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BANACH SPACES WHICH ARE *M*-IDEALS IN THEIR BIDUAL HAVE PROPERTY (u)

by G. GODEFROY and D. LI

1. Introduction.

A Banach X is an M-ideal in its bidual if the relation

$$\|y+t\| = \|y\| + \|t\|$$

holds for every $y \in X^*$ and every $t \in X^{\perp} \subseteq X^{***}$. The spaces $c_0(I)$ – I any set-equipped with their canonical norm belong to this class, which also contains e.g. certain spaces $K(E,F)$ of compact operators between reflexive spaces (see [11]) and certain spaces of the form $C(G)/C_{\Lambda}(G)$ where G is an abelian compact group and Λ is a subset of the discrete dual group (see [5]). This class has been carefully investigated, in particular by A. Lima and by the « West-Berlin school », since the notion of M-ideal was introduced by Alfsen and Effros in 1972 [1].

We will show in this paper that these spaces somehow « look like » c_0; more precisely, that they share the property (u) with this latter space. This solves affirmatively a question that was pending for several years, and provides improvements of some results of [6] and [10].

Our proof uses non-linear arguments. The key lemma is actually a special case of a fundamental lemma ([1], lemma 1.4.) of the original article of Alfsen and Effros.

Notation. – The closed unit ball of a Banach space Z is denoted by Z_1, and its dual by Z^*. The topology defined on Z^* by the pointwise convergence on Z is denoted by ω^*. The canonical injection from a Banach space X into its bidual X^{**} is denoted by i. A sequence (x_m) in X is said to be a weakly unconditionally convergent series –

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in short, w.u.c. series — if for every $y \in X^*$,

$$\sum_{m=0}^{\infty} |y(x_m)| < \infty .$$

If (x_m) is a w.u.c. series, then clearly the sequence

$$s_k = \sum_{m=0}^{k} x_m \quad (k \geqslant 1)$$

is weakly Cauchy and thus it converges in (X^{**}, w^*); we note $\Sigma^* x_m = \lim_{k\to\infty} (s_k)$ in (X^{**}, ω^*). A Banach space X has the property (u) ([14]; see [12], p. 32) if every $z \in X^{**}$ which is in the sequential closure of X in (X^{**}, ω^*) may be written

$$z = \Sigma^* x_m$$

for some w.u.c. series (x_m) in X.

If $\tau : Z_1^* \to \mathbf{R}$ is a real-valued function defined on a dual unit ball Z_1^*, we denote by $\hat{\tau}$ the smallest concave ω^*-u.s.c. function which is greater than τ on Z_1^*. The function $\hat{\tau}$ is the infimum of the affine continuous functions on (Z_1^*, ω^*) which maximize τ on Z_1^*. The reader should consult [2] for a presentation of the basic facts about M-ideals. Similar ideas to those we use in this work are to be found e.g. in [15].

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2. The main result.

We will now prove:

THEOREM 1. – *Let X be a Banach space which is an M-ideal in its bidual. Then X has the property* (*u*).

Proof. – If i denotes the canonical injection from X into X^{**}, i^{**} is an isometric injection from X^{**} into X^{****} which is distinct from the canonical injection if X is not reflexive. We will use a simplified notation that we now define: i^* denotes the canonical projection from X^{***} onto X^* of kernel $i(X)^\perp = X^\perp$, and then of course $i^{**}(z) = z \circ i^*$.

From now on, we assume that X is a real Banach space. By ([2], p. 22), we can do so without loss of generality. The notation $\mathbf{1}_{X_1^\perp}$ denotes the characteristic function of the subset $X_1^\perp$ of the unit ball X_1^{***} of X^{***}. Thus for any $z \in X^{**}$, $(\mathbf{1}_{X_1^\perp}.z \vee 0)$ denotes the supremum of 0 and of the pointwise product of z and $\mathbf{1}_{X_1^\perp}$.

With this notation, we have the following crucial lemma.

LEMMA 2. — *If X is an M-ideal in its bidual X^{**}, then for every $z \in X^{**}$ and every $t \in X_1^{***}$ one has*

$$< z - i^{**}(z), t \rangle = [\widehat{\mathbf{1}_{X_1^\perp}\cdot z} \vee 0](t) - [\widehat{\mathbf{1}_{X_1^\perp}\cdot z} \vee 0](-t).$$

This lemma is actually a special case of ([1], lemma I.4). For sake of completeness, we give a simplified proof of this special case.

