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# FINE TOPOLOGY AND QUASILINEAR ELLIPTIC EQUATIONS

by J. HEINONEN, T. KILPELÄINEN and O. MARTIO

## 1. Introduction.

The fine topology, introduced by H. Cartan, is the coarsest topology which makes all superharmonic functions in $\mathbb{R}^n$ continuous. It turns out that a set $U$ in $\mathbb{R}^n$ is a fine neighborhood of a point $x_0 \in U$ if and only if the complement of $U$ is thin at $x_0$, i.e. the Wiener integral of the complement of $U$ converges at $x_0$. In the nonlinear potential theory this latter condition has been taken as a starting point : a set $U$ in $\mathbb{R}^n$ is called an *$(\alpha,p)$-fine neighborhood* of a point $x_0 \in U$ if the $(\alpha,p)$-Wiener integral of the complement of $U$ converges at $x_0$. Here $\alpha > 0$, $p > 1$ and $\alpha p \leqslant n$. The $(\alpha,p)$-fine topologies associated with Bessel potentials were introduced by N. G. Meyers [M] and later studied for example by D. R. Adams and L. I. Hedberg [AH] ; see also [AL], [AM] and [F1]. The classical fine topology of Cartan is included in the case $\alpha = 1$ and $p = 2$.

In this paper we show that the case $\alpha = 1$ and $1 < p \leqslant n$, which is related to second order elliptic equations and to the Sobolev space $W^1_p$, admits an approach similar to that of Cartan : the $(1,p)$-fine topology is the coarsest topology making all supersolutions of the $p$-Laplace equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$$

in $\mathbb{R}^n$ continuous. In fact, there is a wide class of degenerate elliptic equations, other than the $p$-Laplace equation, with the same property. In particular, there are nonlinear equations whose supersolutions induce the same $(1,2)$-fine topology as superharmonic functions do.

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To fix ideas, consider an operator $\mathscr{A}: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $n \geqslant 2$, $\mathscr{A}(x,h)\cdot h \approx |h|^p$, $1 < p \leqslant n$, and the equation

$$\operatorname{div} \mathscr{A}(x,\nabla u) = 0. \tag{1.1}$$

The precise assumptions on $\mathscr{A}$ are given in Section 2. Continuous weak solutions of (1.1) are called *$\mathscr{A}$-harmonic*. A lower semicontinuous function $u$ in an open set $\Omega$ in $\mathbb{R}^n$, $-\infty < u \leqslant \infty$, is called *$\mathscr{A}$-superharmonic* if it satisfies the comparison principle : for each domain $D \subset\subset \Omega$ and each $\mathscr{A}$-harmonic function $h$ in $D$, $h \in C(\bar{D})$, the condition $h \leqslant u$ in $\partial D$ implies $h \leqslant u$ in $D$. This definition was apparently first used in a nonlinear situation when S. Granlund, P. Lindqvist and O. Martio studied superextremals of certain variational integrals with applications to function theory [GLM1-2]. The general $\mathscr{A}$-superharmonic functions and their (nonlinear) potential theory were investigated in the papers [HK1-3], [K] and [L]. It was shown in [HK1] that all supersolutions of (1.1) are $\mathscr{A}$-superharmonic when properly pointwise redefined ; ther converse is not true.

We define the *$\mathscr{A}$-fine topology* to be the coarsest topology in $\mathbb{R}^n$ making all $\mathscr{A}$-superharmonic functions in $\mathbb{R}^n$ continuous. Our main result, Theorem 3.2, characterizes the $\mathscr{A}$-fine topology by means of the Wiener criterion, whence it coincides with the above mentioned $(1,p)$-fine topology. It follows in particular that the $\mathscr{A}$-fine topology is independent from the operator $\mathscr{A}$ once the number $p$ is fixed. Actually, we obtain the whole spectrum of fine topologies $\tau_p$ such that $\tau_q \subsetneqq \tau_p$ for $1 < p < q \leqslant n$. If $p > n$, the $\mathscr{A}$-fine topology can be similarly defined, but then it always equals the euclidean topology.

Quasiregular mappings and *BLD* mappings (mappings of bounded length distortion) are harmonic morphisms in this nonlinear potential theory, see [GLM1], [MV] and Section 5 below. We study the $\mathscr{A}$-fine limits of quasiregular and *BLD* mappings ; for example we show that if $f$ is a quasiregular mapping in an open set $\Omega \subset \mathbb{R}^n$ omitting a set of positive $n$-capacity and if the complement of $\Omega$ is $n$-thin at $x_0 \in \partial\Omega$, then $f$ has an $n$-fine limit at $x_0$. This result was recently proved for plane analytic functions by B. Fuglede in [F4]. It remains an open problem whether a similar result is true for general lower bounded $\mathscr{A}$-superharmonic functions, cf. [D, pp. 190].

The paper is organized as follows. The $\mathscr{A}$-fine topology is introduced in Section 2 where we consider some of its basic properties ; it is Hausdorff, completely regular and Baire but fails to have countable neighborhood bases. Also, only finite sets are $\mathscr{A}$-finely compact. In Section 3 we prove the Wiener criterion characterization of the $\mathscr{A}$-fine topology. In Section 4 we discuss the theorem of D. R. Adams and J. L. Lewis about local quasiconvexity of the $\mathscr{A}$-fine topology, and in Section 5 we study the $\mathscr{A}$-fine limits of quasiregular and *BLD* mappings.

Our notation is standard. Throughout, $\Omega$ is an open set in $\mathbb{R}^n$, $n \geqslant 2$. For a set $E \subset \mathbb{R}^n$ we let $\complement E$ denote the complement of $E$, $\complement E = \mathbb{R}^n \setminus E$ and $E \subset\subset \Omega$ means that the closure of $E$ is compact in $\Omega$. If $B = B(x,r) = \{y \in \mathbb{R}^n : |x - y| < r\}$ is an open $n$-ball and $\sigma > 0$, then $\sigma B = B(x,\sigma r)$. The Lebesgue $n$-measure of a measurable set $E$ is written as $|E|$. The letter $c$ denotes a positive generic constant whose value is not necessarily the same at each occurrence.

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## 2. Fine topology and $\mathscr{A}$-superharmonic functions.

We assume throughout this paper that $\mathscr{A} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $n \geqslant 2$, is an operator which satisfies the following assumptions for some numbers $1 < p \leqslant n$ and $0 < \alpha \leqslant \beta < \infty$:

(2.1) the function $x \mapsto \mathscr{A}(x,h)$ is measurable for all $h \in \mathbb{R}^n$, and the function $h \mapsto \mathscr{A}(x,h)$ is continuous for a.e. $x \in \mathbb{R}^n$,

for all $h \in \mathbb{R}^n$ and a.e. $x \in \mathbb{R}^n$

$$\mathscr{A}(x,h)\cdot h \geqslant \alpha|h|^p, \tag{2.2}$$

$$|\mathscr{A}(x,h)| \leqslant \beta|h|^{p-1}, \tag{2.3}$$

$$(\mathscr{A}(x,h_1) - \mathscr{A}(x,h_2))\cdot(h_1-h_2) > 0 \tag{2.4}$$

whenever $h_1 \neq h_2$, and

$$\mathscr{A}(x,\lambda h) = \lambda|\lambda|^{p-2}\mathscr{A}(x,h) \tag{2.5}$$

for all $\lambda \in \mathbb{R}$, $\lambda \neq 0$.

A function $u$ in the local Sobolev space $\operatorname{loc} W_p^1(\Omega)$ is a weak solution (supersolution) to the equation (1.1) in $\Omega$ if

$$\int_\Omega \mathscr{A}(x,\nabla u)\cdot\nabla\varphi\, dx = 0 \quad (\geqslant 0) \tag{2.6}$$

for all $\varphi \in C_0^\infty(\Omega)$ ($\varphi \in C_0^\infty(\Omega), \varphi \geqslant 0$). It is well known that each weak solution of (1.1) has a continuous representative; we call continuous weak solutions of (1.1) *$\mathscr{A}$-harmonic*, cf. [HK1]. A function $u$ in $\Omega$ is called *$\mathscr{A}$-superharmonic* if

(i) $u$ is lower semicontinuous (lsc),

(ii) $-\infty < u \leqslant \infty$,

and

(iii) for each domain $D \subset\subset \Omega$ and each $h \in C(\bar{D})$, $\mathscr{A}$-harmonic in $D$, $h \leqslant u$ in $\partial D$ implies $h \leqslant u$ in $D$.

