.3) \quad \{\beta_1, \beta_2\}'(X) = (L_X \pi)(\beta_1, \beta_2).$$

In this section we use the type decomposition of  $\sigma$  in order to indicate a recurrent computational process of the LP cohomology which, in fact, is similar to the one used in [VK] for the case where  $\mathcal{L}$  is a fibration. Take  $Q \in \mathcal{V}^k(M)$ , and decompose it as

$$(2.4) \quad Q = Q^{k,0} + Q^{k-1,1} + \dots + Q^{0,k},$$

where the indices denote the type of the components. Then,  $\sigma Q = 0$  means

$$(2.5) \quad \sigma'' Q^{i,k-i} + \sigma' Q^{i+1,k-i-1} = 0 \quad (i=0, \dots, k).$$

For  $i = k$ , (2.5) gives  $\sigma'' Q^{k,0} = 0$ , and, on the other hand,  $(Q + \tilde{Q})^{k,0} = Q^{k,0}$ ,  $\forall \tilde{Q} \in \mathcal{V}^{k-1}(M)$ . Therefore, there exist homomorphisms

$$(2.6) \quad p_{k,0}: H_{LP}^k(M, \Pi) \rightarrow \mathcal{V}_0^{k,0}(M),$$

where  $\mathcal{V}_0^{k,0}(M)$  is the space of  $\sigma''$ -closed  $k$ -vectors of type  $(k,0)$ , and, furthermore, (2.5) shows that  $\text{im } p_{k,0}$  consists of  $k$ -vectors  $Q^{k,0} \in \mathcal{V}_0^{k,0}(M)$  which satisfy the following sequence of existence conditions of  $k$ -vectors  $Q^{k-1,1}, \dots, Q^{0,k}$  such that

$$\begin{aligned} (c_1) \quad \sigma' Q^{k,0} &= \sigma''\text{-exact} \stackrel{\text{def}}{=} -\sigma'' Q^{k-1,1}, \\ (c_2) \quad \sigma' Q^{k-1,1} &= \sigma''\text{-exact} \stackrel{\text{def}}{=} -\sigma'' Q^{k-2,2}, \\ &\dots\dots\dots \\ (c_k) \quad \sigma' Q^{1,k-1} &= \sigma''\text{-exact} \stackrel{\text{def}}{=} -\sigma'' Q^{0,k}. \end{aligned}$$

In this case we shall say that  $\sigma' Q^{k,0}$  satisfies  $k$  times the  $\sigma''$ -exactness condition, and we shall denote by  $\mathcal{V}_{0(k)}^{k,0}(M)$  the space of such  $Q^{k,0}$ . If we also denote  $\ker p_{k,0} = {}^0 H_{LP}^k(M, \Pi)$  = the space of  $k$ -dimensional LP cohomology classes whose cocycles are (2.4) with  $Q^{k,0} = 0$ , we obtain the result of the first recurrence step

$$(2.7) \quad H_{LP}^k(M, \Pi) \approx {}^0 H_{LP}^k(M, \Pi) \oplus \mathcal{V}_{0(k)}^{k,0}(M).$$

Now, in the next step we have to compute  ${}^0 H_{LP}^k(M, \Pi)$ , and for this purpose we take the subcomplex  ${}^0 \mathcal{L}(M)$  of  $\mathcal{L}(M)$  consisting of multivectors  $Q$  with a vanishing  $(\cdot, 0)$  component, and denote by  $H^k({}^0 \mathcal{L}(M))$  its cohomology spaces. Then  ${}^0 H_{LP}^k(M, \Pi)$  is the image of  $H^k({}^0 \mathcal{L}(M))$  with respect to the inclusion:  ${}^0 \mathcal{L}(M) \subseteq \mathcal{L}(M)$ . It is clear that the complex  $\mathcal{L}(M)/{}^0 \mathcal{L}(M)$  has coboundary zero, therefore,  $H^k(\mathcal{L}/{}^0 \mathcal{L}) = (\mathcal{L}/{}^0 \mathcal{L})^k = \mathcal{V}^{k,0}(M)$ . This gives us the exact sequence

$\mathcal{V}^{k-1,0}(M) \xrightarrow{\sigma} H^k({}^0\mathcal{L}(M)) \xrightarrow{\iota^*} H^k(\mathcal{L}(M))$ , and we get

$$(2.8) \quad {}^0H_{LP}^k(M, \Pi) \approx H^k({}^0\mathcal{L}(M)) / \sigma(\mathcal{V}^{k-1,0}(M)).$$

Hence, the second step will have to consist of an analysis of  $H^k({}^0\mathcal{L}(M))$ , which can be made in the same way as in step 1, and resulting in a formula similar to (2.7), and so on.

