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CHARACTERIZATION OF THE LINEAR PARTIAL DIFFERENTIAL OPERATORS WITH CONSTANT COEFFICIENTS THAT ADMIT A CONTINUOUS LINEAR RIGHT INVERSE

by R. MEISE, B.A. TAYLOR & D. VOGT

In the early fifties L. Schwartz posed the problem of determining when a linear differential operator $P(D)$ has a (continuous linear) right inverse; that is, when does there exist a continuous linear map R such that

$$P(D)R(f) = f \text{ for all } f \in \mathcal{E}(\Omega) \text{ or all } f \in \mathcal{D}'(\Omega) .$$

For example, when Ω is $\mathbb{R}^n$ and $P(D)$ is hyperbolic in some direction, such an operator exists; one can take $R(f)$ to be the unique solution of the Cauchy-Problem $P(D)u = f$ with zero initial data. Negative results for important special classes were given by several authors. For $n \geq 2$ Grothendieck has shown (see e.g. Treves [T1]) that no elliptic operator can have a right inverse on $\mathcal{E}(\Omega)$. The same holds for hypoelliptic operators, as Vogt[V1], [V2] has proved. For parabolic and other operators this had been shown before by Cohoon [C1], [C2].

In the present article we give a fairly complete solution of Schwartz's problem. As one main result we show that for an open set Ω in $\mathbb{R}^n$ and for $P \in \mathbb{C}[z_1, \ldots, z_n]$ the differential operator $P(D)$ has a right inverse on $\mathcal{E}(\Omega)$ if and only if $P(D)$ has a right inverse on $\mathcal{D}'(\Omega)$. This property is further characterized by several other conditions in Theorem 2.7. In particular it is equivalent to the fact that Ω satisfies a very strong form of P-convexity, which we call P-convexity with bounds.

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For the evaluation of these conditions we use different methods. If Ω is P-convex with bounds and has a C^1-boundary, then Holmgren's uniqueness theorem can be used to show that P is hyperbolic with respect to each non-characteristic normal vector to $\partial\Omega$. In particular, a bounded set Ω with C^1-boundary is P-convex with bounds if and only if P is hyperbolic with respect to each non-characteristic direction, and this happens if and only if every open convex set is P-convex with bounds (see Thm. 3.8).

For convex open sets Ω, Fourier analysis can be used to reformulate P-convexity with bounds as a Phragmén-Lindelöf condition for the algebraic variety $V(P) = \{z \in \mathbb{C}^n : P(-z) = 0\}$ (see Thm. 4.5). This Phragmén-Lindelöf condition is related to a different but similar one introduced by Hörmander [HO2]. The evaluation of the condition shows that for $n \geq 3$ there exist non-hyperbolic operators $P(D)$ on $\mathcal{E}(\mathbb{R}^n)$ which do have a right inverse. The case $n = 2$ is exceptional, since a consequence of the Phragmén-Lindelöf condition implies that $P(D)$ has a right inverse on $\mathcal{E}(\mathbb{R}^2)$ if and only if $P(D)$ is hyperbolic.

Parts of the results of the present paper were announced in [MTV1] and [MTV2]. Recently, Palamodov [P] has shown that the splitting of differential complexes with constant coefficients over convex open sets in $\mathbb{R}^n$ is also characterized by the Phragmén-Lindelöf condition for the set of algebraic varieties associated with the complex.

The authors thank A. Kaneko for pointing out to them that one equivalence in Theorem 3.8 had been obtained already by de Christoforis [CR]. They also wish to thank L. Ehrenpreis for informing them that the problem of existence of right inverses was posed by L. Schwartz.

1. Preliminaries.

In this preliminary section we introduce most of the notation which will be used in the article. For undefined notation we refer to Hörmander [HO1], [HO3], and Schwartz [S].

1.1. *Spaces of functions and distributions.* — Let Ω be an open subset of $\mathbb{R}^n$. For $\epsilon > 0$ we define

$$\Omega_\epsilon := \{x \in \Omega : |x| < \frac{1}{\epsilon} \text{ and } \operatorname{dist}(x, \partial\Omega) > \epsilon\} .$$

(1) For $k \in \mathbb{N}_0$ we denote by $C^k(\Omega)$ the space of complex-valued functions on Ω which are continuously differentiable up to the order

k. $C^k(\Omega)$ is a Fréchet space under the semi-norms

$$\| f \|_{\epsilon,k} := \sup_{x\in\Omega_\epsilon} \sup_{|\alpha|\leq k} |f|(\alpha)(x)|, \epsilon > 0 .$$

(2) By $\mathcal{E}(\Omega)$ or $C^\infty(\Omega)$ we denote the space of all C^∞-functions on Ω endowed with the Fréchet space topology induced by the semi-norms $\| \cdot \|_{\epsilon,k}$, $\epsilon > 0$, $k \in \mathbb{N}$. Note that the topology of $\mathcal{E}(\Omega)$ is also induced by the semi-norms

$$|f|_{\epsilon,k} := \Big(\sum_{|\alpha|\leq k} \int_{\Omega_\epsilon} |f|^{(\alpha)}(x)|^2 d\lambda(x)\Big)^{1/2}, \epsilon > 0, k \in \mathbb{N},$$

where λ denotes the Lebesgue measure on $\mathbf{R}^n$.

(3) For a compact set $K \subset \Omega$ we let

$$\mathcal{D}(K) := \{f \in \mathcal{E}(\Omega) \ : \ \operatorname{Supp} f \subset K\}$$

and endow $\mathcal{D}(K)$ with the Fréchet-space topology induced by $\mathcal{E}(\Omega)$. Then

$$\mathcal{D}(\Omega) = \bigcup_{\epsilon>0} \mathcal{D}(\overline{\Omega}_\epsilon)$$

is endowed with its usual inductive limit topology.

(4) For $k \in \mathbb{N}_0$ we define the Sobolev space

$$W^k(\Omega) = \{f \in L_2(\Omega) \ : \ f \text{ is weakly differentiable up to the order } k \text{ and } |f|_k := \Big(\sum_{|\alpha|\leq k} \int |f^{(\alpha)}(x)|^2 d\lambda(x)\Big)^{1/2} < \infty\} .$$

By $W_0^k(\Omega)$ we denote the closure of $\mathcal{D}(\Omega)$ in $W^k(\Omega)$.

(5) If $X(\Omega)$ denotes any of the spaces definied in (1) – (4) then $X'(\Omega)$ or $X(\Omega)'$ denotes the strong dual of $X(\Omega)$. Moreover, for an open subset U of Ω we let

$$X(\Omega, U) := \{f \in X(\Omega) \ : \ f|_U = 0\} .$$

This notation will be used also for $\mathcal{D}'(\Omega, U)$ and $\mathcal{E}'(\Omega, U)$.

1.2. *Polynomials and partial differential operators.*

(1) By $\mathbb{C}[z_1, ..., z_n]$ we denote the ring of all complex polynomials in n variables, which will be also regarded as functions on $\mathbb{C}^n$. For $P \in \mathbb{C}[z_1, ..., z_n]$, $P(z) = \sum_{|\alpha|\leq m} a_\alpha z^\alpha$, with $\sum_{|\alpha|=m} |a_\alpha| \neq 0$ we call

$$P_m \ : \ z \mapsto \sum_{|\alpha|=m} a_\alpha z^\alpha$$

the principal part of P. Note that P_m is a homogeneous polynomial of degree m.

(2) For P as above and for an open set Ω in $\mathbf{R}^n$ we define the linear partial differential operator

$$P(D) : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega) \quad , \quad P(D)f : \sum_{|\alpha| \le m} a_\alpha i^{-|\alpha|} f^{(\alpha)} .$$

By this definition $P(D)$ acts on all subspaces of $\mathcal{D}'(\Omega)$. In particular $P(D) : X(\Omega) \to \mathcal{D}'(\Omega)$ is defined for all spaces $X(\Omega)$ defined in 1.1(1) – (5).

Note that $P(D)$ is a continuous endomorphism on $\mathcal{D}'(\Omega)$ and $\mathcal{E}(\Omega)$.

1.3. *Null spaces.* — For P as in 1.2 and an open set Ω in $\mathbf{R}^n$ we define

$$\mathcal{N}(\Omega) := \{f \in \mathcal{D}'(\Omega) : P(D)f = 0\}$$
$$N(\Omega) := \mathcal{N}(\Omega) \cap \mathcal{E}(\Omega) .$$

1.4. *Right inverses.* — For locally convex spaces E and F we denote

$$L(E,F) := \{A : E \to F : A \text{ is continuous and linear}\} .$$

A map $A \in L(E,F)$ is said to admit a right inverse, if there exists $R \in L(F,E)$ so that $A \circ R = \mathrm{id}_F$.
Note that a topological epimorphism $A \in L(E,F)$ admits a right inverse if and only if there exists $P \in L(E,E)$ with $P^2 = P$ and $\mathrm{im} P = \ker A$, i.e. iff $\ker A$ is a complemented subspace of E.
Obviously, the surjectivity of A is necessary for the existence of a right inverse for A.

The existence of a continuous non-linear right inverse for continuous linear surjective maps between Fréchet spaces is guaranteed by a result of Michael [M].

2. Right inverses on $\mathcal{D}'(\Omega)$ and $\mathcal{E}(\Omega)$.

For an open set Ω in $\mathbf{R}^n$ we characterize in this section the partial differential operators $P(D)$ that admit a right inverse on $\mathcal{D}'(\Omega)$ (resp. on $\mathcal{E}(\Omega)$). In particular we show that $P(D)$ has a right inverse on $\mathcal{D}'(\Omega)$ if and only if $P(D)$ has a right inverse on $\mathcal{E}(\Omega)$. The results of the present section will be evaluated further in the subsequent sections.

Some parts of the following lemma are essentially due to Grothendieck (see Treves [T1]).

2.1. LEMMA. — *Let Ω be an open set in $\mathbf{R}^n$ and let P be a complex polynomial in n variables. Then we have (1)⇒(2)⇒(3)⇒(4) for the following assertions:*

(1) *$P(D) : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ admits a right inverse*

(2) *for each $\epsilon > 0$ there exists $0 < \delta < \epsilon$ so that for each $f \in D'(\Omega, \Omega_\delta)$ there exists $g \in \mathcal{D}'(\Omega, \Omega_\epsilon)$ with $P(D)g = f$*

(3) *for each $\epsilon > 0$ there exists $0 < \delta < \epsilon$ so that for each $\mu \in \mathcal{N}(\Omega_\delta)$ there exists $\nu \in \mathcal{N}(\Omega)$ with $\nu|_{\Omega_\epsilon} = \mu|_{\Omega_\epsilon}$*

(4) *for each $\epsilon > 0$ there exists $0 < \delta_0 < \epsilon$ so that for all $0 < \sigma < \eta < \delta < \delta_0$ and each $\xi \in \Omega_\eta \backslash \overline{\Omega}_\delta$ there exists $E_\xi \in \mathcal{D}'(\mathbf{R}^n)$ so that*

(i) $\operatorname{Supp} E_\xi \subset (\mathbf{R}^n \backslash \Omega_\epsilon) - \xi$

(ii) $P(D)E_\xi = \delta + T_\xi$ where $\operatorname{Supp} T_\xi \subset (\Omega_\sigma \backslash \overline{\Omega}_\eta) - \xi$.

Proof. — (1)⇒(2) : Let $R : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ denote a right inverse for $P(D)$ and let $\epsilon > 0$ be given. Since $\mathcal{D}(\overline{\Omega}_\epsilon)$ is a separable Fréchet space, we can choose a bounded subset B of $\mathcal{D}(\overline{\Omega}_\epsilon)$ which is total. Since B is bounded in $\mathcal{D}(\Omega)$,

$$q_B : \mathcal{D}'(\Omega) \to \mathbf{R} \quad , \quad q_B(\mu) := \sup_{\varphi \in B} |\mu(\varphi)|$$

is a continuous semi-norm on $\mathcal{D}'(\Omega)$. By the continuity of the right inverse R there exist a bounded set C in $\mathcal{D}(\Omega)$ and $L > 0$ so that

$$q_B(R\mu) \leq L q_C(\mu) \quad \text{for all } \mu \in \mathcal{D}'(\Omega) .$$

By Schwartz [S], III, Thm. IV, we may assume that there exist a sequence $(C_m)_{m \in \mathbf{N}}$ of positive numbers and a compact set $Q \supset \overline{\Omega}_\epsilon$ so that

$$C = \{\varphi \in \mathcal{D}(Q) \; : \; \sup_{x \in Q} \sup_{|\alpha| \leq m} |D^\alpha \varphi(x)| \leq C_m \text{ for all } m \in \mathbf{N}\} .$$

Now fix $0 < \delta < \epsilon$ so that $Q \subset \Omega_\delta$ and let $f \in \mathcal{D}'(\Omega, \Omega_\delta)$ be given. Then $g := R(f) \in \mathcal{D}'(\Omega)$ satisfies

$$q_B(g) = q_B(R(f)) \leq L q_C(f) = 0 .$$

Since B is total in $\mathcal{D}(\overline{\Omega}_\epsilon)$, this shows $g \in \mathcal{D}'(\Omega, \Omega_\epsilon)$. Hence (2) holds, because of

$$P(D)g = P(D)R(f) = f .$$

(2)⇒(3) : For a given number $\epsilon > 0$ choose $0 < \delta_0 \leq \epsilon$ according to (2) and fix $0 < \delta < \delta_0$. If $\mu \in \mathcal{N}(\Omega_\delta)$ is given, choose $\varphi \in \mathcal{D}(\Omega_\delta)$ with

$\varphi\,|_{\overline{\Omega}_{\delta_0}} \equiv 1$. Then $P(D)(\varphi\mu)$ is in $\mathcal{D}'(\Omega, \Omega_{\delta_0})$. Hence (2) implies the existence of $f \in \mathcal{D}'(\Omega, \Omega_\epsilon)$ with $P(D)f = P(D)(\varphi\mu)$. Then $\nu := \varphi\mu - f$ is in $\mathcal{N}(\Omega)$ and satisfies $\nu\,|_{\Omega_\epsilon} = \mu\,|_{\Omega_\epsilon}$.