Proof. — If Ψ is a function from X_1^{***} to $\mathbf{R}^+$, we define

$$\mathfrak{G}^-(\Psi) = \{(t,\lambda) \in X_1^{***} \times \mathbf{R}^+ \,|\, 0 \leqslant \lambda \leqslant \Psi(t)\}$$

we let $\tau = (\mathbf{1}_{X_1^\perp}\cdot z \vee 0)$, and

$$B = \{(t, z(t)) \in X_1^\perp \times \mathbf{R}^+ \,|\, z(t) \geqslant 0\}.$$

We clearly have

$$(1) \qquad \mathfrak{G}^-(\hat{\tau}) = \overline{\text{conv}}^* (\mathfrak{G}^-(\tau)).$$

On the other hand,

(2) $X_1^{***} = \text{conv}\,(X_1^* \cup X_1^\perp)$ since X is an M-ideal in X^{**}

(3) if $0 \leqslant \lambda \leqslant z(t)$, $(t,\lambda) \in \text{conv}\,\{(t,0)\,;\,(t,z(t))\}$.

From (1), (2) and (3) follows

$$\begin{aligned}\mathfrak{G}^-(\hat{\tau}) &= \overline{\text{conv}}^* (\mathfrak{G}^-(\tau)) \\ &= \text{conv}\,\{(X_1^{***} \times \{0\}) \cup B\} \\ &= \text{conv}\,\{(X_1^* \times \{0\}) \cup (X_1^\perp \times \{0\}) \cup B\}\end{aligned}$$

since by w^*-compactness we don't need to take w^*-closures.

For every $t \in X_1^{***}$, $(t,\hat{\tau}(t)) \in \mathfrak{G}^-(\hat{\tau})$, hence we may write $(t,\hat{\tau}(t)) = \alpha_1(t_1,0) + \alpha_2(t_2,0) + \alpha_3(t_3,z(t_3))$ with:

$$\begin{cases} t_1 \in X^* \\ t_2 \in X_1^\perp \\ t_3 \in X_1^\perp ; \qquad z(t_3) \geqslant 0 \\ \alpha_1, \alpha_2, \alpha_3 \geqslant 0, \qquad \alpha_1 + \alpha_2 + \alpha_3 = 1 . \end{cases}$$

Since $t = \alpha_1 t_1 + (\alpha_2 t_2 + \alpha_3 t_3)$ is the unique decomposition of t on the direct sum $X^* \oplus X^\perp$ one has

$$t - i^*(t) = \alpha_2 t_2 + \alpha_3 t_3 .$$

Since $z(t_3) \geqslant 0$, one has $\tau(t_3) = z(t_3)$ and $\tau(-t_3) = 0$. Since $\hat{\tau}$ is concave, one has

$$\begin{aligned} \hat{\tau}(t) &= \alpha_3 z(t_3) \\ &\geqslant \sum_{i=1}^{3} \alpha_i \hat{\tau}(t_i) \\ &\geqslant \sum_{i=1}^{3} \alpha_i \tau(t_i) \\ &= \alpha_2 \tau(t_2) + \alpha_3 \tau(t_3) \\ &= \alpha_2 \tau(t_2) + \alpha_3 z(t_3) \end{aligned}$$

hence $\alpha_2 \tau(t_2) \leqslant 0$; if $\alpha_2 = 0$ we may take $t_2 = 0$ as well; if $\alpha_2 > 0$ this implies $\tau(t_2) \leqslant 0$, hence $z(t_2) \leqslant 0$. In both cases, we have $z(-t_2) \geqslant 0$ and thus $z(-t_2) = \tau(-t_2)$.