A typical example of the operators satisfying (2.1)-(2.5) is the $p$-harmonic operator, $\mathscr{A}(x,h) = |h|^{p-2}h$, which is conformally invariant if $p = n$. The fundamental $p$-superharmonic function in $\mathbb{R}^n$ is

$$(2.7) \qquad u(x) = \begin{cases} |x|^{\frac{p-n}{p-1}}, & 1 < p < n, \\ -\log|x|, & p = n. \end{cases}$$

The sum of two $\mathscr{A}$-superharmonic functions is not $\mathscr{A}$-superharmonic in general, but clearly $\lambda u + \mu$ is $\mathscr{A}$-superharmonic whenever $u$ is $\mathscr{A}$-superharmonic, $\lambda \geqslant 0$, and $\mu \in \mathbb{R}$. Also, if $u$ and $v$ are $\mathscr{A}$-superharmonic, then so is $\min(u,v)$.

We recall some fundamental properties of $\mathscr{A}$-superharmonic functions; for the proofs we refer to [HK1]. First, each supersolution $u$ of (1.1) has a unique $\mathscr{A}$-superharmonic representative given by

$$(2.8) \qquad u(x) = \operatorname{ess}\liminf_{y \to x} u(y)$$

or, if $u$ is locally bounded, equivalently by

$$u(x) = \lim_{r \to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y)\, dy.$$

Conversely, if $u$ is $\mathscr{A}$-superharmonic in $\Omega$, then it satisfies (2.8) at each point $x$ in $\Omega$; if, in addition, $u$ is in loc $W_p^1(\Omega)$, then $u$ is a supersolution of (1.1). Locally bounded $\mathscr{A}$-superharmonic functions are always in loc $W_p^1$, whence each $\mathscr{A}$-superharmonic function $u$ is the limit of an increasing sequence of supersolutions, namely $u = \lim_{k \to \infty} \min(u,k)$. On the other hand, an $\mathscr{A}$-superharmonic function need not be locally in the Sobolev space $W_p^1$ as displayed by the examples in (2.7).

2.9. *Comparison principle* [HK1, 3.7]. Suppose that $u$ and $-v$ are $\mathscr{A}$-superharmonic in a bounded open set $\Omega$. If

$$\limsup_{y\to x} v(y) \leqslant \liminf_{y\to x} u(y)$$

for all $x \in \partial\Omega$ and if the left and the right hand sides are not simultaneously $+\infty$ or $-\infty$, then $v \leqslant u$ in $\Omega$.

We are ready to define the fine topology generated by $\mathscr{A}$-superharmonic functions.

2.10. DEFINITION. – *The $\mathscr{A}$-fine topology $\tau_{\mathscr{A}}$ is the coarsest topology in $\mathbb{R}^n$ making all $\mathscr{A}$-superharmonic functions $u$ in $\mathbb{R}^n$ continuous.*

Since $\min(u,\lambda)$, $\lambda \in \mathbb{R}$, is $\mathscr{A}$-superharmonic whenever $u$ is, $\tau_{\mathscr{A}}$ is the coarsest topology in $\mathbb{R}^n$ making all upper bounded $\mathscr{A}$-superharmonic functions in $\mathbb{R}^n$ continuous. In particular, all supersolutions of (1.1) which are redefined via (2.8) produce the same fine topology $\tau_{\mathscr{A}}$.

*The $\mathscr{A}$-fine topology is strictly finer than the euclidean topology* : It follows from [K, 3.2] that for any euclidean ball $B$ there is an $\mathscr{A}$-superharmonic function $u$ in $\mathbb{R}^n$ such that the nonempty set $\{x \in \mathbb{R}^n : u(x) > 0\}$ is contained in $B$ whence $\tau_{\mathscr{A}}$ is finer than the euclidean topology. On the other hand, $\tau_{\mathscr{A}}$ does not coincide with the euclidean topology since there are discontinuous $\mathscr{A}$-superharmonic functions in $\mathbb{R}^n$, see [HK2], [K].

In Section 3 we shall show that the $\mathscr{A}$-fine topology actually depends only on $p$, and therefore we could write $\tau_{\mathscr{A}} = \tau_p$. Moreover, it will be proved that if $1 < p < q \leqslant n$, then $\tau_p$ is strictly finer than $\tau_q$. At this point it is convenient to note that if an operator $\mathscr{A}$ satisfies (2.1)-(2.5) with $p > n$, then the fine topology $\tau_{\mathscr{A}}$ can be defined similarly, but then it coincides with the euclidean topology; indeed, it was noted in [HK1, 3.20] that all $\mathscr{A}$-superharmonic functions are continuous if $p > n$.

A natural subbase of $\tau_{\mathscr{A}}$ consists of sets of the form $\{u > \lambda\}$ or $\{u < \lambda\}$, where $u$ is $\mathscr{A}$-superharmonic and $\lambda \in \mathbb{R}$; the family of the finite intersections of the sets of this form is a base of $\tau_{\mathscr{A}}$. A convenient neighborhood base of a point $x_0$ consists of the sets

$$\bigcap_{i=1}^{k} \{x \in \bar{B} : u_i(x) \leqslant c\},$$

where $k$ is an integer, each $u_i$ is an upper bounded $\mathscr{A}$-superharmonic function with $u_i(x_0) = 0$, $c > 0$, and $B$ is an euclidean ball centered at $x_0$, cf. [D, pp. 167]; these sets are euclidean compact and $\mathscr{A}$-finely closed. Now it is easy to show that the $\mathscr{A}$-fine topology $\tau_{\mathscr{A}}$ is Hausdorff, completely regular, and Baire; see also 3.14 below. All this is standard in the linear case, cf. [B], [D], [LMZ].

Jan Malý has pointed out that $\tau_{\mathscr{A}}$ is also locally connected and quasi-Lindelöf, cf. [D], [F2].

We close this section with two remarks. First, the topology which $\tau_{\mathscr{A}}$ induces to an open set $\Omega \subset \mathbb{R}^n$ is the coarsest topology making all $\mathscr{A}$-superharmonic functions $u$ in $\Omega$ continuous. This follows easily from the fact that an $\mathscr{A}$-superharmonic function $u$ in $\Omega$ can be extended from each ball $B \subset\subset \Omega$ to an $\mathscr{A}$-superharmonic function in $\mathbb{R}^n$ [K, 3.1].

Finally, let $\mathfrak{F}(\mathscr{A})$ stand for the smallest convex cone closed under the min-operation and containing all $\mathscr{A}$-superharmonic functions $u$ in $\mathbb{R}^n$. It is an elementary fact that *$\tau_{\mathscr{A}}$ is the coarsest topology in $\mathbb{R}^n$ making all functions in $\mathfrak{F}(\mathscr{A})$ continuous*, see [LMZ].

## 3. Fine topology and the Wiener criterion.

In this section we show that the $\mathscr{A}$-fine topology can be defined in terms of a Wiener criterion: *a set $U$ is $\mathscr{A}$-finely open if and only if $\complement U$, the complement of $U$, is p-thin at each point $x \in U$.* It follows, in particular, that the $\mathscr{A}$-fine topology which is induced intrinsically by $\mathscr{A}$-superharmonic functions depends only on $p$, the type of the operator $\mathscr{A}$, and coincides with the $(1,p)$-topology studied earlier by N. G. Meyers, D. R. Adams, and L. I. Hedberg [M], [AH], [AM].

We point out that in the case the equation (1.1) is linear our proof for Theorem 3.2 yields the well known characterization of regular boundary points to the Dirichlet problem in terms of the Wiener criterion without any reference to Green functions, cf. [LSW].

First we recall the definition for the *variational p-capacity* of a pair $(E,\Omega)$ where if $E$ is a subset of an open set $\Omega \subset \mathbb{R}^n$. We set

$$\operatorname{cap}_p(E,\Omega) = \inf_{\substack{E \subset G \subset \Omega \\ G \text{ open}}} \operatorname{cap}_{p_*}(G,\Omega)$$

where, for any set $F \subset \Omega$,

$$\mathrm{cap}_{p_*}(F,\Omega) = \sup_{\substack{K \subset F \\ K \text{compact}}} \inf_{u \in W(K;\Omega)} \int_\Omega |\nabla u|^p \, dx$$

and $W(K;\Omega) = \{u \in C_0^\infty(\Omega) : u = 1 \text{ in } K\}$.

