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FREQUENCY FUNCTIONS ON THE HEISENBERG GROUP, THE UNCERTAINTY PRINCIPLE AND UNIQUE CONTINUATION

by N. GAROFALO (*) and E. LANCONELLI

1. Introduction.

The Heisenberg group $\mathbb{H}^n$ of real dimension $N = 2n + 1$, $n \in \mathbb{N}$, is the nilpotent Lie group of step two whose underlying manifold is $\mathbb{R}^{2n+1}$ equipped with the group law

$$(1.1)\quad (x,y,t) \circ (x',y',t') = (x+x', y+y', t+t'+2(x'\cdot y - x\cdot y')),$$

where $x\cdot y$ denotes the usual inner product in $\mathbb{R}^n$. A basis for the Lie algebra of left-invariant vector fields on $\mathbb{H}^n$ is given by

$$(1.2)\quad X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t}, \quad j = 1, \ldots, n, \frac{\partial}{\partial t}\cdot$$

From (1.2) we have for j, $k = 1, \ldots, n$

$$[X_j, X_k] = [Y_j, Y_k] = \left[X_j, \frac{\partial}{\partial t}\right] = \left[Y_j, \frac{\partial}{\partial t}\right] = 0,$$
$$[X_j, Y_k] = -4\delta_{jk}\frac{\partial}{\partial t},$$

which constitute Heisenberg's canonical commutation relations of quantum mechanics for position and momentum, whence the name Heisenberg

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group (see e.g. [He], and also the recent monography [F2] or the expository paper [Ho]). The Kohn-Laplacian on $\mathbb{H}^n$ is

$$\Delta_{\mathbb{H}^n} = \sum_{j=1}^{n} (X_j^2 + Y_j^2). \tag{1.3}$$

Since Hörmander's fundamental work [H1] the study of operators of the type *sum of squares of vector fields* has received a strong impulse and today's literature on the subject is quite large. Much of the development in the field has been connected to the development of analysis on homogeneous nilpotent Lie groups, following a circle of ideas outlined by E. Stein in his address to the 1970 Nice International Congress [S]. Among such groups $\mathbb{H}^n$ and its subelliptic Laplacian (1.3) play a prominent role, see [F3].

In virtue of Hörmander's theorem [H1] the identity $[X_j, Y_k] = -4\delta_{jk}\frac{\partial}{\partial t}$ implies that $\Delta_{\mathbb{H}^n}$ is hypoelliptic. In fact, see (1.13) below, $\Delta_{\mathbb{H}^n}$ is (real) analytic-hypoelliptic and therefore a solution u to $\Delta_{\mathbb{H}^n} u = 0$ cannot vanish to infinite order at one point unless $u \equiv 0$ in the connected component containing that point.

In this paper we are interested in a quantitative version of the above uniqueness property for solutions to the equation

$$-\Delta_{\mathbb{H}^n} u + Vu = 0, \tag{1.4}$$

where on the zero order term V we make suitable assumptions. Specifically, we seek an estimate of the order of vanishing at one point of a solution u to (1.4). Such estimate should in a precise quantitative way only depend on suitable L^2-norms of u, and of $X_j u$, $Y_j u$, $j = 1, \ldots, n$, in a fixed neighborhood of the point in question.

In general, however, there can be no such result even when $V \in C^\infty$. This is a consequence of recent work of Bahouri [Ba].

THEOREM (Bahouri). – *Let $X_0, X_1, \ldots, X_{N-1}$ be C^∞ vector fields in an open set $D \subset \mathbb{R}^N$. Suppose that*

(i) *The vector space generated by $X_1, \ldots, X_{N-1}$ has dimension $N-1$ at every point of D;*

(ii) *The rank of the Lie algebra generated by $X_1, \ldots, X_{N-1}$ is N at every point of D;*

(iii) *There exists a point* $x_0 \in D$ *such that in a neighborhood of* x_0

$$\varepsilon_N \wedge d\varepsilon_N \neq 0, \qquad \varepsilon_N \wedge (d\varepsilon_N)^2 = 0$$

where ε_N *is the one-form that to every vector field* X *associates*

$$\det (X_1, \ldots, X_{N-1}, X) = X_1 \wedge \cdots \wedge X_{N-1} \wedge X.$$

Then there exists an open set Ω, *with* $x_0 \in \Omega \subset D$, *and a function* $V \in C^\infty(\Omega)$ *such that the equation* $-\mathscr{L}u + Vu = 0$, *where* $\mathscr{L} = \sum_{j=1}^{N-1} X_j^2 + X_0$, *admits a nontrivial solution* u *in* Ω *vanishing in an open subset of* Ω.

Assumption (ii) is Hörmander's condition cited above. The conclusion of the above theorem holds without assumption (iii) when the dimension is three or four. In other words, when $N = 3$ or 4 every operator of the above type fails to have the unique continuation property. An interesting example is provided by the operator

$$\mathscr{L} = \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y} - 4x\frac{\partial}{\partial t}\right)^2, \qquad (x,y,t) \in \mathbb{R}^3. \tag{1.5}$$

The construction in [Ba] shows that there exists a neighborhood of the origin Ω, and a $V \in C^\infty(\Omega)$ such that

$$\operatorname{supp} V \subset \{x \geqslant 0\} \cap \Omega, \qquad V \text{ is flat at } \{x=0\}, \tag{1.6}$$

for which the equation $-\mathscr{L}u + Vu = 0$ admits a nonzero solution in Ω flat at $\{x=0\}$ and supported in $\{x \geqslant 0\} \cap \Omega$. The change of variables $(x,y,t) \mapsto (x',y',t')$, where $x' = x$, $y' = y$, $t' = t - 2xy$, maps the plane $\{x=0\}$ into the plane $\{x'=0\}$, and transforms the Kohn-Laplacian in $\mathbb{R}^3$ $\Delta_{\mathbb{H}^1} = \left(\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}\right)^2 + \left(\frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}\right)^2$ into the operator $\mathscr{L}$ in (1.5). Therefore, there exists a neighborhood of the origin Ω and a $V \in C^\infty(\Omega)$ and satisfying (1.6), such that $-\Delta_{\mathbb{H}^1} + V$ fails to have the unique continuation property. When V is real-analytic, then a qualitative result of Bony [B] based on Holmgren's theorem shows that solutions to (1.4) in an open set cannot vanish in an open subset unless they vanish identically.

Is there any positive result when V is not real-analytic? We will answer this question affirmatively by providing a sufficient condition for solutions to (1.4) to have a finite order of vanishing at one point, even when the potential V is allowed strong singularities. In order to state our results we need to introduce some more notation.

Henceforth, we denote by $z = (x,y)$ a generic point of $\mathbb{R}^{2n}$. An easy verification shows that for $u \in C^2(\mathbb{H}^n)$

$$\Delta_{\mathbb{H}^n} u = \Delta_z u + 4|z|^2 \frac{\partial^2 u}{\partial t^2} + 4 \frac{\partial}{\partial t}(Tu), \tag{1.7}$$

where $\Delta_z = \sum_{j=1}^{n} \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} \right)$ is the Laplacian in the variable $z = (x,y)$, and T denotes the vector field

$$T = \sum_{j=1}^{n} \left(y_j \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial y_j} \right). \tag{1.8}$$

An important group of automorphisms of $\mathbb{H}^n$ is given by the so-called *Heisenberg dilations*

$$\delta_\lambda(z,t) = (\lambda z, \lambda^2 t), \qquad \lambda > 0, \ (z,t) \in \mathbb{H}^n. \tag{1.9}$$

It is worth remarking that if $G = (g_{ij})$ is the $(2n+1) \times (2n+1)$ matrix given by: $g_{ij} = \delta_{ij}, i, j = 1, \ldots, 2n+1$, and at least one of the two indices i, j is not $2n+1$, $g_{(2n+1)(2n+1)} = 2$, then

$$\exp[G \log \lambda] = \delta_\lambda. \tag{1.10}$$

The number

$$Q = \text{trace } (G) = 2n + 2$$

is the *homogeneous dimension* of $\mathbb{H}^n$, see [FS]. A function $u : \mathbb{H}^n \to \mathbb{R}$ is said *Heisenberg-homogeneous* of degree $k \in \mathbb{Z}$ if for every $\lambda > 0$

$$u \circ \delta_\lambda = \lambda^k u. \tag{1.11}$$

There exists a distinguished Heisenberg-homogeneous function of degree one, the distance function (see [S], [F1])

$$d(z,t) = (|z|^4 + t^2)^{\frac{1}{4}}. \tag{1.12}$$

It is a remarkable fact that if $\Gamma(z,t)$ denotes the fundamental solution of $-\Delta_{\mathbb{H}^n}$ with singularity at the origin, then

$$\Gamma(z,t) = \frac{c_Q}{d(z,t)^{Q-2}}, \tag{1.13}$$

where $c_Q > 0$ is a number depending only on Q in (1.10). (1.13) was proved by Folland in [F1]. For reasons that will soon be clear we choose and fix c_Q as follows

$$c_Q^{-1} = (Q-2)\int_{\partial\Omega_1} \frac{|z|^2}{|\nabla d(z,t)|}\, dH_{2n}. \tag{1.14}$$

In (1.14) we have denoted by $\partial\Omega_1$ the set $\{(z,t) \in \mathbb{H}^n | d(z,t)=1\}$, and by dH_{2n} the $2n$-dimensional Hausdorff measure in $\mathbb{R}^{2n+1}$. More in general we let

$$\Omega_r = \{(z,t) \in \mathbb{H}^n | d(z,t)<r\}, \qquad \partial\Omega_r = \{(z,t) \in \mathbb{H}^n | d(z,t)=r\}, \tag{1.15}$$

and call these sets respectively the *Heisenberg ball* and *sphere* centered at the origin with radius r. Balls and spheres centered at points other than the origin are defined by left-translation. If $(z_0,t_0) \in \mathbb{H}^n$, then (1.1) yields $(z_0,t_0)^{-1} = (-z_0, -t_0)$. We let $d(z,t;z_0,t_0) = d((z_0,t_0)^{-1} \circ (z,t))$ denote the distance between (z,t) and (z_0,t_0). Then the ball $\Omega_r(z_0,t_0)$ and the sphere $\partial\Omega_r(z_0,t_0)$ centered at (z_0,t_0) with radius r are obtained by replacing $d(z,t)$ in (1.15) with $d(z,t;z_0,t_0)$. Likewise, the fundamental solution $\Gamma(z,t;z_0,t_0)$ of $-\Delta_{\mathbb{H}^n}$ with singularity at (z_0,t_0) is obtained by replacing $d(z,t)$ with $d(z,t;z_0,t_0)$ in (1.13).

Our problem being a local one we work from now on in a fixed Heisenberg ball Ω_{R_0} centered at the origin. We require that the zero order term V in (1.4) satisfy the following assumption: There exist $C > 0$ and an increasing function $f : (0,R_0) \to \mathbb{R}^+$ such that

$$\int_0^{R_0} \frac{f(r)}{r}\, dr < \infty \tag{1.16}$$

and for which

$$|V(z,t)| \leqslant C \frac{f(d(z,t))}{d(z,t)^2}\, \psi(z,t) \text{ for a.e. } (z,t) \in \Omega_{R_0}. \tag{1.17}$$

In (1.17) we have set

$$\psi(z,t) = \frac{|z|^2}{d(z,t)^2}, \qquad (z,t) \neq (0,0). \tag{1.18}$$

The geometric meaning of this function will be explained later on. At this moment we simply remark that: ψ is Heisenberg-homogeneous of degree zero; $0 \leqslant \psi(z,t) \leqslant 1$; $\psi(0,t) \equiv 0$; $\psi(z,0) \equiv 1$.

According to (1.16), (1.17) the potential V is allowed to be quite singular and therefore a notion of solution to (1.4) needs to be specified. Since regularity questions are not the main concern for us throughout this paper we will assume *a priori* that a *solution* to (1.4) is a $u \in C(\Omega_{R_0})$ such that u, $X_j u$, $Y_j u$, $\Delta_{\mathbb{H}^n} u \in L^2(\Omega_{R_0})$. We note that (1.16) implies $\lim_{r \to 0^+} f(r) = 0$. Typical representatives of f's satisfying (1.16) are $f(r) = r^\varepsilon$, $0 < \varepsilon$, $f(r) = \left|\log \frac{1}{r}\right|^{-\alpha}$, $\alpha > 1$.

We need to introduce the following.

DEFINITION 1.1. – *Let u be such that $\psi^{\frac{1}{2}} u \in L^2(\Omega_{R_0})$, where ψ is as in* (1.18). *We say that u* vanishes to infinite order *at the origin if as $r \to 0^+$*

$$\int_{\Omega_r} u^2 \psi \, dz \, dt = O(r^k) \qquad \text{for every } k \in \mathbb{N}.$$

One of the main results in this paper is the following.

THEOREM 1.1. – *Let V satisfy* (1.17) *for some C and f. Let u be a solution to* (1.4) *in Ω_{R_0} and suppose that there exist $C_1 > 0$ and an increasing function $g : (0, R_0) \to \mathbb{R}^+$ satisfying* (1.16) *such that*

$$(1.19) \quad |tTu(z,t)| \leqslant C_1 g(d(z,t)) |z|^2 |u(z,t)|, \text{ for a.e. } (z,t) \in \Omega_{R_0},$$

where T is given by (1.8). *Then, there exist $r_0 = r_0(Q, C, C_1, f, g) > 0$ and $\Gamma = \Gamma(Q, C, C_1, f, g, u) > 0$ such that if $u \not\equiv 0$ in Ω_r for $0 < r < \frac{r_0}{2}$, then*

$$(1.20) \qquad \int_{\Omega_{2r}} u^2 \psi \, dz \, dt \leqslant \Gamma \int_{\Omega_r} u^2 \psi \, dz \, dt$$

for every $r \in \left(0, \frac{r_0}{2}\right)$.

