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ON CONTINUOUS FUNCTIONS WITH NO UNILATERAL DERIVATIVES

by Masayoshi HATA

1. Introduction.

It is known that A. S. Besicovitch in 1925 gave the first example of a continuous function $B(x)$ which has nowhere a unilateral derivative finite or infinite by geometrical process. E. D. Pepper [9] has examined this same function $B(x)$, giving a different exposition. The graph of his function is illustrated in Figure 1. Later, A. N. Singh [12, 13] gave the arithmetical definition of $B(x)$ and constructed an infinite class of such non-differentiable functions. On the other hand, A. P. Morse [8] gave an example of a continuous function $f(x)$ satisfying

$$\liminf_{s \to x^\pm} \left| \frac{f(s) - f(x)}{s - x} \right| < \limsup_{s \to x^\pm} \left| \frac{f(s) - f(x)}{s - x} \right| = \infty$$

respectively, for every $x \in (0,1)$, by arithmetical process.

It seems, however, that their methods are somewhat complicated and inappropriate to the study concerning further properties of such functions. In the present paper we shall develop a simple but powerful method to construct and analyze such singular functions by using certain one-dimensional dynamical systems.

The difficulties of finding such functions may be explained by the fact that the set of functions which have nowhere a unilateral derivative finite or infinite is of only the first category in the space of continuous functions (S. Saks [11]), while the set of functions which have nowhere a finite unilateral derivative is of the second category (S. Banach [1], S. Mazurkiewicz [7] and V. Jarnik [5]).

Key-words: Non-differentiable functions - Knot points - Functional equations.
2. Main Result.

To state our main theorem, we need some definitions and notations. We denote, as usual, the upper and lower derivatives at $x$ of a real-valued function $f(x)$ on the right by $D^+f(x)$, $D_+f(x)$ respectively. Similarly the upper and lower derivatives, on the left, are denoted by $D^-f(x)$, $D_-f(x)$ respectively. A point $x$ is said to be a knot point of $f(x)$ provided that

$$D^+f(x) = D^-f(x) = \infty \quad \text{and} \quad D_+f(x) = D_-f(x) = -\infty.$$ 

The set of knot points of $f(x)$ is denoted by Knot $(f)$. For a measurable
set $E$, we denote by $|E|$ the Lebesgue measure of $E$. Our theorem can now be stated as follows:

**Theorem 2.1.** — For any $\alpha \in [0,1)$ and $\varepsilon \in (0,1)$, there exists a continuous function $\psi_{\alpha, \varepsilon}(x)$ defined on the unit interval $I$ satisfying the following properties:

1. $\psi_{\alpha, \varepsilon}(x)$ has nowhere a unilateral derivative finite or infinite;
2. $|\text{Knot}(\psi_{\alpha, \varepsilon})| = \alpha$;
3. $\psi_{\alpha, \varepsilon}(x)$ satisfies Hölder's condition of order $1 - \varepsilon$.

**Remark.** — K. M. Garg [3] has shown that the set of knot points of Besicovitch's function is of measure zero. He also showed that, for every continuous function defined on $I$ which has nowhere a unilateral derivative finite or infinite, the set of points at which the upper derivative on one side is $+\infty$, the lower derivative on the other side is $-\infty$, and the other two derivatives are finite and equal has a positive measure in every subinterval of $I$; therefore the constant $\alpha$ in our theorem cannot be taken to be 1. Note that the set $\text{Knot}(f)$ is of the second category if $f(x)$ is a continuous function which has nowhere a finite or infinite derivative (W. H. Young [14]).

As a corollary, we have immediately

**Corollary 2.2.** — For any $\alpha \in [0,2\pi)$ and $\varepsilon \in (0,1)$, there exists an absolutely convergent cosine Fourier series

$$\psi_{\alpha, \varepsilon}(x) = \sum_{n=0}^{\infty} a_{\alpha, \varepsilon, n} \cos nx$$

satisfying the following properties:

1. $\psi_{\alpha, \varepsilon}(x)$ has nowhere a unilateral derivative finite or infinite;
2. $|\text{Knot}(\psi_{\alpha, \varepsilon}|_{[0,2\pi)})| = \alpha$;
3. $\sum_{n=1}^{\infty} |a_{\alpha, \varepsilon, n}|^2 n^{2-\varepsilon} < \infty$.

For the proof of Theorem 2.1, we shall introduce a symbol space in section 3 and certain functional equations in section 4. The fundamental properties of the solution are investigated in sections 5 and 6. We then prove Theorem 2.1 in section 7 using Cantor sets of positive measure.
3. Preliminaries.

We first divide the unit interval $I$ into $m$ subintervals

$$I_1 = [c_0, c_1], \ I_2 = [c_1, c_2], \ldots, \ I_m = [c_{m-1}, c_m]$$

where $0 = c_0 < c_1 < c_2 < \ldots < c_m = 1$, $m \geq 2$ and define the address $A(x)$ of a point $x \in I$ by setting $A(x) = j$ for $c_{j-1} \leq x < c_j$, $1 \leq j \leq m$ and $A(c_m) = m$. Let $g_j(x)$ be a strictly monotone, either increasing or decreasing, continuous function defined on the subinterval $I_j$ such that $g_j(I_j) = I$ for $1 \leq j \leq m$. Define the sign $\varepsilon_j$ to be either $+1$ or $-1$ according as $g_j$ is monotone increasing or monotone decreasing on $I_j$. We assume, in addition, that $g_1(x)$ and $g_m(x)$ are monotone increasing; so $\varepsilon_1 = \varepsilon_m = +1$.