A set $E \subset \mathbb{R}^n$ is said to be $p$-*(Wiener-) thin* at a point $x_0$ if

$$\int_0^1 \left( \frac{\mathrm{cap}_p(E \cap B(x_0,t), B(x_0,2t))}{\mathrm{cap}_p(B(x_0,t), B(x_0,2t))} \right)^{\frac{1}{p-1}} \frac{dt}{t} < \infty . \tag{3.1}$$

A set $E$ is $p$-thin at $x_0$ in the sense of (3.1) if and only if $E$ is $(1,p)$-thin in the sense of Meyers [M]; the required comparison between the two different capacities has been made e.g. in [R].

In what follows a set $U$ is called an $\mathscr{A}$-fine neighborhood of a point $x_0$ if there is a set $V$ in $\tau_{\mathscr{A}}$ such that $x_0 \in V \subset U$.

The main result of this paper is

3.2. Theorem. — *A set $U$ is an $\mathscr{A}$-fine neighborhood of a point $x_0$ if and only if $x_0$ is in $U$ and the complement of $U$ is p-thin at $x_0$.*

The following two corollaries are immediate.

3.3. Corollary. — *The fine topology $\tau_{\mathscr{A}}$ depends only on $p$, not on the operator $\mathscr{A}$.*

3.4. Corollary. — *A point $x_0$ is an $\mathscr{A}$-fine limit point of a set $E$ if and only if $E$ is not p-thin at $x_0$.*

Recall that a set $E$ in $\mathbb{R}^n$ is $\mathscr{A}$-*polar* if there is an $\mathscr{A}$-superharmonic function $u$ in $\mathbb{R}^n$, $u \not\equiv \infty$, such that $u = \infty$ in $E$. It was shown in [HK2] (see also [K]) that $E$ is $\mathscr{A}$-polar if and only if $E$ is of $p$-capacity zero, i.e. $\mathrm{cap}_p(E \cap \Omega, \Omega) = 0$ for each open set $\Omega$. By the Kellogg property [HW, Theorem 2] the set $\{x \in E : E$ is $p$-thin at $x\}$ is of $p$-capacity zero. Thus we obtain

3.5. Corollary. — *A set $E$ in $\mathbb{R}^n$ is $\mathscr{A}$-polar if and only if $E$ is $\mathscr{A}$-finely isolated.*

At this point it is convenient to recall that if $E$ is $\mathscr{A}$-polar and $x_0 \notin E$, then there is an $\mathscr{A}$-superharmonic function $u$ in $\mathbb{R}^n$ such that $u = \infty$ in $E$ and $u(x_0) < \infty$ [K].

3.6. COROLLARY. – *A set $E$ in $\mathbb{R}^n$ is $\mathscr{A}$-finely compact if and only if $E$ is finite.*

For the proof of Theorem 3.2 we require two lemmas.

3.7. LEMMA. – *Let $E \subset \mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$. If there is an $\mathscr{A}$-superharmonic function $u$ defined in a neighborhood of $x_0$ such that*

$$\liminf_{\substack{x \to x_0 \\ x \in E\setminus\{x_0\}}} u(x) > u(x_0), \tag{3.8}$$

*then $\complement(E\setminus\{x_0\})$ is an $\mathscr{A}$-fine neighborhood of $x_0$ and $E$ is $p$-thin at $x_0$.*

*Proof.* – We may assume that $u$ is defined in all of $\mathbb{R}^n$, see [K], and that $x_0 \in \bar{E}\setminus E$. Then there is a neighborhood $U$ of $x_0$ such that $u(x) \geqslant \gamma > u(x_0)$ for each $x \in E \cap U$. Since the $\mathscr{A}$-fine neighborhood $V = \{x \in U : u(x) < \gamma\}$ of $x_0$ is contained in $\complement E$, the first assertion is proved.

The second assertion follows from [HK3, 4.1 and 4.3].

For the next lemma recall that if $E$ is a subset of an open set $\Omega$, then the *$\mathscr{A}$-potential* of $E$ in $\Omega$ is the function $\hat{R}^1_E(x) = \hat{R}^1_E(\Omega;\mathscr{A})$ defined as

$$\hat{R}^1_E(x) = \liminf_{y \to x} R^1_E(y),$$

where

$$\begin{aligned} R^1_E &= R^1_E(\Omega;\mathscr{A}) \\ &= \inf\{u : u \geqslant 0 \text{ and } \mathscr{A}\text{-superharmonic in } \Omega, \text{ and } u \geqslant 1 \text{ in } E\}. \end{aligned}$$

The function $\hat{R}^1_E$ is $\mathscr{A}$-superharmonic in $\Omega$ and $\hat{R}^1_E(x_0) = 1$ if $E$ is not $p$-thin at $x_0 \in \Omega$; for these results see [HK2] and [HK3].

3.9. LEMMA. – *Let $B = B(x_0,r)$ be a ball and $0 < \gamma < 1$. Suppose that $E \subset \frac{1}{2}B$ is an open set and let $u = \hat{R}^1_E(B;\mathscr{A})$ be the $\mathscr{A}$-potential of $E$ in $B$. There is a constant $c = c(n,p,\alpha,\beta) > 0$ such that if*

$$\frac{\operatorname{cap}_p(E,B)}{\operatorname{cap}_p\left(\frac{1}{2}B,B\right)} \leqslant c\gamma^{p-1},$$

*then $\inf_{\partial B(x_0,\rho)} u < \gamma$ for each $\rho \in \left(\frac{r}{4},\frac{r}{2}\right)$.*

*Proof.* – Let $E_\gamma = \{x \in B : u(x) \geqslant \gamma\}$ and suppose that $\partial B(x_0, \rho) \subset E_\gamma$ for some $\rho \in \left(\frac{r}{4}, \frac{r}{2}\right)$. Then it follows from [HK3, 3.2] that

$$\begin{aligned} \gamma^{1-p} \operatorname{cap}_p (E,B) &\geqslant c \operatorname{cap}_p (E_\gamma, B) \\ &\geqslant c \operatorname{cap}_p (B(x_0, \rho), B) \\ &\geqslant c \operatorname{cap}_p \left(\frac{1}{2} B, B\right), \end{aligned}$$

where $c = c(n,p,\alpha,\beta) > 0$; the last inequality holds since if $\delta > 1$, then

$$\operatorname{cap}_p (B(x_0, s), B(x_0, \delta s)) = cs^{n-p}$$

where $c$ depends only on $\delta$, $n$, and $p$.

The lemma follows.

*Proof of Theorem* 3.2. – Suppose first that $U$ is an $\mathscr{A}$-fine neighborhood of $x_0$. Then there are $\mathscr{A}$-superharmonic functions $u_1, \ldots, u_k$ defined in a ball $B = B(x_0, r)$ and constants $c_i > u_i(x_0)$, $i = 1, \ldots, k$, such that

$$x_0 \in \bigcap_{i=1}^{k} \{x \in B : u_i(x) < c_i\} \subset U.$$

Therefore

$$\complement U \cap B \subset \bigcup_{i=1}^{k} \{x \in B : u_i(x) \geqslant c_i\},$$

and by Lemma 3.7 the set $\{x \in B : u_i(x) \geqslant c_i\} \cup \complement B$ is $p$-thin at $x_0$ for each $i = 1, \ldots, k$. Consequently, $\complement U$ is $p$-thin at $x_0$ as required.

To prove the converse statement, we use a separation argument which was introduced in [LM] and further exploited in [HK3]. Thus, denote $E = \complement U$ and suppose that $E$ is $p$-thin at $x_0$. We may suppose that $E \subset B\left(x_0, \frac{1}{2}\right)$ and that $E$ is open [HK3]. Let

$$D = \bigcup_{j=1}^{\infty} \left( (E \cap B(x_0, 2^{-j})) \setminus \bar{B}\left(x_0, \frac{4}{3} 2^{-j-1}\right) \right).$$

We shall show that there exists an $\mathscr{A}$-superharmonic function $v_0$ in a neighborhood of $x_0$ and $r > 0$ such that $v_0|_{(D \cap B(x_0, r))} = 1$ while $v_0(x_0) < 1$.

This means by Lemma 3.7 that $\complement D$ is an $\mathscr{A}$-fine neighborhood of $x_0$. A similar construction shows that the complement of the set

$$D' = \bigcup_{j=1}^{\infty}\left((E\cap B(x_0,\tfrac{5}{6}2^{-j}))\setminus \bar{B}(x_0,\tfrac{7}{8}2^{-j-1})\right)$$

is an $\mathscr{A}$-fine neighborhood of $x_0$ as well. Since $E \subset D \cup D'$, this will establish the desired conclusion.

