ANNALES DE L'INSTITUT FOURIER

H. HAUSER

G. MULLER

The trivial locus of an analytic map germ

Annales de l'institut Fourier, tome 39, nº 4 (1989), p. 831-844

<http://www.numdam.org/item?id=AIF_1989__39_4_831_0>

Ann. Inst. Fourier, Grenoble
39, 4 (1989), 831-844.

THE TRIVIAL LOCUS OF AN ANALYTIC MAP GERM

by H. HAUSER and G. MÜLLER

1. Statement of results.

In this paper we shall prove :

THEOREM 1. — *Let $\pi : X \to S$ be a morphism of analytic space germs. For $a \in X$ denote by $X(a)$ the germ in a of the fiber of π through a. There exist analytic space germs $Y \subset X$ and $T \subset S$ with the following properties :*

(i) *The germ of points $a \in X$ for which $X(a)$ is isomorphic to the special fiber $X(0)$ equals the reduction of Y (and is hence analytic). The reduction of T is the image of the reduction of Y under π (which is hence analytic).*

(ii) *The restriction of π to Y is a submersion $\pi_Y : Y \to T$, i.e., its special fiber $Y(0)$ is smooth and $Y \simeq Y(0) \times T$ over T.*

(iii) *The special fiber $X(0)$ of π is isomorphic to $Y(0) \times Z$ for some germ Z.*

(iv) *For any cartesian square*

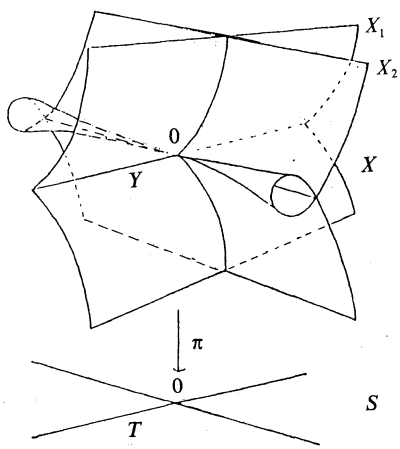
$$\begin{array}{ccc} X' & \to & X \\ \pi' \downarrow & & \downarrow \pi \\ S' & \to & S \end{array}$$

the morphism π' is trivial (ie., $X' \simeq X(0) \times S'$ over S') if and only if the base change $S' \to S$ factors through T.

Key-words : Morphisms of analytic space germs - Cartesian products - Deformations.
A.M.S. Classification : 14 - 32.

We shall call T the *trivial locus* of π. By (iv) it is uniquely determined. Note that the germ Y is only determined up to isomorphism.

Example. — Let $X_1, X_2 \subset (\mathbb{C}^4,0)$ be defined by ideals $I_1 = (x_1^2 - x_2^3, t_1)$, $I_2 = (x_1^2 - (x_2 + t_1^2) \cdot x_2^2, t_2)$ and let $X = X_1 \cup X_2$ with ideal $I = I_1 \cap I_2 = (x_1^2 - (x_2 + t_1^2) \cdot x_2^2,\ t_1 t_2)$. Furthermore let $S \subset (\mathbb{C}^2,0)$ be defined by $t_1 t_2$. The projection $(\mathbb{C}^4,0) \to (\mathbb{C}^2,0)$ restricts to a morphism $\pi : X \to S$. The trivial locus of π is then defined in $(\mathbb{C}^2,0)$ by $(t_1^4, t_1 t_2)$ (cf. the proof of Theorem 1).



Let us state some consequences and special cases of the Theorem.

COROLLARY 1 (Ephraim, [E, Thm. 0.2]). — *Let X be an analytic space germ. For $a \in X$ denote by $X(a)$ the germ in a of X. Then the germ Y of points $a \in X$ with $X(a) \simeq X$ is a smooth analytic germ and $X \simeq Y \times Z$ for some germ Z (case $S = 0$ in the Theorem).*

COROLLARY 2 (Economy of the semi-universal deformation, Teissier, [T, Thm. 4.8.4]). — *Let $X(0)$ be an isolated singularity with semi-universal deformation $\pi : X \to S$. Then no fiber $X(a)$ is isomorphic to the special fiber $X(0)$.*

Indeed, the trivial deformation $X(0) \times T \to T$ can be obtained from π by the base changes $T \hookrightarrow S$ and $T \to 0 \subset S$. The uniqueness of the derivative of the base change gives $T = 0$. Since $X(0)$ is an isolated singularity this implies $Y = 0$.

COROLLARY 3. — *Let $\pi : \mathcal{X} \to \mathcal{S}$ be a morphism of analytic spaces. For $a \in \mathcal{X}$ denote by $\mathcal{X}(a)$ the germ in a of the fiber of π through a.*

Then $\mathscr{X}(a) \simeq \mathscr{X}(b)$ defines an equivalence relation on $\mathscr{X}$ whose equivalence classes are locally closed analytic subsets of $\mathscr{X}$.