Again by concavity of $\hat{\tau}$, one has

$$\begin{aligned} \hat{\tau}(-t) &\geqslant \sum_{i=1}^{3} \alpha_i \hat{\tau}(-t_i) \\ &\geqslant \sum_{i=1}^{3} \alpha_i \tau(-t_i) \\ &= \alpha_2 \tau(-t_2) + \alpha_3 \tau(-t_3) \\ &= \alpha_2 z(-t_2) \end{aligned}$$

hence

$$- \hat{\tau}(-t) \leqslant \alpha_2 z(t_2)$$

and therefore

$$\hat{\tau}(t) - \hat{\tau}(-t) \leqslant \alpha_3 z(t_3) + \alpha_2 z(t_2)$$

and

$$\begin{aligned}\alpha_3 z(t_3) + \alpha_2 z(t_2) &= \langle z, \alpha_2 t_2 + \alpha_3 t_3 \rangle \\ &= \langle z, t - i^*(t) \rangle \\ &= \langle z - i^{**}(z), t \rangle .\end{aligned}$$

Now the functions $\Phi(t) = \hat{\tau}(t) - \hat{\tau}(-t)$ and $(z - i^{**}(z))$ are both odd functions on X_1^{***} and they satisfy $\Phi \leqslant z - i^{**}(z)$; hence necessarily $\Phi = z - i^{**}(z)$ on X_1^{***}. □

We now come back to the proof of theorem 1. By lemma 2, for every $z \in X^{**}$, we can write

$$\forall t \in X_1^{***}, \quad \langle i^{**}(z), t \rangle = (z(t) - \hat{\tau}(t)) + \hat{\tau}(-t)$$

hence if we let

$$\begin{aligned} h_1(t) &= z(t) - \hat{\tau}(t) \\ h_2(t) &= -\hat{\tau}(-t) \end{aligned}$$

we have $i^{**}(z) = h_1 - h_2$ and h_1, h_2 are both l.s.c. on (X_1^{***}, w^*).

We need now a topological argument for going down to (X_1^*, w^*).

LEMMA 3 (Saint-Raymond). – *Let K be a compact topological space and $S : K \to K'$ be a continuous surjection. Let f be a function from K' to* **R** *which is such that $(f \circ S)$ is the difference of two l.s.c. functions on K. Then f is the difference of two l.s.c. functions on K'.*

Proof. – Write $f \circ S = g_1 - g_2$ where g_1, g_2 are l.s.c. on K; we define for $y \in K'$

$$\begin{aligned} \tilde{g}_1(y) &= \inf \{ g_1(t) \mid S(t) = y \} \\ \tilde{g}_2(y) &= \inf \{ g_2(t) \mid S(t) = y \} \end{aligned}$$

the functions $\tilde{g}_i (i=1,2)$ are l.s.c. on K'. Indeed, pick $\alpha < \tilde{g}_i(y)$; this means

(1) $$\forall t \in S^{-1}(y), \quad g_i(t) > \alpha .$$

Since g_i is l.s.c. and $S^{-1}(y)$ is compact, (1) implies that there exists $\varepsilon > 0$ and an open neighbourhood V of $S^{-1}(y)$ such that

(2) $$\forall t \in V, \quad g_i(t) > \alpha + \varepsilon .$$

Again by compactness, there exists a neighbourhood W of y such that $S^{-1}(W) \subseteq V$; by (2) and the definition of $\tilde{g}_i$, this implies

$$\forall y' \in W, \quad \tilde{g}_i(y') \geqslant \alpha + \varepsilon > \alpha$$

and thus $\tilde{g}_i$ is l.s.c.

We show now that $f = \tilde{g}_1 - \tilde{g}_2$; for every $y \in K'$ and $t \in S^{-1}(y)$, one has

$$\begin{aligned} \tilde{g}_1(y) \leqslant g_1(t) &= f \circ S(t) + g_2(t) \\ &= f(y) + g_2(t) \end{aligned}$$

hence by definition of $\tilde{g}_2$

$$\tilde{g}_1(y) \leqslant f(y) + \tilde{g}_2(y).$$

On the other hand,

$$\begin{aligned} f(y) + \tilde{g}_2(y) &\leqslant f(y) + g_2(t) \\ &= f \circ S(t) + g_2(t) \\ &= g_1(t) \end{aligned}$$

and thus by definition of $\tilde{g}_1$,

$$f(y) + \tilde{g}_2(y) \leqslant \tilde{g}_1(y)$$

and this concludes the proof of lemma 3. □

Let us now conclude the proof of the theorem. Since

$$i^{**}(z) = z \circ i^* = h_1 - h_2$$

with h_1 and h_2 l.s.c. on (X_1^{***}, w^*), we may apply lemma 3 with $f = z$, $S = i^*$ and $K' = (X_1^*, w^*)$; this lemma provides us with the l.s.c. functions $\tilde{h}_1$ and $\tilde{h}_2$ on (X_1^*, w^*) such that $z = \tilde{h}_1 - \tilde{h}_2$.