(3)$\Rightarrow$(4) : For a given number $\epsilon > 0$ choose $0 < \delta_0 < \epsilon$ according to (3) and note that the conclusion of (3) holds for all $0 < \delta < \delta_0$. Now fix $0 < \sigma < \eta < \delta < \delta_0$, $\xi \in \Omega_\eta \backslash \overline{\Omega}_\delta$ and $F_\xi \in \mathcal{D}'(\mathbf{R}^n)$ satisfying $P(D)F_\xi = \delta_\xi$. Then $F_\xi\,|_{\Omega_\delta}$ is in $\mathcal{N}(\Omega_\delta)$. Hence (3) implies the existence of $\nu_\xi \in \mathcal{N}(\Omega)$ so that $\nu_\xi\,|_{\Omega_\epsilon} = F_\xi\,|_{\Omega_\epsilon}$. Now choose $\psi \in \mathcal{D}(\Omega_\sigma)$ so that $\psi(x) = 1$ for all x in some neighbourhood of $\overline{\Omega}_\eta$ and define $G_\xi \in \mathcal{D}'(\mathbf{R}^n)$ by $G_\xi := F_\xi - \psi\nu_\xi$. Then we have:

$$\operatorname{Supp} G_\xi \subset \mathbf{R}^n \backslash \Omega_\epsilon$$

$$P(D)G_\xi = \delta_\xi - P(D)(\psi\nu_\xi)$$

$$\operatorname{Supp}(P(D)\psi\nu_\xi) \subset \operatorname{Supp}\psi \backslash \overline{\Omega}_\eta \subset \Omega_\sigma \backslash \overline{\Omega}_\eta\ .$$

Now define $E_\xi \in \mathcal{D}'(\mathbf{R}^n)$ by

$$E_\xi\ :\ \phi \mapsto \langle G_\xi, \varphi(\cdot - \xi)\rangle, \quad \varphi \in \mathcal{D}(\mathbf{R}^n)\ ,$$

and note that E_ξ has all the desired properties.

An easy modification of the arguments used in the proof of Lemma 2.1 shows that the following holds.

2.2. LEMMA. — *Let Ω be an open set in $\mathbf{R}^n$ and let P be a complex polynomial in n variables. Then we have (1) $\Rightarrow$ (2) $\Rightarrow$ (3) for the following assertions:*

(1) *$P(D)$: $\mathcal{E}(\Omega) \to \mathcal{E}(\Omega)$ admits a right inverse*

(2) *for each $\epsilon > 0$ there exists $0 < \delta < \epsilon$ so that for each $f \in \mathcal{E}(\Omega, \Omega_\delta)$ there exists $g \in \mathcal{E}(\Omega, \Omega_\epsilon)$ with $P(D)g = f$.*

(3) *for each $\epsilon > 0$ there exists $0 < \delta < \epsilon$ so that for each $f \in N(\Omega_\delta)$ there exists $g \in N(\Omega)$ with $f\,|_{\Omega_\epsilon} = g\,|_{\Omega_\epsilon}$.*

2.3. LEMMA. — *Let Ω be an open set in $\mathbf{R}^n$ and let P be a complex polynomial in n variables. If the following condition $(*)$ holds :*

(*) *There exists a sequence $(\Omega_k)_{k\in\mathbb{N}}$ of open subsets of Ω satisfying $\Omega_k \subset\subset \Omega_{k+1}$ for all $k \in \mathbb{N}$ and $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$ so that for each $\xi \in \Omega_{k+2}\backslash\overline{\Omega}_{k+1}$ there exist E_ξ and T_ξ in $\mathcal{D}'(\mathbf{R}^n)$ with*

(a) $\operatorname{Supp} E_\xi \subset (\mathbf{R}^n\backslash\Omega_k) - \xi$

(b) $P(D)E_\xi = \delta + T_\xi$ *where* $\operatorname{Supp} T_\xi \subset (\Omega_{k+4}\backslash\overline{\Omega}_{k+3}) - \xi$,

then $P(D)$ admits a continuous linear right inverse on $\mathcal{D}'(\Omega)$ and on $\mathcal{E}(\Omega)$.

Proof. — For $k \in \mathbb{N}$ define $\epsilon_k > 0$ by $\epsilon_1 := \operatorname{dist}(\Omega_3, \Omega\backslash\Omega_4)$ and

$$\epsilon_k = \min(\operatorname{dist}(\Omega_{k-1}, \Omega\backslash\Omega_k)\ ,\ \operatorname{dist}(\Omega_{k+2}, \Omega\backslash\Omega_{k+3}))$$

for $k \geq 2$. Next fix $k \in \mathbb{N}$ and $\xi \in \Omega_{k+2}\backslash\overline{\Omega}_{k+1}$ and let $\mu \in \mathcal{D}'(\mathbf{R}^n)$ with

$$\operatorname{Supp}\mu \subset \xi + \{x \in \mathbf{R}^n\ :\ |x| < \epsilon_k\} =: \xi + B_{\epsilon_k}(0)$$

be given. Then (*) and the choice of ϵ_k imply ($\Omega_0 := \emptyset$)

$$P(D)\mu * E_\xi = \mu * (\delta + T_\xi) = \mu + \mu * T_\xi$$

$$\operatorname{Supp}\mu * E_\xi \subset \xi + B_{\epsilon_k}(0) + (\mathbf{R}^n\backslash\Omega_k) - \xi \subset \mathbf{R}^n\backslash\Omega_{k-1}$$

$$\operatorname{Supp}\mu * T_\xi \subset \xi + B_{\epsilon_k}(0) + (\Omega_{k+4}\backslash\overline{\Omega}_{k+3}) - \xi \subset \Omega\backslash\overline{\Omega}_{k+2}\ .$$

By the compactness of $\overline{\Omega}_{k+2}\backslash\Omega_{k+1}$ we can find $m \in \mathbb{N}$, $\xi_1, \dots, \xi_m \in \Omega_{k+2}\backslash\overline{\Omega}_{k+1}$ and $\varphi_1, \dots, \varphi_m \in \mathcal{D}(\Omega)$ so that

$$\operatorname{Supp}\varphi_j \subset \xi_j + B_{\epsilon_k}(0) \quad \text{for } 1 \leq j \leq m\ ,$$

and $\sum_{j=1}^{m} \varphi_j(x) = 1$ for all x in some neighbourhood of $\overline{\Omega}_{k+2}\backslash\Omega_{k+1}$. Next we define for $f \in \mathcal{D}'(\Omega, \Omega_{k+1})$

$$R_k(f) := \sum_{j=1}^{m} \operatorname{lm}(\varphi_j f) * E_{\xi_j}, \quad F_k(f) := -f + \sum_{j=1}^{m} \operatorname{lm}(\varphi_j f) * (\delta + T_{\xi_j})\ .$$

Then (*) implies

$$P(D)R_k(f) = f + F_k(f) \text{ for each } f \in \mathcal{D}'(\Omega, \Omega_{k+1})\ .$$

Moreover, the preceding considerations imply that

$$\operatorname{Supp} R_k(f) \subset \mathbf{R}^n\backslash\Omega_{k-1}$$
$$\operatorname{Supp} F_k(f) \subset \Omega\backslash\overline{\Omega}_{k+2}$$

for each $f \in \mathcal{D}'(\Omega, \Omega_{k+1})$. Hence we have continuous linear maps

$$R_k\ :\ \mathcal{D}'(\Omega, \Omega_{k+1}) \to \mathcal{D}'(\Omega, \Omega_{k-1}), \quad F_k\ :\ \mathcal{D}'(\Omega, \Omega_{k+1}) \to \mathcal{D}'(\Omega, \Omega_{k+2})$$

which satisfiy

$$(**) \qquad P(D) \circ R_k = \mathrm{id}_{\mathcal{D}'(\Omega,\Omega_{k+1})} + F_k \, .$$

Now we want to use these properties to construct a continuous linear map $R : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ with $P(D) \circ R = \mathrm{id}_{\mathcal{D}'(\Omega)}$. To do this, we choose $\psi \in \mathcal{D}(\Omega_3)$ with $\psi|_{\overline{\Omega}_2} \equiv 1$. Next we fix a fundamental solution E of $P(D)$ and we define

$$R_0 \, : \, \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega), \quad R_0(g) := E * (\psi g)\,|_\Omega$$

and

$$F_0 \, : \, \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega, \Omega_2), \quad F_0(g) := (1-\psi)g \, .$$

Then we have

$$\operatorname{Supp} F_{k-1} \circ \ldots \circ F_0(g) \subset \Omega \backslash \Omega_{k+1} \quad \text{for all} \ \ g \in \mathcal{D}'(\Omega), \ k \in \mathbb{N} \, ,$$

and therefore

$$\operatorname{Supp} R_k \circ F_{k-1} \circ \ldots \circ F_0(g) \subset \Omega \backslash \Omega_{k-1} \text{ for all } g \in \mathcal{D}'(\Omega) \, .$$

Hence the series

$$R(g) := R_0(g) + \sum_{k=1}^{\infty} = (-1)^{k+1} R_k \circ F_{k-1} \circ \ldots \circ F_0(g), \quad g \in \mathcal{D}'(\Omega)$$

has locally finite supports. Consequently it defines a continuous linear map $R : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$. From $(**)$ we get

$$\begin{aligned} P(D)R(g) = & \psi g + F_0(g) + F_1 \circ F_0(g) \\ & + \sum_{k=2}^{\infty} (-1)^{k+1} (F_{k-1} \circ \ldots \circ F_0(g) + F_k \circ \ldots \circ F_0(g)) = g \, . \end{aligned}$$

Hence $R : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ is a right inverse for $P(D)$. An inspection of the proof shows that R maps $\mathcal{E}(\Omega)$ continuously and linearly into $\mathcal{E}(\Omega)$. Hence R also gives a right inverse for $P(D)$ on $\mathcal{E}(\Omega)$.

Before we combine Lemma 2.1 and 2.3 we investigate conditions for the existence of a right inverse for $P(D)$ on $\mathcal{E}(\Omega)$. To do this, we introduce the following notation.

Notation. — For an open set Ω in $\mathbf{R}^n$, $\epsilon > 0$ and $m \in \mathbb{N}_0$ we put

$$B_{\epsilon,m} := \{\mu \in \mathcal{E}'(\Omega) \, : \, \operatorname{Supp} \mu \subset \Omega_\epsilon, |\mu(f)| \leq \|f\|_{\epsilon,m} \text{ for all } f \in \mathcal{E}(\Omega)\} \, .$$

Obviously, $B_{\epsilon,m}$ is a relatively compact subset of $\mathcal{E}'(\Omega)$. Moreover, for each compact set K in $\mathcal{E}'(\Omega)$ there exist $\epsilon > 0$ and $m \in \mathbb{N}$ with $K \subset mB_{\epsilon,m}$.

2.4. LEMMA. — *Let Ω be an open set in $\mathbf{R}^n$ and let P be a complex polynomial in n variables. If $P(D) : \mathcal{E}(\Omega) \to \mathcal{E}(\Omega)$ admits a right inverse then the following condition $(*)$ holds:*

$$(*)\qquad \begin{array}{l} \textit{For each } \epsilon > 0 \textit{ there exists } 0 < \delta < \epsilon \textit{ so that for each } 0 < \eta < \delta \\ \textit{and each } m \in \mathbb{N}_0 \textit{ there exist } k \in \mathbb{N}_0 \textit{ and } C > 0 \textit{ so that for each} \\ \mu \in \mathcal{E}'(\Omega_\epsilon) \textit{ with } (\mu + \operatorname{im} P(D)^t) \cap B_{\eta,m} \neq \emptyset \textit{ there exists } \lambda \in \mathcal{E}'(\Omega_\delta) \\ \textit{so that } \mu + P(D)^t\lambda \in CB_{\delta,k}\ . \end{array}$$

Proof. — Choose a right inverse $R \in L(\mathcal{E}(\Omega))$ for $P(D)$ and note that

$$\pi := (R \circ P(D))^t = P(D)^t \circ R^t$$

is a projection on $\mathcal{E}'(\Omega)$ with $\operatorname{im} \pi = \operatorname{im} P(D)^t$. Hence $Q := \mathrm{id}_{\mathcal{E}'(\Omega)} - \pi$ is a projection on $\mathcal{E}'(\Omega)$ and satisfies

$$\ker Q = \operatorname{im} \pi = \operatorname{im} P(D)^t = \operatorname{im} P(-D)\ .$$

Now let $\epsilon > 0$ be given. Then $(\mathcal{E}(\Omega)/\mathcal{E}(\Omega, \Omega_\epsilon))'$ can be identified canonically with $\mathcal{E}(\Omega, \Omega_\epsilon)^\perp$. Since $\mathcal{E}(\Omega)$ is a Fréchet-Schwartz space, the set $\Delta := \{\delta_x \ :\ x \in \Omega_\epsilon\}$ is total in $\mathcal{E}(\Omega, \Omega_\epsilon)^\perp$ and relatively compact in $\mathcal{E}'(\Omega)$. Hence $Q(\Delta)$ is relatively compact in $\mathcal{E}'(\Omega)$. Consequently, there exists $0 < \delta_0 < \epsilon$ so that

$$\bigcup\{\operatorname{Supp} Q(\delta_x) \ :\ x \in \Omega_\epsilon\} \subset \Omega_{\delta_0}\ .$$

Since $\mathcal{E}(\Omega, \Omega_{\delta_0})^\perp$ is closed in $\mathcal{E}'(\Omega)$, this implies

$$(2)\qquad Q(\mathcal{E}(\Omega, \Omega_\epsilon)^\perp) \subset \mathcal{E}(\Omega, \Omega_{\delta_0})^\perp\ .$$

Now fix $0 < \delta < \delta_0$ and let $0 < \eta < \delta$ and $m \in \mathbb{N}_0$ be given. By the continuity of Q there exist $0 < \zeta < \eta$, $k \in \mathbb{N}_0$ and $C > 0$ with

$$(3)\qquad Q(B_{\eta,m}) \subset CB_{\zeta,k}\ .$$

Next fix $\mu \in \mathcal{E}'(\Omega_\epsilon)$ and assume that for some $\nu \in \mathcal{E}'(\Omega)$ we have $\mu + P(D)^t\nu \in B_{\eta,m}$. Then (1) implies

$$Q(\mu + P(D)^t\nu) = Q\mu + Q(P(D)^t\nu) = Q\mu$$

and therefore (3) gives $Q\mu \in CB_{\zeta,k}$. Moreover, $\operatorname{Supp}\mu \subset \Omega_\epsilon$ and (2) imply $Q\mu \in \mathcal{E}(\Omega, \Omega_{\delta_0})^\perp$. This gives

$$\operatorname{Supp} Q\mu \subset \overline{\Omega}_{\delta_0} \subset \Omega_\delta\ .$$

Hence we can find $\tilde{C}$ depending only on δ_0, δ, k and C so that $Q\mu \in \tilde{C}B_{\delta,k}$. Now define $\lambda := -R^t(\mu)$ and note that

$$(5)\qquad \mu + P(D)^t\lambda = \mu - P(D)^tR^t(\mu) = \mu - \pi(\mu) = Q\mu \in \tilde{C}B_{\delta,k}\ .$$

Hence (4) implies

(6) $\quad \operatorname{Supp} P(-D)\lambda = \operatorname{Supp} P(D)^t\lambda = \operatorname{Supp}(Q\mu - \mu) \subset \Omega_\delta$.