Remark. — For flat morphisms $\pi : X \to S$ the existence of a germ $T \subset S$ with the universal property (iv) of Theorem 1 was also proven by Greuel and Karras [GrK, Lemma 1.4] in case $X(0)$ is an isolated singularity and by Flenner and Kosarew [FlK, Cor. 0.2] for $X(0)$ arbitrary.

Theorem 1 will be derived from :

THEOREM 2. — *Let $\pi : X \to S$ be a morphism of analytic space germs with section $\sigma : S \to X$. For $t \in S$ denote by $X_t = X(\sigma(t))$ the germ in $\sigma(t)$ of the fiber of π over t. There is a unique analytic space germ $T \subset S$ with the following properties :*

(i) *The germ of points $t \in S$ for which X_t is isomorphic to the special fiber X_0 equals the reduction of T (and is hence analytic).*

(ii) *For any cartesian square*

$$\begin{array}{ccc} X' & \to & X \\ \pi' \downarrow & & \downarrow \pi \\ S' & \to & S \end{array}$$

with induced section $\sigma' : S' \to X'$ of π' the morphism π' can be trivialized by an isomorphism $X' \simeq X_0 \times S'$ mapping $\sigma'(S')$ onto $0 \times S'$ if and only if the base change $S' \to S$ factors through T.

We shall call T the *trivial locus of the pair* (π,σ).

Example. — In the example following Theorem 1 the embedding $(\mathbb{C}^2,0) \hookrightarrow 0 \times (\mathbb{C}^2,0) \subset (\mathbb{C}^4,0)$ restricts to a section $\sigma : S \to X$ of $\pi : X \to S$. The trivial locus of (π,σ) is then defined in $(\mathbb{C}^2,0)$ by (t_1^2, t_1t_2).

COROLLARY 4. — *Let $\pi : X \to T$ be a morphism with section and assume that T is reduced. If all fibers X_t are isomorphic to the special fiber X_0 then π is trivial.*

Remarks. — (a) Corollary 4 can be interpreted as follows : The fibers of π form a local analytic family $\{X_t\}_{t \in T}$ of analytic space germs. By assumption there is for every $t \in T$ an isomorphism $\phi_t : X_t \simeq X_0$

which, of course, is far from being unique. For bad choices ϕ_t won't be even continuous in t. But also if there were some canonical choice for ϕ_t it is a priori not at all clear that this ϕ_t will be analytic in t. Corollary 4 asserts that one can always choose a family $\{\phi_t\}_{t \in T}$ which is analytic in t.

(b) Our proof will show that the assertions of Theorems 1 and 2 (except parts (i)) hold true for algebroid spaces (defined by formal power series) in place of analytic space germs. The algebroid counterpart of Corollary 4 was proven by Seidenberg [Se, Thm. 3] in the special case T smooth.

(c) Also, our proof will show that if the given data are algebraic then the trivial locus in Theorem 1 as well as in Theorem 2 is algebraic.

(d) The statement in Corollary 4 is a local analogon of a result of Fischer and Grauert [FiG, Satz] and Schuster [Sc, Satz 4.9]: Let $\pi : \mathscr{X} \to \mathscr{T}$ be a proper flat morphism of analytic spaces and assume that $\mathscr{T}$ is reduced. If the fibers of π are pairwise (globally) isomorphic then π is trivial locally around any point of $\mathscr{T}$.

(e) Proposition 1 of [GaH] is an incorrect quotation of Ephraim's result stated as Corollary 1 above and anticipates the assertion of Corollary 4. We thank G.-M. Greuel for pointing out this error and for stimulating us to prove the present result. Also we are indebted to H. Flenner for valuable suggestions concerning the proof of Theorem 1 (iv).

2. Infinitesimal neighbourhoods and isomorphisms of analytic space germs.

Let us recall the following Approximation Theorem:

THEOREM 3 (Artin [A, Thm. 6.1] in the polynomial case, Pfister, Popescu [PfPo, Thm. 2.5] and Wavrik [W, Thm. 1] in the general case). — *For given $\ell \in \mathbb{N}$ and convergent $f \in \mathbb{C}\{x,y\}^p$ there exists $k \in \mathbb{N}$ such that: If $\bar{y}(x) \in \mathbb{C}[[x]]^m$ with $\bar{y}(0) = 0$ is a formal solution of $f(x,y) = 0$ up to order k:*

$$f(x,\bar{y}(x)) \equiv 0 \mod (x)^{k+1},$$

then there exists a convergent solution $y(x) \in \mathbb{C}\{x\}^m$ approximating $\bar{y}(x)$

up to order ℓ :

$$f(x,y(x)) = 0$$
$$y(x) \equiv \bar{y}(x) \mod (x)^{\ell+1}.$$

Remarks. – (a) If $f \in \mathbb{C}[x,y]^p$ is polynomial then the integer k can be chosen to depend only on ℓ, the numbers n and m of x- and y-variables, and the degree of f, [A, Thm. 6.1]. We do not know whether in the general case there is an integer k only depending on ℓ, n, m, and some numerical invariants of f.