Remarks. – *a*) The constant Γ in (1.20) must depend on u, as easy examples show. *b*) The dependence of r_0 and Γ on the parameters involved can be made very explicit. In particular, the proof of Theorem

1.1 yields a number r_0 such that

$$f(r_0) < C^{-1}\left(\frac{Q-2}{2}\right)^2 \qquad \text{and} \qquad g(r_0) < 2C_1^{-1},$$

while Γ is determined by the formula

$$\Gamma = 2^Q \exp\left\{2 \log 2 \max(1, N(r_0))\left[1+\exp\left(M\int_0^{R_0}[f(t)+g(t)]\frac{dt}{t}\right)\right]\right\}.$$

In the above equation $M > 0$ is a constant depending on Q, on C, f in (1.17), and on C_1, g in (1.19), while $N(r_0)$ brings the dependence on u in Γ according to the equation

$$N(r_0) = \int_{\Omega_{r_0}}\left\{\sum_{j=1}^{n}[(X_ju)^2+(Y_ju)^2]+Vu^2\right\}dz\,dt\Big/\int_{\partial\Omega_{r_0}} u^2\frac{\psi}{|\nabla d|}\,dH_{2n}.$$

c) condition (1.19) is trivially satisfied by all functions such that $Tu \equiv 0$. These are the functions which are invariant w.r.t. the natural action of the torus $\mathbb{T}$ on $\mathbb{H}^n$. If we identify $z = (x,y) \in \mathbb{R}^{2n}$ with $\tilde{z} = (\tilde{z}_1, \ldots, \tilde{z}_n) \in \mathbb{C}^n$, where $\tilde{z}_j = x_j + iy_j$, then $\mathbb{T}$ acts on $\mathbb{H}^n$ by $\varphi_\theta(\tilde{z},t) = (e^{i\theta}\tilde{z},t)$, $\theta \in [0,2\pi]$. It is not difficult to recognize that $Tu \equiv 0$ iff

(1.21) $$u \circ \varphi_\theta = u \qquad \text{for every } \theta \in [0,2\pi].$$

We would like to thank David Catlin for pointing this fact out to us. When $n = 1$ (1.21) is easily seen to be equivalent to the fact that $u(z,t) = u^*(|z|^2,t)$, for some u^*. When $n > 1$, (1.21) is less obvious. For instance every polyradial function satisfies it, i.e., every function which can be written as $u(z,t) = u^*(|\tilde{z}_1|^2,, \ldots, |\tilde{z}_n|^2,t)$ for some u^*.

Theorem 1.1 yields the sought quantitative information on the order of vanishing at the origin of a solution u to (1.4). As a consequence of it we obtain

THEOREM 1.2. — *Under the assumptions of Theorem* 1.1 *if u vanishes to infinite order at the origin, then must be $u \equiv 0$ in Ω_{r_0}, where r_0 is as in the statement of Theorem* 1.1.

We emphasize that given a V satisfying (1.16), (1.17) the conclusion of Theorem 1.1 is simply false if we do not restrict the family of solutions to (1.4). This can be easily seen by choosing $f(r) = r^2$ in

(1.17), so that the latter becomes

$$(1.22)\qquad |V(z,t)| \leqslant C\psi(z,t) \qquad \text{for a.e. } (z,t)\in\Omega_{R_0}.$$

This condition is certainly satisfied by Bahouri's potential since according to (1.6) the latter is flat at $\{x = 0\}$. Yet, the differential inequality (1.19) is only a sufficient condition for (1.20) to hold. According to Proposition 4.2 below the function $u(x,y,t) = x|z|^2 + 2ty$, which solves $\Delta_{\mathbb{H}^1}u = 0$ in $\mathbb{H}^1$, satisfies (1.20), although it does not satisfy (1.19). Whether there exist an optimal condition for (1.20) remains an interesting open problem.

The proof of Theorem 1.1 is based on an approach to unique continuation for elliptic equations found by F. H. Lin and one of us in [GL1] and [GL2]. Indeed, our analysis shows some remarkable similarities with the elliptic case. Yet, in the present sub-elliptic context new and interesting difficulties arise, some of which of a rather subtle geometric nature. Before we outline the plan of the paper we comment on assumptions (1.16), (1.17) on the potential V in (1.4). Henceforth, for a function u on $\mathbb{H}^n$ we set

$$(1.23)\qquad |\nabla_{\mathbb{H}^n}u|^2 = \sum_{j=1}^{n} (X_ju)^2 + (Y_ju)^2,$$

where X_j, Y_j, $j = 1, \ldots, n$ are given by (1.2). With d as in (1.12) and ψ as in (1.18) a computation yields

$$(1.24)\qquad |\nabla_{\mathbb{H}^n}d|^2 = \psi \qquad \text{in } \mathbb{H}^n\backslash\{(0,0)\}.$$

In Euclidean space the presence of the density ψ is outshone by the flat geometry of $\mathbb{R}^N$, which yields $\psi \equiv 1$. Roughly speaking, (1.17) means that we measure $\frac{1}{\psi}V$, rather than V itself, w.r.t. Lebesgue measure. This was suggested to us by the natural occurrence of the measure $\psi dz\, dt$ in Theorems 2.1 and 2.2 below.

Thinking in terms of $\frac{1}{\psi}V$ assumption (1.17) is an ad hoc adaptation to our context of the condition $|V(x)| \leqslant \frac{Cf(|x|)}{|x|^2}$, in the paper [GL2] on strong unique continuation for elliptic operators. We also recall Hörmander's strong uniqueness result in [H2] which was concerned with the assumption $|V(x)| \leqslant \frac{C}{|x|^{2-\varepsilon}}$, for some $0 < \varepsilon < 1$.

Another result in this paper is a theorem of uniqueness in which we make a weaker assumption both on the potential V and in the differential inequality (1.19). Specifically, we request that given $V(z,t) = V^+(z,t) - V^-(z,t)$, where V^+ and V^- respectively denote the positive and the negative part of V, then there exist a constant $C > 0$ and a dimensional constant $\delta = \delta_Q > 0$, such that for a.e. $(z,t) \in \Omega_{R_0}$

$$(1.25) \quad 0 \leqslant V^+(z,t) \leqslant \frac{C}{d(z,t)^2}\psi(z,t), \quad 0 \leqslant V^-(z,t) \leqslant \frac{\delta}{d(z,t)^2}\psi(z,t).$$

We then have

THEOREM 1.3. – *Let u be a solution to* (1.4) *in Ω_{R_0} with a V satisfying* (1.25). *Suppose that there exists $C_1 > 0$ such that*

$$(1.26) \quad |tTu(z,t)| \leqslant C_1|z|^2|uz,t)|, \qquad \text{for a.e. } (z,t) \in \Omega_{R_0}.$$

Then, there exists $r_0 = r_0(Q, C, C_1, \delta) > 0$ such that if as $r \to 0^+$

$$\int_{\Omega_r} u(z,t)^2\psi(z,t)\,dz\,dt = 0\,(\exp[-Ar^{-\alpha}])$$

for some A, $\alpha > 0$, then must be $u \equiv 0$ in Ω_{r_0}.

The main ingredients in the proof of Theorems 1.1 and 1.3 are:

(I) Representation formulas for (smooth) functions on $\mathbb{H}^n$ as integrals on Heisenberg spheres and balls.

(II) A strong form of uncertainty principle for $\mathbb{H}^n$.

(III) A formula for the first variation of the energy integral associated to (1.4).

(IV) A frequency function on $\mathbb{H}^n$ and the study of its growth properties via parts (I), (II) and (III).

Section 2 is devoted to parts (I) and (II). Section 3 to part (III). Section 4 is dedicated to the implementation of part (IV) in the proof of Theorem 1.1. There, we also prove along with Theorems 1.2 and 1.3, other results concerning solutions to (1.4) which are invariant w.r.t. the action of the torus $\mathbb{T}$ on $\mathbb{H}^n$, i.e., solutions satisfying (1.21). One remarkable fact is that when $V(z,t) = \dfrac{C}{d(z,t)^2}\psi(z,t)$, with $C \in \mathbb{R}$, the analogue in the present context of the *inverse square potentiel* $V(x) = \frac{C}{|x|^2}$, then the frequency of a solution to (1.4) satisfying (1.21) is strictly increasing. As a consequence of our results and of unique continuation results for elliptic theory we prove in section 4

THEOREM 1.4. — *Let u be a solution in* $\mathbb{H}^n$ *to the equation*

$$- \Delta_{\mathbb{H}^n} u + \frac{C}{d^2} \psi u = 0, \qquad C \in \mathbb{R},$$

and suppose that u satisfies (1.21). *If u vanishes to infinite order at the origin, then must be* $u \equiv 0$ *in* $\mathbb{H}^n$.

At the end of the section we give an example which proves that the inverse square potential $V = \frac{C}{d^2} \psi$ constitues a threshold for our results to hold. For every $\varepsilon > 0$ we provide a nontrivial solution to the equation $- \Delta_{\mathbb{H}^n} u + \frac{C}{d^{2+\varepsilon}} \psi u = 0$ which vanishes to infinite order at the origin and for which conditions (1.19) or (1.26) are trivially satisfied.

Finally, we would like to thank Luis Caffarelli, David Catlin, Carlos Kenig, Fang-Hua Lin and Xavier Saint-Raymond for their interest in the results of this paper and for stimulating conversations.

2. Sub-elliptic mean value formulas and a Hardy-type inequality.

We begin this section by establishing some representation formulas for (smooth) functions on $\mathbb{H}^n$. These formulas generalize classical results involving functions and their Laplacians in Euclidean space. Gaveau [Ga] proved the following result : *Let u be such that* $\Delta_{\mathbb{H}^n} u = 0$ *in* $\mathbb{H}^n$. *Then*

$$(2.1) \qquad u(0,0) = (Q-2) C_Q \int_{\partial \Omega_1} u(z,t) \frac{|z|^2}{|\nabla d(z,t)|} dH_{2n},$$

where C_Q *is given by* (1.14).

For ψ given by (1.18) we now define

$$(2.2) \qquad |\Omega_r|_{\mathbb{H}^n} = \int_{\Omega_r} \psi(z,t) \, dz \, dt .$$

Using the polar coordinates adapted to $\mathbb{H}^n$ introduced by Greiner [Gr], see also [GrK], it is easy to recognize that there exists $\alpha_Q > 0$ depending only on $Q = 2n + 2$ such that

(2.3) $$|\Omega_r|_{\mathbb{H}^n} = \alpha_Q r^Q.$$

We now recall Federer's co-area formula [Fe], Theorem 3.2.12, p. 249 : *Let* $f \in L^1(\mathbb{R}^N)$ *and* $g \in \mathrm{Lip}(\mathbb{R}^N)$. *Then*

(2.4) $$\int_{\mathbb{R}^N} f(x)\,dx = \int_{-\infty}^{+\infty} ds \left(\int_{\{g=s\}} \frac{f(x)}{|\nabla g(x)|}\,dH_{N-1} \right),$$

provided that ∇g *does not vanish on the set* $\{g=s\}$ *for a.e.* $s \in \mathbb{R}$.

From the definition (1.15) of Ω_r and (2.4) we can write

(2.5) $$|\Omega_r|_{\mathbb{H}^n} = \int_0^r \left(\int_{\partial\Omega_\rho} \frac{\psi(z,t)}{|\nabla d(z,t)|}\,dH_{2n} \right) d\rho.$$

(2.5) yields upon differentiation

(2.6) $$\frac{d}{dr}|\Omega_r|_{\mathbb{H}^n} = \int_{\partial\Omega_r} \frac{\psi(z,t)}{|\nabla d(z,t)|}\,dH_{2n} \overset{\text{def}}{=} |\partial\Omega_r|_{\mathbb{H}^n}.$$

Comparison of (2.6), (2.3) gives

(2.7) $$|\partial\Omega_r|_{\mathbb{H}^n} = Q\alpha_Q r^{Q-1}.$$

THEOREM 2.1. – *Let* $v \in c^\infty(\mathbb{H}^n)$, *then for every* $r > 0$ *we have*

(2.8) $$\frac{1}{|\partial\Omega_r|_{\mathbb{H}^n}} \int_{\partial\Omega_r} v(z,t)\frac{\psi(z,t)}{|\nabla d(z,t)|}\,dH_{2n} = v(0,0) + \int_{\Omega_r} \Delta_{\mathbb{H}^n} v(z,t)\left[\Gamma(z,t) - \frac{C_Q}{r^{Q-2}}\right] dz\,dt.$$

Also, we have

(2.9) $$\frac{1}{|\Omega_r|_{\mathbb{H}^n}} \int_{\Omega_r} v(z,t)\psi(z,t)\,dz\,dt = v(0,0) + \frac{Q}{r^Q}\int_0^r \rho^{Q-1}\left\{ \int_{\Omega_\rho} \Delta_{\mathbb{H}^n} v(z,t)\left[\Gamma(z,t) - \frac{C_Q}{\rho^{Q-2}}\right] dz\,dt \right\} d\rho.$$

In (2.8), (2.9) $\Gamma(z,t)$ is given by (1.13).

Remarks. – *a*) Theorem 2.1 holds unchanged if the origin is replaced by any other point $(z_0, t_0) \in \mathbb{H}^n$. For instance, by left-translation we obtain from (2.8)

$$\frac{1}{|\partial\Omega_r(z_0,t_0)|_{\mathbb{H}^n}} \int_{\partial\Omega_r(z_0,t_0)} v(z,t) \frac{\psi(z,t;z_0,t_0)}{|\nabla d(z,t;z_0,t_0)|} dH_{2n} = v(z_0,t_0)$$

$$+ \int_{\Omega_r(z_0,t_0)} \Delta_{\mathbb{H}^n} v(z,t) \left[\Gamma(z,t;z_0,t_0) - \frac{C_Q}{r^{Q-2}} \right] dz\, dt,$$

where we have let $\psi(z,t;z_0,t_0) = \psi((z_0,t_0)^{-1} \circ (z,t)) = \dfrac{|z-z_0|^2}{d(z,t;z_0,t_0)^2}$, and then $|\partial\Omega_r(z_0,t_0)|_{\mathbb{H}^n} = \displaystyle\int_{\partial\Omega_r(z_0,t_0)} \frac{\psi(z,t;z_0,t_0)}{|\nabla d(z,t;z_0,t_0)|} dH_{2n}$.

b) If $\Delta_{\mathbb{H}^n} v = 0$ in $\mathbb{H}^n$ from (2.8) and (1.14) we obtain Gaveau's mean value formula (2.1).