For  $k = 1$ , we get easily

$$(2.9) \quad {}^0H_{LP}^1(M, \Pi) = \{X \in \mathcal{V}^{0,1}(M) / \sigma'' X = 0\} / \sigma''(\mathcal{V}^0(M)).$$

For  $k = 2$ , we have first

$$(2.10) \quad H^2({}^0\mathcal{L}(M)) = \frac{\{Q^{1,1} + Q^{0,2} / \sigma'' Q^{1,1} = 0, \sigma'' Q^{0,2} + \sigma' Q^{1,1} = 0\}}{\{\sigma'' X^{0,1}\}},$$

and the analysis which gave (2.7) now yields

$$(2.11) \quad H^2({}^0\mathcal{L}(M)) \approx {}''H^2(\mathcal{L}^{0,*}(M)) \oplus \mathcal{V}_{0(1)}^{1,1}(M),$$

where  $\mathcal{L}^{0,*}(M) = \bigoplus_k \mathcal{V}^{0,k}(M)$ , and  ${}''H$  is its cohomology with respect to  $\sigma''$ , and

$$(2.12) \quad \mathcal{V}_{0(1)}^{1,1}(M) = \{Q^{1,1} / \sigma'' Q^{1,1} = 0 \text{ and } \sigma' Q^{1,1} = \sigma''\text{-exact}\}.$$

(We shall see in Section 3 that, if the foliation  $\mathcal{S}$  is either transversally Riemannian or transversally symplectic, then

$${}''H^i(\mathcal{L}^{0,*}(M)) \approx H^i(M, \Phi^0(\mathcal{S})),$$

where  $\Phi^0(\mathcal{S})$  is the sheaf of germs of functions which are constant along the leaves of  $\mathcal{S}$ .) Summing up the results we get

$$(2.13) \quad H_{LP}^2(M, \Pi) \approx ({}''H^2(\mathcal{L}^{0,*}(M)) \oplus ((\mathcal{V}_{0(1)}^{1,1}(M)) / \sigma(\mathcal{V}^{1,0}(M)))) \oplus \mathcal{V}_{0(2)}^{2,0}(M),$$

Etc.

### 3. The spectral sequence.

In this section we continue to refer to a regular Poisson manifold  $(M, \Pi)$ , and use the notation introduced in Section 2, while we are focussing on the spectral sequence  $E_r^{p,q}(M, \Pi)$  defined at the end of Section 1. We have :



PROPOSITION 3.1. — *The first terms of the LP Serre-Hochschild spectral sequence of a regular Poisson manifold  $(M, \Pi)$  are given by*

$$(3.1) \quad \begin{aligned} E_0^{pq}(M, \Pi) &= E_1^{pq}(M, \Pi) = \mathcal{V}^{q,p}(M), \\ E_2^{pq}(M, \Pi) &= H^p(\oplus \mathcal{V}^{q,*}, \sigma''). \end{aligned}$$

The reader can prove this by noticing that the  $h$ -filtering subcomplex of  $\mathcal{L}(M)$  as defined in Section 1 is equal to  $S_h(M) = \bigoplus_{i \geq h} \bigoplus_p \mathcal{V}^{p,i}(M)$ , and then following the usual definition of  $E_r^{pq}$ . Here, we just prefer to observe that  $\{\mathcal{L}(M) = \bigoplus \mathcal{W}^{i,j}(M), \sigma = \Sigma d_{hk}\}$ , where  $\mathcal{W}^{i,j}(M) = \mathcal{V}^{j,i}(M)$ , and the terms of  $\sigma$  are  $d_{01} = 0, d_{10} = \sigma'', d_{2,-1} = \sigma'$ , is a double semipositive cochain complex in the sense of [V1], p. 76-77, and then (3.1) follows from this reference.

Now, let  $G$  be a metric of the vector bundle  $\mathcal{S}'^*$  of Section 2, and let  $\tilde{\#} \stackrel{\text{def}}{=} \#_G \oplus \# : \mathcal{S}'^* \oplus T^*\mathcal{S} \rightarrow \mathcal{S}' \oplus T\mathcal{S}$  be the corresponding musical isomorphism also extended to  $\Lambda^k(M) \rightarrow \mathcal{V}^k(M)$ . Then, if  $\lambda$  is a differential form of type  $(p, q)$ ,  $\lambda^{\tilde{\#}}$  is a multivector of the same type, and we have

$$(3.2) \quad \begin{aligned} (\tilde{\#}^{-1}\sigma''\lambda^{\tilde{\#}})(X_0, \dots, X_{p-1}, Y_0, \dots, Y_q) \\ = (-1)^{q+1}(\sigma''\lambda^{\tilde{\#}})(\#_G^{-1}X_0, \dots, \#_G^{-1}X_{p-1}, \#^{-1}Y_0, \dots, \#^{-1}Y_q). \end{aligned}$$