(b) If $f \in \mathbb{C}[[x,y]]^p$ is formal then the analogous statement holds yielding a formal solution $y(x) \in \mathbb{C}[[x]]^m$, [PfPo, Thm. 2.5], [W, Thm. I_n].

Theorem 3 will allow us to reduce the problem of checking isomorphy of space germs to the comparison of infinitesimal neighbourhoods of their special points and thus to the comparison of finite dimensional $\mathbb{C}$-algebras. This can also be applied to the case of relative space germs $X \to T$: Let X be an analytic space germ with local ring $\mathcal{O}_X$ and $Z \subset X$ a germ with ideal $I_Z \subset \mathcal{O}_X$. For $k \in \mathbb{N}$ denote by $Z_X^{(k)}$ the k-th infinitesimal neighbourhood of Z in X with local ring $\mathcal{O}_X/I_Z^{k+1}$. If Z is the special point 0 of X we write $Z_X^{(k)} = 0_X^{(k)}$. For a morphism $\pi : X \to T$ with section $\sigma : T \to X$ we shall identify T with $\sigma(T) \subset X$ and then write $T_X^{(k)}$ instead of $\sigma(T)_X^{(k)}$. The composition $\sigma^{(k)} : T \to \sigma(T) \subset T_X^{(k)}$ gives a section of the restriction $\pi^{(k)} :\ T_X^{(k)} \to T$ of π.

THEOREM 4. – *Let* $\pi : X \to T$ *and* $\tau : Y \to T$ *be morphisms with sections* $\sigma : T \to X$ *and* $\rho : T \to Y$. *Then* X *and* Y *are isomorphic over* T *and the sections* σ, ρ *if and only if for all* $k \in \mathbb{N}$ *the infinitesimal neighbourhoods* $T_X^{(k)}$ *and* $T_Y^{(k)}$ *are isomorphic over* T *and the sections* $\sigma^{(k)}$, $\rho^{(k)}$. (*Here, isomorphic over* T *and* σ, ρ *means that there is an isomorphism* $X \simeq Y$ *over* T *mapping* $\sigma(T)$ *onto* $\rho(T)$.)

In particular, two absolute analytic space germs X and Y are isomorphic if and only if for all k the fat points $0_X^{(k)}$ and $0_Y^{(k)}$ are isomorphic.

Remark. – The following version of Theorem 4 seems, at first view, to be stronger but is actually equivalent: For given π, τ, σ, ρ there is $k \in \mathbb{N}$ such that X and Y are isomorphic over T and σ, ρ if and only if $T_X^{(k)}$ and $T_Y^{(k)}$ are isomorphic over T and $\sigma^{(k)}$, $\rho^{(k)}$. As the proof will show the integer k is obtained by an application of Theorem 3. Hence this second version would be useful if we could explicitly calculate k from the given data (cf. remark (a) after Theorem 3).

Proof of Theorem 4. – Choose embeddings X, $Y \subset (\mathbb{C}^n,0) \times T$ over T, [Fi, 0.35]. Composing with an automorphism of $(\mathbb{C}^n,0) \times T$ we can assume $\sigma(T) = 0 \times T = \rho(T)$. Choose an embedding $T \subset (\mathbb{C}^m,0)$ and let (x,t) be coordinates on $(\mathbb{C}^{n+m},0)$. Moreover choose map germs G, H: $(\mathbb{C}^{n+m},0) \to (\mathbb{C}^p,0)$ such that G defines X in $(\mathbb{C}^n,0) \times T$ by $X = G^{-1}(0) \cap ((\mathbb{C}^n,0) \times T)$, and analogously for Y and H.

One implication in Theorem 4 being obvious, let us suppose that $\psi_k : T_X^{(k)} \simeq T_Y^{(k)}$ is an isomorphism over T and the sections. By an argument similar to [D, Prop. 1.2] ψ_k can be extended to an automorphism ϕ_k of $(\mathbb{C}^{n+m},0)$ over $(\mathbb{C}^m,0)$ and the sections. Write $\phi_k(x,t) = (y_k(x,t),t)$ with some $y_k \in \mathcal{O}_{n+m}^n$. Since ϕ_k maps $T_X^{(k)}$ onto $T_Y^{(k)}$ there is a $p \times p$-matrix $U_k(x,t)$ with entries in $\mathcal{O}_{n+m}$ such that

$$H(y_k(x,t),t) \equiv U_k(x,t) \cdot G(x,t) \mod (x)^{k+1} + I_{(\mathbb{C}^n,0)\times T}.$$