Now note that the surjectivity of $P(D)$ on $\mathcal{E}(\Omega)$ implies that Ω is P - convex in the sense of Hörmander [HO1], Def. 3.5.1. Hence (6) and Hörmander [HO1], Thm. 3.5.2, imply $\operatorname{Supp}\lambda \subset \Omega_\delta$.

2.5. LEMMA. — *Let Ω be an open set in $\mathbb{R}^n$ and let P be a complex polynomial in n variables. If $(*)$ is satisfied*

$(*)$ *For each $\epsilon > 0$ there exists $0 < \delta < \epsilon$ so that for each $0 < \eta < \delta$ there exist $m \in \mathbb{N}_0, k \in \mathbb{N}_0$ and $C > 0$ so that for each $\mu \in \mathcal{E}'(\Omega_\epsilon)$ with $(\mu + \mathrm{im}P(D)^t) \cap B_{\eta,m} \neq \emptyset$ there exist $\lambda \in \mathcal{E}'(\Omega_\delta)$ so that $\mu + P(D)^t\lambda \in CB_{\delta,k}$*

then the following assertions hold:

(1) *Ω is P - convex.*

(2) *For each $\epsilon > 0$ there exists $0 < \delta < \epsilon$ so that for each $0 < \eta < \delta$ there exists $l \in \mathbb{N}$ so that for each $f \in C^l(\Omega, \Omega_\delta)$ there exists $g \in \mathcal{D}'(\Omega_\eta, \Omega_\epsilon)$ so that $P(D)g = f\,|_{\Omega_\eta}$ holds in $\mathcal{D}'(\Omega_\eta)$.*

Proof. — (1) To show that Ω is P - convex in the sense of Hörmander [HO1], Def. 3.5.1, let K be a given compact subset of Ω. Then there exists $\epsilon > 0$ with $K \subset \Omega_\epsilon$. Choose $0 < \delta < \epsilon$ according to $(*)$, fix $0 < \eta < \delta$ and choose $m, k \in \mathbb{N}_0$ and $C > 0$ according to $(*)$. Next fix $\varphi \in \mathcal{D}(\Omega)$ with

$$\operatorname{Supp} P(-D)\varphi \subset K$$

and let $\mu := -P(-D)\varphi = -P(D)^t\varphi \in \mathcal{E}'(\Omega)$. Then

$$\mu + P(D)^t\varphi = \mu + P(-D)\varphi = 0 \in B_{\eta,m}$$

implies that for each $s \in]0, 1]$ we have

$$\frac{1}{s}(\mu + P(D^t)\varphi) \in B_{\eta,m} .$$

Hence $(*)$ implies the existence of $\lambda_s \in \mathcal{E}'(\Omega_\delta)$ so that

$$P(D)^t\Big[-\frac{1}{s}\varphi + \lambda_s\Big] = \frac{1}{s}\mu + P(D)^t\lambda_s \in CB_{\delta,k} .$$

Now note that $P(D)$ is surjective on $\mathcal{E}(\mathbb{R}^n)$ (see Hörmander [HO1], Thm. 3.5.1). Hence $P(D)^t = P(-D) : \mathcal{E}'(\mathbb{R}^n) \to \mathcal{E}'(\mathbb{R}^n)$ is an injective topological homomorphism. Therefore, there exist $l \in \mathbb{N}$, $D > 0$ and a bounded open set ω so that

$$P(-D)^{-1}(CB_{\delta,k}) \subset DB_{\omega,l}$$

where

$$B_{\omega,l} = \{\nu \in \mathcal{E}'(\mathbf{R}^n) \ : \ \operatorname{Supp}\nu \subset \omega, \ |\nu(f)| \le \|f\|_{\omega,l} \text{ for all } f \in \mathcal{E}(\mathbf{R}^n)\} \ .$$

This implies

$$-\frac{1}{s}\varphi + \lambda_s \in DB_{\omega,l} \quad \text{for all } s \in]0,1]$$

and consequently $\varphi = \lim_{s\to 0} s\lambda_s$ in $\mathcal{E}'(\mathbf{R}^n)$. Since $\operatorname{Supp}\lambda_s \subset \Omega_\delta$ for each $s \in]0,1]$, this proves $\operatorname{Supp}\varphi \subset \overline{\Omega}_\delta$.

(2) For a given number $\epsilon > 0$ choose $0 < \delta < \epsilon$ according to (*) and fix $0 < \eta < \delta$. Then choose $m, k \in \mathbb{N}_0$ and $C \ge 1$ according to (*) and note that without loss of generality we can assume $m \le k$. Since Ω is P-convex by (1), $P(-D) = P(D)^t \ : \mathcal{E}'(\Omega) \to \mathcal{E}'(\Omega)$ is an injective topological homomorphism. Hence there exist $l \in \mathbb{N}$, $L > 0$ and $0 < \zeta < \eta$ so that

$$P(-D)^{-1}(B_{\eta,k}) \subset LB_{\zeta,l} \ .$$

For a given $\nu \in \mathcal{E}'(\Omega)$ with $P(-D)\nu \in B_{\eta,k}$ we therefore have

$$\nu = P(-D)^{-1}(P(-D)\nu) \in LB_{\zeta,l} \ .$$

Since Ω is P-convex, this together with Hörmander [HO1], Thm. 3.5.2 and Lemma 3.4.3, implies $\operatorname{Supp}\nu \subset \Omega_\eta$. Hence we get

$$\text{(3)} \qquad \frac{1}{L}P(-D)^{-1}(B_{\eta,k}) \subset \tilde{B} := B_{\zeta,l} \cap \{\mu \in \mathcal{E}'(\Omega) \ : \ \operatorname{Supp}\mu \subset \Omega_\eta\} \ .$$

Now let

$$X := \operatorname{span}\{(P(-D)\mathcal{E}'(\Omega)) \cap B_{\eta,k}, \mathcal{E}'(\Omega_\epsilon)\} \subset \mathcal{E}'(\Omega)$$

and fix $f \in C^l(\Omega, \Omega_\delta)$. Note that for $\nu \in \mathcal{E}'(\Omega)$ satisfying $P(-D)\nu \in B_{\eta,k}$, we have $\nu \in L\tilde{B}$ because of (3). Therefore $\langle \nu, f\rangle$ is defined for such $\nu \in \mathcal{E}'(\Omega)$. Now we define $F : X \to \mathbb{C}$ by

$$F \ : \ P(-D)\nu + \mu \mapsto \begin{cases} 0 & \text{if } \operatorname{Supp}(P(-D)\nu + \mu) \subset \Omega_\delta \\ \langle \nu, f\rangle & \text{otherwise} \end{cases}$$

for $P(-D)\nu \in \operatorname{span}((P(-D)\mathcal{E}'(\Omega)) \cap B_{\eta,k})$ and $\mu \in \mathcal{E}'(\Omega_\epsilon)$. To show that F is well-defined, assume that

$$P(-D)\nu_1 + \mu_1 = P(-D)\nu_2 + \mu_2$$

for $\mu_1, \mu_2 \in \mathcal{E}'(\Omega_\epsilon)$ and $P(-D)\nu_1, P(-D)\nu_2 \in \operatorname{span} B_{\eta,k}$. Then we have

$$\operatorname{Supp} P(-D)(\nu_1 - \nu_2) \subset \operatorname{Supp}(\mu_2 - \mu_1) \subset \Omega_\epsilon \ .$$

Since Ω is P-convex, this implies

$$\operatorname{Supp}(\nu_1 - \nu_2) \subset \Omega_\epsilon$$

and consequently

$$F(P(-D)\nu_1 + \mu_1) = F(P(-D)\nu_2 + \mu_2) \ .$$

Using the P-convexity of Ω_δ and discussing several cases one shows that F is a linear functional on X.

Next we denote by E_0 the normed space which is generated by the bounded absolutely convex subset $B_{\eta,m}$ of $\mathcal{E}'(\Omega)$. We claim that $F|_{X\cap E_0}$ is continuous. To show this, fix $P(-D)\nu + \mu \in X \cap B_{\eta,m}$. Then $(*)$ implies the existence of $\lambda \in \mathcal{E}'(\Omega_\delta)$ satisfying

$$P(-D)\lambda + \mu \in CB_{\delta,k} \ . \tag{4}$$

From this we get (assuming $C \geq 1$)

$$\begin{aligned} P(-D)(\nu - \lambda) &= (P(-D)\nu + \mu) - (P(-D)\lambda + \mu) \\ &\in B_{\eta,m} + CB_{\delta,k} \subset 2CB_{\eta,k} \ . \end{aligned} \tag{5}$$

Since $P(-D)\nu$ is in $\operatorname{span} B_{\eta,k}$, this implies $P(-D)\lambda \in \operatorname{span} B_{\eta,k}$, so that $P(-D)\lambda + \mu \in X$. (3) and (5) imply

$$\nu - \lambda \in 2CL\tilde{B} \ .$$

From this, (4) and (5) we get by the definition of F

$$\begin{aligned} F(P(-D)\nu + \mu) &= F(P(-D)\lambda + \mu) + F(P(-D)(\nu - \lambda)) \\ &= F(P(-D)(\nu - \lambda)) = \langle \nu - \lambda, f \rangle \end{aligned}$$

and hence

$$|F(P(-D)\nu + \mu)| \leq 2CL\|f\|_\zeta \ .$$

Since $P(-D)\nu + \mu$ was an arbitrary element of $X \cap B_{\eta,m}$, this proves that F is bounded on $X \cap B_{\eta,m}$. Hence the theorem of Hahn – Banach implies the existence of $\tilde{F} \in E_0'$ satisfying $\tilde{F}\,|_{X\cap E_0} = F$.

Next let Φ: $\mathcal{D}(\Omega_\eta) \to \mathcal{E}'(\Omega_\eta)$ denote the canonical injection, defined by

$$\Phi(\varphi) \ : \ h \mapsto \int_{\Omega_\eta} \varphi(x)h(x)dx, \quad \varphi \in \mathcal{D}(\Omega_\eta),\ h \in \mathcal{E}(\Omega_\eta) \ .$$

It is easily seen that Φ maps $\mathcal{D}(\Omega_\eta)$ continuously into E_0. Therefore $g := \tilde{F} \circ \Phi = \Phi^t(\tilde{F})$ is in $\mathcal{D}'(\Omega_\eta)$. Since Ω_δ is P-convex and since f vanishes on Ω_δ, the definition of $\tilde{F}$ and F gives for each $\varphi \in \mathcal{D}(\Omega_\eta)$

$$\begin{aligned} \langle P(D)g, \phi \rangle &= \langle g, P(-D)\varphi \rangle = \tilde{F}(\Phi(P(-D)\varphi)) = F(P(-D)\Phi(\varphi)) \\ &= \langle \Phi(\varphi), f \rangle = \langle f, \varphi \rangle \ . \end{aligned}$$

Furthermore, $\varphi \in \mathcal{D}(\Omega_\epsilon)$ implies $\Phi(\varphi) \in \mathcal{E}'(\Omega_\epsilon)$ so that

$$\langle g, \varphi \rangle = \tilde{F}(\Phi(\varphi)) = F(\Phi(\varphi)) = 0 \ .$$

Hence g is in $\mathcal{D}'(\Omega_\eta, \Omega_\epsilon)$ and satisfies $P(D)g = f\,|_{\Omega_\eta}$.

2.6. LEMMA. — *Let Ω be an open set in $\mathbf{R}^n$ and let P be a complex polynomial. If condition 2.5(2) holds then condition 2.1(4) holds, too.*

Proof. — For a given number $\epsilon > 0$ choose $0 < \delta_0 < \epsilon$ according to 2.5(2). Then fix $0 < \zeta < \eta < \delta < \delta_0$ and choose $l \in \mathbf{N}$ so large that 2.5(2) holds with η replaced by ζ. Next fix $\xi \in \Omega_\eta \backslash \overline{\Omega}_\delta$ and choose $M \in \mathbf{N}$ so large that the equation $\Delta^M F_\xi = \delta_\xi$ has a solution $F_\xi \in C^l(\mathbf{R}^n)$ (see Hörmander [HO1], Thm. 3.2.1). Also choose $\varphi_\xi \in \mathcal{D}(\Omega_\eta \backslash \overline{\Omega}_\delta)$ so that $\varphi_\xi(x) = 1$ for all x in a neighbourhood of ξ. Then

$$f_\xi := \varphi_\xi F_\xi \in C^l(\Omega, \Omega_\delta) \subset C^l(\Omega, \Omega_{\delta_0})$$

and

$$\Delta^M f_\xi = \delta_\xi + h_\xi, \quad h_\xi \in C^\infty(\Omega, \Omega_{\delta_0}) .$$

Therefore condition 2.5(2) with η replaced by ζ implies the existence of $g_\xi, H_\xi \in \mathcal{D}'(\Omega_\zeta, \Omega_\epsilon)$ satisfying

$$P(D)g_\xi = f_\xi\,|_{\Omega_\zeta} \quad \text{and} \quad P(D)H_\xi = h_\xi\,|_{\Omega_\zeta} .$$

Now choose an open set ω with $\overline{\Omega}_\eta \subset \omega \subset \overline{\omega} \subset \Omega_\zeta$, fix $\psi \in \mathcal{D}(\Omega_\zeta)$ with $\psi\,|_\omega \equiv 1$ and let

$$G_\xi := \psi(\Delta^M g_\xi - H_\xi) .$$

Then we have

$$\operatorname{Supp} G_\xi \subset \Omega_\zeta \backslash \Omega_\epsilon$$

$$P(D)G_\xi\,|_\omega = P(D)(\Delta^M g_\xi - H_\xi)\,|_\omega = \Delta^M f_\xi\,|_\omega - h_\xi\,|_\omega = \delta_\xi$$

and hence

$$P(D)G_\xi = \delta_\xi + S_\xi, \text{ where } \operatorname{Supp} S_\xi \subset \Omega_\zeta \backslash \overline{\Omega}_\eta .$$

As in the proof of 2.1(4), this implies condition 2.1(4).

2.7. THEOREM. — *For an open set Ω in $\mathbf{R}^n$ and for a complex polynomial P in n variables the following assertions are equivalent :*

(1) *$P(D)\colon \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ admits a right inverse*

(2) *$P(D)\colon \mathcal{E}(\Omega) \to \mathcal{E}(\Omega)$ admits a right inverse*

(3) *for each compact set $K \subset \Omega$ there exists a compact set L with $K \subset L \subset \Omega$ so that for each open set $\omega \subset\subset \Omega$ with $L \subset \omega$ there exist $s \in \mathbf{N}_0$ and $D > 0$ so that for each $\nu \in \mathcal{E}'(\omega)$ satisfying $P(-D)\nu\,|_{\omega\backslash K} \in B^0$ we have*

$\nu\ |_{\omega\setminus L} \in DB^{-s}(\omega\setminus L)$, where B^0 denotes the closed unit ball of $L^2(\mathbf{R}^n)$, while $B^{-s}(\omega\setminus L)$ denotes the closed unit ball of $W^{-s}(\omega\setminus L) = W_0^s(\omega\setminus L)'$.