Proof of Theorem 2.1. – We begin with proving (2.8). To this end it will be convenient to represent $\Delta_{\mathbb{H}^n}$ as a divergence form operator

$$\Delta_{\mathbb{H}^n} = \operatorname{div}(A(z)\nabla), \tag{2.10}$$

where

$$A(z) = \left(\begin{array}{c|c} I_{\mathbb{R}^{2n}} & \begin{matrix} 2y_1 \\ \vdots \\ 2y_n \\ -2x_1 \\ \vdots \\ -2x_n \end{matrix} \\ \hline 2y_1 \cdots 2y_n \; -2x_1 \cdots -2x_n & 4|z|^2 \end{array} \right), \tag{2.11}$$

and $I_{\mathbb{R}^{2n}}$ denotes the identify matrix in $\mathbb{R}^{2n}$. Let $0 < \varepsilon < r$ be fixed and consider the open set with smooth boundary $D = \Omega_r \backslash \bar{\Omega}_\varepsilon$. If $u \in C^\infty(\bar{D})$ and v is as in the statement of the theorem we have by (2.10) and the divergence theorem

$$\int_D (u\Delta_{\mathbb{H}^n} v - v\Delta_{\mathbb{H}^n} u)\, dz\, dt = \int_{\partial D} (uA\nabla v \cdot \vec{n} - vA\nabla u \cdot \vec{n})\, dH_{2n}, \tag{2.12}$$

where we have denoted by $\vec{n}$ the exterior unit normal to ∂D. Letting

$u \equiv 1$ in (2.12) we obtain

$$\int_D \Delta_{\mathbb{H}^n} v \, dz \, dt = \int_{\partial D} A \nabla v \cdot \vec{n} \, dH_{2n}. \tag{2.13}$$

Now we apply (2.12) to the functions v and $u = \Gamma$ obtaining

$$\begin{aligned} \int_{\Omega_r \setminus \Omega_\varepsilon} \Gamma \Delta_{\mathbb{H}^n} v \, dz \, dt &= \frac{C_Q}{r^{Q-2}} \int_{\partial \Omega_r} A \nabla v \cdot \vec{n} \, dH_{2n} + \frac{C_Q}{\varepsilon^{Q-2}} \int_{\partial \Omega_\varepsilon} A \nabla v \cdot \vec{n} \, dH_{2n} \\ &= \int_{\partial \Omega_r} v A \nabla \Gamma \cdot \vec{n} \, dH_{2n} - \int_{\partial \Omega_\varepsilon} v A \nabla \Gamma \cdot \vec{n} \, dH_{2n}. \end{aligned} \tag{2.14}$$

In the first two terms in the r.h.s. of (2.14) we have used the fact that $\Gamma = \dfrac{C_Q}{\rho^{Q-2}}$ on $\partial \Omega_\rho$, $\rho > 0$. Next, we observe that $\vec{n} = \dfrac{\nabla d}{|\nabla d|}$ on $\partial \Omega_r$, while $\vec{n} = -\dfrac{\nabla d}{|\nabla d|}$ on $\partial \Omega_\varepsilon$. Using this observation and (2.13) for the first integral in the r.h.s. of (2.14) we obtain

$$\begin{aligned} \int_{\Omega_r \setminus \Omega_\varepsilon} \Gamma \Delta_{\mathbb{H}^n} v \, dz \, dt &= \frac{C_Q}{r^{Q-2}} \int_{\Omega_r} \Delta_{\mathbb{H}^r} v \, dz \, dt \\ &+ \frac{C_Q}{\varepsilon^{Q-2}} \int_{\partial \Omega_\varepsilon} A \nabla v \cdot \vec{n} \, dH_{2n} - \int_{\partial \Omega_r} v \frac{A \nabla \Gamma \cdot \nabla d}{|\nabla d|} dH_{2n} \\ &+ \int_{\partial \Omega_\varepsilon} v \frac{A \nabla \Gamma \cdot \nabla d}{|\nabla d|} dH_{2n}. \end{aligned} \tag{2.15}$$

(1.13) yields

$$\nabla \Gamma(z,t) = -(Q-2) C_Q d(z,t)^{1-Q} \nabla d(z,t), \tag{2.16}$$

so that using the notation (1.23)

$$\int_{\partial \Omega_\varepsilon} v \frac{A \nabla \Gamma \cdot \nabla d}{|\nabla d|} dH_{2n} = -\frac{(Q-2) C_Q}{\varepsilon^{Q-1}} \int_{\partial \Omega_\varepsilon} v \frac{|\nabla_{\mathbb{N}^n} d|^2}{|\nabla d|} dH_{2n}. \tag{2.17}$$

We want to show that (1.24) holds. (1.12) yields

$$\nabla d(z,t) = \frac{1}{4 d(z,t)^3} \begin{pmatrix} 4|z|^2 x \\ 4|z|^2 y \\ 2t \end{pmatrix}, \tag{2.18}$$

so that applying (2.11) we obtain

$$(2.19)\quad A(z)\nabla d(z,t) = \frac{1}{4d(z,t)^3}\begin{pmatrix} 4|z|^2x+4ty \\ 4|z|^2y-4tx \\ 8|z|^2t \end{pmatrix}$$

$$= \frac{1}{d(z,t)}\left[\frac{|z|^2}{d(z,t)^2}\begin{pmatrix} x \\ y \\ 2t \end{pmatrix} + \frac{t}{d(z,t)^2}\begin{pmatrix} y \\ -x \\ 0 \end{pmatrix}\right].$$

Now we define

$$(2.20)\qquad X = \sum_{j=1}^{n}\left(x_j\frac{\partial}{\partial x_j} + y_j\frac{\partial}{\partial y_j}\right) + 2t\frac{\partial}{\partial t}$$

$$(2.21)\qquad \varphi(z,t) = \frac{t}{d(z,t)^2}.$$

We remark that φ is Heisenberg-homogeneous of degree zero. Recalling (1.8), (1.18) we can rewrite (2.19) as follows

$$(2.22)\qquad A\nabla d = \frac{1}{d}\{\psi X+\varphi T\} \qquad \text{in } \mathbb{H}^n\backslash\{0,0\}.$$

This formula plays an important role in the sequel. From (1.8), (2.18) it is readily verified that

$$(2.23)\qquad Td \equiv 0 \qquad \text{in } \mathbb{H}^n\backslash\{(0,0)\},$$

which shows that the vector field T is tangent to the Heisenberg spheres $\partial\Omega_r$, $r > 0$. Moreover, (2.18), (2.20) give

$$(2.24)\qquad Xd = d \qquad \text{in } \mathbb{H}^n\backslash\{(0,0)\}.$$

This fact will play a crucial role in section 3. (2.22), (2.23) and (2.24) immediately give (1.24). If we replace (1.24) in (2.17), recalling (1.14) and (2.7) we obtain

$$(2.25)\quad \lim_{\varepsilon\to 0^+}\int_{\partial\Omega_\varepsilon} v\frac{A\nabla\Gamma\cdot\nabla d}{|\nabla d|}\,dH_{2n}$$

$$= -\lim_{\varepsilon\to 0^+}\frac{1}{|\partial\Omega_\varepsilon|_{\mathbb{H}^n}}\int_{\partial\Omega_\varepsilon} v\frac{\psi}{|\nabla d|}\,dH_{2n} = -v(0,0).$$

Furthermore, (2.7) gives

$$(2.26)\qquad \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^{Q-2}}\int_{\partial\Omega_\varepsilon} A\nabla v\cdot\vec{n}\,dH_{2n} = 0.$$

Finally, since by (1.13) $\Gamma \in L^1(\Omega_r)$ (to see it use Greiner's polar coordinates, see [GrK]), using (2.25), (2.26) when passing to the limit as $\varepsilon \to 0^+$ in (2.15), we obtain (2.8).

To complete the proof we change r with ρ in (2.8), multiply both sides by ρ^{Q-1} and integrate in ρ between 0 and r. The co-area formula (2.4) allows us to conclude that (2.9) holds. □

Before we proceed we pause for a moment to further elucidate the role of the density ψ. (2.22) and a direct computation give

$$\Delta_{\mathbb{H}^n} d = \frac{Q-1}{d}\psi \qquad \text{in } \mathbb{H}^n \backslash \{(0,0)\}. \tag{2.27}$$

Observing that if u is a *radial* function on $\mathbb{H}^n$, i.e., if $u(z,t) = f(d(z,t))$ for some f on $\mathbb{R}^+$, then

$$\Delta_{\mathbb{H}^n} u = (|\nabla_{\mathbb{H}^n} d|^2) f''(d) + (\Delta_{\mathbb{H}^n} d) f'(d), \tag{2.28}$$

replacing (1.24), (2.27) in (2.28) we find the remarkable formula

$$\Delta_{\mathbb{H}^n} u = \psi \left[f''(d) + \frac{Q-1}{d} f'(d) \right], \tag{2.29}$$

see also [FS]. From (2.29) it is immediate to guess that the fundamental solution of $-\Delta_{\mathbb{H}^n}$ should be given by (1.13).

Our next task is to establish an uncertainty principle for the Heinsenberg group. In [GL2] an important role was played by the following *a priori* inequality. Let $B_r = \{x \in \mathbb{R}^N \mid |x| < r\}$. Then for every $u \in C_0^\infty(\mathbb{R}^N \backslash \{0\})$

$$\int_{B_r} \frac{u(x)^2}{|x|^2} dx \leqslant \left(\frac{2}{N-2}\right)^2 \left\{ \left(\frac{N-2}{2}\right) \frac{1}{r} \int_{\partial B_r} u^2(x)\, d\sigma + \int_{B_r} |\nabla u(x)|^2\, dx \right\}. \tag{2.30}$$

By density one immediately obtains from (2.30) for every $u \in H^1(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} \frac{u(x)^2}{|x|^2} dx \leqslant \left(\frac{2}{N-2}\right)^2 \int_{\mathbb{R}^n} |\nabla u(x)|^2\, dx, \tag{2.31}$$

which is an inequality of Hardy, see [KSWW], Lemma 1. (2.31) and Schwarz's inequality imply

$$(2.32)\quad \left(\int_{\mathbb{R}^N} |x|^2 u(x)^2\,dx\right)\left(\int_{\mathbb{R}^N} |\nabla u(x)|^2\,dx\right) \geqslant \left(\frac{N-2}{2}\right)^2 \left(\int_{\mathbb{R}^N} u(x)^2\,dx\right)^2,$$

which via Plancherel's theorem for the Fourier transform yields the harmonic analysis formulation of Heisenberg's uncertainty principle, see [He], Chap. 2, (with $\frac{N-2}{2}$ replaced by $\frac{N}{2}$ equality is attained in (2.32) iff $u(x) = A\exp(-\alpha|x|^2)$, for some $A \in \mathbb{R}$, $\alpha > 0$).

In our context we have

THEOREM 2.2. — *For every* $u \in C_0^\infty(\mathbb{H}^n \backslash \{(0,0)\})$ *and every* $r > 0$

$$(2.33)\quad \int_{\Omega_r} \frac{u(z,t)^2}{d(z,t)^2}\psi(z,t)\,dz\,dt \leqslant \left(\frac{2}{Q-2}\right)^2 \left\{\left(\frac{Q-2}{2}\right)\frac{1}{r}\int_{\partial\Omega_r} u(z,t)^2 \frac{\psi(z,t)}{|\nabla d(z,t)|^2}\,dH_{2n} + \int_{\Omega_r} |\nabla_{\mathbb{H}^n} u(z,t)|^2\,dz\,dt\right\}.$$

COROLLARY 2.1 (Hardy-type inequality for $\mathbb{H}^n$). — *Let* $u \in C_0^\infty(\mathbb{H}^n \backslash \{(0,0)\})$, *then*

$$\int_{\mathbb{H}^n} \frac{u(z,t)^2}{d(z,t)^2}\psi(z,t)\,dz\,dt \leqslant \left(\frac{2}{Q-2}\right)^2 \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u(z,t)|^2\,dz\,dt.$$

COROLLARY 2.2 (Uncertainty principle for $\mathbb{H}^n$). — *For every* $u \in C_0^\infty(\mathbb{H}^n \backslash \{(0,0)\})$

$$\left(\int_{\mathbb{H}^n} d(z,t)^2 u(z,t)^2 \psi(z,t)\,dz\,dt\right)\left(\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u(z,t)|^2\,dz\,dt\right) \geqslant \left(\frac{Q-2}{2}\right)^2 \left(\int_{\mathbb{H}^n} u(z,t)^2 \psi(z,t)\,dz\,dt\right)\cdot$$

One should compare Theorem 2.2 and Corollaries 2.1 and 2.2 with (2.30)-(2.32) above. Also, it should be noticed that with $\frac{Q-2}{2}$ replaced by $\frac{Q}{2}$ equality is attained in Corollary 2.2 iff $u(z,t) = A \exp(-\alpha d(z,t)^2)$, for some $A \in \mathbb{R}$, $\alpha > 0$.

Proof of Theorem 2.2. – By formula (2.4) we have

$$\text{(2.34)} \quad \int_{\Omega_r} \frac{u(z,t)^2}{d(z,t)^2} \psi(z,t)\, dz\, dt$$

$$= \int_0^r \left(\int_{\partial\Omega_\rho} \frac{u(z,t)^2}{d(z,t)^2} \frac{\psi(z,t)}{|\nabla d(z,t)|} dH_{2n} \right) d\rho$$

$$= \int_0^r \left(-\frac{1}{\rho} \right)' \left(\int_{\partial\Omega_\rho} u(z,t)^2 \frac{\psi(z,t)}{|\nabla d(z,t)|\, dH_{2n}} \right) d\rho$$

$$= \text{(integrating by parts)} - \frac{1}{r} \int_{\partial\Omega_r} u(z,t)^2 \frac{\psi(z,t)}{|\nabla d(z,t)|} dH_{2n}$$

$$+ \int_0^r \frac{1}{\rho} \frac{d}{d\rho} \left(\int_{\partial\Omega_\rho} u(z,t)^2 \frac{\psi(z,t)}{|\nabla d(z,t)|} dH_{2n} \right) d\rho .$$

We now apply formula (2.8) to the function $v = u^2$ obtaining

$$\int_0^r \frac{1}{\rho} \frac{d}{d\rho} \left(\int_{\partial\Omega_\rho} u(z,t)^2 \frac{\psi(z,t)}{|\nabla d(z,t)|} dH_{2n} \right) d\rho$$

$$= \int_0^r \frac{1}{\rho} \frac{d}{d\rho} \left(\frac{\rho^{Q-1}}{(Q-2)C_Q} \left\{ \int_{\Omega_\rho} \Delta_{\mathbb{H}^n}(u^2)(z,t) \right. \right.$$

$$\text{(2.35)} \quad \left. \left. \left[\Gamma(z,t) - \frac{C_Q}{\rho^{Q-2}} \right] dz\, dt + u(0,0)^2 \right\} \right) d\rho$$

$$= (Q-1) \int_0^r \frac{1}{\rho^2} \int_{\partial\Omega_\rho} u(z,t)^2 \frac{\psi(z,t)}{|\nabla d(z,t)|} dH_{2n}$$

$$+ \frac{1}{(Q-2)C_Q} \int_0^r \rho^{Q-2} \frac{d}{d\rho} \left(\int_{\Omega_\rho} \Delta_{\mathbb{H}^n}(u^2)(z,t) \right.$$

$$\left. \left[\Gamma(z,t) - \frac{C_Q}{\rho^{Q-2}} \right] dz\, dt \right) d\rho .$$