By a standard trick (cf. [M, 2.3]) $U_k(x,t)$ can be chosen to be invertible: $U_k(x,t) \in \mathrm{GL}_p(\mathcal{O}_{n+m})$. Since ϕ_k maps $0 \times (\mathbb{C}^m,0)$ onto itself there is $V_k(x,t) \in \mathrm{GL}_n(\mathcal{O}_{n+m})$ such that

$$y_k(x,t) = V_k(x,t) \cdot x.$$

If k was sufficiently large Theorem 3 with $\ell = 1$ yields $y(x,t) \in \mathcal{O}_{n+m}^n$ and matrices $U(x,t) \in \mathrm{GL}_p(\mathcal{O}_{n+m})$, $V(x,t) \in \mathrm{GL}_n(\mathcal{O}_{n+m})$ such that

$$H(y(x,t),t) \equiv U(x,t) \cdot G(x,t) \mod I_{(\mathbb{C}^n,0)\times T}$$
$$y(x,t) = V(x,t) \cdot x.$$

Then $\phi : (\mathbb{C}^{n+m},0) \to (\mathbb{C}^{n+m},0)$ given by $\phi(x,t) = (y(x,t),t)$ is an automorphism of $(\mathbb{C}^{n+m},0)$ over $(\mathbb{C}^m,0)$ and the sections. Its restriction to X is an isomorphism $X \simeq Y$ over T and the sections. This concludes the proof.

In the sequel we shall need a more explicit version of Theorem 4. Suppose that we are in the situation established at the beginning of the preceding proof. Define maps γ, η: $(\mathbb{C}^m,0) \to \mathcal{O}_n^p$ by $\gamma(t)(x) = G(x,t)$ and $\eta(t)(x) = H(x,t)$. Composition with the natural map

$\mathcal{O}_n^p \to (\mathcal{O}_n/\mathfrak{m}_n^{k+1})^p = \mathbb{V}_k$ gives analytic map germs

$$\gamma_k : (\mathbb{C}^m,0) \to (\mathbb{V}_k, g_k) \qquad \text{and} \qquad \eta_k : (\mathbb{C}^m,0) \mapsto (\mathbb{V}_k, \mathrm{h}_k).$$

Now consider the semi-direct product

$$\mathbb{K}_k = \mathrm{GL}_p(A_k) \times \mathrm{Aut}\,(0^{(k)}_{(\mathbb{C}^n,0)})$$

where $A_k = \mathcal{O}_n/\mathfrak{m}_n^{k+1}$ is the local ring of $0^{(k)}_{(\mathbb{C}^n,0)}$. This is an algebraic group acting rationally on the finite dimensional vector space $\mathbb{V}_k$. Finally set $K_k = (\mathbb{K}_k, 1)$, the germ of $\mathbb{K}_k$ in 1.

COROLLARY 5. — *The space germs X and Y are isomorphic over T and the sections if and only if for all $k \in \mathbb{N}$ there is a morphism of space germs $\Phi_k : T \to K_k$ such that the diagram*

$$\begin{array}{ccc} T & \xrightarrow{(\Phi_k,\ \gamma_k|T)} & K_k \times (\mathbb{V}_k, g_k) \\ & \searrow^{\eta_k|T} & \downarrow \\ & & (\mathbb{V}_k, h_k) \end{array}$$

commutes. Here the vertical arrow is induced by the action of $\mathbb{K}_k$ on $\mathbb{V}_k$.

Proof. — Every analytic map germ $B : (\mathbb{C}^{n+m},0) \to \mathbb{C}$ induces a map germ $\beta : (\mathbb{C}^m,0) \to \mathcal{O}_n$ by $\beta(t)(x) = B(x,t)$, hence an analytic map germ $\beta_k : (\mathbb{C}^m,0) \to \mathcal{O}_n/\mathfrak{m}_n^{k+1} = A_k$. Conversely, every analytic map germ $(\mathbb{C}^m,0) \to A_k$ is obtained in this way. A given B is contained in $(x)^{k+1} + \mathrm{I}_{(\mathbb{C}^n,0)\times T}$ if and only if the coefficients of β_k (considered as a polynomial in x) are contained in I_T, i.e., $\beta_k|T = 0$. Using these observations, similar arguments as in the proof of Theorem 4 yield the assertion.

3. Proof of Theorem 2.

We need a simple result on algebraic group actions:

PROPOSITION 1. — *Let $\mathbb{V}$ be a finite dimensional complex vector space with germ $V = (\mathbb{V},v)$ in a fixed point v, $\mathbb{G} \subset \mathrm{GL}(\mathbb{V})$ an algebraic subgroup with germ $\mathrm{G} = (\mathbb{G},1)$. Let $\gamma : S \to V$ be a morphism of analytic space germs.*

(i) *The orbit $\mathbb{G}\cdot v$ is a smooth algebraic subvariety of $\mathbb{V}$. Let $G\cdot v$ be its germ in v and $T = \gamma^{-1}(G\cdot v) \subset S$ the inverse image (with possibly non-reduced structure).*

(ii) *There is a morphism* $\Phi : T \to G$ *such that the diagram*

$$\begin{array}{ccc} & & G \\ & \overset{\Phi}{\nearrow} & \downarrow \\ T & \xrightarrow[\gamma|T]{} & G\cdot v \end{array}$$

commutes. Here the vertical arrow denotes the orbit map.