To start the construction, let $\varepsilon > 0$; we shall specify $\varepsilon$ later. Choose $j_0$ such that

$$\sum_{j=1}^{\infty} a_j^{\frac{1}{p-1}} < \varepsilon, \tag{3.10}$$

where

$$a_j = \frac{\operatorname{cap}_p(E\cap B_j, B_{j-1})}{\operatorname{cap}_p(B_j, B_{j-1})},$$

$r_j = 2^{-j_0-j}$, and $B_j = B(x_0, r_j)$; note that the Wiener sum (3.10) and the Wiener integral (3.1) converge simultaneously. Next, let $D_j = D \cap B_j$ and let $u_j = \hat{R}^1_{D_j}(B_{j-1};\mathscr{A})$ be the $\mathscr{A}$-potential of $D_j$ in $B_{j-1}$. If $S_j$ is the boundary of the ball $\frac{7}{6}B_{j+1}$, then $S_j \subset B_j\setminus\bar{B}_{j+1}$ and

$$\frac{\operatorname{dist}(S_j,\bar{D})}{\operatorname{dist}(S_j,x_0)} \geqslant c > 0. \tag{3.11}$$

By Lemma 3.9 there is $x$ on $S_j$ such that

$$u_j(x) \leqslant c a_j^{\frac{1}{p-1}}. \tag{3.12}$$

Since $u_j$ is $\mathscr{A}$-harmonic in $B_j\setminus\bar{D}_j$, see [HK2], then (3.11), (3.12), and Harnack's inequality yield

$$u_j < c a_j^{\frac{1}{p-1}} = b_j \quad \text{on} \quad S_j,$$

where $c$ is independent of $j$.

To this end, choose $\varepsilon > 0$ in such a way that $\sum_{j=1}^{\infty} b_j < \frac{1}{2}$. We show that $v_0 = u_1$ is the desired $\mathscr{A}$-superharmonic function. Indeed, since $D_1$ is open, $u_1 = 1$ in $D_1$ [HK2], and it remains to show that $u_1(x_0) < 1$.

Let

$$v_1 = \frac{u_1 - b_1}{1 - b_1}$$

and

$$w_1 = \begin{cases} \min(v_1, u_2) & \text{in } \frac{7}{6}B_2 \\ v_1 & \text{in } B_0 \setminus \frac{7}{6}B_2. \end{cases}$$

Since $v_1 < 0$ on $S_1$, $w_1$ is lsc, and it follows from the comparison principle that $w_1$ is $\mathscr{A}$-superharmonic in $B_0$. Since $v_1$ is the minimal $\mathscr{A}$-superharmonic function in $B_0$ lying above the function $\psi_1 = \dfrac{\chi_{D_1} - b_1}{1 - b_1}$, where $\chi_{D_1}$ is the characteristic function of $D_1$, we obtain $w_1 \geqslant v_1$ in $B_0$. In particular, $u_2 \geqslant v_1$ in $\frac{7}{6}B_2$, and hence

$$u_1 - b_1 < u_2 < b_2 \quad \text{on} \quad S_2.$$

We continue in this way; write

$$v_2 = \frac{v_1 - b_2}{1 - b_2}$$

and observe that $v_2$ is the minimal $\mathscr{A}$-superharmonic function lying above $\psi_2 = \dfrac{\psi_1 - b_2}{1 - b_2}$. Therefore the function

$$w_2 = \begin{cases} \min(v_2, u_3) & \text{in } \frac{7}{6}B_3, \\ v_2 & \text{in } \mathrm{B}_0 \setminus \frac{7}{6}\mathrm{B}_3, \end{cases}$$

satisfies $w_2 \geqslant v_2$. Thus

$$v_1 - b_2 < u_3 < b_3 \quad \text{on} \quad S_3$$

or

$$u_1 < b_1 + b_2 + b_3 \quad \text{on} \quad S_3.$$

Repeating this argument, we arrive at the estimate

$$u_1 < \sum_{j=1}^{k} b_j \quad \text{on} \quad S_k,$$

and therefore

$$u_1(x_0) \leqslant \sum_{j=1}^{\infty} b_j < \frac{1}{2}$$

as desired.

This completes the construction for $v_0$ and Theorem 3.2 is thereby proved.

From now on we shall write $\tau_p$ instead of $\tau_{\mathscr{A}}$ and call $\tau_p$ the *p-fine topology*. Similarly we use the expressions *p-fine neighborhood*, *p-fine limit*, etc. As mentioned earlier, it follows from Theorem 3.2 that the fine topology $\tau_p$ coincides with the $(1,p)$-fine topology introduced in [M].

It was proved in [HK3, Section 4] that if $p > n - 1$, then a set $E$ is $p$-thin at $x_0$ if and only if there is an $\mathscr{A}$-superharmonic function $u$ defined in a neighborhood of $x_0$ satisfying (3.8). Thus we obtain Cartan's theorem at least for $p > n - 1$, cf. [D, p. 168].

3.13. THEOREM. – *Let $E$ be a set in $\mathbb{R}^n$, $x_0 \in \bar{E} \setminus E$, and $p > n - 1$. Then $x_0$ is not a p-fine limit point of $E$ if and only if there is an $\mathscr{A}$-superharmonic function $u$ in a neighborhood of $x_0$ such that*

$$\liminf_{\substack{x \to x_0 \\ x \in E}} u(x) > u(x_0).$$

It was conjectured in [HK3] that the above mentioned characterization of $p$-thin points via $\mathscr{A}$-superharmonic functions is valid for all $p \in (1,n]$.

3.14. *Remarks.* – (*a*) It follows from Theorem 3.2 that no non-empty $p$-finely open set has zero $p$-capacity; what is more, each non-empty $p$-finely open set has positive (outer) $n$-measure. In particular, no countable set is $p$-finely open and each countable set is $p$-finely closed. Therefore

(i) $(\mathbb{R}^n, \tau_p)$ is not separable;

(ii) a sequence $x_j$ converges to $x$ in the $p$-fine topology if and only if $x_j = x$ but finitely many $j$;

(iii) no point $x$ in $\mathbb{R}^n$ has a countable neighborhood base in $\tau_p$ whence $(\mathbb{R}^n, \tau_p)$ is not metrizable.

(*b*) Another consequence of Theorem 3.2 is that the $p$-fine topology $\tau_p$ is strictly contained in the *density topology* of $\mathbb{R}^n$; it follows, in particular, that each euclidean domain is $p$-finely connected, cf. [AL].

(*c*) If $p = 2$, then $\mathscr{A}$-superharmonic functions generate the classical fine topology of Cartan regardless of the operator $\mathscr{A}$; note that the equation (1.1) may still be nonlinear. For a proof in the classical linear case see [LSW].

Theorem 3.2 and the Kellogg property imply the inclusion relations among $p$-fine topologies. The following result also follows from Theorem A in [AH].

3.15. THEOREM. – *If $p < q$, then $\tau_p$ strictly includes $\tau_q$.*

*Proof.* – We first show that $\tau_p \backslash \tau_q$ is not empty. Let $K$ be a compact set such that $\operatorname{cap}_p K = 0$ and $\operatorname{cap}_q K > 0$. By the Kellogg property, see [HW], there is a point $x_0 \in K$ such that $K\backslash\{x_0\}$ is not $q$-thin at $x_0$. Thus $\complement(K\backslash\{x_0\})$ is not a $q$-fine neighborhood of $x_0$ although it is trivially a $p$-fine neighborhood of $x_0$.

That $\tau_p$ includes $\tau_q$ is an immediate consequence of Theorem 3.2 and the following elementary lemma.

3.16. LEMMA. – *Let $1 < p < q \leqslant n$, $r > 0$, and $E \subset B = B(x,r)$. Then there is a constant $c = c(n,p,q) > 0$ such that*

$$\left(\frac{\operatorname{cap}_p(E,2B)}{\operatorname{cap}_p(B,2B)}\right)^{\frac{1}{p-1}} \leqslant c\left(\frac{\operatorname{cap}_q(E,2B)}{\operatorname{cap}_q(B,2B)}\right)^{\frac{1}{q-1}}.$$

*Proof.* – We may clearly assume that $E$ is compact. Let $u \in C_0^\infty(2B)$ such that $u \geqslant 1$ on $E$. Then Hölder's inequality yields

$$\int_{2B} |\nabla u|^p \, dx \leqslant c r^{\frac{n(q-p)}{q}} \left(\int_{2B} |\nabla u|^q \, dx\right)^{\frac{p}{q}}.$$

Since $\operatorname{cap}_s(B,2B) = c r^{n-s}$, $1 < s \leqslant n$, we obtain

$$\left(\frac{\operatorname{cap}_p(E,2B)}{\operatorname{cap}_p(B,2B)}\right)^{\frac{1}{p-1}} \leqslant c\left(\frac{\operatorname{cap}_q(E,2B)}{\operatorname{cap}_q(B,2B)}\right)^{\frac{p}{q(p-1)}}$$

$$\leqslant c\left(\frac{\operatorname{cap}_q(E,2B)}{\operatorname{cap}_q(B,2B)}\right)^{\frac{1}{q-1}}$$

as desired.