Moreover, (1), (2) and (3) are also equivalent to each of the following conditions: 2.1(2), 2.1(3), 2.1(4), 2.3(∗), 2.4(∗), 2.5(∗) and 2.5(2).

Proof. — We first note that 2.1(4) implies 2.3(∗). To show this, fix $\epsilon_1 > 0$ so that $\Omega_{\epsilon_1} \neq \emptyset$. Then choose $(\epsilon_k)_{k\in\mathbf{N}}$ inductively so that

$$0 < \epsilon_{k+1} < \min\left(\frac{1}{k}, \delta_0(\epsilon_k)\right)$$

where $\delta_0(\epsilon_k)$ denotes the number $0 < \delta_0 < \epsilon_k$ which exists by 2.1(4) if we choose $\epsilon = \epsilon_k$. Next define $\Omega_k := \Omega_{\epsilon_k}$ and note that $\Omega_k \subset\subset \Omega_{k+1}$ and $\Omega = \bigcup_{k\in\mathbf{N}} \Omega_k$ by the choice of the sequence $(\epsilon_k)_{k\in\mathbf{N}}$. Choosing $\sigma = \epsilon_{k+4}$, $\eta = \epsilon_{k+3}$ and $\delta = \epsilon_{k+2}$ in 2.1(4), we get that all the other requirements of condition 2.3(∗) are fulfilled, too.

Therefore, Lemma 2.1 and 2.3 (resp. Lemma 2.4, 2.5 and 2.6) show that the following implications hold :
(1) ⇒ 2.1(2) ⇒ 2.1(3) ⇒ 2.1(4) ⇒2.3(∗) ⇒ (2) and (1);
(2) ⇒ 2.4(∗) ⇒ 2.5(∗) ⇒ 2.5(2) ⇒ 2.1(4) ⇒ 2.3(∗) ⇒ (1) and (2).

To prove the equivalence of (2) and (3) we claim that Lemma 2.4 and Lemma 2.5 remain true if the sets $B_{\epsilon,m}$ are defined as

$$B_{\epsilon,m} = \{\mu \in \mathcal{E}'(\Omega) \ : \ \operatorname{Supp}\mu \subset \Omega_\epsilon,\ |\mu(f)| \leq |f|_{\epsilon,m} \text{ for all } f \in \mathcal{E}(\Omega)\}$$

for $\epsilon > 0$ and $m \in \mathbf{N}_0$. This can be checked by going through the corresponding proofs again. Therefore (2) is equivalent to 2.4(∗) and also to 2.5(∗) with the new meaning of $B_{\epsilon,m}$. Consequently, (2) is equivalent to

for each $\epsilon > 0$ there exists $0 < \delta < \epsilon$ so that for each $0 < \eta < \delta$ there exist $k \in \mathbf{N}_0$ and $C > 0$ so that for each $\mu \in \mathcal{E}'(\Omega_\epsilon)$ with

$$(\mu + \operatorname{im} P(D)^t) \cap B_{\eta,0} \neq \emptyset \tag{4}$$

there exists $\lambda \in \mathcal{E}'(\Omega_\delta)$ so that $\mu + P(D)^t\lambda \in CB_{\delta,k}$.

Hence the proof is complete if we show (4) ⇔ (3).

(4) ⇒ (3) : If K is a given compact subset of Ω, we choose $\epsilon > 0$ so that $K \subset \Omega_\epsilon$. Then we choose $0 < \delta < \epsilon$ according to (4) and let $L := \overline{\Omega}_\delta$. Next we fix an open set $\omega \subset\subset \Omega$ with $L \subset \omega$ and we choose $0 < \eta < \delta$ with $\omega \subset \Omega_\eta$. Then we choose $k \in \mathbf{N}_0$ and $C > 0$ according to (4) and we fix $\nu \in \mathcal{E}'(\omega)$ so that $P(-D)\nu\ |_{\omega\setminus K}$ is in B^0. To show that the conclusion of (3) holds for suitable $s \in \mathbf{N}_0$ and $D > 0$ (not depending on ν), we choose

$\varphi \in \mathcal{D}(\Omega_\epsilon)$ so that $\varphi \equiv 1$ in a neighbourhood of K. Then $\mu := -\varphi P(-D)\nu$ is in $\mathcal{E}'(\Omega_\epsilon)$ and we have

$$\mu + P(-D)\nu = (1-\varphi)P(-D)\nu \in \mathcal{E}'(\Omega_\eta) \cap B^0 = B_{\eta,0} \ .$$

Therefore, (4) implies the existence of $\lambda \in \mathcal{E}'(\Omega_\delta)$ so that

$$T := \mu + P(-D)\lambda \in CB_{\delta,k} \ .$$

Hence $P(-D)\lambda = T - \mu$ is in $\mathcal{E}'(\Omega_\delta)$. By the L^2-version of Lemma 2.5, (4) implies that Ω is P-convex. Therefore, Hörmander [HO1], Thm. 3.5.2, implies $\lambda \in \mathcal{E}'(\Omega_\delta)$. Consequently we have

$$(5) \qquad (\nu - \lambda)\,|_{\omega\setminus L} = (\nu - \lambda)\,|_{\omega\setminus\overline{\Omega}_\delta} = \nu\,|_{\omega\setminus L} \ .$$

Next note that

$$P(-D)(\nu - \lambda) = (\mu + P(-D)\nu) - (\mu + P(-D)\lambda) \in \mathcal{E}'(\Omega_\eta) \cap (1+C)B_{\eta,k} \ .$$

Since we have seen already that Ω is P-convex, it follows as in the first part of the proof of 2.5(2), that there exist $D > 0$ and $s \in \mathbb{N}_0$, $s \geq k$, so that $\nu - \lambda$ is in $DB_{\eta,s}$. Because of (5), this implies (3).

(3) $\Rightarrow$ (4) : Let us first show that (3) implies P-convexity of Ω. If K is a given compact set in Ω, choose L according to (3). Next let $\varphi \in \mathcal{D}(\Omega)$ be given, so that $\operatorname{Supp} P(-D)\varphi \subset K$. Then choose an open set $\omega \subset\subset \Omega$ with $\operatorname{Supp}\varphi \subset \omega$ and note that for each $t > 0$ we have $P(-D)(t\varphi)\,|_{\omega\setminus K} = 0 \in B^0$. Hence (3) implies $t\varphi\,|_{\omega\setminus L} \in DB^{-s}$ for all $t > 0$, which proves $\operatorname{Supp}\varphi \subset L$.

To show that (4) holds, let $\epsilon > 0$ be given and let $K := \overline{\Omega}_\epsilon$. Next choose L according to (3) and find $0 < \delta_1 < \epsilon$ with $L \subset \Omega_\delta$. Then choose $0 < \delta < \delta_1$, let $0 < \eta < \delta$ be given and assume that for some $\mu \in \mathcal{E}'(\Omega_\epsilon)$ and some $\nu \in \mathcal{E}'(\Omega)$ we have

$$\mu + P(-D)\nu = \mu + P(D)^t\nu \in B_{\eta,0} \ .$$

This implies $\operatorname{Supp} P(-D)\nu \subset \Omega_\eta$. Since Ω is P-convex, Ω_η is P-convex, too. Therefore, $\operatorname{Supp}\nu \subset \Omega_\eta$ and we have

$$P(-D)\nu\,|_{\Omega_\eta\setminus K} = (\mu + P(-D)\nu)\,|_{\Omega_\eta\setminus K} \in B^0 \ .$$

Now (3) with $\omega = \Omega_\eta$ implies

$$\nu\,|_{\Omega_\eta\setminus L} \in DB^{-s}(\Omega_\eta\setminus L) \ .$$

Next choose $\varphi \in \mathcal{D}(\Omega_\delta)$ so that $\varphi \equiv 1$ in a neighbourhood of L and we let $\lambda := \varphi\nu$. Then λ is in $\mathcal{E}'(\Omega_\delta)$ and satisfies

$$\mu + P(-D)\lambda = \mu + P(-D)\nu - P(-D)(1-\varphi)\nu \ .$$

This shows that for suitable $E > 0$ and $m = \deg P$ we have

$$\mu + P(-D)\lambda \in B^0 + EDB^{-s-m}(\Omega_\eta) \quad \text{and} \quad \operatorname{Supp}(\mu + P(-D)\lambda) \subset \Omega_{\delta_1} .$$

Since E depends only on P and φ, but not on μ, this implies (4).

2.8. DEFINITION. — *Let Ω be an open set in $\mathbf{R}^n$ and let P be a complex polynomial in n variables. Ω is called P-convex with bounds if one of the equivalent conditions in Theorem 2.7 holds.*

2.9. *Remark.* — (a) Lemma 2.1 and 2.3 hold – mutatis mutandis – also for differential operators (even for ultradifferential operators admitting a fundamental solution) on the spaces $\mathcal{D}'_{(\omega)}(\Omega)$ and $\mathcal{D}'_{\{\omega\}}(\Omega)$ of ultradistributions. In particular they hold for all non-quasianalytic Gevrey-classes. For more details we refer to our forthcoming paper [MTV5] (see also Meise and Vogt [MV] for the case of one variable).

(b) From Theorem 2.7 and the proof of Lemma 2.3 it follows that a differential operator that admits a right inverse on $\mathcal{D}'(\Omega)$ also admits a right inverse on any non-quasianalytic class $\mathcal{E}_*$ of functions on Ω which has partitions of unity, which is an algebra with continuous multiplication and on which distributions act continuously by convolution. Moreover, $P(D)$ has also a right inverse on $\mathcal{D}'_*(\Omega)$, the associated class of ultradistributions on Ω. In particular, $P(D)$ has a right inverse on all non-quasianalytic Gevrey-classes (and Gevrey ultradistributions) whenever $P(D)$ has a right inverse on the distributions.
(c) The conditions 2.2(2) and 2.2(3) are equivalent to Ω being P-convex with bounds. This is shown in [MTV6].

2.10. COROLLARY. — *Let P be a complex polynomial in n variables and let $(\Omega_i)_{i\in I}$ be a family of open sets in $\mathbf{R}^n$ for which $\Omega := \cap_{i\in I}\Omega_i \neq \emptyset$ is open. If Ω_i is P-convex with bounds for each $i \in I$ then Ω is P-convex with bounds.*

Proof. — To show that condition 2.5(2) holds, let $\epsilon > 0$ be given. Then $\overline{\Omega}_\epsilon$ is compact in Ω_i for each $i \in I$. Hence there exists $\epsilon_i > 0$ so that $\overline{\Omega}_\epsilon \subset (\Omega_i)_{\epsilon_i}$ for $i \in I$. Since Ω_i is P-convex with bounds, we get from 2.7 the existence of $0 < \delta_i < \epsilon_i$ so that 2.1(2) holds for Ω_i. Now note that $\bigcap_{i\in I} \overline{(\Omega_i)_{\delta_i}}$ is contained in Ω. Hence there exists $0 < \delta < \epsilon$ so that

$$\bigcap_{i\in I} \overline{(\Omega_i)_{\delta_i}}) \subset \Omega_\delta .$$

Next fix $0 < \eta < \delta$ and let $f \in C(\Omega, \Omega_\delta)$ be given. Then $(\Omega \backslash \overline{(\Omega_i)_{\delta_i}})_{i \in I}$ is an open cover of the compact set $\overline{\Omega}_\eta \backslash \Omega_\delta$. Therefore we can find $m \in \mathbb{N}$ and $i_j \in I$ for $1 \leq j \leq m$ as well as $f_{i_j} \in C(\Omega, (\Omega_{i_j})_{\delta_{i_j}})$, $1 \leq j \leq m$, so that

$$f \mid_{\overline{\Omega}_\eta} = \sum_{j=1}^{m} f_{i_j} \mid_{\overline{\Omega}_\eta} .$$

By 2.1(2) there exist $g_{i_j} \in \mathcal{D}'(\Omega_{i_j}, (\Omega_{i_j})_{\epsilon_{i_j}})$ so that

$$P(D) g_{i_j} = f_{i_j} \quad \text{in } \mathcal{D}'(\Omega_{i_j}) \text{ for } 1 \leq j \leq m .$$

Consequently $g := \sum_{j=1}^{m} g_{i_j} \mid_{\Omega_\eta}$ is in $\mathcal{D}'(\Omega_\eta)$ and vanishes on

$$\Omega \cap \bigcap_{j=1}^{m} (\Omega_{i_j})_{\epsilon_{i_j}} \supset \Omega_\epsilon .$$

Hence $g \in \mathcal{D}'(\Omega_\eta, \Omega_\epsilon)$ and

$$P(D) g = \sum_{j=1}^{m} P(D) g_{i_j} \mid_{\Omega_\eta} = \sum_{j=1}^{m} f_{i_j} \mid_{\Omega_\eta} = f \mid_{\Omega_\eta} .$$

From Theorem 2.7 we can derive the following result of Vogt [V1], [V2], which extends a theorem of Grothendieck.

2.11. COROLLARY. — *Let P be a hypoelliptic polynomial in n ($n \geq 2$) variables. Then each open set Ω in $\mathbb{R}^n$ is not P-convex with bounds.*

Proof. — To argue by contradiction, we may assume that there exists an open set Ω in $\mathbb{R}^n$ which is P-convex with bounds. Then Theorem 2.7 implies that condition 2.1(3) holds. Next we fix $\epsilon > 0$ with $\Omega_\epsilon \neq \emptyset$ and choose $0 < \delta < \epsilon$ according to 2.1(3). Then we note that the hypoellipticity of P implies by Hörmander [HO1], 4.1.3, that for each open set Ω in $\mathbb{R}^n$ the space $\mathcal{N}(\Omega)$ is a nuclear Fréchet space which is contained in $\mathcal{E}(\Omega)$. Therefore we can define

$$\mathcal{N}B(\Omega_\delta) := \{ f \in \mathcal{N}(\Omega_\delta) \ : \ f \text{ is bounded on } \Omega_\delta \} .$$

endowed with the supremum norm. Moreover, we let

$$\mathcal{N}B(\Omega_\delta, \Omega_\epsilon) := \mathcal{N}B(\Omega_\delta) \cap \mathcal{E}(\Omega_\delta, \Omega_\epsilon) \quad \text{and} \quad \mathcal{N}(\Omega, \Omega_\epsilon) = \mathcal{N}(\Omega) \cap \mathcal{E}(\Omega, \Omega_\epsilon) .$$

Now it is easy to check that because of 2.1(3) the restriction map ρ: $\mathcal{N}(\Omega) \to \mathcal{N}B(\Omega_\delta)$ induces an isomorphism

$$\bar{\rho} \ : \ \mathcal{N}(\Omega)/\mathcal{N}(\Omega, \Omega_\epsilon) \to \mathcal{N}B(\Omega_\delta)/\mathcal{N}B(\Omega_\delta, \Omega_\epsilon) .$$

Hence the nuclear Fréchet space $\mathcal{N}(\Omega)/\mathcal{N}(\Omega,\Omega_\epsilon)$ is a Banach space and consequently finite dimensional. However, this is a contradiction, since all exponential solutions

$$f_z \,:\, x \mapsto \exp(-i\langle x,n\rangle), \quad z \in \mathbb{C}^n,\ P(z)=0$$

are linearly independent in $\mathcal{N}(\Omega)/\mathcal{N}(\Omega,\Omega_\epsilon)$.