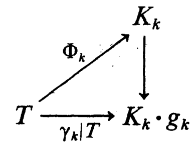
Proof. – Consider the homogeneous manifold $\mathbb{G}/\mathbb{G}_v$ where $\mathbb{G}_v$ denotes the stabilizer of v in $\mathbb{G}$, [V, Thm. 2.9.4]. By [H, 8.3] the orbit $\mathbb{G}\cdot v$ is a smooth locally closed algebraic subvariety of $\mathbb{V}$. The orbit map $\mathbb{G}/\mathbb{G}_v \to \mathbb{G}\cdot v$ is an isomorphism of analytic manifols, [V, Thm. 2.9.7]. By [V, Thm. 2.9.5] there is a germ of an analytic section $(\mathbb{G}/\mathbb{G}_v, 1\cdot\mathbb{G}_v) \to (\mathbb{G},1) = G$. Composition with $T \overset{\gamma|T}{\longrightarrow} G\cdot v = (\mathbb{G}\cdot v, v) \simeq (\mathbb{G}/\mathbb{G}_v, 1\cdot\mathbb{G}_v)$ yields the desired morphism $T \to G$.

Let us now turn to the situation of Theorem 2. First observe that the uniqueness of T is clear by the universal property. As in the proof of Theorem 4 we choose embeddings $X \subset (\mathbb{C}^n,0) \times S$ over S with $\sigma(S) = 0 \times S$ and $S \subset (\mathbb{C}^m,0)$. Choose $G : (\mathbb{C}^{n+m},0) \to (\mathbb{C}^p,0)$ such that G defines X in $(\mathbb{C}^n,0) \times S$ by $X = G^{-1}(0) \cap ((\mathbb{C}^n,0) \times S)$. Consider the map germ $\gamma : (\mathbb{C}^m,0) \to (\mathcal{O}_n^p, g)$ given by $\gamma(t)(x) = G(x,t)$. Composition with the natural map from $\mathcal{O}_n^p$ to $\mathbb{V}_k = (\mathcal{O}_n/\mathfrak{m}_n^{k+1})^p$ gives a morphism of space germs $\gamma_k : (\mathbb{C}^m,0) \to (\mathbb{V}_k, g_k) = V_k$. Consider again $A_k = \mathcal{O}_n/\mathfrak{m}_n^{k+1}$ and $\mathbb{K}_k = \mathrm{GL}_p(A_k) \rtimes \mathrm{Aut}\,(0^{(k)}_{(\mathbb{C}^n,0)})$ acting on $\mathbb{V}_k$. Set $K_k = (\mathbb{K}_k, 1)$.

We now define a sequence T_k of space germs in S by $T_k = \gamma_k^{-1}(K_k \cdot g_k) \cap S$. Obviously $T_{k+1} \subset T_k$. As $\mathcal{O}_S$ is Noetherian the sequence becomes stationary, say $T_k = T$ for $k \gg 0$. We shall show that T has the properties stated in Theorem 2. Let T^0 denote the germ of points $t \in S$ with $X_t \simeq X_0$. For $t \in S$ the fiber X_t is defined in $(\mathbb{C}^n,0)$ by $\gamma(t)$. Hence by Corollary 5 applied to the absolute space germs X_t and X_0 we have $T^0 \subset \mathrm{red}\, T_k$ for all k, hence $T^0 \subset \mathrm{red}\, T$. Consider the cartesian square

$$\begin{array}{ccc} X' & \to & X \\ \downarrow\pi' & & \downarrow\pi \\ T & \to & S \end{array}$$

and let $\sigma' : T \to X'$ be the induced section of π'. By Proposition 1 there are morphisms $\Phi_k : T \to K_k$ such that the diagrams



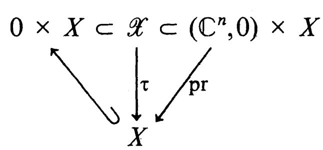
commute. Corollary 5 yields an isomorphism $X' \simeq X_0 \times T$ over T and the sections. This implies red $T \subset T^0$ and « if » in (ii).

To prove « only if » assume that X' and $X_0 \times S'$ are isomorphic over S' and the sections. By Corollary 5 the morphisms $S' \to S \xrightarrow{\gamma_k} V_k$ factor through $K_k \cdot g_k$. Hence the base change $S' \to S$ factors through $T_k = \gamma_k^{-1}(K_k \cdot g_k)$. This completes the proof of Theorem 2.

4. Proof of Theorem 1.

(a) It is not possible to deduce Theorem 1 directly from Theorem 2, for, in general, π does not admit a section. Instead, we shall associate to π a morphism $\mathscr{X} \to X$ which does admit a section, apply Theorem 2 to it and then transfer the obtained assertions back to π.