If now $z = \lim_{n\to\infty} x_n$ in (X^{**}, w^*), where (x_n) is a sequence in X, we let

$$Y = \overline{\operatorname{span}}\,\{x_n | n \geqslant 1\}$$

and we call Q the canonical quotient map from X^* onto Y^*; since $z \in Y^{\perp\perp} = Q^*(Y^{**})$, there is $z' \in Y^{**}$ such that $z = z' \circ Q$; again by lemma 3, there exist two l.s.c. functions $\tilde{\tilde{h}}_1$ and $\tilde{\tilde{h}}_2$ on (Y_1^*, w^*) such that

$$z' = \tilde{\tilde{h}}_1 - \tilde{\tilde{h}}_2.$$

But since Y is separable, the w^*-topology on Y_1^* is defined by a metric d, and then classically the sequences $f_n^i (i=1,2)$ defined for $y \in Y_1^*$ and $n \geqslant 1$ by

$$f_n^i(y) = \inf \{\tilde{\tilde{h}}_i(y') + nd(y,y') \mid y' \in Y_1^*\}$$

are increasing sequences of continuous functions on (Y_1^*, w^*) which converge pointwise to $\tilde{\tilde{h}}_i$. Now the sequence $u_n (n \geqslant 0)$ of continuous functions on (Y_1^*, w^*) defined by

$$\begin{aligned} u_0 &= f_1^1 - f_1^2 \\ u_n &= f_{n+1}^1 + f_n^2 - f_n^1 - f_{n+1}^2 \quad (n \geqslant 1) \end{aligned}$$

satisfies

$$\sum_{n=0}^{\infty} |u_n(y)| < \infty, \quad \forall y \in Y_1^*$$

and

$$\sum_{n=0}^{\infty} u_n(y) = z'(y), \quad \forall y \in Y_1^*.$$

But we still have

$$z'(y) = \lim_{n \to \infty} x_n(y), \quad \forall y \in Y_1^*$$

in this situation, a classical lemma of Pelczynski [14] (see [12], p. 32), which relies on a convex combination argument, shows that there is a sequence $(c_n)_{n \geqslant 0}$ in Y with

$$\sum_{n=0}^{\infty} |c_n(y)| < \infty, \quad \forall y \in Y_1^*$$

and

$$\sum_{n=0}^{\infty} c_n(y) = z'(y), \quad \forall y \in Y_1^*$$

and since $z = Q^*(z')$ and $c_n = Q^*(c_n)$, this shows that

$$z = \Sigma^* c_n$$

and (c_n) is a w.u.c. series in X. □

Before mentioning a few applications of our result, we would like to mention that the proof provides an explicit expression of $z \in X^{**}$

as a difference of two l.s.c. functions on (X_1^*, w^*) ; indeed, if we define for $y \in X_1^*$

$$v(y) = \inf \{z(t) - [\widehat{\mathbf{1}_{X_1^\perp} \cdot z} \vee 0](t) \mid t \in X_1^{***}, i^*(t) = y\}$$

then the functions v and $(v - z)$ are both l.s.c. on (X_1^*, w^*).

3. Applications.

We gather in this section a few consequences of our result.

3.1. P. Saab and the first-named author showed in ([6], Theorem 1) that if X is an M-ideal in its bidual then X has the property (V) of Pelczynski ; the proof uses « pseudo-balls » ([3]) and the local reflexivity principle. Since such an X does not contain $\ell^1(N)$, our result is an improvement of ([6], Theorem 1), and of course also of the fact ([10]) that non-reflexive M-ideals in their bidual contain $c_0(N)$.