In this relation, and in the sequel, we agree that type  $X = (1, 0)$  and type  $Y = (0, 1)$ . Furthermore, in order to compute  $\sigma''\lambda^{\tilde{\#}}$  by (2.2) we establish first

$$\{\#^{-1}Y_i, \#^{-1}\tilde{Y}_j\}^{\#} = \{\#^{-1}Y_i, \#^{-1}Y_j\}^{\#} = [Y_i, Y_j]$$

(remember that  $\{\alpha, \beta\}^{\#} = [\alpha^{\#}, \beta^{\#}]$  [BV]), and using (1.1))

$$\{\#_G^{-1}X_i, \#^{-1}Y_j\}(X) = - (L_{Y_j}G^*)(X_i, X) - G^*([Y_j, X_i], X),$$

where  $G^*$  is the dual metric of  $G$  on  $\mathcal{S}'$ . If these formulas are used, and the result is compared with the formula of the  $\mathcal{S}$ -leafwise exterior differential  $d_f$  [V1], p. 184, one gets

$$(3.3) \quad \begin{aligned} (\tilde{\#}^{-1}\sigma''\lambda^{\tilde{\#}})(X_0, \dots, X_{p-1}, Y_0, \dots, Y_q) \\ = - (d_f\lambda)(X_0, \dots, X_{p-1}, Y_0, \dots, Y_q) \\ + \sum_{i=0}^{p-1} \sum_{j=0}^q (-1)^{p+i+j} \lambda([(L_{Y_j}G^*)(X_i, \cdot)]^{\#G}, \\ X_0, \dots, \hat{X}_i, \dots, X_{p-1}, Y_0, \dots, \hat{Y}_j, \dots, Y_q). \end{aligned}$$

*Remark.* – The same result holds if  $G$  is a symplectic structure on  $\mathcal{S}'^*$ .

This computation leads to

**PROPOSITION 3.2.** – *If the symplectic foliation  $\mathcal{S}$  of the regular Poisson manifold  $(M, \Pi)$  is either transversally Riemannian or transversally symplectic, one has*

$$(3.4) \quad E_2^{pq}(M, \Pi) = E_1^{pq}(\mathcal{S}) = H^p(M, \Phi^q(\mathcal{S}))$$

where  $E_r^{pq}(\mathcal{S})$  is the spectral sequence of the foliation  $\mathcal{S}$  (e.g., [KT]), and  $\Phi^q(\mathcal{S})$  is the sheaf of germs of  $\mathcal{S}$ -foliated  $q$ -forms of  $M$  (e.g., [V1]). Particularly, (3.4) holds if  $\mathcal{S}$  is a fibration.

Indeed, under the hypotheses,  $L_Y G = 0$  in (3.3), and in view of (3.1) we get an isomorphism  $E_2^{pq}(M, \Pi) = H^p(\oplus \Lambda^{q,*}(M), d_f)$ . But then (3.4) is known [V1], p. 216, 222, 77. (Remember that an  $\mathcal{S}$ -foliated  $q$ -form is a  $q$ -form which, locally, is the pull-back of a form of a local transversal manifold of the foliation  $\mathcal{S}$ .)

Now, let us define an interesting special class of Poisson manifolds. A vector field  $V$  of  $M$  is  $\mathcal{S}$ -foliated if it sends leaves to leaves or, equivalently,  $\forall Y \in T\mathcal{S}, [V, Y] \in T\mathcal{S}$ . For instance, this happens if  $V$  is an infinitesimal automorphism of  $\Pi$  i.e.,  $L_V \Pi = 0$ , a condition which is easily seen to be equivalent to each of the following two conditions, where  $f, g \in C^\infty(M)$ ,