Choose embeddings $X \subset (\mathbb{C}^n,0) \times X$ over S and $S \subset (\mathbb{C}^m,0)$. Also choose H: $(\mathbb{C}^{n+m},0) \to (\mathbb{C}^p,0)$ such that H defines X in $(\mathbb{C}^n,0) \times S$ by $X = H^{-1}(0) \cap ((\mathbb{C}^n,0) \times S)$. For $a = (a_1, a_2) \in X$ the fiber $X(a)$ of π through a is defined by the morphism $\gamma(a) : (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$ given by $\gamma(a)(x) = H(x+a_1, a_2)$. Consider $G: (\mathbb{C}^n,0) \times (\mathbb{C}^{n+m},0) \to (\mathbb{C}^p,0)$, $G(x,a) = H(x+a_1,a_2)$ and let $\mathscr{X} = G^{-1}(0) \cap ((\mathbb{C}^n,0) \times X)$. As $0 \times X \subset G^{-1}(0)$ we get a commutative diagram



In particular, the inclusion $X \subset 0 \times X \subset \mathscr{X}$ defines a section $\sigma : X \to \mathscr{X}$ of τ. For $a \in X$ let $\mathscr{X}_a$ denote the germ in $(0,a) = \sigma(a)$ of the fiber of τ over a. It is defined in $(\mathbb{C}^n,0)$ by $\gamma(a)$. Thus we have constructed a morphism τ with section σ such that the fibers of π (taken as germs in points varying over whole X) are just the fibers of τ along the section σ. Let $Y \subset X$ be the trivial locus of the pair (τ, σ) as described in Theorem 2. Then its reduction red Y satisfies part (i) of Theorem 1.

(b) We now show that there is a germ $T \subset S$ such that π induces a submersion $\pi_Y : Y \to T$. This is the most technical part of the proof. By [Fi, 2.19] it is equivalent to say that $T(Y/S)$ is a vector bundle over Y. Here $T(Y/S)$ is the relative tangent space of Y over S considered as a linear fiber space over Y, [Fi, 1.4 and 2.7].

Look at the map germ $\gamma : (\mathbb{C}^{n+m}, 0) \to (\mathcal{O}_n^p, g)$, $\gamma(a)(x) = G(x,a) = H(x + a_1, a_2)$. By composition with the natural maps $j_k : \mathbb{V} = \mathcal{O}_n^p \to \mathbb{V}_k = (\mathcal{O}_n/\mathfrak{m}_n^{k+1})^p$ we obtain morphisms of space germs $\gamma_k : (\mathbb{C}^{n+m}, 0) \to (\mathbb{V}_k, g_k) = V_k$. The construction of $Y \subset X$ in the proof of Theorem 2 yields $Y = \gamma_k^{-1}(K_k \cdot g_k) \cap X$ for $k \gg 0$. Here again $K_k \cdot g_k$ denotes the germ of the orbit of K_k through g_k. Thus $T(Y/S) = \Delta_k^{-1}\gamma_k^* T(K_k \cdot g_k)$, where $T(K_k \cdot g_k) \subset K_k \cdot g_k \times \mathbb{V}_k$ is the tangent space of $K_k \cdot g_k$, γ_k^* denotes the pull back of linear fiber spaces via the base change $\gamma_k :\ Y \to K_k \cdot g_k$ and $\Delta_k :\ Y \times \mathbb{C}^n \to Y \times \mathbb{V}_k$ is the homomorphism of linear fiber spaces over Y defined by $\Delta_k(a, w) = (a, w \cdot \partial_{a_1}\gamma_k(a))$. For $w \in \mathbb{C}^n$ consider the map $w \cdot \partial_x :\ \mathbb{V} \to \mathbb{V}$. Since $w \cdot \partial_x(\mathfrak{m}_n^{k+2}) \subset \mathfrak{m}_n^{k+1}$ it induces a commutative diagram

$$\begin{array}{ccc} \mathbb{V} & \xrightarrow{w \cdot \partial_x} & \mathbb{V} \\ {\scriptstyle j_{k+1}}\downarrow & & \downarrow{\scriptstyle j_k} \\ \mathbb{V}_{k+1} & \xrightarrow{w \cdot \partial_x} & \mathbb{V}_k \end{array}$$

The equality $\partial_{a_1} G(x,a) = \partial_x H(x + a_1, a_2) = \partial_x G(x,a)$ implies

$$w \cdot \partial_{a_1}\gamma_k(a) = j_k(w \cdot \partial_x(\gamma(a))) = w \cdot \partial_x(\gamma_{k+1}(a)).$$

Therefore

$$T(Y/S) = \gamma_{k+1}^* \nabla_{k+1}^{-1} j_k^* T(K_k \cdot g_k)$$

where we have used the base changes $j_k :\ K_{k+1} \cdot g_{k+1} \to K_k \cdot g_k$ (induced by the natural map $V_{k+1} \to V_k$) and $\gamma_{k+1} :\ Y \to K_{k+1} \cdot g_{k+1}$ as well as the homomorphism of linear fiber spaces

$$\nabla_{k+1} : K_{k+1} \cdot g_{k+1} \times \mathbb{C}^n \to K_{k+1} \cdot g_{k+1} \times \mathbb{V}_k$$
$$(\mathrm{f}_{k+1}, w) \to (f_{k+1}, w \cdot \partial_x(\mathrm{f}_{k+1})).$$