Using again formula (2.4), the fact that $\Gamma = \frac{C_Q}{\rho^{Q-2}}$ on $\partial\Omega_\rho$, (2.10) and the divergence theorem we have

$$\frac{d}{d\rho}\left(\int_{\Omega_\rho} \Delta_{\mathbb{H}^n}(u^2)(z,t)\left[\Gamma(z,t)-\frac{C_Q}{\rho^{Q-2}}\right]dz\,dt\right)$$

$$= \frac{(Q-2)C_Q}{\rho^{Q-1}}\int_{\Omega_\rho} \Delta_{\mathbb{H}^n}(u^2)(z,t)\,dz\,dt$$

$$= \frac{(Q-2)C_Q}{\rho^{Q-1}}\int_{\partial\Omega_\rho} A(z)\nabla(u^2)(z,t).\frac{\nabla d(z,t)}{|\nabla d(z,t)|}\,dH_{2n}.$$

Using this information in the second integral in the r.h.s. of (2.35) we obtain from (2.34)

$$(2.36)\quad \int_{\Omega_r}\frac{u(z,t)^2}{d(z,t)^2}\psi(z,t)\,dz\,dt = -\frac{1}{r}\int_{\partial\Omega_r} u(z,t)^2\frac{\psi(z,t)}{|\nabla d(z,t)|}\,dH_{2n}$$
$$+ (Q-1)\int_{\Omega_r}\frac{u(z,t)^2}{d(z,t)^2}\psi(z,t)\,dz\,dt$$
$$+ 2\int_{\Omega_r}\frac{u(z,t)}{d(z,t)}A(z)\nabla u(z,t).\nabla d(z,t)\,dz\,dt.$$

Since the matrix $A(z)$ in (2.11) is symmetric and positive semi-definite, (2.36), Schwarz's inequality and (1.24) give

$$(2.37)\quad (Q-2)\int_{\Omega_r}\frac{u(z,t)^2}{d(z,t)^2}\psi(z,t)\,dz\,dt$$
$$\leqslant \frac{1}{r}\int_{\partial\Omega_r} u(z,t)^2\frac{\psi(z,t)}{|\nabla d(z,t)|}\,dH_{2n}$$
$$+ 2\left(\int_{\Omega_r}\frac{u(z,t)^2}{d(z,t)^2}|\nabla_{\mathbb{H}^n} d(z,t)|^2\,dz\,dt\right)^{\frac{1}{2}}\left(\int_{\Omega_r}|\nabla_{\mathbb{H}^n} u(z,t)|^2\,dz\,dt\right)^{\frac{1}{2}}$$
$$\leqslant \frac{1}{r}\int_{\partial\Omega_r} u(z,t)^2\frac{\psi(z,t)}{|\nabla d(z,t)|}\,dH_{2n}$$
$$+ \varepsilon\int_{\Omega_r}\frac{u(z,t)^2}{d(z,t)^2}\psi(z,t)\,dz\,dt$$
$$+ \frac{1}{\varepsilon}\int_{\Omega_r}|\nabla_{\mathbb{H}^n} u(z,t)|^2\,dz\,dt,$$

where $\varepsilon > 0$ arbitrary. Choosing $\varepsilon = \frac{Q-2}{2}$, we obtain (2.33) from (2.37). □

3. Conformal vector fields and sub-elliptic first variation formulas.

The aim of this section is to compute the first variation of the energy integral associated to (1.4). First variation estimates play an important role in calculus of variations and geometric measure theory. The standard strategy to achieve them is to perform a so-called *radial deformation* and then use the minimizing properties of the energy integral. The effectiveness of this procedure is deeply related to the existence, at least locally, of conformal vector fields. We recall that on a Riemannian manifold of dimension N, (M,g_{ij}), a vector field Z is said *conformal* if

$$Z_{i,j} + Z_{j,i} = \frac{2}{N}(\mathrm{div}_M Z)g_{ij}, \qquad i,j = 1,\dots,N,$$

where we have denoted by $Z_{i,j}$ the covariant derivative of the i-th component of Z w.r.t. the j-th local coordinate. In Euclidean flat space there exists a distinguished conformal vector field, namely $Z = r\frac{\partial}{\partial r} = x$, $x \in \mathbb{R}^N$. The fact that this vector field is orthogonal to the level sets of the fundamental solution of Laplace's operator with pole at the origin has important consequences. One of them is clearly illustrated by the first variation formula for the Dirichlet integral of a function in $\mathbb{R}^N$. Let $B_r = \{x \in \mathbb{R}^N \mid |x| < r\}$, then

$$(3.1)\quad \int_{\partial B_r} |\nabla u(x)|^2 \, d\sigma = \frac{N-2}{r}\int_{B_r} |\nabla u(x)|^2 \, dx + 2\int_{\partial B_r}\left(\frac{\partial u}{\partial n}(x)\right)^2 d\sigma - \frac{2}{r}\int_{B_r} x\cdot\nabla u(x)\Delta u(x)\, dx,$$

where we have let $\frac{\partial u}{\partial n}(x) = \nabla u(x)\cdot\frac{x}{|x|}\cdot$ It is noticeable in (3.1) the absence of terms involving tangential derivatives of u.

There exists on the Heisenberg group a vector field which plays much the same fundamental role of the conformal vector field $Z = r\frac{\partial}{\partial r}$ in $\mathbb{R}^N$, namely the vector field X introduced in (2.20). This vector field can be thought of as conformal in the following sense. Let $G = (g_{ij})$ be the $(2n+1)\times(2n+1)$ matrix that generates the Heisenberg dilations, see (1.10). Then we have

$$(3.2) \quad X_{i,j} + X_{j,i} = \frac{2}{Q}(\operatorname{div} X) g_{ij}, \qquad i,j = 1, \ldots, 2n+1,$$

where $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$. In (3.2) we have denoted by $X_{i,j}$ the partial $\frac{\partial X_i}{\partial \xi_j}$, where we have let for $(x,y,t) \in \mathbb{H}^n$ $\xi_i = x_i$, $\xi_{n+i} = y_i$, $i = 1, \ldots, n$, $\xi_{2n+1} = t$.

Unlike its Euclidean relative $r\frac{\partial}{\partial r}$ the vector field X in (2.20) is not orthogonal to the level sets of the fundamental solution (1.13), i.e., the Heisenberg spheres $\partial\Omega_r$. Yet, it displays the important redeeming feature (2.24) : Its projection Xd along the direction orthogonal to the sphere $\partial\Omega_r$ has constant value r on the sphere itself. We exploit this fact to obtain the following remarkable sub-elliptic first-variation formula.

THEOREM 3.1. — *Let u be a function such that u, $X_j u$, $T_j u$, $j = 1, \ldots, n$, and $\Delta_{\mathbb{H}^n} u \in L^2(\mathbb{H}^n)$. Then we have for a.e. $r > 0$,*

$$(3.3) \quad \int_{\partial\Omega_r} |\nabla_{\mathbb{H}^n} u(z,t)|^2 \frac{1}{|\nabla d(z,t)|} dH_{2n} = \frac{Q-2}{r} \int_{\Omega_r} |\nabla_{\mathbb{H}^n} u(z,t)|^2 \, dz\, dt$$
$$+ 2 \int_{\partial\Omega_r} \left(\frac{Xu(z,t)}{r}\right)^2 \frac{\psi(z,t)}{|\nabla d(z,t)|} dH_{2n}$$
$$+ 2 \int_{\partial\Omega_r} \left(\frac{Xu(z,t)}{r}\right)\left(\frac{Tu(z,t)}{r}\right) \frac{\varphi(z,t)}{|\nabla d(z,t)|} dH_{2n}$$
$$- \frac{2}{r} \int_{\Omega_r} (Xu(z,t))\, \Delta_{\mathbb{H}^n} u(z,t)\, dz\, dt.$$

In (3.3) T is defined by (1.8), X by (2.20), and φ by (2.21) (recall that φ is Heisenberg-homogeneous of degree zero). According to (2.23) the vector field T is tangential to the sphere $\partial\Omega_r$. Therefore, the third term in the r.h.s. of (3.3) represents a novelty w.r.t. the Euclidean case, and an unpleasant one, indeed.

Proof of Theorem 3.1. — In what follows for the sake of simplicity we will drop the independent variable (z,t) in all the integrands. Moreover, we agree to let $\xi_i = x_i$, $\xi_{n+i} = y_i$, $i = 1, \ldots, n$, $\xi_{2n+1} = t$, and denote by u_i, $X_{i,j}$, $a_{ij,k}$ respectively the partials $\frac{\partial u}{\partial \xi_i}, \frac{\partial X_i}{\partial \xi_j}, \frac{\partial a_{ij}}{\partial \xi_k}$, where

$(a_{ij}) = A$ is the matrix in (2.11). Also we denote by $(A\nabla u)_i$ the i-th component of $A\nabla u$, so that $(A\nabla u)_{i,j}$ denotes the partial $(A\nabla u)_i$ w.r.t. ξ_j. Throughout, we will use the summation convention over repeated indices.

(2.24), the divergence theorem and the fact that

$$\operatorname{div} X = Q$$

yield

$$(3.4)\quad \int_{\partial\Omega_r} |\nabla_{\mathbb{H}^n} u|^2 \frac{1}{|\nabla d|}\, dH_{2n} = \frac{1}{r}\int_{\partial\Omega_r} |\nabla_{\mathbb{H}^n} u|^2 \frac{Xd}{|\nabla d|}\, dH_{2n}$$

$$= \frac{1}{r}\int_{\Omega_r} \operatorname{div}(|\nabla_{\mathbb{H}^n} u|^2 X)\, dz\, dt = \frac{Q}{r}\int_{\Omega_r} |\nabla_{\mathbb{H}^n} u|^2\, dz\, dt$$

$$+ \frac{1}{r}\int_{\Omega_r} X(|\nabla_{\mathbb{H}^n} u|^2)\, dz\, dt$$

$$= \frac{Q}{r}\int_{\Omega_r} |\nabla_{\mathbb{H}^n} u|^2\, dz\, dt + \frac{1}{r}\int_{\Omega_r} X_i((A\nabla u)_j u_{,j})_{,i}\, dz\, dt$$

$$= \frac{Q}{r}\int_{\Omega_r} |\nabla_{\mathbb{H}^n} u|^2\, dz\, dt + \frac{1}{r}\int_{\Omega_r} X_i(A\nabla u)_{j,i} u_{,j}\, dz\, dt$$

$$+ \frac{1}{r}\int_{\Omega_r} X_i(A\nabla u)_j u_{,ji}\, dz\, dt$$

$$= \frac{Q}{r}\int_{\Omega_r} |\nabla_{\mathbb{H}^n} u|^2\, dz\, dt + \frac{1}{r}\int_{\Omega_r} X_i(a_{jk}u_{,k})_{,i} u_{,j}\, dz\, dt$$

$$+ \frac{1}{r}\int_{\Omega_r} X_i((A\nabla u)_j u_{,i})_{,j}\, dz\, dt - \frac{1}{r}\int_{\Omega_r} X_i u_{,i}((A\nabla u)_{j,j}\, dz\, dt.$$

Integrating by parts in the third integral in the r.h.s. of (3.4) we obtain

$$(3.5)\quad \frac{1}{r}\int_{\Omega_r} X_i((A\nabla u)_j u_{,i})_{,j}\, dz\, dt = \frac{1}{r}\int_{\partial\Omega_r} X_i u_{,i}(A\nabla u)_j n_j\, dH_{2n}$$

$$- \frac{1}{r}\int_{\Omega_r} X_{i,j}(A\nabla u)_j u_{,i}\, dz\, dt$$

$$= \frac{1}{r}\int_{\partial\Omega_r} (Xu)(A\nabla u\cdot\vec{n})\, dH_{2n} - \frac{1}{r}\int_{\Omega_r} X_{i,j}(A\nabla u)_j u_{,i}\, dz\, dt.$$

As for the second integral in the r.h.s. of (3.4) we have

$$(3.6)\quad \frac{1}{r}\int_{\Omega_r} X_i(a_{jk}u_{,k})_{,i}u_{,j}\,dz\,dt = \frac{1}{r}\int_{\Omega_r} X_i u_{,ki}a_{jk}u_{,j}\,dz\,dt$$

$$+\frac{1}{r}\int_{\Omega_r} X_i a_{jk,i}u_{,k}u_{,j}\,dz\,dt$$

$$=\frac{1}{r}\int_{\Omega_r} X_i u_{,ki}(A\nabla u)_k\,dz\,dt + \frac{1}{r}\int_{\Omega_r} X_i a_{jk,i}u_{,k}u_{,j}\,dz\,dt$$

$$=\frac{1}{r}\int_{\Omega_r} X_i((A\nabla u)_k u_{,i})_{,k}\,dz\,dt - \frac{1}{r}\int_{\Omega_r} X_i u_{,i}(A\nabla u)_{k,k}\,dz\,dt$$

$$+\frac{1}{r}\int_{\Omega_r} X_i a_{jk,i}u_{,k}u_{,j}\,dz\,dt$$

$$=\frac{1}{r}\int_{\partial\Omega_r} X_i u_{,i}(A\nabla u)_k n_k\,dH_{2n} - \frac{1}{r}\int_{\Omega_r} X_{i,k}(A\nabla u)_k u_{,i}\,dz\,dt$$

$$-\frac{1}{r}\int_{\Omega_r} Xu\,\Delta_{\mathbb{H}^n}u\,dz\,dt + \frac{1}{r}\int_{\Omega_r} X_i a_{jk,i}u_{,k}u_{,j}\,dz\,dt.$$

Substituting (3.5), (3.6) in (3.4) we obtain

$$(3.7)\quad \int_{\partial\Omega_r} |\nabla_{\mathbb{H}^n}u|^2\frac{1}{|\nabla d|}\,dH_{2n}$$

$$=\frac{Q}{r}\int_{\Omega_r}|\nabla_{\mathbb{H}^n}u|^2\,dz\,dt - \frac{2}{r}\int_{\Omega_r} X_{i,j}(A\nabla u)_j u_{,i}\,dz\,dt$$

$$+\frac{1}{r}\int_{\Omega_r} X_i a_{jk,i}u_{,k}u_{,j}\,dz\,dt + \frac{2}{r}\int_{\partial\Omega_r}(Xu)(A\nabla u\cdot\vec{n})\,dH_{2n}$$

$$-\frac{2}{r}\int_{\Omega_r} Xu\,\Delta_{\mathbb{H}^n}u\,dz\,dt.$$

From (3.2), the symmetry of the matrix $(X_{i,j})$ and the fact that $\operatorname{div} X = Q$ we infer $X_{i,j} = g_{ij}$. On the other hand, (2.11) gives with obvious meaning of the notation

$$A\nabla u = \begin{pmatrix} \nabla_x u + 2\dfrac{\partial u}{\partial t}y \\ \nabla_y u - 2\dfrac{\partial u}{\partial t}x \\ 2Tu + 4|z|^2\dfrac{\partial u}{\partial t}\end{pmatrix}.$$