$$(3.5) \quad V\{f, g\} = [V, X_f](g) - [V, X_g](f),$$

$$(3.6) \quad [V, X_f] = X_{V(f)}.$$

A regular Poisson structure  $\Pi$  of  $M$  will be called *transversally constant* if  $\mathcal{S}$  has a transversal distribution  $\mathcal{S}'$  such that every local foliate vector field  $V \in \mathcal{S}'$  is a local infinitesimal automorphism of  $\Pi$ . For instance, if  $M = S \times N$ , and  $\Pi$  is defined by a symplectic structure of  $S$ , the distribution  $\mathcal{S}' = TN$  has this property. Particularly, the existence of the local canonical coordinates of  $\Pi$  in the sense of [L] p. 256-257, shows that every regular Poisson manifold is locally transversally constant. Another example is the *Dirac bracket* defined as follows. Let  $(M, \omega)$  be a symplectic manifold endowed with a foliation  $\mathcal{F}$  such that  $\omega$  induces symplectic structures of its leaves. These induced structures yield a Poisson bivector  $\Pi$  such that  $\mathcal{S}(\Pi) = \mathcal{F}$ , and  $\{ , \}_\Pi$  is the Dirac bracket of  $(M, \omega, \mathcal{F})$ . It follows that every  $\mathcal{F}$ -foliate vector

field  $V$  which is  $\omega$ -orthogonal to  $\mathcal{F}$  is an infinitesimal automorphism of  $\Pi$ . Indeed, for such  $V$ , (3.5) is equivalent to  $(L_V\omega)(X_f, X_g) = 0$ , and this is an easy consequence of  $d\omega = 0$ . Using this definition, we have

PROPOSITION 3.3. — *If  $\Pi$  is transversally constant,  $\sigma' = 0$ , and*

$$(3.7) \quad H_{LP}^k(M, \Pi) = \bigoplus_{k=0}^q E_2^{k-q, q}(M, \Pi).$$

*Proof.* — Of course, the proposition refers to  $\sigma'$  of (2.1) taken with respect to the distribution  $\mathcal{S}'$  involved in the definition of a transversally constant Poisson structure. Let us use the notation of (2.1), and evaluate there  $\{\beta_i, \beta_j\}'_p(X_p)$  ( $p \in M, X_p \in \mathcal{S}'_p$ ). This may be done by extending  $X_p$  to a local foliate (1,0)-vector field  $X$ , and using (2.3). Since  $\Pi$  is transversally constant,  $L_X\Pi = 0$  and we get  $\sigma' = 0$ . Then, (3.7) follows from (3.1). Q.e.d.

We shall finish by giving various corollaries of Propositions 3.1, 3.2, 3.3.

COROLLARY 3.1. — *If  $(M, \Pi)$  is a transversally constant Poisson manifold whose symplectic foliation is either transversally Riemannian or transversally symplectic, one has*

$$(3.8) \quad H_{LP}^k(M, \Pi) = \bigoplus_{q=0}^k E_1^{q, k-q}(\mathcal{S}) = \bigoplus_{q=0}^k H^q(M, \Phi^{k-q}(\mathcal{S})).$$

COROLLARY 3.2. — *Let  $\Pi$  be a Dirac bracket of a symplectic manifold  $(M, \omega)$  endowed with a leafwise symplectic foliation  $\mathcal{S}$ , and its  $\omega$ -orthogonal distribution  $\mathcal{S}'$ . Assume that the bihomogeneous components of  $\omega$  with respect to the decomposition  $TM = \mathcal{S}' \oplus T\mathcal{S}$  are closed. Then, again, formula (3.8) holds good.*

*Proof.* — Being a Dirac bracket,  $\Pi$  is transversally constant. On the other hand, if  $\omega = \omega_{(2,0)} + \omega_{(0,2)}$ ; the hypothesis  $d\omega_{(2,0)} = 0$  implies  $(L_Y\omega_{(2,0)})(X_1, X_2) = 0$  for  $(Y \in T\mathcal{S}, X_{1,2} \in \mathcal{S}')$ , and we see that  $\omega_{(2,0)}$  defines a transversal symplectic structure of  $\mathcal{S}$ . Q.e.d.

COROLLARY 3.3 [X]. — *Let  $\Pi$  be the Poisson structure defined on  $M = S \times N$  by a fixed symplectic structure of  $S$ , and assume that  $S$  has finite Betti numbers. Then one has*

$$(3.9) \quad H_{LP}^k(M, \Pi) = \bigoplus_{q=0}^k [H^q(S, \mathbb{R}) \otimes \Lambda^{k-q}(N)].$$

This result follows from (3.8) and from

PROPOSITION 3.4. — *Let  $\mathcal{F}$  be the foliation of  $M = F \times N$  by the leaves  $F \times \{x\}$  ( $x \in N$ ), and assume that  $F$  has finite Betti numbers. Then*

$$(3.10) \quad H^q(M, \Phi^p(\mathcal{F})) = H^q(F, \mathbb{R}) \otimes \Lambda^p(N).$$