We close this section by giving an alternative characterization for $p$-finely continuous functions, cf. [D, p. 179]. Let $E^f = E^f_p$ be the set of $p$-fine limit points of a set $E$ in $\mathbb{R}^n$. Then, by Corollary 3.4,

$$E^f = \{x \in \mathbb{R}^n : E \text{ is not } p\text{-thin at } x\} \subset \bar{E}$$

and for any $x_0 \in \mathbb{R}^n$,

$$E^f = (E \setminus \{x_0\})^f .$$

3.17. THEOREM. — *Suppose that* $E \subset \mathbb{R}^n$, $x_0 \in E^f \setminus E$, *and* $g : E \to \bar{\mathbb{R}} = [-\infty, +\infty]$. *Then*

$$\tau_p - \lim_{\substack{x \to x_0 \\ x \in E}} g(x) = \lambda$$

*if and only if there is a p-fine neighborhood* $V$ *of* $x_0$ *such that*

$$\lim_{\substack{x \to x_0 \\ x \in E \cap V}} g(x) = \lambda .$$

*Proof.* — For convenience we assume that $\lambda \in \mathbb{R}$. If

$$\tau_p - \lim_{\substack{x \to x_0 \\ x \in E}} g(x) = \lambda ,$$

it follows from Theorem 3.2 that the set

$$E_j = \left\{ x \in E : |g(x) - \lambda| \geqslant \frac{1}{j} \right\},$$

$j = 1, 2, \ldots$, is $p$-thin at $x_0$. It is easily seen that there is a sequence of positive numbers $r_j$ such that

$$E_\infty = \bigcup_{j=1}^{\infty} (E_j \cap B(x_0, r_j))$$

is $p$-thin at $x_0$, cf. [M, Proposition 3.1]. The set $V = \complement E_\infty$ is the desired $p$-fine neighborhood of $x_0$ since

$$\lim_{\substack{x \to x_0 \\ x \in E \cap V}} g(x) = \lambda .$$

To prove the converse, fix $\varepsilon > 0$. Then there is $\delta > 0$ such that $|g(x) - \lambda| < \varepsilon$ whenever $x \in E \cap V \cap B(x_0, \delta)$. Since $V \cap B(x_0, \delta)$ is a $p$-fine neighborhood of $x_0$, the assertion follows.

A function $g$ is called *p-continuous* at a point $x_0 \in \mathbb{R}^n$ if there is a set $E \subset \mathbb{R}^n \setminus \{x_0\}$ such that $E$ is $p$-thin at $x_0$ and $g|_{\complement E}$ is continuous at $x_0$. Now Theorem 3.17 implies

3.18. COROLLARY. – *A function $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ is p-finely continuous at $x_0 \in \mathbb{R}^n$ if and only if $g$ is p-continuous at $x_0$.*

## 4. Arcwise connectedness of *p*-fine topology.

The following important result is due to D. R. Adams and J. L. Lewis [AL] : *If $U$ is p-finely open and $x_0 \in U$, then there is a p-finely open neighborhood $V$ of $x_0$ such that any two points $x,y \in V$ can be joined by a coordinate path $\gamma$ in $U$ of length at most $c|x-y|$, $c = c(n,p) > 0$. In particular, each p-finely open and p-finely connected set is arcwise connected.*

By a coordinate path we mean a countable union of (possibly degenerated) line segments parallel to the coordinate axes.

Adams and Lewis, in their proof, worked with Bessel capacities, but their ingenious proof could be written in terms of $p$-capacities and $\mathscr{A}$-potentials. Unfortunately, the proof in [AL] leaves it open whether the path $\gamma$ can omit the central point $x_0$. This property is required in Section 5.

4.1. LEMMA. – *Suppose that $U$ is p-finely open and that $x_0 \in U$. Then there is a p-finely open neighborhood $V$ of $x_0$ such that any two points $x,y \in V \setminus \{x_0\}$ can be joined by a coordinate path $\gamma$ in $U \setminus \{x_0\}$ of length at most $c|x-y|$, $c = c(n,p) > 0$.*

*Proof.* – Let $V$ be a $p$-finely open neighborhood of $x_0$ in $U$ with the aforementioned coordinate path property of [AL]. Pick $x, y \in V \cap B(x_0,r_0)$ where $r_0 > 0$ will be determined later. We may suppose that $x_0 = 0$ and $|x| \leqslant |y|$. Let $L_1$ be the line segment from $y$ to $0$ and let $z$ be the point on $L_1$ with $|z| = |x|$. Let $L_2$ be the shortest arc from $z$ to $x$ on the sphere $\partial B(0,|x|)$; then $L_2$ is in the two dimensional plane spanned by $x$, $y$ and $0$. Note that we may have $z=y$ or $z=x$. Choose points $z_0=y$ ; $z_1, \ldots, z_k=z$ ; $z_{k+1}, \ldots, z_{\ell+1}=x$ on $L_1 \cup L_2$ and positive numbers $\rho_0, \rho_1, \ldots, \rho_{\ell+1}$ inductively as follows : Let $z_0 = y$ and always set $\rho_j = \dfrac{|z_j|}{4c}$ where $c$ is the constant of [AL].

If $z_0, \ldots, z_j$ are chosen then either $z \in B(z_j, \rho_j)$ or $z \notin B(z_j, \rho_j)$ ; in the first case set $z_{j+1} = z$ and in the latter case let $z_{j+1}$ be the point in $L_1$ with $|z_{j+1}| = |z_j| - \rho_j$. We eventually obtain $z = z_k$ for some integer $k \geqslant 0$. If $x \in B(z_k, \rho_k)$, then set $z_{k+1} = x$ and stop ; in this case $\ell = k$. If $x \notin B(z_k, \rho_k)$, then for $j = 1, 2, \ldots$ let $z_{k+j}$ be the point on $L_2$ with $|z_{k+j} - z_{k+j-1}| = \rho_{k+j-1}$. Let $z_\ell$ be the first point with $x \in B(z_\ell, \rho_\ell)$ ; then set $z_{\ell+1} = x$ and stop.

We shall next show that if $r_0$ is originally chosen small enough, depending only on $x_0$ and $V$, then each ball $B_j = B(z_j, \rho_j)$, $j = 0, \ldots, \ell + 1$, contains points from $V$. Indeed, if $B_j \subset \complement V$, then

$$(4.2) \qquad \frac{|\complement V \cap B(0, 2|z_j|)|}{|B(0, 2|z_j|)|} \geqslant \frac{|B_j|}{|B(0, 2|z_j|)|} \geqslant c > 0$$

where $c$ depends only on $n$ and $p$. On the other hand, since $\complement V$ is $p$-thin at $0$, the $n$-density of $\complement V$ is zero at $0$ ; hence choosing $r_0 \geqslant |z_j|$ small enough contradicts (4.2). Thus $B_j \cap V \neq \varnothing$ for each $j = 0, \ldots, \ell + 1$.

To this end, pick $w_j \in B_j \cap V, j = 0, \ldots, \ell + 1, w_0 = y, w_{\ell+1} = x$ and let $\gamma_j$ be a coordinate path joining $w_j$ and $w_{j+1}$ in $U$ such that *length* $(\gamma_j) \leqslant c|w_j - w_{j+1}|$. Since $|w_j - w_{j+1}| < 2\rho_j + \rho_{j+1} < \frac{1}{c}(|w_j| + |w_{j+1}|)$, we see that $\gamma_j$ does not go through $0$. Thus $\gamma = \bigcup_{j=0}^{\ell} \gamma_j$ is a coordinate path which joins $y$ and $x$ in $U \backslash \{x_0\}$ with

$$\text{length}\,(\gamma) \leqslant c \sum_{j=0}^{\ell} |w_j - w_{j+1}| \leqslant c|L_1 \cup L_2| \leqslant c|x - y| .$$

This completes the proof of the lemma.