3. Right inverses and hyperbolicity.

In this section we investigate how properties of the boundary of an open set Ω in $\mathbb{R}^n$ are related with the conditions for P-convexity with bounds which were derived in Theorem 2.7. In doing this we assume throughout the entire section that P always denotes a non-constant polynomial in n variables.

From Hörmander [HO1], 5.4.1, we recall that a complex polynomial P on $\mathbb{C}^n$ is called hyperbolic with respect to $N \in \mathbb{R}^n\backslash\{0\}$ if N is non-characteristic, i.e. $P_m(N) \neq 0$ and if there exists $\tau_0 \in \mathbb{R}$ so that for each $\xi \in \mathbb{R}^n$ and each $\tau < \tau_0$ we have $P(\xi + i\tau N) \neq 0$. P is called hyperbolic if P is hyperbolic with respect to some $N \in \mathbb{R}^n$.

3.1. LEMMA. — *Let $\Omega \subset \mathbb{R}^n$ be P-convex with bounds and let $N \in \mathbb{R}^n$ be non-characteristic for P. If there exists $x_0 \in \partial\Omega$ so that $\partial\Omega$ is continuously differentiable in a neighbourhood of x_0 and if N is normal to $\partial\Omega$ at x_0 then P is hyperbolic with respect to N.*

Proof. — After an appropriate change of variables we can assume $x_0 = 0$ and $N = (0,\dots,0,1)$. Further, we can assume that for a suitable zero-neighbourhood V in $\mathbb{R}^{n-1}$ and for some $a > 0$ there exists a C^1-function g: $V \to]-a,a[$ so that

$$\Omega \cap (V\times]-a,a[) = \{(x',x_n) \in V\times]-a,a[\,:\, x_n < g(x')\}$$
$$\partial\Omega \cap (V\times]-a,a[) = \{(x',g(x')) \,:\, x' \in V\}\ .$$

Since N is not characteristic for P, there exists $0 < \alpha < \pi/4$ so that for the closed cone

$$\Gamma_\alpha(N) := \{L \in \mathbb{R}^n \,:\, |L|\cos\alpha \leq \langle L,N\rangle\}$$

every non-zero vector in $\Gamma_\alpha(N)$ is not characteristic for P. Since g is a C^1-function with a vanishing derivative at zero, we can use the mean-value theorem to find $0 < R < 1$ so that

$$(1) \qquad U := \{x \in \mathbb{R}^n \,:\, \|x\|_\infty \leq R\} \subset V\times]-a,a[$$

(2) $$|g(x')| < \frac{R}{4} \sin\alpha \quad \text{for all} \quad x' \in \mathbb{R}^{n-1} \quad \text{with} \quad \|x'\|_\infty \le R\,.$$

Then

$$K := \{(x', x_n) \in U \ : \ -R \le x_n \le g(x') - \frac{R}{4} \sin\alpha\}$$

is a compact subset of Ω, hence there exists $\epsilon > 0$ with $K \subset \Omega_\epsilon$. Since Ω is P-convex with bounds, condition 2.1(2) holds. Hence there exists $0 < \delta < \epsilon$ so that for each $f \in \mathcal{D}'(\Omega, \Omega_\delta)$ there exists $h \in \mathcal{D}'(\Omega, \Omega_\epsilon)$ with $P(D)h = f$. Now choose $0 < \eta < \frac{R}{8} \sin\alpha$ so that

(3) $$U \cap \Omega_\delta \subset \{(x', x_n) \in U \ : \ -R \le x_n < g(x') - 2\eta\}$$

and choose $0 < r < \eta$ so that

$$\{(x', -\eta) \in \mathbb{R}^n \ : \ |x'| \le r\} \subset \{(x', x_n) \in U \ : \ g(x') - 2\eta < x_n < g(x')\}\,.$$

Next denote by Γ the open cone

$$\Gamma := \{L \in \mathbb{R}^n \ : \ |L| \cos(\frac{\pi}{2} - \alpha) < -\langle L, N\rangle\}$$

and note that every characteristic hyperplane through the origin intersects $\bar{\Gamma}$ not only at the origin, because of our choice of α. Now let $\sigma := \eta + r \sin\alpha$, define $\xi := (0, ..., 0, -\sigma)$ and note that $\xi \in \Omega \backslash \Omega_\delta$ because of (3). Therefore there exists $T \in \mathcal{D}'(\Omega, \Omega_\epsilon)$ with $P(D)T = \delta_\xi$.

Next fix $x' \in \mathbb{R}^{n-1}$ with $|x'| = r \cos\alpha$, put $y := (x', -\eta)$ and look at the open cone $y + \Gamma$. Then we get from (2)

$$\begin{aligned}(\operatorname{Supp} T) \cap U \subset U \backslash K &\subset \{(x', x_n) \in U \ : \ x_n > g(x') - \frac{R}{4} \sin\alpha\} \\ &\subset \{(x', x_n) \in U \ : \ x_n > -\frac{R}{2} \sin\alpha\}\,.\end{aligned}$$

By the construction we have

$$\{x \in \mathbb{R}^n \ : \ x_n = -\frac{R}{2} \sin\alpha\} \cap \overline{(y + \Gamma)} \subset \mathring{U}$$

Therefore we can define $T_y \in \mathcal{D}(y + \Gamma)$ by

$$T_y|_{(y+\Gamma)\cap U} = T|_{(y+\Gamma)\cap U} \quad \text{and} \quad T_y|_{(y+\Gamma)\backslash K} = 0\,.$$

Since $P(D)T \equiv 0$, it follows from Hörmander [HO1], Cor. 5.3.3, that $T_y = 0$. Since this holds for all $y = (x', -\eta)$, $|x'| = r\cos\alpha$ we get

$$\begin{aligned}&(\operatorname{Supp} T) \cap \{(x', x_n) \in \mathbb{R}^n : |X_n + \sigma| \ge r \sin\alpha, |x'| \ge r \cos\alpha\} \\ (4) \qquad &\subset \{(x', x_n) \in \mathbb{R}^n : -\sigma \ge x_n \ge -\sigma + r \sin\alpha, |x'| \ge (x_n + \sigma) \tan\alpha\} \\ &= \Gamma_1\,.\end{aligned}$$

Now choose $\chi \in \mathcal{D}(\xi + U_t(0))$, where $t = \frac{r}{2} \sin \alpha$ and

$$U_t(0) = \{x \in \mathbf{R}^n \ : \ |x| < t\}$$

which satisfies $\chi|_{\xi + U_{t/2}(0)} \equiv 1$. Then $T_0 := \chi T$ satisfies

$$P(D)T_0 = \delta_\xi - S_0$$

where $\operatorname{Supp} S_0 \subset \Gamma_1 \backslash (U_{t/2}(0) + \xi)$. After a suitable translation which moves ξ into the origin, we obtain from this a distribution T_1 which satisfies $P(D)T_1 = \delta - S_1$ and has support in a closed convex cone Γ_2 satisfying

$$\Gamma_2 \cap \{x \in \mathbf{R}^n \ : \ x_n \leq 0\} = \{0\} \ .$$

Moreover

$$\operatorname{Supp} S_1 \subset \Gamma_2 \backslash U_{t/2}(0) \ .$$

Now define $S_1^0 := \delta$ and $S_1^j := S_1 * \ldots * S_1$ (j-times) and note that the series $\sum_{j=0}^{\infty} S_1^j$ converges in $\mathcal{D}'(\mathbf{R}^n)$ since it is locally finite. From $\operatorname{Supp}\left(\sum_{j=0}^{\infty} S_1^j\right) \subset$ Γ_2 it follows that we can define

$$E := T_1 * (\sum_{j=0}^{\infty} S_1^j) \ .$$

Then

$$P(D)E = (P(D)T_1) * \left(\sum_{j=0}^{\infty} S_1^j\right) = (\delta - S_1) * \sum_{j=0}^{\infty} S_1^j = \delta$$

shows that E is a fundamental solution for $P(D)$. Since $\operatorname{Supp} E$ is contained in Γ_2, it follows from Hörmander [HO1], Thm. 5.6.2, that P is hyperbolic with respect to N.

To give a first application of Lemma 3.1 we denote by $H_+(N)$ (resp. $H_-(N)$) the positive (resp. negative) open half space determined by a vector $N \in \mathbf{R}^n \backslash \{0\}$, i.e.

$$H_\pm(N) := \{x \in \mathbf{R}^n \ : \ \pm\langle x, N\rangle > 0\} \ .$$

3.2. PROPOSITION.. — *Let $N \in \mathbf{R}^n$ be non-characteristic for P. Then $H_\pm(N)$ is P-convex with bounds if and only if P is hyperbolic with respect to N.*

Proof. — $\Longrightarrow$: Lemma 3.1.

$\Longleftarrow$: For $j \in \mathbb{N}_0$ define U_j by

$$U_0 := \{x \in \mathbf{R}^n \ : \ \langle x, N\rangle > \frac{1}{2}\}$$

$$U_j := \{x \in \mathbf{R}^n \ : \ \frac{1}{j+2} < \langle x, N\rangle < \frac{1}{j}\} \quad , \quad j \in \mathbb{N} .$$

Then $(U_j)_{j\in\mathbb{N}_0}$ is an open cover of $H_+(N)$. Hence we can choose a C^∞-partition of unity $(\varphi_j)_{j\in\mathbb{N}}$, subordinate to $(U_j)_{j\in\mathbb{N}_0}$. Since P is also hyperbolic with respect to $-N$ (see Hörmander [HO1], Thm. 5.5.1) there exist fundamental solutions E_+ resp. E_- with supports in closed cones which are contained in $H_+(N)$ resp. $H_-(N)$ except for the origin. Then for each $f \in \mathcal{E}(H_+(N))$ the series

$$R(f) := E_+ * (\varphi_0 f) + \sum_{j=1}^{\infty} E_- * (\varphi_j f)$$

is locally finite and converges in $\mathcal{E}(H_+(N))$. Moreover, it is easily checked that $R : \mathcal{E}(H_+(N)) \to \mathcal{E}(H_+(N))$ is a right inverse for $P(D)$. Therefore, $H_+(N)$ is P-convex with bounds by Theorem 2.7.

3.3. COROLLARY. — *Let Ω be an open convex polyhedron in $\mathbf{R}^n$ with faces whose normal vectors are non-characteristic for P. Then Ω is P-convex with bounds if and only if P is hyperbolic with respect to all vectors which are normal to some face of Ω.*

Proof. — $\Longrightarrow$: Lemma 3.1

$\Longleftarrow$: Since Ω is a finite intersection of translations of open half spaces, this follows from Proposition 3.2 and Corollary 2.10.

3.4. LEMMA. — *Let Ω be an open convex polyhedron in $\mathbf{R}^n$ with faces whose normal vectors are non-characteristic for P. If Ω is P-convex with bounds then the following condition holds :*

for each $\epsilon > 0$ there exists $0 < \delta < \epsilon$ so that for each

(*) $\quad f \in \mathcal{D}'(\mathbf{R}^n, \Omega_\delta)$ *there exists* $g \in \mathcal{D}'(\mathbf{R}^n, \Omega_\epsilon)$ *with*

$P(D)g = f$.

Proof. — Without loss of generality we can assume $0 \in \Omega$. Then $\{t\overline{\Omega} \ : \ 0 < t < 1\}$ is a fundamental system of compact subsets of Ω. Hence for $\epsilon > 0$ there exists $0 < t < 1$ so that $\Omega_\epsilon \subset t\overline{\Omega}$. Next choose $0 < \delta < \epsilon$ with $t\overline{\Omega} \subset \Omega_\delta$ and let $f \in \mathcal{D}'(\mathbf{R}^n, \Omega_\delta)$ be given. If we denote by $N_1, \ldots, N_m$ the outer normals to the faces of Ω then we have

$$\mathbf{R}^n \backslash t\overline{\Omega} = \bigcup_{j=1}^{m} H_j$$

where for suitable $b_j \in \mathbb{R}$

$$H_j = \{x \in \mathbb{R}^n \ : \ \langle x, N_j \rangle > b_j\}, \quad 1 \le j \le m \ .$$

Using a suitable partition of unity, we therefore have

$$f = \sum_{j=1}^{m} f_j \text{ with } \operatorname{Supp} f_j \subset H_j, \ 1 \le j \le m \ .$$

By Corollary 3.3 and Hörmander [HO1], Thm. 5.6.1, the hypothesis implies the existence of fundamental solutions E_j for $P(D)$ where $\operatorname{Supp} E_j$ is contained in a closed cone which is contained in $H_+(N_j)$, except for the origin. Therefore,

$$g := \sum_{j=1}^{m} E_j * f_j$$

has all the required properties.