Hence it will be sufficient to show that the linear fiber space

$$L_{k+1} = \nabla_{k+1}^{-1} j_k^* T(K_k \cdot g_k)$$

is actually a vector bundle over $K_{k+1} \cdot g_{k+1}$. But $K_{k+1} \cdot g_{k+1}$ is smooth and so it is enough to show that L_{k+1} has constant fiber dimension. [Fi, Prop. 1.8]. For this purpose fix an $f_{k+1} \in K_{k+1} \cdot g_{k+1}$. We can write $f_{k+1} = j_{k+1}(\Phi \cdot g)$ with $\Phi \in \mathbb{K} = \mathrm{GL}_p(\mathcal{O}_n) \times \mathrm{Aut}\,(\mathbb{C}^n,0)$, the contact group acting on $\mathbb{V} = \mathcal{O}_n^p$. A vector $w \in \mathbb{C}^n$ is contained in the fiber of L_{k+1} over f_{k+1} if and only if

$$j_k(w \cdot \partial_x(\Phi \cdot g)) = w \cdot \partial_x(f_{k+1}) \in T_{j_k(\Phi \cdot g)}(K_k \cdot g_k) = T_{j_k(\Phi \cdot g)}(K_k \cdot j_k(\Phi \cdot g)).$$

By [M, Prop. 7.4] the tangent space to the $\mathbb{K}_k$-orbit equals for $f \in \mathbb{V}$:

$$T_{j_k(f)}(K_k \cdot j_k(f) = j_k[I(f) \cdot \mathcal{O}_n^p + m_n \cdot J(f)],$$

where $I(f)$ is the ideal of $\mathcal{O}_n$ generated by the components of f and $J(f) \subset \mathcal{O}_n^p$ is the $\mathcal{O}_n$-submodule generated by the partial derivatives of f. Thus we see that the fiber of L_{k+1} over f_{k+1} equals the vector space

$$W(\Phi) = \{w \in \mathbb{C}^n, w \cdot \partial_x(\Phi \cdot g) \in I(\Phi \cdot g) \cdot \mathcal{O}_n^p + m_n \cdot J(\Phi \cdot g) + m_n^{k+1} \cdot \mathcal{O}_n^p\}.$$

Write $\Phi = (u, \phi)$. Chain and product rule give : $w \in W(\Phi)$ if and only if $w \cdot (\partial_x \phi \circ \phi^{-1})(0) \in W(\mathbb{1}_{\mathbb{K}})$, the fiber of L_{k+1} over g_{k+1}. As $(\partial_x \phi \circ \phi^{-1})(0)$ is an invertible matrix, L_{k+1} has constant fiber dimension. This proves (ii) of Theorem 1.

(c) Since the description of red T in part (i) is obvious we are left to show (iii) and (iv). We already know that $Y(0) \simeq (\mathbb{C}^d,0)$ where d is the fiber dimension of $T(Y/S)$ which is the dimension of

$$\{w \in \mathbb{C}^n, w \cdot \partial_x(g) \in I(g) \cdot \mathcal{O}_n^p + m_n \cdot J(g) + m_n^{k+1} \cdot \mathcal{O}_n^p\}$$

for $k \gg 0$. Krull's Intersection Theorem gives

$$d = \dim_{\mathbb{C}}\{w \in \mathbb{C}^n, w \cdot \partial_x(g) \in I(g) \cdot \mathcal{O}_n^p + m_n \cdot J(g)\}.$$

As g defines $X(0)$ in $(\mathbb{C}^n,0)$ there are thus d vectorfields $\xi_1, \ldots, \xi_d$ on $X(0)$ with $\xi_1(0), \ldots, \xi_d(0)$ linearly independant. A theorem of Rossi [Fi, 2.12] gives $X(0) \simeq (\mathbb{C}^d,0) \times Z$ for some germ Z. This proves (iii).

(d) We finally show the universal property of the trivial locus T of π. As in part (a) of the proof consider $\mathcal{X} \xrightarrow{\tau} X$ with section σ together with the trivial locus $Y \subset X$ of the pair (τ, σ) as given by Theorem 2. Let

$$\begin{array}{ccc} X' & \xrightarrow{\alpha'} & X \\ \downarrow \pi' & & \downarrow \pi \\ S' & \xrightarrow{\alpha} & S \end{array}$$

be a cartesian square and assume that π' is trivial. Choose a section $\rho' : S' \to X'$ of π' along which π' is trivial. By the claim below the composition $\alpha'\rho' : S' \to X$ factors through Y. Therefore $\alpha = \pi\alpha'\rho'$ factors through $\pi(Y) = T$.