4.3. *Remarks.* — (*a*) If $p \leqslant n - 1$, it is not true that each arcwise connected $p$-finely open set is $p$-finely connected ; the example of B. Fuglede [F3] can be easily modified to cover all values $1 < p \leqslant n - 1$. It can be shown that an arcwise connected $p$-finely open set is $p$-finely connected for $n - 1 < p \leqslant n$ ; for $p = n = 2$ see [F3].

(*b*) Let $U$ be a $p$-fine neighborhood of $x_0$ and suppose that $\Omega = U \backslash \{x_0\}$ is open in the usual topology. Then the $p$-finely open neighborhood $V$ of $x_0$ in Lemma 4.1 can be chosen so that $V \backslash \{x_0\}$ is, in addition, open.

To see this let $V$ first be a $p$-finely open neighborhood of $x_0$ given by Lemma 4.1. Put

$$V' = \bigcup_{x \in V \setminus \{x_0\}} B(x, d(x, \partial\Omega)/2).$$

Then $V' \subset \Omega$ is open and $V' \cup \{x_0\} \supset V$; hence $V' \cup \{x_0\}$ is a $p$-fine neighborhood of $x_0$. To prove the coordinate path property of $V'$ let $z_1, z_2 \in V'$. Pick $x_j \in V \setminus \{x_0\}$ such that $z_j \in B(x_j, d_j/2)$, $d_j = d(x_j, \partial\Omega)$, $j = 1, 2$. Let $\gamma'$ be a coordinate path joining $x_1$ to $x_2$ in $\Omega$ with *length* $(\gamma') \leqslant c|x_1 - x_2|$ where $c$ is the constant of Lemma 4.1. Suppose first that $\max(d_1, d_2) \leqslant 2|z_1 - z_2|$. Let $\gamma$ be a coordinate path from $z_1$ to $z_2$ consisting of $\gamma'$ and of two coordinate paths $\gamma_j$ from $x_j$ to $z_j$ in $B(z_j, d_j/2)$ with *length* $(\gamma_j) \leqslant \sqrt{n}|x_j - z_j|$, $j = 1,2$. Then

$$\text{(4.4)} \quad \begin{aligned} \textit{length}\,(\gamma) &\leqslant c(|x_1 - z_1| + |z_1 - z_2| + |x_2 - z_2|) + \sqrt{n}(d_1 + d_2)/2 \\ &\leqslant (3c + 2\sqrt{n})|z_1 - z_2|. \end{aligned}$$

Hence $\gamma$ has the desired property in this case. If $\max(d_1, d_2) > 2|z_1 - z_2|$, then we may assume that $d_1 = \max(d_1, d_2)$. Now $z_1, z_2 \in B(x_1, d_1)$ and hence $z_1$ and $z_2$ can be joined in $\Omega$ by a coordinate path $\gamma$ with *length* $(\gamma) \leqslant \sqrt{n}|z_1 - z_2|$. Thus in both cases $z_1$ and $z_2$ can be joined in $\Omega$ so that (4.4) holds.

(*c*) Let $U \bigcup \{x_0\}$ be a $p$-finely open neighborhood of $x_0$. Then there exists a unique euclidean component $U_0$ of $U$ such that $U_0 \cup \{x_0\}$ is a $p$-finely open neigborhood of $x_0$. To prove the existence of $U_0$ let $V$ be a $p$-finely open neighborhood of $x_0$ in $U$ with the coordinate path property. Choose $x_1 \in V \setminus \{x_0\}$ and let $U_0$ be the $x_1$-component of $U$. Since $U_0 \cup \{x_0\} \supset V$, $U_0 \cup \{x_0\}$ is a $p$-finely open neighborhood of $x_0$. The uniqueness is due to the fact that $\{x_0\}$ is not $p$-finely open.

Note that if $p > n - 1$, then $\{x_0\}$ is component of $\complement(U_0 \setminus \{x_0\})$ because there is a sequence of radii $r_i \to 0$ such that $\partial B(x_0, r_i) \subset U_0$, cf. [HK, 3.4]. For $p \leqslant n - 1$, this is not true.

## 5. Fine limits of QR and BLD maps.

In this section we study the behavior of maps of bounded length distortion and quasiregular maps at thin boundary points.

5.1. *BLD and QR maps.* – Let $\Omega$ be an open set in $\mathbb{R}^n$. A continuous map $f : \Omega \to \mathbb{R}^n$ *is quasiregular*, abbreviated *QR*, if $f$ is in loc $W^1_n(\Omega)$ and $|f'(x)|^n \leqslant KJ(x,f)$ a.e. in $\Omega$ for some $K \geqslant 1$; $f$ is of *bounded length distortion*, abbreviated *BLD*, if $f$ is in loc $W^1_1(\Omega)$, for some $L \geqslant 1$ and a.e. $x \in \Omega$, $|h|/L \leqslant |f'(x)h| \leqslant L|h|$ for all $h \in \mathbb{R}^n$ and $J(x,f) \geqslant 0$. Here $J(x,f)$ is the Jacobian determinant and $f'(x)$ is the (formal) derivative of $f$ at $x$. A homeomorphic *QR* map of $\Omega$ onto $f(\Omega)$ is called *quasiconformal.* Note that a *BLD* map is *QR* but the converse is not true.

The classes of *QR* and *BLD* maps are closely connected to the quasilinear operators $\mathscr{A}$ of Section 2. Let $\mathfrak{U}_p$, $1 < p \leqslant n$, be the class of operators $\mathscr{A}$ satisfying (2.1)-(2.5) for some $0 < \alpha \leqslant \beta < \infty$. If $\mathscr{A} \in \mathfrak{U}_p$, then a *BLD* map $f : \Omega \to \mathbb{R}^n$ induces an operator $f^\sharp\mathscr{A}$ in $\Omega \times \mathbb{R}^n$; this operator can be extended to $\mathbb{R}^n \times \mathbb{R}^n$ and the extended operator, denoted again by $f^\sharp\mathscr{A}$, belongs to $\mathfrak{U}_p$. If $f : \Omega \to \mathbb{R}^n$ is *QR* and if $\mathscr{A} \in \mathfrak{U}_n$, then $f^\sharp\mathscr{A} \in \mathfrak{U}_n$. In both cases $u \circ f$ is $f^\sharp\mathscr{A}$-harmonic or $f^\sharp\mathscr{A}$-superharmonic whenever $u$ is $\mathscr{A}$-harmonic or $\mathscr{A}$-superharmonic, respectively. For these and other properties of *BLD* and *QR* maps see [GLM1] and [MV].

5.2. *Fine limits of BLD maps.* – A function $u$ of $\Omega$ into $\mathbb{R}^m$, $m \geqslant 1$, is said to be locally *L-lipschitzian* if

$$|u(x)-u(y)| \leqslant L|x-y| \tag{5.3}$$

whenever the line segment from $x$ to $y$ lies in $\Omega$.

5.4. LEMMA. – *Suppose that $u : \Omega \to \mathbb{R}^m$ is locally L-lipschitzian. If $\complement\Omega$ is p-thin at $x_0 \in \partial\Omega$, then $u$ has a p-fine limit at $x_0$.*

*Proof.* – Let $U = \Omega \cup \{x_0\}$. Then $U$ is a $p$-fine neighborhood of $x_0$ and by Lemma 4.1 there is a $p$-finely open neighborhood $V \subset U$ of $x_0$ such that any two points $x, y \in V\backslash\{x_0\}$ can be joined by a coordinate path $\gamma \subset \Omega$ of length at most $c|x-y|$. Let $\varepsilon > 0$ and let $x, y \in B\left(x_0, \dfrac{\varepsilon}{2cL}\right) \cap (V\backslash\{x_0\})$. Choose such a coordinate path $\gamma$ joining $x$ to $y$ in $\Omega$. Then $\gamma = \cup\gamma_i$ where each $\gamma_i = [x_i, y_i]$ is a line segment from $x_i$ to $y_i$ in $\Omega$. Thus (5.3) yields

$$\begin{aligned}|u(x)-u(y)| &\leqslant \Sigma|u(x_i)-u(y_i)| \\ &\leqslant L\Sigma|x_i-y_i| \leqslant cL|x-y|\end{aligned}$$

and since $x, y \in B(x_0, \varepsilon/2cL)$, we obtain $|u(x)-u(y)| < \varepsilon$. Hence $u(x)$ has a limit as $x$ approaches $x_0$ in $V$. The lemma follows.

By [MV, Lemma 2.3] a *BLD* map is locally lipschitzian. Hence Lemma 5.4 yields.