3.5. LEMMA. — *Let $(\Omega_j)_{j \in \mathbb{N}}$ be an increasing sequence of open subsets of $\mathbb{R}^n$. If Ω_j satisfies condition 3.4($*$) for each $j \in \mathbb{N}$ then $\Omega := \bigcup_{j \in \mathbb{N}} \Omega_j$ is P-convex with bounds.*

Proof. — To show that condition 2.5(2) holds, let $\epsilon > 0$ be given. Then there exists $k \in \mathbb{N}$ with $\overline{\Omega}_\epsilon \subset \Omega_k$. Hence there exists $\epsilon' > 0$ with $\overline{\Omega}_\epsilon \subset (\Omega_k)_{\epsilon'}$. Now choose $0 < \delta' < \epsilon'$ so that condition 3.4($*$) holds for Ω_k and choose $0 < \delta < \epsilon$ so that $\overline{(\Omega_k)_{\delta'}} \subset \Omega_\delta$. Then fix $0 < \eta < \delta$ and choose $\varphi \in \mathcal{D}(\Omega)$ with $\varphi \,|_{\overline{\Omega}_\eta} \equiv 1$. Next let $f \in \mathcal{D}'(\Omega, \Omega_\delta)$ be given. Since φf is in $\mathcal{D}'(\mathbb{R}^n, (\Omega_k)_{\delta'})$, 3.4($*$) implies the existence of $g \in \mathcal{D}'(\mathbb{R}^n, (\Omega_k)_{\epsilon'})$ with

$$P(D) g = \varphi f \ .$$

By our choices $g \,|_{\Omega_\eta}$ is in $\mathcal{D}'(\Omega_\eta, \Omega_\epsilon)$ and satisfies

$$P(D)(g \,|_{\Omega_\eta}) = f \,|_{\Omega_\eta} \ .$$

3.6. PROPOSITION. — *Let $N \in S^{n-1}$ be characteristic for P. If $N = \lim_{k \to \infty} N_k$, where P is hyperbolic with respect to $N_k \in \mathbb{R}^n$ for all $k \in \mathbb{N}$ then $H_+(N)$ is P-convex with bounds.*

Proof. — For a non-characteristic vector M of P let $\Gamma(P, M)$ denote the component of the set

$$\{\Theta \in S^{n-1} \ : \ P_m(\Theta) \neq 0\}$$

which contains M. Using this notation, the hypothesis implies the existence of $M \in S^{n-1}$ so that P is hyperbolic with respect to M and so that

$N \in \overline{\Gamma(P,M)}^{S^{n-1}}$. Note that $\Gamma(P,M)$ is an open subset of S^{n-1} and that by Hörmander [HO3], 12.4.5, P is hyperbolic with respect to each vector $K \in S^{n-1}$ with $\pm K \in \Gamma(P,M)$. Therefore we can find an increasing sequence $(\Omega_j)_{j\in\mathbb{N}}$ of parallelepipeds so that P is hyperbolic with respect to each normal vector to every face of Ω_j and so that

$$H_+(N) = \bigcup_{j\in\mathbb{N}} \Omega_j \ .$$

Hence the result follows from 3.3, 3.4 and 3.5.

The following examples show that for characteristic vectors N in general $H_+(N)$ is not P-convex with bounds.

3.7 *Example.* — Let $P \in \mathbb{C}[z_1, z_2[$ be defined by

$$P(z_1, z_2) = z_1^2 - iz_2 \ .$$

Obviously $N = (0,1)$ is a characteristic vector for P. Since P is hypoelliptic, $P(D)$ does not have a right inverse on $\mathcal{E}(H_-(N))$ by Vogt [V1], [V2] (see also 2.11). Another example is $Q(z_1, z_2, z_3) = (z_1^2 + z_2^2 - z_3^2)(z_1^2 + z_2^2 + z_3^2)$. Since Q has an elliptic factor, it follows from 2.11 that Q does not have a right inverse on $\mathcal{E}(H_\pm(N))$, where $N = (1,0,1)$.

For a further evaluation of Lemma 3.1 we recall the following notation: Let Ω be an open subset of $\mathbb{R}^n$ with C^1-boundary. Then the Gauss-map $G\colon \partial\Omega \to S^{n-1}$ is defined by $G(x) := N_x$, where N_x denotes the outer unit normal to $\partial\Omega$ at x.

3.8. THEOREM. — *For a non-constant polynomial P on $\mathbb{C}^n$ the following conditions are equivalent :*

(1) *there exists an open bounded subset $\Omega \neq \emptyset$ of $\mathbb{R}^n$ with C^1-boundary which is P-convex with bounds*

(2) *there exists an open subset $\Omega \neq \emptyset$ of $\mathbb{R}^n$ with C^1-boundary and surjective Gauss-map which is P-convex with bounds*

(3) *P is hyperbolic with respect to every non-characteristic direction*

(4) *P and its principle part P_m are equally strong and P_m is proportional to a product of m linear functions with real coefficients*

(5) *each open convex subset of $\mathbb{R}^n$ is P-convex with bounds.*

Proof. — (1)$\Rightarrow$(2) : This is obvious. (2)$\Rightarrow$(3) : Lemma 3.1. (3)$\Leftrightarrow$(4) : This holds by Lanza de Christoforis [CR] Thm. 1. (3)$\Rightarrow$(5) : Let Ω be an open convex subset of $\mathbb{R}^n$. Then the present hypothesis implies the existence of an increasing sequence $(\Omega_j)_{j\in\mathbf{N}}$ of open convex polyhedra with non-characteristic faces so that each Ω_j is P-convex with bounds and so that

$$\Omega = \bigcup_{j\in\mathbf{N}} \Omega_j \ .$$

Therefore (5) follows from 3.3, 3.4 and 3.5. (5)$\Rightarrow$(1) : This is obvious.

There are many open sets Ω in $\mathbf{R}^n$ for which no differential operator has a right inverse on $\mathcal{E}(\Omega)$. To show this, we introduce the following definition.

3.9. DEFINITION. — *Let Ω be an open set in $\mathbf{R}^n$ and let $N \in \mathbf{R}^n\backslash\{0\}$ be given. A point $x_0 \in \partial\Omega$ is called a point of inner support for N if there exists a compact neighbourhood U of x_0 so that for $\gamma := \langle x_0, N\rangle$ we have*

$$U \cap \{x \in \mathbf{R}^n \ : \ \langle x, N\rangle < \gamma\} \subset \Omega \quad \textit{and}$$
$$\partial U \cap \{x \in \mathbf{R}^n \ : \ \langle x, N\rangle \leq \gamma\} \subset \Omega \ .$$

3.10. LEMMA. — *Let Ω be an open set in $\mathbf{R}^n$, let P be a non-constant polynomial on $\mathbf{C}^n$ and let $x_0 \in \partial\Omega$ be a point of inner support for $N \in \mathbf{R}^n$. If $P(D) \ : \ \mathcal{E}(\Omega) \to \mathcal{E}(\Omega)$ is surjective then N is not characteristic for P.*

Proof. — Assume that $N \in S^{n-1}$ is characteristic for P and choose U and γ according to 3.9. Next choose $\varphi \in \mathcal{D}(\overset{\circ}{U})$ so that $\varphi(x) = 1$ for all x in a neighbourhood of the set

$$\partial\Omega \cap \{x \in \mathbf{R}^n \ : \langle x, N\rangle = \gamma\} \ .$$

Then denote by K the closure of the set

$$\{x \in \mathbf{R}^n \ : \ \varphi^2(x) \neq \varphi(x) \text{ and } \langle x, N\rangle \leq \gamma\}$$

and note that K is a compact subset of Ω by the properties of U and φ.

Since N is characteristic for P it follows from Hörmander [HO1], 5.2.2, that for each $\delta < \gamma$ we can find $f_\delta \in \mathcal{E}(\mathbf{R}^n)$ which satisfies $P(-D)f_\delta = 0$, $x_0 - (\gamma - \delta)N \in \operatorname{Supp}(f_\delta)$ and

$$\operatorname{Supp} f_\delta \subset \{x \in \mathbf{R}^n \ : \ \langle x, N\rangle \leq \delta\} \ .$$

Now define $g_\delta := \varphi f_\delta$ and note that

$$\operatorname{Supp} g_\delta \subset \{x \in U \ : \ \langle x, N\rangle \leq \delta\}$$

so it is compact in Ω. Furthermore we have $\operatorname{Supp} P(-D)g_\delta \subset K$ and

$$\operatorname{dist}(x_0, \operatorname{Supp} g_\delta) = \gamma - \delta$$

for δ sufficiently close to γ. This shows that Ω is not $P(D)$-convex. By Hörmander [HO1], Def. 3.5.1 and Cor. 3.5.2, this contradicts the hypothesis that $P(D)$ is surjective on $\mathcal{E}(\Omega)$.

3.11. PROPOSITION. — *Let Ω be an open subset of $\mathbb{R}^n$ for which $\mathbb{R}^n \backslash \Omega$ has a compact component. Then Ω is not P-convex with bounds for each non-constant polynomial P on $\mathbb{C}^n$.*

Proof. — Let P be a non-constant polynomial on $\mathbb{C}^n$. Since the surjectivity of $P(D)$ on $\mathcal{E}(\Omega)$ is necessary for the existence of a right inverse, assume that $P(D)$ is surjective. Let K denote a compact component of $\mathbb{R}^n \backslash \Omega$ and let $N \in \mathbb{R}^n \backslash \{0\}$ be given. Then there exists t_0 so that

$$K \cap \{x \in \mathbb{R}^n \ : \ \langle x, N \rangle \leq t_0\} \neq \emptyset \, .$$

Define

$$\gamma := \sup\{t < t_0 \ : \ K \cap \{x \in \mathbb{R}^n \ : \ \langle x, N \rangle \leq t\} = \emptyset\}$$

and pick

$$x_0 \in \partial\Omega \cap \{x \in \mathbb{R}^n \ : \ \langle x, N \rangle \leq \gamma\} \neq \emptyset \, .$$

Then it is easily checked that x_0 is a point of inner support for N. Hence Lemma 3.10 implies that N is not characteristic for P. Since $N \in \mathbb{R}^n \backslash \{0\}$ was arbitrarily chosen, the polynomial P is elliptic. However, then $P(D)$ does not admit a right inverse on any open set, as Grothendieck has shown (cf. 2.11).

3.12. *Examples.* — (1) Let Ω be an open set in $\mathbb{R}^n$ with C^2-boundary. Then, for $x_0 \in \partial\Omega$ there exists a neighbourhood V of x_0 and $\varphi \in C^2(V)$ so that

$$V \cap \Omega = \{x \in V \ : \ \varphi(x) < 0\} \quad , \quad V \cap \partial\Omega = \{x \in V \ : \ \varphi(x) = 0\}$$

and $\operatorname{grad} \varphi(x) \neq 0$ for all $x \in V \cap \partial\Omega$. If

$$H_\varphi(x_0) = \Big(\frac{\partial^2 \phi}{\partial x_j \partial x_k}(x_0)\Big)_{j,k=1}^n$$

is negative definite then x_0 is a point of inner support for $N := \operatorname{grad} \varphi(x_0)$.

(2) For $t > 0$ define

$$\Omega(t) := \{x \in \mathbb{R}^n \ : \ x_n > 0 \text{ and } \sum_{j=1}^{n-1} x_j^2 + (x_n - \frac{t}{2})^2 > t^2\} \, .$$

Then (1) implies that each $N \in S^{n-1}$ with $N_n \geq 0$ admits a point of inner support in $\partial\Omega(t)$. Hence Lemma 3.10 implies that only elliptic operators $P(D)$ are surjective on $\mathcal{E}(\Omega(t))$. Therefore $\Omega(t)$ is not P-convex with bounds for each non-constant polynomial P on $\mathbb{C}^n$.

4. The Phragmén-Lindelöf condition.

In this section we use Fourier analysis in order to characterize when a convex open subset Ω of $\mathbb{R}^n$ is P-convex with bounds in terms of a Phragmén-Lindelöf condition on the zero variety of P.

Notation. — Let Ω be an open convex subset of $\mathbb{R}^n$ which is not empty. For sufficiently small $\epsilon > 0$ then Ω_ϵ is convex and not empty, too. By $h_\epsilon \colon \mathbb{R}^n \to \mathbb{R}$ we denote the support functional of $\overline{\Omega}_\epsilon$, i.e.

$$h_\epsilon(x) = \sup_{y \in \overline{\Omega}_\epsilon} \langle x, y\rangle = \sup_{y \in \overline{\Omega}_\epsilon} \sum_{j=1}^{n} x_j y_j \ .$$

4.1. DEFINITION.. — *Let P be a non-constant polynomial on $\mathbb{C}^n$, let $\Omega \neq \emptyset$ be a convex open subset of $\mathbb{R}^n$ and let*

$$V = V(P) := \{z \in \mathbb{C}^n \ : \ P(-z) = 0\} \ .$$

We say that P (resp. $V(P)$) satisfies the Phragmén-Lindelöf condition $PL(\Omega)$, if for each $\epsilon > 0$ there exists $0 < \delta < \epsilon$ so that for each $0 < \eta < \delta$ there exists $B > 0$ so that for each plurisubharmonic (psh.) function u on $\mathbb{C}^n$ the following two conditions :

(a) $u(z) \leq h_\epsilon(\mathrm{Im})z + O(\log(1 + |z|^2))$ *for all* $z \in \mathbb{C}^n$

(b) $u(z) \leq h_\epsilon(\mathrm{Im}z)$ *for all* $z \in V(P)$

imply

(c) $u(z) \leq h_\epsilon(\mathrm{Im}z) + B(\log(1 + |z|^2)) + B$ *for all* $z \in V(P)$.

We say that P (resp $V(P)$) satisfies the analytic Phragmén-Lindelöf condition $APL(\Omega)$ if the above holds for all $u = \log|f|$, where f is an entire function on $\mathbb{C}^n$.

For a comprehensive study of the Phragmén-Lindelöf condition we refer to our paper [MTV4], the results of which we are going to use in this section. Before we explain how $APL(\Omega)$ and $PL(\Omega)$ are related with the existence of a right inverse for $P(D)$, we first note :

4.2. LEMMA. — *Let $P = P_1 \ldots P_k$, where $P_1, \ldots, P_k$ are polynomials on $\mathbb{C}^n$ and let Ω be an open subset of $\mathbb{R}^n$. Then $P(D)$ has a right inverse on $\mathcal{E}(\Omega)$ if and only if each $P_j(D)$ has a right inverse on $\mathcal{E}(\Omega)$.*

Proof. — If R is a right inverse for $P(D)$ then $P_2(D) \circ \ldots \circ P_k(D) \circ R$ is a right inverse for $P_1(D)$. Obviously, $R_k \circ \ldots \circ R_1$ is a right inverse for $P(D)$, whenever R_j is a right inverse for $P_j(D)$.