*Proof.* — For  $q = 0$  the result was proven in [E] by a spectral sequence argument. Generally, we have the following straightforward argument. By the foliated de Rham theorem [V1], p. 216, we have

$$(3.11) \quad H^q(M, \Phi^p(\mathcal{F})) = \frac{\ker [d_f : \Lambda^{p,q}(M) \rightarrow \Lambda^{p,q+1}(M)]}{\text{im} [d_f : \Lambda^{p,q-1}(M) \rightarrow \Lambda^{p,q}(M)]}.$$

In our case,  $\Lambda^{p,q}(M)$  is isomorphic to the space  $\Lambda^q(F, \Lambda^p(N))$  of  $\Lambda^p(N)$ -valued  $q$ -forms on  $F$  by the mapping which sends  $\lambda \in \Lambda^{p,q}(M)$  to  $\tilde{\lambda} \in \Lambda^q(F, \Lambda^p(N))$  defined by

$$(\tilde{\lambda}_y(Y_1, \dots, Y_q))_x(X_1, \dots, X_p) = (-1)^p \lambda_{(x,y)}(X_1, \dots, X_p, Y_1, \dots, Y_q),$$

$y \in F$ ,  $x \in N$ ,  $Y_i \in T_y F$ ,  $X_j \in T_x N$ . Moreover, this isomorphism sends  $d_f$  to the exterior differential of  $\Lambda^p N$ -valued forms. Hence (3.11) becomes

$$H^q(M, \Phi^p(\mathcal{F})) = H^q(F, \Lambda^p(N)) = H^q(F, \mathbb{R}) \otimes \Lambda^p(N),$$

where the last equality follows from the hypothesis on  $F$ . Q.e.d.

*Remark.* — If  $M = S \times N$  of Corollary 3.3 is given a Poisson structure  $\Pi$  which has the symplectic foliation  $S \times \{x\}$  ( $x \in N$ ), but where each leaf has a different symplectic structure (e.g., the structure studied in [X]),  $\Pi$  is no more transversally constant, but we may use Propositions 3.2. and 3.4, and get

$$(3.12) \quad E_2^{p,q}(M, \Pi) = H^p(S, \mathbb{R}) \otimes \Lambda^q(N).$$

COROLLARY 3.4. — *Let  $(M, \Pi)$  be an arbitrary regular Poisson manifold. Then every  $x \in M$  has a connected open neighbourhood  $Y$  such that*

$$(3.13) \quad H_{LF}^k(U, \Pi|_U) = \Gamma(\Phi^k(\mathcal{S}|_U)),$$

*i.e., the space of the  $\mathcal{S}$ -foliated  $k$ -forms over  $U$ .*

Indeed, we may take  $U = S \times N$  where  $S$  is contractible, and such that the product coordinates are canonical for  $\Pi$  in the sense of [L], p. 257. Then Corollary 3.3 holds on  $U$ , and we get (3.13). We shall say that such a neighbourhood  $U$  is LP-simple.

**COROLLARY 3.5** (*The LP Poincaré Lemma* [L]). — *Let  $(M, \Pi)$  be a regular Poisson manifold, and  $x \in M$ . Then, there exists an open neighbourhood  $U$  of  $x$  in  $M$  such that, if  $Q \in \mathcal{V}^k(U)$  and  $\sigma Q = 0$ , one has  $Q = A + \sigma B$  for some  $B \in \mathcal{V}^{k-1}(U)$  and a  $k$ -vector field  $A$  over  $U$  which is projectable to a  $k$ -vector field of a local transversal submanifold of  $\mathcal{S}$  in  $U$ .*

*Proof.* — Take  $U$  LP-simple, and with  $\Pi$ -canonical coordinates. The latter define a bigrading, and we may write  $Q = \sum_{p=0}^k (\lambda^{p,k-p})^{\tilde{\#}}$ , where  $\lambda$  are differential forms, and  $\tilde{\#}$  is like in (3.2). The use of the canonical coordinates makes  $\Pi|_U$  transversally constant and transversally Riemannian hence, by Proposition 3.3 and formula (3.3),  $\sigma = \sigma''$ , and  $\sigma Q = 0$  is equivalent to  $d_f \lambda^{p,k-p} = 0$  ( $k=0, \dots, p$ ). But  $d_f$  satisfies a local Poincaré lemma [V1], p. 215, hence, there are local forms  $\mu$  such that  $\lambda^{p,k-p} = d_f \mu^{p,k-p-1}$  for  $k-p > 0$ , while  $\lambda^{k,0}$  is a foliate form. The conclusion follows by using again (3.3). Q.e.d

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