Another result of [6] is a structural result (Corollary 6) for certain spaces E such that $K(E)$ is an M-ideal in $L(E)$. The proof uses Banach algebras techniques that require to work with complex Banach spaces. This is not needed any more, and our result together with the proofs of ([6], Theorem 4 and Corollary 6) implies for instance the

PROPOSITION 4. — *Let E be a separable reflexive space with A.P. such that $K(E)$ is an M-ideal in $L(E)$. Then E is complemented in a reflexive space with an unconditional finite dimensional decomposition.*

There are some similarities between the techniques of [6] and of the present work ; the main difference is that instead of using l.s.c. affine functions on a non-symmetric convex set — namely, the state space of a Banach algebra — we employ l.s.c. convex functions on a symmetric convex set — namely, a dual unit ball.

3.2. A Banach space Y is said to have to property (X) [7] if the following holds : $z \in Y^{**}$ belongs to Y if and only if for every w.u.c. series (y_n) in Y^*,

$$z(\Sigma_* y_n) = \Sigma z(y_n)$$

where $(\Sigma_* y_n)$ denotes the limit of the sequence $\left\{s_k = \sum_{n=1}^{k} y_n \mid k \geqslant 1\right\}$ in (Y^*, w^*). This condition roughly means that an abstract Radon-

Nikodym theorem is available for deciding which elements of Y^{**} belong to Y. Property (X) is equivalent to saying that $Y < \ell^1(N)$ for Edgar's ordering of Banach spaces [4]. For more details about this property, the reader may consult the recent survey [8].

Let us recall now the following easy

Claim. — If X is separable, does not contain $\ell^1(N)$ and has the property (u), then X^* has the property (X).

Proof of the claim. — We must show that every $t \in X^{***}$ such that $t(\Sigma_* z_n) = \Sigma t(z_n)$ for every w.u.c. series in X^{**} belongs to X^*. We can write $t = y + t'$ with $y \in X^*$ and $t' \in X^\perp$; since $y(\Sigma_* z_n) = \Sigma y(z_n)$ by w^*-continuity of y, we also have $t'(\Sigma_* z_n) = \Sigma t'(z_n)$.

Since X is separable, does not contain $\ell^1(N)$ and has (u), every $z \in X^{**}$ can be written $z = \Sigma^* x_n = \Sigma_* i(x_n)$ for some w.u.c. series in X; but since $t'(x_n) = 0$ for every n, this implies $t'(z) = 0$, hence $t' = 0$ and $t = y \in X^*$.

Now this claim, together with theorem 1, shows:

PROPOSITION 5. — *If a separable Banach space X is an M-ideal in its bidual, then X^* has the property (X).*

By ([8], Theorem VII.8) this implies the following

COROLLARY 6. — *Let X be a separable Banach space X which is an M-ideal in its bidual, and let Y be an arbitrary Banach space. Let T : $X^{**} \to Y^*$ be a bounded linear operator. The following are equivalent :*

(1) *there is an operator $T_0 = Y \to X^*$ such that $T_0^* = T$,*

(2) ker (T) *and $T(X_1^{**})$ are w^*-closed,*

(3) *T is $(w^* - w^*)$-Borel,*

(4) *T is $(w^* - w^*)$-strongly Baire measurable.*

Let us conclude this work with a few natural questions.

Question 3.4. — Does there exist a separable Asplund space with property (u) which is not isomorphic to an M-ideal in its bidual? It looks reasonable to believe that this question has a positive answer; a candidate example is the space $K(L^p)(1<p<\infty, p\neq 2)$ which has (u) ([12], Th. 3) but is not M-ideal of its bidual for its canonical norm [11].

Let us also mention that a separable $\mathscr{L}^{\infty}$-space which is isomorphic to an M-ideal in its bidual is in fact isomorphic to $c_0(N)$ [9]. We do not know whether any isomorphic predual of $\ell^1(N)$ which has property (u) is isomorphic to $c_0(N)$.

Question 3.5. – A reformulation of Proposition 5 is that if Y is a separable space such that there exists a projection $\pi : Y^{**} \to Y$ with:

(*a*) $\|z\| = \|\pi(z)\| + \|z-\pi(z)\|, \ \forall z \in Y^{**}$

(*b*) $(\ker \pi)$ w^*-closed,

then Y has the property (X). It is not known whether the assumption (*b*) can be removed, or whether (*a*) alone implies the weaker property (V*) (see [14]), or at least that Y contains a complemented copy of $\ell^1(N)$ if it is not reflexive.

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