4.3 *Some Fourier analysis.* — Let $P = P_1 \cdots P_k$, where $P_1 \ldots P_k$ are irreducible polynomials on $\mathbb{C}^n$ so that P_j is not proportional to P_k for $j \neq k$. Furthermore, let Ω be an open convex subset of $\mathbb{R}^n$. We denote by $A(V)$ the space of holomorphic functions on $V = \{z \in \mathbb{C}^n : P(-z) = 0\}$ and we define

$$A_\Omega(V) = \{f \in A(V) \ : \ \text{there exist } \epsilon > 0 \text{ and } k \in \mathbb{N} \text{ so that}$$
$$\|f\|_{\epsilon,k} := \sup_{z \in V} |f(z)| \exp(-h_\epsilon(\mathrm{Im} z) - k \log(1 + |z|^2)) < \infty\}$$

and we endow $A_\Omega(V)$ with its natural inductive limit topology. Then it is easy to check that $A_\Omega(V)$ is a (DFS)-space and that for each bounded set B in $A_\Omega(V)$ there exist $\lambda > 0$, $0 < \epsilon < 1$ and $k \in \mathbb{N}$ with $B \subset \lambda L_{\epsilon,k}$, where

$$L_{\epsilon,k} = \{f \in A_\Omega(V) \ : \ \|f\|_{\epsilon,k} \leq 1\} .$$

From the fundamental work of Ehrenpreis and Palamodov (see e.g. Hansen [H]) it is well-known that the Fourier-Laplace transform

$$F \ : \ N(\Omega)' \to A_\Omega(V) \quad , \quad F(\mu) \ : \ z \mapsto \mu_x(e^{-i\langle x,z\rangle}) \ , \quad z \in V$$

is a linear topological isomorphism.

If we let

$$U_{\epsilon,k} := \{\chi \in N(\Omega) \ : \ \|\chi\|_{\epsilon,k} \leq 1\} \quad 0 < \epsilon < 1, \quad k \in \mathbb{N}$$

then each bounded set of $N(\Omega)'$ is contained in a multiple of some set $U^o_{\epsilon,k}$. Hence we have

$$\text{(1)} \qquad \begin{aligned} &\text{for each } 0 < \eta < 1 \text{ there exist } < 0 < \sigma < 1, \ m \in \mathbb{N} \\ &\text{and } E > 0 \text{ so that } L_{\eta,0} \subset \mathcal{F}(EU^o_{\sigma,m}) . \end{aligned}$$

A sharper version of (1) can be derived from Hansen [H], Thm. 2.3, if we let

$$V_{\epsilon,k} := \{\chi \in N(\Omega) \ : \ |\chi|_{\epsilon,k} \leq 1\}, \quad 0 < \epsilon < 1, \quad k \in \mathbb{N} .$$

Then we get

$$\text{(2)} \qquad \begin{aligned} &\text{for each } 0 < \delta < 1 \text{ and each } q \in \mathbb{N} \text{ there exist} \\ &k \in \mathbb{N} \text{ and } D > 0 \text{ so that } L_{\delta,q} \subset \mathcal{F}(DV^o_{\delta,k}) . \end{aligned}$$

4.4. Lemma. — *Let P be a non-constant polynomial on $\mathbf{C}^n$ and let Ω be an open convex set in $\mathbf{R}^n$. If $P(D) : \mathcal{E}(\Omega) \to \mathcal{E}(\Omega)$ admits a right inverse then P satisfies the condition $APL(\Omega)$.*

Proof. — Let $P = P_1^{m_1} \cdots P_k^{m_k}$ where $P_1 \ldots P_k$ are irreducible polynomials so that P_j is not proportional to P_l for $j \neq l$ and let $Q := \prod_{i=1}^{k} P_i$. By 4.2, $P(D)$ has a right inverse on $\mathcal{E}(\Omega)$ if and only if $Q(D)$ has one. Since $V(Q) = V(P)$ we can therefore assume without loss of generality that $P = Q$ so that 4.3 applies. Since $Q(D) : \mathcal{E}(\Omega) \to \mathcal{E}(\Omega)$ admits a right inverse, Lemma 2.4 implies that condition 2.4(*) holds. To show that this implies $APL(\Omega)$, let $\epsilon > 0$ be given. Then fix $0 < \epsilon_1 < \epsilon$ and choose $0 < \delta < \epsilon_1$ according to 2.4(*) with ϵ replaced by ϵ_1. Next let $0 < \eta < \delta$ be given and choose $0 < \sigma < \eta$, $m \in \mathbf{N}$ and $E > 0$ so that 4.3(1) holds. Furthermore fix $0 < \zeta < \sigma$ and choose $k \in \mathbf{N}$ and $C > 0$ so that 2.4(*) holds with $\epsilon_1, \delta, \zeta$ and m and put $B := \max(k, \log CE)$. Now fix $f \in A(\mathbf{C}^n)$ which satisfies

$$\begin{array}{lll} \text{(a)} & \log|f(z)| \le h_\epsilon(\operatorname{Im} z) + O(\log(1+|z|^2)) & \text{for all } z \in \mathbf{C}^n \\ \text{(b)} & \log|f(z)| \le h_\eta(\operatorname{Im} z) & \text{for all } z \in V(P)\ . \end{array}$$

Then the theorem of Paley-Wiener-Schwartz and (a) imply that there exists $\mu \in \mathcal{E}(\Omega_{\epsilon_1})'$ so that

$$f(z) = \mu_x(e^{-i\langle x,z\rangle}) \quad \text{for all } z \in \mathbf{C}^n\ .$$

Moreover, $\nu := \mu\,|_{N(\Omega)}$ is in $N(\Omega)'$ and (b) implies that $\mathcal{F}(\nu)$ is in $L_{\eta,0}$. Therefore 4.3(1) shows that ν is in $EU^o_{\sigma,m}$. Because of $0 < \zeta < \sigma$, this and the theorem of Hahn-Banach imply the existence of $\tilde\nu \in \mathcal{E}'(\Omega)$ with $\tilde\nu \in EB_{\zeta,m}$ (using the notion from 2.4). Therefore

$$\tilde\nu - \mu \in N(\Omega)^\perp = (\ker P(D))^\perp = \operatorname{im} P(D)^t$$

implies

$$\tilde\nu = \mu + (\tilde\nu - \mu) \in (\mu + \operatorname{im} P(D)^t) \cap EB_{\zeta,m}\ .$$

Hence 2.4(*) gives the existence of $\lambda \in \mathcal{E}'(\Omega_\delta)$ so that $\mu + P(D)^t\lambda \in CEB_{\delta,k}$. Because of $P(D)^t\lambda \in N(\Omega)^\perp$, this implies

$$\text{(1)} \qquad |\mu(\chi)| \le CE\|\chi\|_{\delta,k} \quad \text{for all } \chi \in N(\Omega)\ .$$

Now fix $z \in V(P)$ and note that $\chi_z : x \mapsto \exp(-i\langle x,z\rangle)$ is in $N(\Omega)$ and that

$$\|\chi_z\|_{\delta,k} = \sup_{x\in\Omega_\delta}\ \sup_{|\alpha|\le k} |(-iz)^\alpha e^{-i\langle x,z\rangle}| \le (1+|z|^2)^k \exp(h_\delta(\operatorname{Im} z))\ .$$

Hence we get from (1)

$$\log|f(z)| = \log|\mu(\chi_z)| \leq \log\|\chi_z\|_{\delta,k} + \log CE$$
$$\leq h_\delta(\mathrm{Im} z) + B\log(1+|z|^2) + B$$

for each $z \in V(P)$. This proves that f satisfies condition (c) of $APL(\Omega)$.

4.5. THEOREM. — *Let $\Omega \neq \emptyset$ be an open convex subset of $\mathbf{R}^n$ and let P be a non-constant polynomial on $\mathbf{C}^n$. Then the following conditions are equivalent:*

(1) *Ω is P-convex with bounds*

(2) *P satisfies $APL(\Omega)$*

(3) *P satisfies $PL(\Omega)$.*

Proof. — (1) $\Longleftrightarrow$ (2) : Because of Lemma 4.4 it suffices to show that (2) implies (1). To do this, note that without loss of generality we can assume (as in the proof of 4.4) that $P = P_1 \cdots P_k$, where $P_1, \ldots, P_k$ are irreducible polynomials so that P_j is not proportional to P_l for $j \neq l$. In order to show that condition 2.5(*) holds, let $0 < \epsilon < 1$ be given. Then choose $0 < \delta_1 < \epsilon$ according to $APL(\Omega)$, fix $0 < \eta < \delta < \delta_1$ and choose $B = B(\eta) > 0$ according to $APL(\Omega)$. Next choose $q \in \mathbf{N}$ with $q \geq B$ and apply 4.3(2) to find $k \in \mathbf{N}$, $C > 0$ so that

$$L_{\delta_1,q} \subset \mathcal{F}(DV^o_{\delta_1,k}) \ . \tag{4}$$

Furthermore, choose $A > 0$ so that (in the notion of 4.3) we have $U_{\delta_1,k} \subset AV_{\delta_1,k}$. Then let $m := 0$ and $C := e^B AD$. To show that 2.5(*) holds with these choices, let $\mu \in \mathcal{E}'(\Omega_\epsilon)$ be given and assume that for some $\nu \in \mathcal{E}'(\Omega)$ we have

$$\mu + P(D)^t\nu \in B_{\eta,0} \ . \tag{5}$$

Then the theorem of Paley-Wiener-Schwartz implies that

$$\hat{\mu} \ : \mapsto \mu_x(\exp(-i\langle x,z\rangle)) \ , \ z \in \mathbf{C}^n$$

is an entire function on $\mathbf{C}^n$ which satisfies

$$\log|\hat{\mu}(z)| \leq h_\epsilon(\mathrm{Im}\, z) + O(\log(1+|z|^2)) \quad \text{for all } z \in \mathbf{C}^n \ . \tag{a}$$

Next note again that for $z \in V(P)$ the function $\chi_z \ : x \mapsto \exp(-i\langle x,z\rangle)$ is in $N(\mathbf{R}^n)$ and satisfies

$$\|\chi_z\|_{\eta,0} = \sup_{x\in\Omega_\eta} |\exp(-i\langle x,z\rangle)| = \exp(h_\eta(\mathrm{Im}\, z)) \ .$$

Therefore, (5) and the definition of $B_{\eta,0}$ imply for each $z \in V$

$$(b) \quad |\hat{\mu}(z)| = |\mu(\chi_z)| = |(\mu + P(D)^t\nu)[\chi_z]| \leq \|\chi_z\|_{\eta,0} \leq \exp(h_\eta(\operatorname{Im} z)) \ .$$

Hence we conclude from $APL(\Omega)$ and $q \geq B$:

$$(c) \qquad \log|\hat{\mu}(z)| \leq h_{\delta_1}(\operatorname{Im} z) + q\log(1+|z|^2) + B \quad \text{for all } \ z \in V \ .$$

Because of (4), this shows

$$e^{-B}\mathcal{F}(\mu\,|_{N(\Omega)}) \in L_{\delta_1,q} \subset \mathcal{F}(DV^o_{\delta_1,k}) \ .$$

By the choice of A this implies

$$\mu\,|_{N(\Omega)} \in e^B ADU^o_{\delta_1,k} \ .$$

Now the theorem of Hahn-Banach shows that there exists $\lambda \in \mathcal{E}'(\Omega)$ with $\mu\,|_{N(\Omega)} = \lambda\,|_{N(\Omega)}$ so that $\lambda \in CB_{\delta,k}$. Since $P(D)$ is surjective on $\mathcal{E}(\Omega)$, we have

$$\lambda - \mu \in N(\Omega)^\perp = (\ker P(D))^\perp = \operatorname{im} P(D)^t \ .$$

Hence we have shown that 2.5(∗) holds.

(2) $\iff$ (3) : Obviously (3) implies (2). The converse implication is shown in [MTV3].

An easy scaling argument proves the following corollary.

4.6. COROLLARY. — *For a non-constant complex polynomial P on $\mathbb{C}^n$ the operator $P(D)$ has a right inverse on $\mathcal{E}(\mathbb{R}^n)$ and/or $\mathcal{D}'(\mathbb{R}^n)$ if and only if the following Phragmén-Lindelöf condition (PL) holds:*
There exists $R > 1$ so that for each $\rho > R$ there exists $B > 0$ so that for each psh. function u on $\mathbb{C}^n$ which satisfies

$$\begin{array}{llll} \text{(a)} & u(z) \ \leq & |\operatorname{Im} z| + O(|\log(1+|z|^2)) & \textit{for all } z \in \mathbb{C}^n \\ \text{(b)} & u(z) \ \leq & \rho|\operatorname{Im} z| & \textit{for all } z \in V(P) \end{array}$$

we have

$$\begin{array}{llll} \text{(c)} & u(z) \ \leq & R|\operatorname{Im} z| + B\log(1+|z|^2) + B & \textit{for all } z \in V(P) \ . \end{array}$$

Specializing 4.6 to homogeneous polynomials, we get from [MTV4] (see also [MTV2], 5.6) :

4.7. THEOREM. — *Let $P \in \mathbb{C}[z_1, ..., z_n]$ be homogeneous and non-constant. Then $P(D)$ has a right inverse in $\mathcal{E}(\mathbb{R}^n)$ and/or $\mathcal{D}'(\mathbb{R}^n)$ if and only if the following two conditions are satisfied:*

(1) $\dim_{\mathbb{R}} V(Q) \cap \mathbb{R}^n = n-1$ *for each irreducible factor Q of P*

(2) *for each $\xi \in V(P) \cap \mathbf{R}^n$ with $|\xi| = 1$ there exist $0 < \epsilon_1 < \epsilon_2 < \epsilon_3$ and $A > 0$ so that for each psh. function u on the set $\{x \in \mathbf{C}^n : |x - \xi| < \epsilon_3\}$ with $0 \leq u \leq 1$, which satisfies $u(\zeta) \leq 0$ for all $\zeta \in V(P) \cap \mathbf{R}^n$ with $|\zeta - \xi| < \epsilon_2$ we have $u(\zeta) \leq A|\mathrm{Im}\zeta|$ for all $\zeta \in V(P)$ with $|\zeta - \xi| < \epsilon_1$.*

4.8. COROLLARY. — *Let P be an irreducible homogeneous polynomial on $\mathbf{C}^n$ that has real coefficients and satisfies*

$$(*) \qquad \mathrm{grad}P(\xi) \neq 0 \text{ for each } \xi \in V(P) \cap \mathbf{R}^n, |\xi| = 1 .$$

Then P is either elliptic or $P(D)$ admits a right inverse on $\mathcal{E}(\mathbf{R}^n)$ and $\mathcal{D}'(\mathbf{R}^n)$.