We conclude

$$(3.8)\quad X_{i,j}(A\nabla u)_j u_{,i} = g_{ij}(A\nabla u)_j u_{,i} = |\nabla_{\mathbb{H}^n} u|^2 + 2\frac{\partial u}{\partial t} Tu + 4|z|^2\left(\frac{\partial u}{\partial t}\right)^2 .$$

The final step is the computation of the sum $X_i a_{jk,i} u_{,k} u_{,j}$. We note that from (2.11) the only non-zero terms in $a_{jk,i}$ occur when either $j = 2n+1$ or $k = 2n+1$. Moreover, since a_{jk} does not depend on $\xi_{2n+1} = t$ we have $a_{jk,(2n+1)} = 0$ for every j, k. Therefore, since $a_{ij} = a_{ji}$

$$\begin{aligned}(3.9)\quad X_i a_{jk,i} u_{,k} u_{,j} &= 2\sum_{i=1}^{2n} X_i\left(\sum_{k=1}^{2n} a_{(2n+1)k,i} u_{,k}\right) u_{,(2n+1)} \\ &\quad + u_{,(2n+1)}^2 \sum_{i=1}^{2n} X_i a_{(2n+1)(2n+1),i} \\ &= 2\frac{\partial u}{\partial t}\sum_{i=1}^{n} X_i\left[\sum_{k=1}^{n} a_{(2n+1)k,i} u_{,k} + \sum_{k=1}^{n} a_{(2n+1)(n+k),i} u_{,(n+k)}\right] \\ &\quad + 2\frac{\partial u}{\partial t}\sum_{i=1}^{n} X_{n+i}\left[\sum_{k=1}^{n} a_{(2n+1)k,(n+i)} u_k + \sum_{k=1}^{n} a_{(2n+1)(n+k),(n+i)} u_{,(n+k)}\right] \\ &\quad + \left(\frac{\partial u}{\partial t}\right)^2 \sum_{i=1}^{2n} X_i a_{(2n+1)(2n+1),i} .\end{aligned}$$

If $i = 1, \dots, n$ and $k = 1, \dots, n$ we have

$$a_{(2n+1)k,i} = 2\frac{\partial \xi_{n+k}}{\partial \xi_i} = 0, \qquad a_{(2n+1)(n+k),i} = -2\frac{\partial \xi_k}{\partial \xi_i} = -2\delta_{ki},$$

$$a_{(2n+1)k,(n+i)} = 2\frac{\partial \xi_{n+k}}{\partial \xi_{n+i}} = 2\delta_{ki}, \qquad a_{(2n+1)(n+k),(n+i)} = -2\frac{\partial \xi_k}{\partial \xi_{n+i}} = 0,$$

whereas $a_{(2n+1)(2n+1),i} = \dfrac{\partial}{\partial \xi_i}(4|z|^2) = 8\xi_i$, $i = 1, \dots, 2n$.

We then infer from (3.9)

$$\begin{aligned}(3.10)\quad X_i a_{jk,i} u_{,k} u_{,j} &= 4\frac{\partial u}{\partial t}\sum_{i=1}^{n} [X_{n+i} u_{,i} - X_i u_{,(n+i)}] \\ &\quad + 8\left(\frac{\partial u}{\partial t}\right)^2 \sum_{i=1}^{n} [X_i \xi_i + X_{n+i}\xi_{n+i}] \\ &= 4\frac{\partial u}{\partial t} Tu + 8|z|^2\left(\frac{\partial u}{\partial t}\right)^2 .\end{aligned}$$

Substituting (3.8), (3.10) in (3.7) we finally obtain

$$(3.11)\quad \int_{\partial\Omega_r} |\nabla_{\mathbb{H}^n} u|^2 \frac{1}{|\nabla d|}\, dH_{2n}$$
$$= \frac{Q-2}{r}\int_{\Omega_r} |\nabla_{\mathbb{H}^n} u|^2\, dz\, dt + \frac{2}{r}\int_{\partial\Omega_r} (Xu)(A\nabla u\cdot\vec{n})\, dH_{2n}$$
$$- \frac{2}{r}\int_{\Omega_r} Xu\, \Delta_{\mathbb{H}^n} u\, dz\, dt\,.$$

Since $A = A^*$ we have $A\nabla u\cdot\vec{n} = \dfrac{\nabla u\cdot A\nabla d}{|\nabla d|} = \dfrac{1}{d}\dfrac{[(Xu)\psi + (Tu)\varphi]}{|\nabla d|}$, where in the last equality we have used (2.22). From this observation and (3.11), (3.3) now follows. □

4. Frequency functions on $\mathbb{H}^n$ and unique continuation.

The aim of this section is to prove Theorems 1.1-1.4. We do so by studying the growth properties of certain quotients of variational integrals which are naturally related to equation (1.4). Henceforth, we will work in a fixed Heisenberg ball Ω_{R_0} centered at the origin.

For a solution u to (2.4) in Ω_{R_0} and $r < R_0$ we define its *height* in Ω_r as follows

$$(4.1)\qquad H(r) = \int_{\partial\Omega_r} u(z,t)^2 \frac{\psi(z,t)}{|\nabla d(z,t)|}\, dH_{2n}\,.$$

We also let

$$(4.2)\qquad D(r) = \int_{\Omega_r} |\nabla_{\mathbb{H}^n} u(z,t)|^2\, dz\, dt\,,$$

$$I(r) = \int_{\Omega_r} [|\nabla_{\mathbb{H}^n} u(z,t)|^2 + V(z,t)u(z,t)^2]\, dz\, dt\,,$$

and call these quantities respectively the *Dirichlet integral* and the *total energy* of u in Ω_r.

LEMMA 4.1. – *Let u be a solution to* (1.4) *in* Ω_{R_0}. *Then, for a.e.* $r \in (0, R_0)$

$$(4.3)\qquad H'(r) = \frac{Q-1}{r} H(r) + 2I(r)\,.$$

Proof. – A density argument, (1.14) and (2.8) of Theorem 2.1 yield

$$(4.4)\quad H(r) = \frac{r^{Q-1}}{(Q-2)C_Q}\left\{\int_{\Omega_r} \Delta_{\mathbb{H}^n}(u^2)(z,t)\cdot \left[\Gamma(z,t)-\frac{C_Q}{r^{Q-2}}\right] dz\, dt + u(0,0)^2\right\}.$$

Differentiating (4.4) w.r.t. r and using the co-area formula (2.4) to evaluate the derivative of the integral within curly brackets we obtain

$$(4.5)\qquad H'(r) = \frac{Q-1}{r} H(r) + \int_{\Omega_r} \Delta_{\mathbb{H}^n}(u^2)(z,t)\, dz\, dt\,.$$

Now we observe that

$$(4.6)\qquad \Delta_{\mathbb{H}^n}(u^2) = 2u\, \Delta_{\mathbb{H}^n} u + 2|\nabla_{\mathbb{H}^n} u|^2.$$

If we substitute (4.6) in (4.5) and use (1.4) we obtain (4.3). □

LEMMA 4.2. – *Let u be a solution to (1.4) on Ω_{R_0}. There exists $r_0 > 0$, depending only on Q, and on C and f in* (1.17), *such that either $u \equiv 0$ in Ω_{r_0} or $H(r) \neq 0$ for every $r \in (0,r_0)$.*

Proof. – Suppose that for some $r_0 < R_0$ $H(r_0) = 0$. Then from (4.6) and the divergence theorem

$$(4.7)\quad D(r_0) = I(r) - \int_{\Omega_{r_0}} V(z,t)u(z,t)^2\, dz\, dt \leqslant \frac{1}{2}\int_{\Omega_{r_0}} \Delta_{\mathbb{H}^n}(u^2)(z,t)\, dz\, dt$$
$$+ \int_{\Omega_{r_0}} |V(z,t)|u(z,t)^2\, dz\, dt = \int_{\partial\Omega_{r_0}} u(z,t)A(z)\nabla u(z,t)\cdot \vec{n}\, dH_{2n}$$
$$+ \int_{\Omega_{r_0}} |V(z,t)|u(z,t)^2\, dz\, dt = \int_{\partial\Omega_{r_0}} |V(z,t)|u(z,t)^2\, dz\, dt\,.$$

Next, we use (1.17) and (2.33) of Theorem 2.2 to obtain the bound

$$(4.8)\quad \int_{\Omega_{r_0}} |V(z,t)|u(z,t)^2\, dz\, dt \leqslant Cf(r_0)\int_{\Omega_{r_0}} \frac{u(z,t)^2}{d(z,t)^2}\psi(z,t)\, dz\, dt$$
$$\leqslant C\left(\frac{2}{Q-2}\right)^2 f(r_0)\left\{\frac{Q-2}{2}\frac{H(r_0)}{r_0} + D(r_0)\right\} = C\left(\frac{2}{Q-2}\right)^2 f(r_0)D(r_0)\,.$$

Since by (1.16) $\lim_{r\to 0^+} f(r) = 0$, we obtain a contradiction from (4.7), (4.8) unless $D(r_0) = 0$, which forces $u \equiv 0$ in Ω_{r_0}.

From the proof of Lemma 4.2 it is obvious that the dependence of r_0 on Q, C and f is such that $Cf(r_0) < \left(\frac{Q-2}{2}\right)^2$. Lemma 4.2 allows us to divide out by $H(r)$ in (4.3) obtaining

$$\text{(4.9)} \qquad \frac{H'(r)}{H(r)} = \frac{Q-1}{r} + 2\frac{I(r)}{H(r)} \qquad \text{for a.e. } r \in (0,r_0).$$

We now introduce the following

DEFINITION 4.1. – *Let u be a solution to* (1.4) *in Ω_{R_0} and let r_0 be as in Lemma* 4.2. *The quantity*

$$\text{(4.10)} \qquad N(r) = \frac{rI(r)}{H(r)}, \qquad r \in (0,r_0)$$

is called the frequency of u in Ω_r.

We stress the invariance of (4.10) w.r.t. the Heisenberg dilations (1.9).

Remark. – The function $N(r)$ introduced in (4.10) is the analogue of that introduced in [GL2] for solutions to uniformly elliptic equations of the type $-\operatorname{div}(A(x)\nabla u) + V(x)u = 0$. For harmonic functions in $\mathbb{R}^N$ the frequency was first introduced by F. Almgren [A], who proved its increasingness w.r.t. r. He called the quotient $N(r) = r\int_{B_r} |\nabla u|^2\, dx / \int_{\partial B_r} u^2 d\sigma$ the frequency of u in B_r because of the observation that when $u = \operatorname{Im} z^k$, $z \in \mathbb{C}$, $k \in \mathbb{N}$, then $N(r) = k$ for every $r > 0$. In view of this remark our definition is justified by the following

PROPOSITION 4.1. – *Let $V \equiv 0$ in Ω_{R_0}, $0 < R_0 \leqslant \infty$, so that $\nabla_{\mathbb{H}^n} u \equiv 0$ in Ω_{R_0}. If u is Heisenberg-homogeneous of degree $k \in \mathbb{N}$, then $N(r) = k$ for every $r \in (0,R_0)$.*

Proof. – Since $V \equiv 0$ we have $I(r) = D(r)$. From the definition (4.2), equation (4.6), the divergence theorem, the self-adjointness of A and

(2.22) we have

$$(4.11)\quad D(r) = \frac{1}{2}\int_{\Omega_r} \Delta_{\mathbb{H}^n}(u^2)(z,t)\,dz\,dt$$
$$= \int_{\partial\Omega_r} u(z,t)\,\frac{\nabla u(z,t)\cdot A(z)\nabla d(z,t)}{|\nabla d(z,t)|}\,dH_{2n}$$
$$= \frac{1}{r}\int_{\partial\Omega_r} u(z,t)\,Xu(z,t)\,\frac{\psi(z,t)}{|\nabla d(z,t)|}\,dH_{2n}$$
$$+ \frac{1}{r}\int_{\partial\Omega_r} u(z,t)\,Tu(z,t)\,\frac{\varphi(z,t)}{|\nabla d(z,t)|}\,dH_{2n}.$$

Since by (2.23) T is tangent to the smooth manifold without boundary $\partial\Omega_r$, Stokes' theorem and the fact that T is divergence free imply

$$(4.12)\qquad \int_{\partial\Omega_r} u(z,t)\,Tu(z,t)\,\frac{\varphi(z,t)}{|\nabla d(z,t)|}\,dH_{2n} = 0.$$

On the other hand, the fact that u is Heisenberg-homogeneous of degree k and Euler's formula yield

$$(4.13)\qquad Xu = ku.$$

Replacing (4.12), (4.13) into (4.11) finally gives

$$D(r) = \frac{k}{r}\int_{\partial\Omega_r} u(z,t)^2\,\frac{\psi(z,t)}{|\nabla d(z,t)|}\,dH_{2n} = \frac{k}{r}H(r).$$

This proves Proposition 4.1. □

We return to the general case $V \not\equiv 0$. As a corollary of Lemma 4.2 we have

LEMMA 4.3. – *The function $r \mapsto N(r)$ is absolutely continuous on the interval $(0,r_0)$. In particular, $N'(r)$ exists for a.e. $r \in (0,r_0)$.*

At this point we introduce the set

$$(4.14)\qquad \Lambda_{r_0} = \{r \in (0,r_0) \mid N(r) > \max(1, Nr_0))\}.$$

Because of Lemma 4.3 Λ_{r_0} is an open set, therefore

$$(4.15)\qquad \Lambda_{r_0} = \bigcup_{j=1}^{\infty}(a_j,b_j), \qquad a_j,\, b_j \notin \Lambda_{r_0}.$$

We emphasize that by the definition (4.14) we have

$$\frac{H(r)}{r} < I(r) \qquad \text{for every } r \in \Lambda_{r_0}, \tag{4.16}$$

thus, in particular, $I(r) \neq 0$ for every $r \in \Lambda_{r_0}$, see Lemma 4.2. By this observation and by Lemma 4.3 we have

$$\frac{N'(r)}{N(r)} = \frac{I'(r)}{I(r)} + \frac{1}{r} - \frac{H'(r)}{H(r)} \qquad \text{for a.e. } r \in \Lambda_{r_0}. \tag{4.17}$$