5.5. THEOREM. — *Suppose that $f\colon \Omega \to \mathbb{R}^n$ is a BLD map and that $x_0 \in \partial\Omega$. If $\complement\Omega$ is p-thin at $x_0$, then $f$ has a p-fine limit at $x_0$.*

5.6. *Remarks.* — (*a*) Let $u : \Omega \to \mathbb{R}^m$ be a locally lipschitzian function and suppose that $\complement\Omega$ is $p$-thin at $x_0 \in \partial\Omega$. By Lemma 5.4 and Theorem 3.17 there is a $p$-fine neighborhood $V$ of $x_0$ such that

$$\lim_{\substack{x \to x_0 \\ x \in V}} u(x) = a \in \mathbb{R}^n$$

exists. By Remark 4.3., $V \setminus \{x_0\}$ can be chosen to be a domain in $\Omega$.

(*b*) Let $\complement\Omega$ be $p$-thin at $x_0 \in \partial\Omega$. A *BLD* map $f\colon \Omega \to \mathbb{R}^n$, and hence a locally lipschitzian map, need not have a limit at $x_0$. Simple examples exist due to the fact that $\Omega$ can have a sequence of components $\Omega_i$ converging to $x_0$. Here we show that the limit need not exist even if $\Omega$ is a domain. We produce an example for $n = 2$; there are similar examples in all dimensions, see also Remark 5.11. Let $\alpha : [0,\infty) \to \bar{Q}$ be a half open $C^2$-arc of infinite length in the closure of the cube $Q = \{x \in \mathbb{R}^2 : |x_j| < 1, j=1,2\}$. Assume, furthermore, that $\alpha$ is parametrized by arc length and that $\alpha(0) = -e_1$, $\alpha'(0) = e_1$, $\alpha(t) \in Q$ for $t \in (0,\infty)$ and $\alpha(t) \to 0$ as $t \to \infty$. For $x \in \alpha$ let $N(x)$ be the line through $x$, orthogonal to $\alpha$. Then it is not difficult to see that there is a neighborhood $U$ of $\alpha$ such that if $\tilde{N}(x)$ is the $x$-component of $N(x) \cap U$, then

(i) $Q \cap U = \bigcup_{t>0} \tilde{N}(\alpha(t))$

and

(ii) $\tilde{N}(x) \cap \tilde{N}(y) = \varnothing$ for $x \neq y$.

Write $G = U \cup (\mathbb{R}^2 \setminus \bar{Q})$ and define a continuous map $g\colon G \to \mathbb{R}^2$ as follows: $g(x) = x$ for $x \in \mathbb{R}^2 \setminus \bar{Q}$ and for $x \in \alpha$, $g|_{\tilde{N}(x)}$ is an isometry into the line orthogonal to $x_1$-axis at the point $(t-1)e_1$; here $t \geqslant 0$ is such that $x = \alpha(t)$. Then $g$ is a *BLD* map which is unbounded in $U$. To complete the construction choose points $x^j$ on the positive real axis and numbers $r_j > 0$ such that

(*a*) $x^j \to 0$, $r_j \to 0$,

(*b*) the cubes $\bar{Q}_j$ centered at $x^j$ and of side length $2r_j$ are disjoint and

(*c*) the set $\cup \bar{Q}_j$ is *p*-thin at 0.

Fix $j$ and let $\varphi_j(x) = r_j(x+x^j)$. Write $U_j = \varphi_j(U)$. Then

$$\Omega = \mathbb{R}^2\setminus\Big(\{0\} \cup \bigcup_j (\bar{Q}_j\setminus U_j)\Big)$$

is a domain and $\mathbb{R}^2\setminus\Omega$ is *p*-thin at $0 \in \partial\Omega$. The map $f\colon \Omega \to \mathbb{R}^2$,

$$f(x) = \begin{cases} x, & x \in \mathbb{R}^2\setminus \cup \bar{Q}_j \\ \varphi_j \circ g \circ \varphi_j^{-1}(x), & x \in \bar{Q}_j \cap U_j, \end{cases}$$

is *BLD*, has a *p*-fine limit 0 at 0 but does not have a limit 0 since $f$ is unbounded at every neighborhood of 0 in $\Omega$.

If $f$ is a *BLD* homeomorphism in a domain and if $p > n - 1$, then Theorem 5.5 can be strengthened. We first prove a lemma.

5.7. LEMMA. — *Suppose that $G$ is a domain in $\mathbb{R}^n$ such that $\complement G$ is p-thin at $x_0 \in \partial G$ and that $f$ is a homeomorphism of $G$ into $\mathbb{R}^n$ having a p-fine limit at $x_0$. If $p > n - 1$, then $f$ has a limit at $x_0$.*

*Proof.* — By Theorem 3.17 there is a *p*-fine neighborhood $V$ of $x_0$ such that

$$\lim_{\substack{x\to x_0 \\ x \in V\setminus\{x_0\}}} f(x) = a. \tag{5.8}$$

We may assume that $a \in \mathbb{R}^n$. Since $p > n - 1$, [HK3, 3.4] implies that there is a sequence of radii $r_i \searrow 0$ such that $\partial B(x_0, r_i) \subset V$. Write $E_i = f\partial B(x_0, r_i)$ and let $F_i$ be the unbounded component of $\complement E_i$. If $F_{j+1} \supset F_j$ for some $j$, then

$$F_{i+1} \supset F_i, \qquad i \geqslant j. \tag{5.9}$$

To see this note that since $f$ is a homeomorphism in the domain $G$, $E_{j+1}$ separates $E_j$ and $E_{j+2}$ in $\mathbb{R}^n$. Thus $E_{j+2} \subset \complement F_{j+1}$; this implies $F_{j+2} \supset F_{j+1}$ and we can proceed by induction to obtain (5.9) for all $i \geqslant j$.

Next we show that for some $j$, $F_{j+1} \supset F_j$. Note first that $F_{i+1} \subset F_i$ for all $i$ is impossible because by (5.8), diam $(E_i) \to 0$. Pick $k$ such that $F_{k+1}$ is not included in $F_k$. If $F_{k+1} \supset F_k$ we may take $j = k$.

Suppose that $F_{k+1} \not\supset F_k$. Now $E_{k+1} \subset F_k$ and $E_k \subset F_{k+1}$ and hence $E_{k+2}$ must lie in $\complement F_{k+1}$ because $E_{k+1}$ separates $E_k$ and $E_{k+2}$ in $\mathbb{R}^n$. Thus $F_{k+2} \supset F_{k+1}$ and we may choose $j = k + 1$.

The inclusion (5.9) means that $f(G \cap B(x_0, r_i)) \subset \complement F_i$ for $i \geqslant j$ and hence $\operatorname{diam}(f(G \cap B(x_0, r_i))) = \operatorname{diam}(E_i)$. Since $\operatorname{diam}(E_i) \to 0$, this implies that $f(x) \to a$ as $x$ tends to $x_0$ in $G$.

Now Theorem 5.5 and Lemma 5.7 yield.

5.10. THEOREM. — *Suppose that $f$ is a BLD homeomorphism of a domain $G$ into $\mathbb{R}^n$ and that $\complement G$ is $p$-thin at $x_0 \in \partial G$. If $p > n - 1$, then $f$ has a limit at $x_0$.*

5.11. *Remark.* — If $p \leqslant n - 1$ and if $\complement G$ is $p$-thin at $x_0 \in \partial G$, then $\complement G$ can contain a set similar to the well known Lebesgue spine. Now a slightly modified construction of Remark 5.6. (*b*) can be used to produce a *BLD* homeomorphism of a domain $G$ into $\mathbb{R}^n$ without limit at $x_0$. Thus the assumption $p > n - 1$ is necessary in Theorem 5.10.

5.12. *Fine limits of QR maps.* — A *QR* map $f: \Omega \to \mathbb{R}^n$ need not have an $n$-fine limit at a point $x_0 \in \partial\Omega$ where $\complement\Omega$ is $n$-thin. A counterexample is the plane analytic function $f(z) = e^{1/z}$ in $\Omega = B(0,1)\backslash\{0\}$ with $x_0 = 0$. Similar examples exist in all dimensions. However, if $f$ omits a set of positive $n$-capacity, then the situation is different. The following theorem was recently proved by B. Fuglede for plane analytic functions [F4].