Proof. — Assume that P is not elliptic. Then there exists $\xi \in V(P) \cap \mathbf{R}^n$ with $\xi \neq 0$. Since P is homogeneous, we can assume $|\xi| = 1$. Therefore, the hypothesis on P implies $\mathrm{grad}P(-\xi) = \pm\mathrm{grad}P(\xi) \neq 0$. Since P has real coefficients the implicit function theorem for $\mathbf{R}^n$ implies $\dim_{\mathbf{R}} V(P) \cap \mathbf{R}^n = n - 1$, i.e. condition 4.7(1) holds. Hence the proof is complete if we show that also condition 4.7(2) holds. To do this, fix $\xi \in V(P) \cap \mathbf{R}^n$ with $|\xi| = 1$. Then (*) implies that we can assume that $V(P)$ near ξ is the graph of an analytic function g which without loss of generality depends on the first $n - 1$ variables. More precisely, there exist $\epsilon > 0$ and $\delta > 0$ so that on $U := \{z' \in \mathbf{C}^{n-1} : |z' - \xi'| < \epsilon\}$ there exists an analytic function $g : U \to \mathbf{C}$ so that

$$V(P) \cap U \times \{\lambda \in \mathbf{C} : |\lambda - \xi_n| < \delta\} = \{(z', g(z')) : z' \in U\} .$$

Now choose $\epsilon_3 > 0$ so that

$$B_{\epsilon_3}(\xi) := \{z \in \mathbf{C}^n : |z - \xi| < \epsilon_3\} \subset U \times \{\lambda \in \mathbf{C} : |\lambda - \xi_n| < \delta\} .$$

Then fix $0 < \epsilon_2 < \epsilon_3$ and choose $0 < \epsilon_1 < \epsilon_2$ so that for

$$W := \{z' \in \mathbf{C}^{N-1} : |z' - \xi'| < 2\epsilon_1\}$$

we have $(z', g(z')) \in B_{\epsilon_3}(\xi)$ for all $z' \in W$.

Next fix a function u which is psh. on $B_{\epsilon_3}(\xi)$ and has all the properties stated in the hypothesis of 4.7(2). Then

$$\varphi : W \to [0,1] \quad , \quad \varphi(z') := u(z', g(z'))$$

is a psh. function on W. Since P has real coefficients, $g\,|_{\mathbf{R}^{n-1}} \cap W$ has values in $\mathbf{R}$, which implies $u\,|_{W \cap \mathbf{R}^{n-1}} \leq 0$. Now fix $\zeta = (\zeta', \zeta_n) \in V(P)$ with $|\zeta - \xi| < \epsilon_1$ and assume that $\mathrm{Im}\zeta' \neq 0$ (otherwise there is nothing to prove). Then define

$$v : \tau \mapsto \varphi\Big(\mathrm{Re}\zeta' + \tau \frac{\mathrm{Im}\zeta'}{|\mathrm{Im}\zeta'|}\epsilon_1\Big) .$$

For $\tau \in \mathbf{C}$ with $|\tau| \leq 1$, we have

$$\left|\mathrm{Re}\zeta' + \tau \frac{\mathrm{Im}\zeta'}{|\mathrm{Im}\zeta'|}\epsilon_1 - \xi'\right| \leq |\mathrm{Re}(\zeta' - \xi')| + \epsilon_1 < 2\epsilon_1 \ .$$

Hence v is a subharmonic function in a neighbourhood of the closed unit disk, which satisfies $0 \leq v \leq 1$ and $v(x) \leq 0$ for all $x \in \mathbf{R}$, $|x| \leq 1$. Hence we get from the proof of Ahlfors [A], Thm. 3.4 :

$$u(\zeta) = \varphi(\zeta') = v\Big(\frac{|\mathrm{Im}\zeta'|}{\epsilon_1}\Big) \leq \frac{4}{\pi\epsilon_1}|\mathrm{Im}\zeta'| \leq \frac{4}{\pi\epsilon_1}|\mathrm{Im}\zeta| \ .$$

This shows that 4.7(2) holds with $A = 4(\pi\epsilon_1)^{-1}$.

4.9. *Example.* — For $n, m \in \mathbf{N}$ with $n \geq 2$ consider the homogeneous polynomials of degree m which are of the form

$$P(x_1, \ldots, x_n) = \sum_{k=1}^{n} a_k x_k^m, \quad a_k \in \mathbf{R}\backslash\{0\} \text{ for } 1 \leq k \leq n \ .$$

(a) $\mathbf{R}^2$ is P-convex with bounds for such a polynomial if and only if $m = 1$, or $m = 2$ and $\operatorname{sign} a_1 \neq \operatorname{sign} a_2$.

(b) $\mathbf{R}^n$ $(n \geq 3)$ is P-convex with bounds for such a polynomial if and only if m is odd or there exist j, l with $\operatorname{sign} a_j \neq \operatorname{sign} a_l$.

To show this, we argue as follows:
(a) : The case $m = 1$ is obvious. For $m \geq 2$ we can assume without loss of generality that $a_1 = 1$. Then we choose $w \in \mathbf{C}$ with $w^m = (-1)^m a_2$ and note that

$$P(x_1, x_2) = x_1^m + a_2 x_2^m = \prod_{k=0}^{m-1} (x_1 - w \exp(2\pi i \frac{k}{m}) x_2) \ .$$

Hence P is a product of m linear factors which are pairwise not proportional. Moreover, P contains an elliptic factor, except for the case $m = 2$ and sign $a_2 = -1$. Because of our normalization assumption and because of 2.10 this proves (a).
(b) : From the considerations in (a) and an easy inductive application of Eisenstein's theorem (see e.g. Van der Waerden [VA], p.27) it follows that each polynomial P above is irreducible. Obviously we have $\operatorname{grad} P(\xi) \neq 0$ for all $\xi \in \mathbf{R}^n$ with $|\xi| = 1$. Hence Corollary 4.8 shows that $\mathbf{R}^n$ is P-convex with bounds whenever the condition in (b) is satisfied. The complementary case is that m is even and that $\operatorname{sign} a_j = \operatorname{sign} a_l$ for all $1 \leq j, l \leq n$. Then P is elliptic and therefore condition 4.7(1) is violated. Hence $\mathbf{R}^n$ is not P-convex with bounds in this case.

As a consequence of (b) we get that for $n \geq 3$ there are differential operators $P(D)$ which admit a right inverse on $\mathcal{E}(\mathbf{R}^n)$ and which are not hyperbolic. For $= 3$ and $n = 4$ we have the examples $(\partial_j := \frac{\partial}{\partial x_j})$

$$\partial_1^3 + \partial_2^3 + \partial_3^3 \quad \text{and} \quad \partial_1^2 + \partial_2^2 - \partial_3^2 - \partial_4^2 \, .$$

To show that the case $n = 2$ is different from the case $n \geq 3$, we recall from [MTV4] (see also [MTV2], 5.3):

4.10. LEMMA. — *Let P be a non-constant polynomial on $\mathbf{C}^n$ and let P_m denote its principal part. If $P(D)$ has a right inverse on $\mathcal{E}(\mathbf{R}^n)$ then $P_m(D)$ has a right inverse on $\mathcal{E}(\mathbf{R}^n)$ and the following holds :*

$$\operatorname{dist}(z, V(P_m)) = O(1) \quad \textit{for } |z| \to \infty \textit{ and } z \in V(P) \, .$$

4.11. THEOREM. — *Let P be a non-constant polynomial on $\mathbf{C}^2$. Then the following conditions are equivalent:*

(1) *P is hyperbolic*

(2) *$P(D)$ has a right inverse on $\mathcal{E}(\mathbf{R}^2)$ or $\mathcal{D}'(\mathbf{R}^2)$*

(3) *each irreducible factor of P is hyperbolic with respect to each non-characteristic direction*

(4) *P is hyperbolic with respect to each non-characteristic direction*

(5) *$P(D)$ has a right inverse on $\mathcal{E}(\Omega)$ and $\mathcal{D}'(\Omega)$ for each open convex subset Ω of $\mathbf{R}^2$.*

Proof. — (1)$\Rightarrow$(2) : This can be shown as in the proof of Proposition 3.2.

(2)$\Rightarrow$(3) : Because of 4.2 we can assume that P is irreducible. If $N \in \mathbf{R}^2$ is non-characteristic for P we can assume – after a real linear change of variables – that $N = (1, 0)$. Because of this, we can furthermore assume that

$$P(s, w) = s^m + \sum_{j=0}^{m-1} e_j(w) s^j \quad \text{for all } (s, w) \in \mathbf{C}^2 \, . \tag{6}$$

Since P_m is a homogeneous polynomial of degree m in two variables, we find $k \in \mathbf{N}_0$ with $0 \leq k \leq m$ and $0 \neq \alpha_j \in \mathbf{C}$ for $1 \leq j \leq k$ (if $k > 0$) so that

$$P_m(s, w) = s^{m-k} \prod_{j=1}^{k} (s - \alpha_j w) \quad \text{for all } (s, w) \in \mathbf{C}^2 \, . \tag{7}$$

In the sequel we shall assume $1 \le k < m$; the cases $k = 0$ and $k = m$ are treated in the same way. Since $P(D)$ has a right inverse on $\mathcal{E}(\mathbb{R}^2)$, we get from 4.10 that this also holds for $P_m(D)$. By 4.7(1) this implies $\alpha_j \in dR$ for $1 \le j \le k$. Now put $\alpha_0 := 0$ and let

$$L_j := \{(\alpha_j w, w) \ : \ w \in \mathbb{C}\} \quad , \quad 0 \le j \le k$$

Then (7) implies

$$V(P_m) = \bigcup_{j=0}^{k} L_j \ . \tag{8}$$

Next note that the solutions of $P(z) = 0$ can be described by a Puiseux expansion. More precisely: there exists $B > 0$ so that

$$V(P) \cap \{(s,w) \in \mathbb{C}^2 \ : \ |w| \le B\}$$

is compact and so that for each branch W of

$$V(P) \cap \{(s,w) \in \mathbb{C}^2 \ : \ |w| > B\}$$

there exists $q \in \mathbb{N}$ so that

$$W = \{(s(w),w) \ : \ |w| > B\}$$

where $s(w) = \sum_{l=-\infty}^{q} a_l w^{l/q}$. Now note that by 4.10 there exist $C > 0$ and $D > B$ so that

$$\operatorname{dist}((s(w),w), V(P_m)) \le C \quad \text{for} |w| > D \ . \tag{9}$$

From this and (8) it follows easily that for some j with $0 \le j \le k$ we have $a_q = \alpha_j$. Furthermore, (9) and (8) imply that $a_l = 0$ for $1 \le l \le q-1$. Hence we have

$$s(w) = \alpha_j w + \sum_{l=-\infty}^{0} a_l w^{l/q} \ .$$

Since α_j is real and since this holds for each branch W, we get the existence of $M > 0$ so that

$$|\operatorname{Im} s| \le M(1 + |\operatorname{Im} w|) \quad \text{for all } (s,w) \in V(P) \ .$$

Since the vector $(1,0)$ is not characteristic for P, this implies that P is hyperbolic with respect to $(1,0)$.

(3)⇒(4) : Let $N \in \mathbb{R}^2$ be non-caracteristic for P. Then N is also non-characteristic for each irreducible factor Q of P. Because of (3) this implies that Q is hyperbolic with respect to N for each irreducible factor Q of P. It is easy to check that this implies that P is hyperbolic with respect to N.

(4)$\Rightarrow$(5)$\Rightarrow$(1) : This holds by Theorem 3.8.

The condition $PL(\Omega)$ used above is related to a different Phragmén-Lindelöf condition which was introduced by Hörmander [HO2] to characterize the surjectivity of operators $P(D)$ on the space $\mathcal{A}(\Omega)$ of real-analytic functions on a convex open set Ω in $\mathbb{R}^n$. In concluding this section we show that $PL(\Omega)$ implies Hörmander's Phragmén-Lindelöf condition. This is an immediate consequence of Theorem 4.5 and the following proposition.

4.12. PROPOSITION. — *Let Ω be a convex open subset of $\mathbb{R}^n$ and let P be a non-constant polynomial in n variables. If $P(D)$ admits a right inverse on $\mathcal{E}(\Omega)$ then the following condition $HPL(\Omega)$ holds:*
For each $\epsilon > 0$ there exist $0 < \delta < \epsilon$ and $\lambda > 0$ so that for each psh. function u on $\mathbb{C}^n$ the following two conditions :

(a) $u(z) \leq h_\epsilon(\mathrm{Im} z) + \lambda|z|$ *for all* $z \in \mathbb{C}^n$

(b) $u(\xi) \leq 0$ *for all* $\xi \in \mathbb{R}^n$ *with* $P_m(\xi) = 0$

imply

(c) $u(z) \leq h_\delta(\mathrm{Im} z)$ *for all* $z \in \mathbb{C}^n$ *with* $P_m(z) = 0$.

In particular, $P(D) : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$ is surjective.

Proof. — To prove that $HPL(\Omega)$ holds, it suffices, by Hörmander [HO2], Thm. 1.1, to show the following:

(1) For each convex compact set $K \subset \Omega$ and for each $f \in \mathcal{A}(\Omega)$ there exists a complex neighbourhood $\hat{K}$ of K so that for each open set ω with $K \subset \omega \subset\subset \Omega$ there is $u \in C(\omega)$ with $P(D)u = f$ so that $u\,|_K$ can be extended analytically to $\hat{K}$.

To prove this, we first recall from 2.2(2) that the hypothesis implies

(2) For each $\epsilon > 0$ there exists $0 < \delta < \epsilon$ so that for each $f \in \mathcal{E}(\Omega, \Omega_\delta)$ there exists $g \in \mathcal{E}(\Omega, \Omega_\epsilon)$ with $P(D)g = f$.

Now fix $K \subset \Omega$ compact and convex. Then there exists $\epsilon > 0$ with $K \subset \Omega_\epsilon$. Choose $0 < \delta < \epsilon$ according to (2) and let $f \in \mathcal{A}(\Omega)$ be given. Next let

$$\hat{K} := \{z \in \mathbb{C}^n \ : \ \mathrm{dist}(z, \Omega_\delta) < \sigma\}$$

where $\sigma > 0$ is so small that $\hat{K} \cap \mathbb{R}^n \subset\subset \Omega$ and that f can be extended analytically to a holomorphic function F on $\hat{K}$. Now observe that by a result of Malgrange (see e.g. Treves [2], Thm. 9.4) there exists a holomorphic function H on V which satisfies

$$P(\frac{1}{i}\frac{\partial}{\partial z_1}, \ldots, \frac{1}{i}\frac{\partial}{\partial z_n})H = F \ .$$

Let $h := H\,|_{\hat{K}\cap\mathbb{R}^n}$ and choose $\varphi \in \mathcal{D}(\hat{K}\cap\mathbb{R}^n)$ with $\varphi\,|_{\Omega_\delta}\equiv 1$. Then

$$f_0 := f - P(D)(\varphi h)$$

is in $C^\infty(\Omega,\Omega_\delta)$. Hence (2) implies the existence of $g_0 \in C^\infty(\Omega,\Omega_\epsilon)$ with $P(D)g_0 = f_0$. Consequently,

$$u := \varphi h + g_0$$

is in $C^\infty(\Omega)$ and satisfies $P(D)u = f$ as well as

$$u\,|_K = h\,|_K = H\,|_K \quad .$$

Obviously, this implies (1). By Hörmander [HO2], Thm. 1.3, condition (1) implies that $P(D) : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$ is surjective.

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