We have already computed $H'(r)/H(r)$, see (4.9). By means of Theorem 3.1, and since $\Delta_{\mathbb{H}^n} u = Vu$, we have

LEMMA 4.4. — *For a.e. $r \in \Lambda_{r_0}$ we have*

$$\frac{I'(r)}{I(r)} = \frac{Q-2}{r} + 2\frac{\int_{\partial\Omega_r}\left(\frac{Xu}{r}\right)^2 \frac{\psi}{|\nabla d|}\,dH_{2n}}{I(r)} + 2\frac{\int_{\partial\Omega_r}\left(\frac{Xu}{r}\right)\left(\frac{Tu}{r}\right)\frac{\varphi}{|\nabla d|}\,dH_{2n}}{I(r)}$$
$$+ \frac{\int_{\partial\Omega_r} Vu^2 \frac{1}{|\nabla d|}\,dH_{2n}}{I(r)} - \frac{Q-2}{r}\frac{\int_{\Omega_r} Vu^2\,dz\,dt}{I(r)} - \frac{2}{r}\frac{\int_{\Omega_r}(Xu)Vu\,dz\,dt}{I(r)}.$$

(4.9), (4.17) and Lemma 4.4 yield

LEMMA 4.5. — *For a.e. $r \in \Lambda_{r_0}$*

$$\frac{N'(r)}{N(r)} = 2\frac{\int_{\partial\Omega_r}\left(\frac{Xu}{r}\right)^2 \frac{\psi}{|\nabla d|}\,dH_{2n}}{I(r)} - 2\frac{I(r)}{H(r)} \tag{4.18}$$
$$+ 2\frac{\int_{\partial\Omega_r}\left(\frac{Xu}{r}\right)\left(\frac{Tu}{r}\right)\frac{\varphi}{|\nabla d|}\,dH_{2n}}{I(r)}$$
$$+ \frac{\int_{\partial\Omega_r} Vu^2 \frac{1}{|\nabla d|}\,dH_{2n}}{I(r)} - \frac{Q-2}{r}\frac{\int_{\Omega_r} Vu^2\,dz\,dt}{I(r)} - \frac{2}{r}\frac{\int_{\Omega_r}(Xu)Vu\,dz\,dt}{I(r)}.$$

We need one more lemma.

LEMMA 4.6. — *There exists a constant* $B = B(Q,C,f) > 0$ *such that every* $r \in \Lambda_{r_0}$ *we have*

$$D(r) \leqslant BI(r). \tag{4.19}$$

Proof. — As in the proof of (4.8) we have

$$\begin{aligned} D(r) &\leqslant I(r) + \int_{\Omega_r} |V(z,t)| u(z,t)^2 \, dz \, dt \\ &\leqslant I(r) + \left(\frac{2}{Q-2}\right)^2 Cf(r) \left\{\frac{Q-2}{2}\frac{H(r)}{r} + D(r)\right\} \\ &\leqslant \left[1 + \left(\frac{2}{Q-2}\right) Cf(r)\right] I(r) + \left(\frac{2}{Q-2}\right)^2 Cf(r)\, D(r), \end{aligned}$$

where in the last inequality we have used (4.16). Recalling that in the proof of Lemma 4.2 we have chosen $r_0 > 0$ such that $\left(\frac{2}{Q-2}\right)^2 Cf(r_0) < 1$, from the above inequality we obtain (4.19) with $B = \frac{1 + A_Q Cf(r_0)}{1 - A_Q^2 Cf(r_0)}$, where we have set $A_Q = \frac{2}{Q-2}$. □

We are now ready to prove the main result of this section.

THEOREM 4.1. — *Let* u *be a solution to* (1.4) *and suppose that* u *satisfies* (1.19) *for some* C_1 *and* g. *Then, there exists a constant* $M = M(Q, C, C_1, f, g) > 0$ *such that for a.e.* $r \in \Lambda_{r_0}$ *we have*

$$\frac{N'(r)}{N(r)} \geqslant - M\left[\frac{f(r)+g(r)}{r}\right]. \tag{4.20}$$

Proof. — Our starting point is (4.18) in Lemma 4.5. Assumption (1.17) on V and (4.16) give

$$\begin{aligned} \left|\int_{\partial\Omega_r} V(z,t) u(z,t)^2 \frac{1}{|\nabla d(z,t)|}\, dH_{2n}\right| \tag{4.21} \\ \leqslant \frac{Cf(r)}{r^2} \int_{\partial\Omega_r} u(z,t)^2 \frac{\psi(z,t)}{|\nabla d(z,t)|}\, dH_{2n} \\ = \frac{Cf(r)}{r}\frac{H(r)}{r} < \frac{Cf(r)}{r} I(r). \end{aligned}$$

Likewise (see the proof of (4.8)) with $A_Q = \dfrac{2}{Q-2}$

$$(4.22)\quad \left|\frac{Q-2}{r}\int_{\Omega_r} V(z,t)u(z,t)^2\,dz\,dt\right| \leqslant \frac{(Q-2)}{r} A_Q^2 Cf(r)\left\{A_Q^{-1}\frac{H(r)}{r} + D(r)\right\}$$

(by (4.16) and (4.19))

$$\leqslant (Q-2)A_Q^2 C(A_Q^{-1}+B)\frac{f(r)}{r} I(r).$$

Now we would like to get a bound for the last integral in the r.h.s. of (4.18). We have from (1.17)

$$(4.23)\quad \left|\frac{2}{r}\int_{\Omega_r} Xu(z,t)V(z,t)u(z,t)\,dz\,dt\right| \leqslant \frac{2Cf(r)}{r}\int_{\Omega_r} |Xu(z,t)|\,\frac{|u(z,t)|}{d(z,t)^2}\psi(z,t)\,dz\,dt$$

$$= \frac{2Cf(r)}{r}\int_{\Omega_r}\left|\frac{1}{d(z,t)}\psi(z,t)Xu(z,t)\right|\frac{|u(z,t)|}{d(z,t)}\,dz\,dt$$

$$= \frac{2Cf(r)}{r}\int_{\Omega_r}\left|A(z)\nabla d(z,t)\cdot\nabla u(z,t) - \frac{1}{d(z,t)}\varphi(z,t)Tu(z,t)\right|\frac{|u(z,t)|}{d(z,t)}\,dz\,dt$$

$$\leqslant \frac{2Cf(r)}{r}\int_{\Omega_r}|A(z)\nabla d(z,t)\cdot\nabla u(z,t)|\frac{|u(z,t)|}{d(z,t)}\,dz\,dt$$

$$+ \frac{2Cf(r)}{r}\int_{\Omega_r}\frac{|\varphi(z,t)Tu(z,t)|}{d(z,t)}\frac{|u(z,t)|}{d(z,t)}\,dz\,dt,$$

where in the second equality we have used (2.22). By the self-adjointness and the positive semi-definiteness of $A(z)$, Schwarz's inequality, and (1.24) we obtain

$$(4.24)\quad \int_{\Omega_r}|A(z)\nabla d(z,t)\cdot\nabla u(z,t)|\,\frac{|u(z,t)|}{d(z,t)}\,dz\,dt$$

$$\leqslant \left(\int_{\Omega_r}\frac{u(z,t)^2}{d(z,t)^2}|\nabla_{\mathbb{H}^n} d(z,t)|^2\,dz\,dt\right)^{\frac{1}{2}}\left(\int_{\Omega_r}|\nabla_{\mathbb{H}^n} u(z,t)|^2\,dz\,dt\right)^{\frac{1}{2}}$$

$$= \left(\int_{\Omega_r} \frac{u(z,t)^2}{d(z,t)^2}\psi(z,t)\,dz\,dt\right)^{\frac{1}{2}} \left(\int_{\Omega_r} |\nabla_{\mathbb{H}^n} u(z,t)|^2\,dz\,dt\right)^{\frac{1}{2}}$$

$$\leqslant A_Q^2\left[A_Q^{-1}\frac{H(r)}{r}+D(r)\right]^{\frac{1}{2}} D(r)^{\frac{1}{2}}$$

$$\leqslant \text{(by (4.16) and (4.19))}\ A_Q[(A_Q^{-1}+B)B]^{\frac{1}{2}}\, I(r).$$

Now we bound the second integral in the r.h.s. of (4.23). By Schwarz's inequality and (1.19) we obtain

$$(4.25)\quad \int_{\Omega_r} \frac{|\varphi(z,t)Tu(z,t)|}{d(z,t)}\frac{|u(z,t)|}{d(z,t)}\,dz\,dt$$

$$\leqslant \left(\int_{\Omega_r} \frac{|\varphi(z,t)Tu(z,t)|^2}{d(z,t)^2\psi(z,t)}\,dz\,dt\right)^{\frac{1}{2}} \left(\int_{\Omega_r} \frac{u(z,t)^2}{d(z,t)^2}\psi(z,t)\,dz\,dt\right)^{\frac{1}{2}}$$

$$\leqslant C_1 g(r)\int_{\Omega_r} \frac{u(z,t)^2}{d(z,t)^2}\psi(z,t)\,dz \quad \text{(by (2.33) of Theorem 2.2)}$$

$$\leqslant C_1 A_Q^2(A_Q^{-1}+B)g(r)I(r).$$

(4.23)-(4.25) finally yield

$$(4.26)\qquad \left|\frac{2}{r}\int_{\Omega_r} Xu(z,t)V(z,t)u(z,t)\,dz\,dt\right| \leqslant G\frac{f(r)}{r}I(r),$$

where $G = G(Q,C,C_1,f,g) > 0$.

(4.18) of Lemma 4.5 and the estimates (4.21), (4.22), (4.26) allow us to conclude that for a.e. $r \in \Lambda_{r_0}$

$$(4.27)\quad \frac{N'(r)}{N(r)} = 2\frac{\displaystyle\int_{\partial\Omega_r}\left(\frac{Xu}{r}\right)^2\frac{\psi}{\nabla d}\,dH_{2n}}{I(r)} - 2\frac{I(r)}{H(r)} + 2\frac{\displaystyle\int_{\partial\Omega_r}\left(\frac{Xu}{r}\right)\left(\frac{Tu}{r}\right)\frac{\varphi}{|\nabla d|}\,dH_{2n}}{I(r)} + 0\left(\frac{f(r)}{(r)}\right),$$

where $0\left(\frac{f(r)}{r}\right)$ is a function whose absolute value is bounded by $L\frac{f(r)}{r}$, where $L = L(Q,C,C_1,f,g) > 0$.

Schwarz's inequality again and (1.19) yield

$$(4.28)\quad \left|\int_{\partial\Omega_r}\left(\frac{Xu(z,t)}{r}\right)\left(\frac{Tu(z,t)}{r}\right)\frac{\varphi(z,t)}{|\nabla d(z,t)|}\,dH_{2n}\right|$$

$$\leqslant \left(\int_{\partial\Omega_r}\left(\frac{Xu(z,t)}{r}\right)^2\frac{\psi(z,t)}{|\nabla d(z,t)|}\,dH_{2n}\right)^{\frac{1}{2}}\cdot\left(\frac{1}{r^2}\int_{\partial\Omega_r}\frac{|\varphi(z,t)Tu(z,t)|^2}{\psi(z,t)|\nabla d(z,t)|}\,dH_{2n}\right)^{\frac{1}{2}}$$

$$\leqslant C_1\frac{g(r)}{r}\left(\int_{\partial\Omega_r}\left(\frac{Xu(z,t)}{r}\right)^2\frac{\psi(z,t)}{|\nabla d(z,t)|}\,dH_{2n}\right)^{\frac{1}{2}}\cdot\left(\int_{\partial\Omega_r}u(z,t)^2\frac{\psi(z,t)}{|\nabla d(z,t)|}\,dH_{2n}\right)^{\frac{1}{2}}.$$

We now distinguish two possibilities:

$$\text{(a)}\quad \int_{\partial\Omega_r}\left(\frac{Xu(z,t)}{r}\right)^2\frac{\psi(z,t)}{|\nabla d(z,t)|}\,dH_{2n}\int_{\partial\Omega_r}u(z,t)^2\frac{\psi(z,t)}{|\nabla d(z,t)|}\,dH_{2n}\leqslant 2I(r)^2$$

$$\text{(b)}\quad \int_{\partial\Omega_r}\left(\frac{Xu(z,t)}{r}\right)^2\frac{\psi(z,t)}{|\nabla d(z,t)|}\,dH_{2n}\int_{\partial\Omega_r}u(z,t)^2\frac{\psi(z,t)}{|\nabla d(z,t)|}\,dH_{2n}> 2I(r)^2.$$

If (a) occurs, then we simply conclude from (4.28) that

$$(4.29)\quad \left|\int_{\partial\Omega_r}\left(\frac{Xu(z,t)}{r}\right)\left(\frac{Tu(z,t)}{r}\right)\frac{\varphi(z,t)}{|\nabla d(z,t)|}\right|\leqslant\sqrt{2}\,C_1\frac{g(r)}{r}\,I(r).$$

Substituting (4.29) in (4.27) we obtain

$$(4.30)\quad \frac{N'(r)}{N(r)}=2\frac{\displaystyle\int_{\partial\Omega_r}\left(\frac{Xu}{r}\right)^2\frac{\psi}{|\nabla d|}\,dH_{2n}}{I(r)}-2\frac{I(r)}{H(r)}+0\left(\frac{f(r)+g(r)}{r}\right),$$

where

$$\left|0\left(\frac{f(r)+g(r)}{r}\right)\right|\leqslant M\left[\frac{f(r)+g(r)}{r}\right],$$

with $M=M(Q,C,C_1,f,g)>0$. We now claim that

$$(4.31)\qquad I(r)=\int_{\partial\Omega_r}u(z,t)\left(\frac{Xu(z,t)}{r}\right)\frac{\psi(z,t)}{|\nabla d(z,t)|}\,dH_{2n}.$$

Taking (4.31) for granted for a moment we see that, in virtue of it and of Schwarz's inequality, the difference of the first two terms in the r.h.s. of (4.30) is $\geqslant 0$. We conclude that (4.20) is true at those points of Λ_{r_0} at which (a) occurs.