Conversely, consider the cartesian square

$$\begin{array}{ccc} X' & \hookrightarrow & X \\ \downarrow \pi' & & \downarrow \pi \\ T & \hookrightarrow & S \end{array}$$

As $\pi_Y : Y \to T$ is a submersion there is a section $\rho' : T \to X'$ which factors through $Y \subset X'$. The claim implies the triviality of π' and concludes the proof of Theorem 1.

Claim. – Let $\pi : X \to S$, $\tau : \mathscr{X} \to X$, σ and Y be as before. For any cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{\alpha'} & X \\ \downarrow \pi' & & \downarrow \pi \\ S' & \xrightarrow{\alpha} & S \end{array}$$

with section $\rho' : S' \to X'$ of π', π' is trivial along ρ' if and only if $\alpha'\rho' : S' \to X$ factors through Y.

Proof. – One checks that the square

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{\varkappa} & X \\ \downarrow \tau & & \downarrow \pi \\ X & \xrightarrow{\pi} & S \end{array}$$

with $\varkappa(x,a) = (x+a_1, a_2)$ is cartesian (notation as in (a)). Combining with the cartesian square

$$\begin{array}{ccc} \mathscr{X}' & \xrightarrow{\beta} & \mathscr{X} \\ \downarrow \tau' & & \downarrow \tau \\ S' & \xrightarrow{\alpha'\rho'} & X \end{array}$$

induced from the base change $\alpha'\rho'$ gives a cartesian square

$$\begin{array}{ccc} \mathscr{X}' & \xrightarrow{\varkappa\beta} & X \\ \downarrow \tau' & & \downarrow \pi \\ S' & \xrightarrow{\pi\alpha'\rho'} & S. \end{array}$$

Since $\pi\alpha'\rho' = \alpha$ the uniqueness of the fiber product allows to assume that $\mathscr{X}' = X'$, $\tau' = \pi'$ and $\varkappa\beta = \alpha'$. As $\rho' : S' \to X'$ is the section of π' induced from σ, Theorem 2 applies to the cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{\beta} & \mathscr{X} \\ \downarrow \pi' & & \downarrow \tau \\ S' & \xrightarrow{\alpha'\rho'} & X \end{array}$$

proving the claim and (iv) of Theorem 1.

BIBLIOGRAPHY

[A] M. Artin, Algebraic approximation of structures over complete local rings, Publ. Math. IHES, 36 (1969), 23-58.

[D] I. F. Donin, Complete families of deformations of germs of complex spaces, Math. USSR-Sbornik, 18 (1972), 397-406.

[E] R. Ephraim, Isosingular loci and the cartesian product structure of complex analytic singularities, Trans. Am. Math. Soc., 241 (1978), 357-371.

[Fi] G. Fischer, Complex analytic geometry, Springer Lect. Notes, 538, 1976.

[FiG] W. FISCHER, H. GRAUERT, Lokal-triviale Familien kompakter komplexer Mannig-faltigkeiten, Nachr. Akad. Wiss. Göttingen, Math.-Phys. Kl. II, 6 (1965), 89-94.

[FlK] H. FLENNER, S. KOSAREW, On locally trivial deformations, Publ. Res. Inst. Math. Sci., 23 (1987), 627-665.

[GaH] T. GAFFNEY, H. HAUSER, Characterizing singularities of varieties and of mappings, Invent. Math., 81 (1985), 427-447.

[GrK] G.-M. GREUEL, U. KARRAS, Families cf varieties with prescribed singularities, Compos. Math., 69 (1989), 83-110.

[H] J. E. HUMPHREYS, Linear algebraic groups, Springer, 1975.

[M] J. N. MATHER, Stability of C^{∞}-mappings, III : Finitely determined map germs, Publ. Math. IHES, 35 (1968), 127-156.

[PfPo] G. PFISTER, D. POPESCU, Die strenge Approximationseigenschaft lokaler Ringe, Invent. Math., 30 (1975), 145-174.

[Se] A. SEIDENBERG, Analytic products, Am. J. Math., 91 (1969), 577-590.

[Sc] H. W. SCHUSTER, Zur Theorie der Deformationen kompakter komplexer Räume, Invent. Math., 9 (1970), 284-294.

[T] B. TEISSIER, The hunting of invariants in the geometry of discriminants, In : Real and complex singularities, Oslo 1976, 565-677. Holm, P., (ed.), Sijthoff and Noordhoff, 1977.

[V] V. S. VARADARAJAN, Lie groups, Lie algebras, and their representations, Prentice Hall, 1974.

[W] J. J. WAVRIK, A theorem on solutions of analytic equations with applications to deformations of complex structures, Math. Ann., 216 (1975), 127-142.

Manuscrit reçu le 22 décembre 1988.

H. HAUSER,
Institut für Mathematik
Universität Innsbruck
A-6020 Innsbruck (Austria)

&

G. MÜLLER,
Fachbereich Mathematik
Universität Mainz
D-6500 Mainz (Fed. Rep. Germany).