5.13. THEOREM. — *Suppose that $f: \Omega \to \mathbb{R}^n$ is a QR map and that $\complement\Omega$ is $n$-thin at $x_0 \in \partial\Omega$. If the set $\complement f(\Omega)$ has positive $n$-capacity, i.e. it is not $n$-polar, then $f$ has an $n$-fine limit at $x_0$.*

Contrary to the *BLD* cases studied in Theorem 5.5 and 5.10, the $n$-fine limit of $f$ at $x_0$ in Theorem 5.13 may be $\infty$. Hence we shall use the compactified space $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ endowed with the spherical metric

$$q(a,b) = |a-b|(1+|a|^2)^{-\frac{1}{2}}(1+|b|^2)^{-\frac{1}{2}}$$

for $a \neq \infty \neq b$ and $q(a,\infty) = (1+|a|^2)^{-\frac{1}{2}}$.

We recall that a closed proper subset $F$ of $\mathbb{R}^n$ has positive $n$-capacity, abbreviated $\operatorname{cap}_n F > 0$, if $\operatorname{cap}_n(E, \complement F) > 0$ for some non-degenerate continuum $E$ in $\complement F$, cf. [MRV2, p. 6]. We need the following lemma.

5.14. LEMMA [MRV2, 3.11]. – *Suppose that $E$ is a closed proper subset of $\mathbb{R}^n$ with $\operatorname{cap}_n E > 0$. Then for every $\varepsilon > 0$ there is $\delta > 0$ such that $\operatorname{cap}_n(F, \complement E) < \delta$ implies $q(F) > \varepsilon$ whenever $F$ is a continuum in $\complement E$.*

For Theorem 5.13 we still need a lemma.

5.15. LEMMA. – *Suppose that $\complement\Omega$ is $n$-thin at $x_0 \in \partial\Omega$. Then there is a domain $D$ in $\Omega$ and a sequence of positive numbers $r_i \to 0$ such that*

(i) *$D \cup \{x_0\}$ is an $n$-fine neighborhood of $x_0$,*

(ii) *$D_i = B(x_0, r_i) \cap D$ is connected, and*

(iii) *for each $i = 1, 2, \ldots$ and every continuum $F \subset D_i$,*

$$\operatorname{cap}_n (F, \Omega) < 1/i.$$

*Proof.* – We first choose a domain $\Omega_0 \subset \Omega$ such that $\Omega_0 \cup \{x_0\}$ is an $n$-fine neighborhood of $x_0$, cf. Remark 4.3. (*c*). By enlarging $E = \complement\Omega_0$ slightly we may assume that $E$ is not $n$-thin at any $x \in E \backslash \{x_0\}$, see [LM, 3.5]. It follows from Theorem 3.13 that there are $0 < r_1 < r_0$ such that

$$u(x_0) < 1$$

where $u$ is the $n$-potential (i.e. the $\mathcal{A}$-potential for the $n$-harmonic operator, $\mathcal{A}(x,h) = |h|^{n-2}h)$ of $B(x_0, r_1) \cap E$ in $B = B(x_0, r_0)$. Furthermore, we may assume that $\partial B \subset \Omega_0$. Choose $\gamma$ such that $u(x_0) < \gamma < 1$ and let

$$D = \{x \in B \cap \Omega_0 : u(x) < \gamma\}.$$

Then $D \subset \Omega_0$ is open and $D \cup \{x_0\}$ is $n$-finely open. We show that $D$ is connected: let $D'$ be the component of $D$ which touches $\partial B$. If $D''$ is another component of $D$, then $D'' \subset\subset B$. Define

$$\tilde{u}(x) = \begin{cases} u(x), & x \in B \cap \Omega_0 \backslash D'' \\ \gamma, & x \in D''; \end{cases}$$

it follows from the comparison principle that $\tilde{u}$ is $n$-subharmonic in $B \cap \Omega_0$ since $u$ is $n$-harmonic there. Moreover, since $u = 1$ in $B(x_0, r_0) \cap E \backslash \{x_0\}$, $u - \tilde{u} \in W^1_{n,0}(B \cap \Omega_0)$ whence [HK1, 2.7] implies that $u = \tilde{u}$ in $B \cap \Omega_0$. In particular, $D'' = \emptyset$ and $D$ is therefore connected.

To complete the proof choose a sequence of radii $r_i \to 0$ such that $\partial B_i \subset D$ where $B_i = B(x_0, r_i)$, cf. [LM, 3.16] or [HK3, 3.4]. Fix $\varepsilon > 0$.

Since

$$\int_B |\nabla u|^n \, dm < \infty ,$$

there is an integer $i = i(\varepsilon)$ such that $B_i \subset B$ and

$$\int_{B_i} |\nabla u|^n \, dm < 2^{-n-1}(1-\gamma)^n \varepsilon. \tag{5.16}$$

Next choose $k = k(\varepsilon) > i$ so large that

$$\operatorname{cap}_n (B_k, B_i) < 2^{-n-1}\varepsilon . \tag{5.17}$$

Define

$$v = \min\left(\frac{1-u}{1-\gamma}, w\right)$$

where $w$ is the $n$-potential of $B_k$ in $B_i$, i.e.

$$w(x) = \begin{cases} 1 - \dfrac{\log\left(\dfrac{|x-x_0|}{r_k}\right)}{\log\left(\dfrac{r_i}{r_k}\right)}, & x \in B_i \backslash B_k \\ 1, \quad x \in B_k . \end{cases}$$

Let $F$ be a continuum in $D \cap B_k$. Then $v \in W^1_{n,0}(B_i)$ is continuous in $B_i \backslash \{x_0\}$ and $v \geqslant 1$ in $F$. Thus it follows from (5.16) and (5.17) that

$$\begin{aligned} \operatorname{cap}_n (F,\Omega) &\leqslant \operatorname{cap}_n (F, B_i \cap \Omega_0) \\ &\leqslant \int_{B_i \cap \Omega_0} |\nabla v|^n \, dm \\ &\leqslant (1-\gamma)^{-n} 2^n \int_{B_i} |\nabla u|^n \, dm + 2^n \int_{B_i} |\nabla w|^n \, dm \\ &< \varepsilon . \end{aligned}$$

Choosing inductively an increasing sequence of integers $k_i$ corresponding to $k(\varepsilon)$ for $\varepsilon = 1/i$ and putting $D_i = B_{k_i} \cap \Omega_0$ completes the proof since $\partial B_{k_i} \subset D$.

*Proof for Theorem* 5.13. — Let $D$ and $r_i \to 0$ be as in Lemma 5.15. We may assume that $f$ is nonconstant in $D$. Since the sets $D_i$ are connected the cluster set of $f$ at $x_0$ along $D$,

$$C(f,x_0,D) = \bigcap_i \overline{f(D_i)},$$

is a compact and connected set of $\overline{\mathbb{R}}^n$; here the closures of $f(D_i)$ are taken in $\overline{\mathbb{R}}^n$. Let $\varepsilon > 0$. Since $E = \mathbb{R}^n \backslash f(D)$ is a proper closed subset of $\mathbb{R}^n$ with $\mathrm{cap}_n E > 0$, Lemma 5.14 gives $\delta > 0$ such that $\mathrm{cap}_n(F',f(D)) < \delta$ implies $q(F') < \varepsilon$ whenever $F'$ is a continuum in $f(D)$.

Let $K_I(f)$ denote the inner dilatation of $f$, see [MRV1, p. 14], and fix $i$ such that $K_I(f)/i < \delta$. Let $F$ be a continuum in $D_i$. Then $F' = f(F)$ is a continuum in $f(D)$ and the fundamental capacity inequality [MRV1, 7.1] and Lemma 5.15 yield

$$\mathrm{cap}_n(F',f(D)) \leqslant K_I(f)\,\mathrm{cap}_n(F,D) \leqslant \frac{K_I(f)}{i} < \delta .$$

Hence $q(F') < \varepsilon$ and since this holds for all continua $F \subset D_i$ and $D_i$ is connected, we obtain

$$q(\overline{f(D_i)}) \leqslant \varepsilon .$$

This shows that $C(f,x_0,D)$ is a single point $a \in \overline{\mathbb{R}}^n$ and thus $f(x) \to a$ as $x \to x_0$ in $D$. The theorem follows.

5.18. *Remark.* – If $f$ is a quasiconformal map of a domain $G$ into $\mathbb{R}^n$, i.e. a $QR$ homeomorphism, and if $\complement G$ is $n$-thin at $x_0 \in \partial G$, then $f$ has a (not necessarily finite) limit at $x_0$. This was proved in [MS, 4.1]; it follows also easily from Theorem 5.13 and Lemma 5.7.

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