If (b) occurs we argue in a different way. We obtain from (4.28) and Young's inequality

$$\text{(4.32)}\quad \left|\int_{\partial\Omega_r}\left(\frac{Xu(z,t)}{r}\right)\left(\frac{Tu(z,t)}{r}\right)\frac{\varphi(z,t)}{|\nabla d(z,t)|}\,dH_{2n}\right| \leqslant \frac{C_1}{2}g(r)\int_{\partial\Omega_r}\left(\frac{Xu(z,t)}{r}\right)^2\frac{\psi(z,t)}{|\nabla d(z,t)|}\,dH_{2n} + \frac{C_1}{2}\frac{g(r)}{r}\frac{H(r)}{r} < \frac{C_1}{2}g(r)\int_{\partial\Omega_r}\left(\frac{Xu(z,t)}{r}\right)^2\frac{\psi(z,t)}{|\nabla d(z,t)|}\,dH_{2n} + \frac{C_1}{2}\frac{g(r)}{r}I(r)$$

where in the last equality we have used the fact that for $r\in\Lambda_{r_0}$ (4.16) holds. At this point we restrict, if needed, the interval $(0,r_0)$ in such a way that r_0 satisfies the two constraints

$$\text{(4.33)}\qquad \left(\frac{2}{Q-2}\right)^2 Cf(r_0) < 1\,, \qquad 2-\frac{C_1}{2}g(r_0) > 1\,,$$

at once. We emphasize that from Lemma 2.1 r_0 has already been chosen to satisfy the first inequality in (4.33). Substitution of (4.32) in (4.27) gives

$$\text{(4.34)}\quad \frac{N'(r)}{N(r)} \geqslant \left[2-\frac{C_1}{2}g(r)\right]\frac{\int_{\partial\Omega_r}\left(\frac{Xu}{r}\right)^2\frac{\psi}{|\nabla d|}\,dH_{2n}}{I(r)} - 2\frac{I(r)}{H(r)} - M\left[\frac{f(r)+g(r)}{r}\right],$$

for a positive constant $M = M(Q, C, C_1, f, g)$. By the increasingness of g, the second inequality in (4.33) and (b), we conclude from (4.34) that (4.20) holds also at those $r\in\Lambda_{r_0}$ at which (b) occurs.

In order to conclude the proof of Theorem 4.1 we are left with verifying (4.31). (4.2), (4.6) and an integration by parts yield

$$I(r) = \frac{1}{2}\int_{\Omega_r}\Delta_{\mathbb{H}^n}(u^2)(z,t)\,dz\,dt = \int_{\partial\Omega_r}u(z,t)\frac{\nabla u(z,t)\cdot A(z)\nabla d(z,t)}{|\nabla d(z,t)|}\,dH_{2n}\,.$$

Using now formulas (2.22) and (4.12) we see that (4.31) holds. □

With Theorem 4.1 in hand we are now ready to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. – Our starting point is identity (4.9) which, using (4.10), we rewrite as

$$\frac{d}{dt}\left[\log\left(\frac{H(t)}{t^{Q-1}}\right)\right] = 2\frac{N(t)}{t}, \quad t \in (0,r_0). \tag{4.35}$$

We now integrate (4.35) between r and $2r$, where $r < \frac{r_0}{2}$, obtaining

$$\begin{aligned}\log\left[\frac{H(2r)}{H(r)}2^{1-Q}\right] &= 2\int_r^{2r} N(t)\frac{dt}{t} \\ &\leqslant 2\int_{(r,2r)\cap\Lambda r_0} N(t)\frac{dt}{t} + 2\int_{Jr} N(t)\frac{dt}{t},\end{aligned} \tag{4.36}$$

where we have let $J_r = \{t \in (r,2r) \mid t \notin \Lambda_{r_0}, N(t) \geqslant 0\}$. Since on J_r we have $0 \leqslant N \leqslant \max(1,N(r_0))$ (see (4.14))

$$\int_{J_r} N(t)\frac{dt}{t} \leqslant \max(1,N(r_0))\int_r^{2r}\frac{dt}{t} = \max(1,N(r_0))\log 2. \tag{4.37}$$

On the other hand, by integrating (4.20) on (r,b_j), where $r \in (a_j,b_j)$, and (a_j,b_j) is an arbitrary interval in the decomposition (4.15), we obtain

$$\log\frac{N(b_j)}{N(r)} = \int_r^{b_j}\frac{N'(t)}{N(t)}dt \geqslant -M\int_0^{r_0}[f(t)+g(t)]\frac{dt}{t} \geqslant -M\int_0^{R_0}[f(t+g(t)]\frac{dt}{t}\cdot$$

From this inequality, recalling that $b_j \notin \Lambda_{r_0}$, we infer

$$N(r) \leqslant \exp\left[M\int_0^{R_0}(f(t)+g(t))\frac{dt}{t}\right]\max(1,N(r_0)) \tag{4.38}$$

for every $r \in \Lambda_{r_0}$. (4.38) yields

$$\begin{aligned}\int_{(r,2r)\cap\Lambda_{r_0}} N(t)\frac{dt}{t} \\ \leqslant \exp\left[M\int_0^{R_0}(f(t)+g(t))\frac{dt}{t}\right]\max(1,N(r_0))\log 2.\end{aligned} \tag{4.39}$$

Using (4.37), (4.39) in (4.36) we finally obtain

$$\begin{aligned}H(2r) &\leqslant 2^{Q-1} \\ &\exp\left\{2\log 2\max(1,N(r_0)\left[1+\exp\left(M\int_0^{R_0}(f(t)+g(t))\frac{dt}{t}\right)\right]\right\}H(r).\end{aligned} \tag{4.40}$$

Integrating (4.40) w.r.t. r and using the co-area formula (2.4) we finally obtain (1.20). □

Proof of Theorem 1.2. – The argument is quite standard. We include it for the sake of completeness. Let r_0 be as in Theorem 1.1. We obtain after k interations of (1.20)

$$(4.41)\quad \int_{\Omega_{r_0}} u(z,t)^2\psi(z,t)\,dz\,dt \leqslant \cdots \leqslant \Gamma^k \int_{\Omega_{r_0 2^{-k}}} u(z,t)^2\psi(z,t)\,dz\,dt$$

$$\Gamma^k |\Omega_{r_0 2^{-k}}|^{\beta}_{\mathbb{H}^n} \frac{1}{|\Omega_{r_0 2^{-k}}|^{\beta}_{\mathbb{H}^n}} \int_{\Omega_{r_0 2^{-k}}} u(z,t)^2\psi(z,t)\,dz\,dt,$$

where for $\rho > 0\, |\Omega_\rho|_{\mathbb{H}^n}$ is defined by (2.2) and $\beta > 0$ is a number to be suitably chosen. By (2.3) we have $\Gamma^k |\Omega_{r_0 2^{-k}}|^{\beta}_{\mathbb{H}^n} = \alpha_Q^{\beta} r_0^{\beta Q} \left(\frac{\Gamma}{2^{\beta Q}}\right)^k$ · We now choose β such that $\Gamma/2^{\beta Q} = 1$. Then, (4.41) becomes

$$(4.42)\quad \int_{\Omega_{r_0}} u(z,t)^2\psi(z,t)\,dz\,dt$$

$$\leqslant |\Omega_{r_0}|^{\beta}_{\mathbb{H}^n} \frac{1}{|\Omega_{r_0 2^{-k}}|^{\beta}_{\mathbb{H}^n}} \int_{\Omega_{r_0 2^{-k}}} u(z,t)^2\psi(z,t)\,dz\,dt.$$

If we now let $k \to \infty$ the r.h.s. of (4.42) goes to zero since, by assumption, u vanishes to infinite order at the origin, see Definition 1.1. We conclude that must be $u \equiv 0$ in Ω_{r_0}. □

At this point we briefly sketch the proof of Theorem 1.3. The main steps are similar to those in the proof of Theorem 1.1. We recall that we no longer have the smallness available from (1.17), but assumption (1.25) on V^- takes its place. Lemma 4.2, e.g., carries over as follows (see (4.7)): Let $H(r_0) = 0$, then

$$D(r_0) = I(r_0) - \int_{\Omega_{r_0}} V^+(z,t)u(z,t)^2\,dz\,dt + \int_{\Omega_{r_0}} V^-(z,t)u(z,t)^2\,dz\,dt$$

$$\leqslant \int_{\Omega_{r_0}} V^-(z,t)u(z,t)^2\,dz\,dt \leqslant \delta \int_{\Omega_{r_0}} \frac{u(z,t)^2}{d(z,t)^2}\psi(z,t)\,dz\,dt$$

$$\leqslant \delta A_Q^2 \left\{ A_Q^{-1} \frac{H(r_0)}{r_0} + D(r_0) \right\} = \delta A_Q^2 D(r_0),$$

where we have used (1.25), (2.33), and we have set $A_Q = \frac{2}{Q-2}\cdot$ It is now obvious that if δ is such that $1 - \delta A_Q^2 > 0$, i.e., $0 < \delta < \left(\frac{Q-2}{2}\right)^2$, then we conclude $D(r_0) = 0$, and therefore $u \equiv 0$ in Ω_{r_0}.

As a consequence of Lemma 4.2, Lemma 4.3 holds, along with Lemmas 4.4 and 4.5. The proof of Lemma 4.6 is similar to that of Lemma 4.2. The constant $B > 0$ is now given by $B = \frac{1 + \delta A_Q}{1 - \delta A_Q^2}\cdot$ We thus come to the crucial point in the proof of Theorem 1.3, the analogue of Theorem 4.1. In the present situation we have

THEOREM 4.2. — *Under the assumption of Theorem* 1.3 *there exists a constant* $M = M(Q, C, \delta, Q_1) > 0$ *such that for a.e.* $r \in \Lambda_{r_0}$ *we have*

$$\frac{N'(r)}{N(r)} \geqslant -\frac{M}{r}\cdot \tag{4.43}$$

Proof. — Using (1.25), (1.26) we see that, as in the proof of Theorem 4.1, we can bound the l.h.s. of (4.21), (4.22) and (4.26) with $\frac{L}{r} I(r)$, where $L = L(Q, C, \delta, C_1) > 0$. We therefore obtain the identity (4.27) with the term $0\left(\frac{f(r)}{r}\right)$ replaced by $0\left(\frac{1}{r}\right)$, where $\left|0\left(\frac{1}{r}\right)\right| \leqslant \frac{L}{r}\cdot$ Using (1.26) we can see that (4.28) now becomes

$$(4.28)' \quad \left|\int_{\partial\Omega_r} \left(\frac{Xu(z,t)}{r}\right)\left(\frac{Tu(z,t)}{r}\right) \frac{\varphi(z,t)}{|\nabla d(z,t)|}\, dH_{2n}\right|$$
$$\leqslant \frac{C_1}{r}\left(\int_{\partial\Omega_r} \left(\frac{Xu(z,t)}{r}\right)^2 \frac{\psi(z,t)}{|\nabla d(z,t)|}\, dH_{2n}\right)^{\frac{1}{2}} \left(\int_{\partial\Omega_r} u(z,t)^2 \frac{\psi(z,t)}{|\nabla d(z,t)|}\, dH_{2n}\right)^{\frac{1}{2}}.$$

Again we distinguish the two possibilities (a), (b). If (a) occurs we bound the l.h.s. of (4.29) with $\frac{\sqrt{2}C_1}{r} I(r)$, estimate which along with the previously established replacement of (4.21), (4.22), (4.26) yields (4.30), with the term $0\left(\frac{f(r)+g(r)}{r}\right)$ replaced by $0\left(\frac{1}{r}\right)\cdot$ (4.30) and Schwarz's inequality give (4.43). If (b) occurs we argue as in (4.32), but we have to make up for the lack of smallness. We have by Young's inequality with $\varepsilon > 0$ arbitrary

$$(4.32)' \quad \left| \int_{\partial\Omega_r} \left(\frac{Xu(z,t)}{r}\right)\left(\frac{Tu(z,t)}{r}\right) \frac{\varphi(z,t)}{|\nabla d(z,t)|} dH_{2n} \right|$$

$$\leqslant \varepsilon C_1 \int_{\partial\Omega_r} \left(\frac{Xu(z,t)}{r}\right)^2 \frac{\psi(z,t)}{|\nabla d(z,t)|} dH_{2n} + \frac{C_1}{\varepsilon}\frac{H(r)}{r^2}$$

$$< \varepsilon C_1 \int_{\partial\Omega_r} \left(\frac{Xu(z,t)}{r}\right)^2 \frac{\psi(z,t)}{|\nabla d(z,t)|} dH_{2n} + \frac{C_1}{\varepsilon r} I(r)$$

(we have used the fact that $r \in \Lambda_{r_0}$ so that (4.16) holds). Choosing $\varepsilon > 0$ such that $2 - \varepsilon C_1 > 1$ we see that (4.34) becomes

$$\frac{N'(r)}{N(r)} \geqslant \frac{\displaystyle\int_{\partial\Omega_r} \left(\frac{Xu}{r}\right)^2 \frac{\psi}{|\nabla d|} dH_{2n}}{I(r)} - 2\frac{I(r)}{H(r)} - \frac{M}{r},$$

from which (4.43) follows since (b) holds. This completes the proof of Theorem 4.2. □

Integrating (4.43) on an interval (r, b_j), where $r \in (a_i, b_j)$ and (a_j, b_j) is an arbitrary interval in the decomposition (4.15) we see that

$$(4.38)' \qquad N(r) \leqslant \frac{\max(1, N(r_0)) r_0^M}{r^M} \qquad \text{for every } r \in \Lambda_{r_0}.$$

Integration of (4.38)′ as in (4.39) yields

$$(4.39)' \qquad \int_{(r,2r)\cap\Lambda_{r_0}} N(t)\frac{dt}{t} \leqslant \frac{\max(1, N(r_0)) r_0^M K}{r^{M+2}},$$

where $K = K(Q, C, \delta, C_1) > 0$. Proceeding as in the proof of Theorem 1.1 we finally obtain

THEOREM 4.3. — *Under the assumption of Theorem* 1.3 *there exists* $B = B(Q, C, \delta, C_1, u) > 0$, $\beta = \beta(Q, C, \delta, C_1) > 0$, $\Gamma = \Gamma(Q) > 0$ and $r_0 = r_0(Q, C, \delta, C_1) > 0$ *such that if* $u \not\equiv 0$ *in* Ω_r *for* $r \in \left(0, \frac{r_0}{2}\right)$, *then we have*

$$\int_{\Omega_{2r}} u(z,t)^2 \psi(z,t) dz\, dt \leqslant \Gamma \exp\left(\frac{B}{r^\beta}\right) \int_{\Omega_r} u(z,t)^2 \psi(z,t) dz\, dt.$$

From Theorem 4.3 the proof of Theorem 1.3 follows along the lines of the proof of Theorem 1.2 and we leave out the details.

From Proposition 4.1 and formula (4.35) we immediately infer

PROPOSITION 4.2. — *Let* $0 < R_0 \leqslant \infty$ *and let* u *be a solution to* $\Delta_{\mathbb{H}^n} u = 0$ *in* Ω_{R_0}. *If* u *is Heisenberg-homogeneous of degree* $k \in \mathbb{N}$, *we have for every* $r \in \left(0, \frac{R_0}{2}\right)$

$$\int_{\Omega_{2r}} u(z,t)^2 \psi(z,t) dz\, dt = 2^{Q+2k} \int_{\Omega_r} u(z,t)^2 \psi(z,t) dz\, dt .$$

Remark. — Proposition 4.2 shows that (1.19) is only a sufficient condition for (1.20) to hold. In fact, the function $u(x,y,t) = x|z|^2 + 2ty$ satisfies the assumption of Proposition 4.2 with $k = 3$, yet it does not satisfy a condition like (1.19). Whether there is a condition which is necessary and sufficient for (1.20) is an interesting open problem.

Theorem 3.1 has the following interesting corollary.

THEOREM 4.2. — *Let* u *be a solution to* $\Delta_{\mathbb{H}^n} u = 0$ *in* Ω_{R_0} *and suppose that* u *is invariant w.r.t. the action of the torus* $\mathbb{T}$ on $\mathbb{H}^n$, *i.e., that* (1.21) *holds. Then*

$$(4.43) \qquad D'(r) = \frac{Q-2}{r} D(r) + 2 \int_{\partial\Omega_r} \left(\frac{Xu(z,t)}{r}\right)^2 \frac{\psi(z,t)}{|\nabla d(z,t)|} dH_{2n} .$$

In particular, the function $r \mapsto \frac{D(r)}{r^{Q-2}}$ *is increasing on* $(0, R_0)$, *as well as the function* $r \mapsto N(r)$.

Proof. — Formula (4.43) follows immediately from (3.3) using $\Delta_{\mathbb{H}^n} u \equiv 0$ and the fact that $Tu \equiv 0$ since (1.21) holds. From (4.43), (4.10) and (4.9) (it should be kept in mind that now $I(r) = D(r)$) we infer

$$\frac{N'(r)}{N(r)} = 2 \frac{\displaystyle\int_{\partial\Omega_r} \left(\frac{Xu}{r}\right)^2 \frac{\psi}{|\nabla d|} dH_{2n}}{D(r)} - 2 \frac{D(r)}{H(r)} .$$

The increasingness of N now follows from the above identity, the fact that $D(r) = \int_{\partial\Omega_r} u\left(\frac{Xu}{r}\right)\frac{\psi}{|\nabla d|}\,dH_{2n}$ (integration by parts) and Schwarz's inequality. □

COROLLARY 4.1. – *If u is a solution to $\Delta_{\mathbb{H}^n}u = 0$ in Ω_{R_0} satisfying* (1.21) *we have for every $r \in \left(0, \frac{R_0}{2}\right)$*

$$(4.44) \qquad \int_{\Omega_{2r}} u(z,t)^2\psi(z,t)dz\,dt \leqslant \Gamma \int_{\Omega_r} u(z,t)^2\psi(z,t)dz\,dt$$

with

$$\Gamma = 2^Q \exp\left[2\log 2\left(R_0 \int_{\partial\Omega R_0} |\nabla_{\mathbb{H}^n}u|^2 dz\,dt \Big/ \int_{\partial\Omega R_0} u^2 \frac{\psi}{|\nabla d|}\,dH_{2n}\right)\right].$$

It was shown in [GL1] that the *inverse square potential* $V(x) = \frac{C}{|x|^2}$ has the remarkable property that the frequency associated to the operator $-\Delta + \frac{C}{|x|^2}$ has the same increasing character of the frequency of a harmonic function. We will show below that if we consider the natural inverse square potential for the Heisenberg group, namely $V(z,t) = \frac{C}{d(z,t)^2}\psi(z,t)$, then for the frequency associated to a solution of $-\Delta_{\mathbb{H}^n}u + Vu = 0$ satisfying (1.21) a result similar to Theorem 4.2 holds.

THEOREM 4.3. – *Let u be a solution to*

$$(4.45) \qquad -\Delta_{\mathbb{H}^n} u + \frac{C}{d(z,t)^2}\psi(z,t)\,u = 0 \qquad in\ \Omega_{R_0},$$

with $C \in \mathbb{R}$, and suppose that u satisfies (1.21). *Then we have for every $r \in (0, R_0)$*

$$(4.46) \quad I'(r) = \frac{Q-2}{r} I(r) + 2\int_{\partial\Omega_r} \left(\frac{Xu(z,t)}{r}\right)^2 \frac{\psi(z,t)}{|\nabla d(z,t)|}\,dH_{2n}.$$

In particular, the function $r \mapsto \frac{I(r)}{r^{Q-2}}$ is increasing on $(0, R_0)$, and so is the function $r \mapsto N(r)$.

Proof. — Since by (1.21) we have $Tu \equiv 0$ we obtain from the formula in Lemma 4.4 and equation (4.45)

$$I'(r) = \frac{Q-2}{r} I(r) + 2\int_{\partial\Omega_r} \left(\frac{Xu(z,t)}{r}\right)^2 \frac{\psi(z,t)}{|\nabla d(z,t)|} dH_{2n}$$

$$(4.47) \qquad + \int_{\partial\Omega_r} \frac{V(z,t)u(z,t)^2}{|\nabla d(z,t)|} dH_{2n} - \frac{Q-2}{4}\int_{\Omega_r} V(z,t)u(z,t)^2 dz\, dt$$

$$-\frac{2C}{r}\int_{\Omega_r} Xu(z,t)\frac{u(z,t)}{d(z,t)^2}\psi(z,t)dz\, dt.$$

Now we have integrating by parts

$$(4.48) \qquad \frac{2C}{r}\int_{\Omega_r} Xu(z,t)\frac{u(z,t)}{d(z,t)^2}\psi(z,t)dz\, dt$$

$$= \frac{C}{r}\lim_{\varepsilon\to 0}\int_{\Omega_r\setminus\bar{\Omega}_\varepsilon} \frac{\psi(z,t)}{d(z,t)^2} X(u^2)(z,t)dz\, dt$$

$$= \frac{C}{r}\lim_{\varepsilon\to 0}\left\{\int_{\partial\Omega_r} \frac{\psi(z,t)}{d(z,t)^2}\frac{Xd(z,t)}{|\nabla d(z,t)|} u(z,t)^2 dH_{2n}\right.$$

$$- \int_{\partial\Omega_\varepsilon} \frac{\psi(z,t)}{d(z,t)^2}\frac{Xd(zt)}{|\nabla d(z,t)|} u(z,t)^2 dH_{2n}$$

$$\left. - \int_{\Omega_r\setminus\bar{\Omega}_\varepsilon} \operatorname{div}\left(\frac{\psi}{d^2}X\right)(z,t)u(z,t)^2 dz\, dt\right\}.$$

Recalling (2.24) and (2.7) we see that

$$(4.49) \quad \frac{C}{r}\int_{\partial\Omega_r} \frac{\psi(z,t)}{d(z,t)^2}\frac{Xd(z,t)}{|\nabla d(z,t)|} u(z,t)^2 dH_{2n}$$

$$= \int_{\partial\Omega_r} \frac{C}{d(z,t)^2}\psi(z,t)\frac{u(z,t)}{|\nabla d(z,t)|} dH_{2n}$$

$$= \int_{\partial\Omega_r} \frac{V(z,t)u(z,t)^2}{|\nabla d(z,t)|} dH_{2n}$$

$$(4.50) \qquad \lim_{\varepsilon\to 0^+}\int_{\partial\Omega_\varepsilon} \frac{\psi(z,t)}{d(z,t)^2}\frac{Xd(zt)}{|\nabla d(z,t)|} u(z,t)^2 dH_{2n} = 0.$$

We compute the solid integral in the r.h.s. of (4.48)

$$(4.51)\quad \int_{\Omega_r\setminus\bar\Omega_\varepsilon} \operatorname{div}\left(\frac{\psi}{d^2}X\right)(z,t)u(z,t)^2dz\,dt = Q\int_{\Omega_r\setminus\bar\Omega_\varepsilon}\frac{\psi(z,t)}{d(z,t)^2}u(z,t)^2dz\,dt + \int_{\Omega_r\setminus\bar\Omega_\varepsilon} X(\psi d^{-2})(z,t)u(z,t)^2dz\,dt,$$

where we have used the fact $\operatorname{div} X = Q$. Now by (2.24) we have in $\Omega_r\setminus\bar\Omega_\varepsilon$

$$(4.52)\quad X(\psi d^{-2}) = d^{-2}(X\psi) - 2d^{-3}\psi Xd = d^{-2}X\psi - 2d^{-2}\psi.$$

Recalling (1.18) and (2.20) a direct computation yields

$$X\psi = d^{-2}X|z|^2 - 2\psi = 0.$$

Using this information in (4.52) and then (4.52) in (4.51) gives

$$\int_{\Omega_r\setminus\bar\Omega_\varepsilon} \operatorname{div}\left(\frac{\psi}{d^2}X\right)(z,t)u(z,t)^2\,dz\,dt = (Q-2)\int_{\Omega_r\setminus\bar\Omega_\varepsilon}\frac{\psi(z,t)}{d(z,t)^2}u(z,t)^2\,dz\,dt.$$

Substituting this identity along with (4.49) in (4.48), taking the limit as $\varepsilon\to 0^+$ ans using (4.50) we finally obtain (4.46) from (4.47).

At this point the increasingness of $r\mapsto \dfrac{I(r)}{r^{Q-2}}$ immediately follows, whereas that of $r\mapsto (r)$ follows from the formula

$$\frac{N'(r)}{N(r)} = 2\frac{\displaystyle\int_{\partial\Omega_r}\left(\frac{Xu}{r}\right)^2\frac{\psi}{|\nabla d|}dH_{2n}}{I(r)} - 2\frac{I(r)}{H(r)},$$

the fact that $I(r) = \displaystyle\int_{\partial\Omega_r}\left(\frac{Xu}{r}\right)\frac{\psi}{|\nabla d|}dH_{2n}$ (divergence theorem), and Schwarz's inequality, as in the proof of Theorem 4.2. □

COROLLARY 4.2. — *Under the assumption of Theorem* 4.3 *inequality* (4.44) *holds with a constant* Γ *given by*

$$\Gamma = 2^Q\exp\left\{2\log 2\left[R_0\int_{\Omega_{R_0}}\left(|\nabla_{\mathbb{H}^n}u|^2 + \frac{C}{d^2}\psi u^2\right)dz\,dt\Big/\int_{\partial\Omega_{r_0}}u^2\frac{\psi}{|\nabla d|}dH_{2n}\right]\right\}.$$

In particular, nonzero solutions of (4.45) *satisfying* (1.21) *have a finite order of vanishing at the origin.*

COROLLARY 4.3. – *Let* u *be a solution to* (4.45) *in* $\mathbb{H}^n$ *and suppose that* u *satisfies* (1.21). *If* u *vanishes to infinite order at the origin, then must be* $u \equiv 0$ *in* $\mathbb{H}^n$.

Proof. – By Theorem 4.3 and Corollary 4.2 we infer that $u \equiv 0$ in Ω_{R_0}. Away from the origin the potential $V(z,t) = \dfrac{C}{d(z,t)^2}\psi(z,t)$ satisfies an estimate of the type $|V(z,t)| \leqslant C'\psi(z,t)$. Since $Tu \equiv 0$ we can apply Theorem 1.1 to infer that for some $r_0 > 0$ must be $u \equiv 0$ in $C_{r_0} = \{(z,t) \in \mathbb{H}^n \mid |z| < r_0, t \in \mathbb{R}\}$. However, in the complementary of C_{r_0} u is a solution, satisfying (1.21), of the equation

$$\Delta_z u + 4|z|^2 \frac{\partial^2 u}{\partial t^2} = \frac{C}{d(z,t)^2}\psi(z,t)u = Vu,$$

where now $V \in L^\infty_{\text{loc}}(\mathbb{H}^n \backslash C_{r_0})$. Since the l.h.s. is an elliptic operator in $\mathbb{H}^n \backslash C_{r_0}$, by the unique continuation theorem for elliptic operators with a bounded potential, see e.g. [GL1], we infer $u \equiv 0$ in $\mathbb{H}^n \backslash C_{r_0}$. □

We conclude by showing by means of an example that the inverse square potential $V(z,t) = \dfrac{C}{d(z,t)^2}\psi(z,t)$ represents a threshold for the results of this paper to hold.

Consider the equation in $\mathbb{H}^n$

$$-\Delta_{\mathbb{H}^n} u + \frac{C}{d(z,t)^{2+\varepsilon}}\psi(z,t)u = 0, \tag{4.53}$$

with $C > 0$, $\varepsilon > 0$, and choose $n \in \mathbb{N}$ such that $\dfrac{Q-2}{\varepsilon} = \dfrac{2n}{\varepsilon} \notin \mathbb{N}$. If we look for solutions of (4.53) of the type $u(z,t) = f(d(z,t))$, then according to (2.29) f must satisfy the ode

$$f''(d) + \frac{Q-1}{d} f'(d) - \frac{C}{d^{2+\varepsilon}} f = 0, \qquad d > 0.$$

Because of the assumption $\dfrac{Q-2}{\varepsilon} \notin \mathbb{N}$ as in [GL1] one can prove that the function

$$u(z,t) = d(z,t)^{\frac{Q-2}{\varepsilon}} K_{\frac{Q-2}{\varepsilon}}\left(\frac{2\sqrt{C}}{\varepsilon}\, d(z,t)^{-\frac{\varepsilon}{2}}\right), \tag{4.54}$$

where $K_{\frac{Q-2}{\varepsilon}}$ is the modified Bessel function of the third kind and order $\nu = \frac{Q-2}{\varepsilon}$, is a solution to (4.53). Using the asymptotic behavior of $K_\nu(r)$ for large positive r, i.e., $K_\nu(r) \cong \frac{\pi}{2} r^{-1/2} e^{-r}$ as $r \mapsto +\infty$, we see that (4.54) vanishes to infinite order at the origin. Moreover, since u depends only on $|z|$ and t we clearly have $Tu \equiv 0$. This example shows that the power two is best possible for Theorems 1.1, 1.2, 4.2, and 4.3 and Corollaries 4.1, 4.2 and 4.3 to hold.

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