Let $\Sigma = \{1, 2, \ldots, m\}^\mathbb{N}$ be the one-sided symbol space endowed with the metric

$$d(w, z) = \sum_{n=1}^{\infty} 2^{-n} |w_n - z_n| \quad \text{for} \quad w = (w_n), \ z = (z_n) \in \Sigma.$$ 

It is known that $\Sigma$ is a totally disconnected compact metric space. Let $G(x) = g_{A(x)}(x)$ for brevity. Note that the function $G: I \to I$ is not necessarily continuous. We then define the itinerary $v(x)$ of a point $x \in I$ by setting

$$v(x) = (A_0(x), A_1(x), \ldots, A_n(x), \ldots)$$

where $A_n(x) = A(G^n(x))$ for $n \geq 0$. Put $e_0 = \{0, 1\}$ and define the set $e_{n+1}$ inductively by setting $e_{n+1} = \{0 < x < 1; G(x) \in e_n\}$ for $n \geq 0$. Obviously $\# e_n = (m^{n-1}(m-1))$ for $n \geq 1$. Let $e = \bigcup_{n \geq 0} e_n$. Then it is easily verified that the set of discontinuity points of $v$ is precisely equal to the set $e - e_0$.

Put $\Lambda_0 = \{v(x); x \in e_0\}$. For $N \geq 1$, let $\Lambda_N$ be the set of words $w = (w_n) \in \Sigma$ such that either $w_n = 1$ for $n > N$, $w_N \neq 1$ or $w_n = m$ for $n > N$, $w_N \neq m$. Let $\Lambda = \bigcup_{n \geq 0} \Lambda_n$. Then it is easily seen that for $x \in e - e_0$ there exist the limits

$$\lim_{\varepsilon \to 0 \pm} v(x + \varepsilon) = (A_0(x \pm), A_1(x \pm), \ldots)$$
in $\Lambda - \Lambda_0$ respectively. Note that $v(x)$ is equal to either $v(x^+)$ or $v(x^-)$. Thus the set $\Lambda_n$ consists of the following $2m^n(m-1)$ distinct words:

$$\{v(x^+); x \in e_n\} + \{v(x^-); x \in e_n\}$$

for $n \geq 1$. Therefore we have $\Lambda = \Lambda_0 + \Sigma_+ + \Sigma_-$, where $\Sigma_+ = \{v(x^+); x \in e-e_0\}$ and $\Sigma_- = \{v(x^-); x \in e-e_0\}$.

We assume further that each function $h_j = g_j^{-1}: I \to I_j$ is a contraction; namely the Lipschitz constant

$$\text{Lip}(h_j) = \sup_{x \neq y \in I} \left| \frac{h_j(x) - h_j(y)}{x-y} \right|$$

satisfies $\text{Lip}(h_j) < 1$. Let $\gamma = \max_{1 \leq j \leq m} \text{Lip}(h_j) \in [1/m, 1)$. We then define the mapping $\mu: \Sigma \to I$ by setting

$$\mu(w) = \lim_{n \to \infty} h_{w_1} \circ h_{w_2} \circ \cdots \circ h_{w_n}(I)$$

for $w = (w_n) \in \Sigma$.

Clearly $\mu$ is continuous. Then it follows that $X = \mu(\Sigma)$ is a compact subset of $I$ and satisfies the following equality:

$$X = h_1(X) \cup h_2(X) \cup \cdots \cup h_m(X).$$

It is known that the above equation possesses a unique non-empty compact solution [4, p. 384]; thus we have $\mu(\Sigma) = X = I$, since $h_j(I) = I_j$ for $1 \leq j \leq m$. It also follows that the set $e$ is a dense subset of $I$; therefore the mapping $v$ is one to one.

Let $S_n = \bigcup_{0 \leq j \leq n} e_j$ for $n \geq 1$ and let

$$H_{n,x}(y) = h_{A_0(x)} \circ h_{A_1(x)} \circ \cdots \circ h_{A_{n-1}(x)}(y)$$

for $n \geq 1$ and $x, y \in I$. Obviously $H_{n,x}$ is a contraction satisfying $\text{Lip}(H_{n,x}) \leq \gamma^n$. We first consider an arbitrary point $x \in I - e$. Put $K_{n,x} = H_{n,x}(I)$ for $n \geq 1$. Since $K_{n,x}$ is the connected component of $I - S_n$ containing $x$ and $|K_{n,x}| \leq \gamma^n$, we have

$$\lim_{n \to \infty} K_{n,x} = x;$$

that is, $\mu \circ v(x) = x$. Thus $v$ maps $I - e$ homeomorphically onto
\(v(I-e)\). We next consider an arbitrary point \(x \in e_N\), \(N \geq 1\). Put \(K_{n,x}^+ = H_{n,x+}(I)\) for \(n \geq N\), respectively. Since \(K_{n,x}^+\) are the two consecutive connected components of \(I - S_n\) such that the left end point of \(K_{n,x}^+\) is \(x\) and the right end point of \(K_{n,x}^-\) is also \(x\), we have

\[
\lim_{n \to \infty} K_{n,x}^+ = \lim_{n \to \infty} K_{n,x}^- = x;
\]

so \(\mu \circ v(x) = \mu \circ v(x \pm) = x\). Similarly we can define \(K_{n,0}^+\) and \(K_{n,1}^-\) for \(n \geq 1\); thus \(\mu \circ v(0) = 0\) and \(\mu \circ v(1) = 1\). Then we have

**Lemma 3.1.** \(v(I-e) = \Sigma - \Lambda\); namely, \(w = (w_n) \in v(I-e)\) if and only if

\[
\# \{n \geq 1; w_n \neq 1\} = \infty = \# \{n \geq 1; w_n \neq m\}.
\]

**Proof.** Suppose that \(w = v(x) \in \Lambda\) for some \(x \in I - e\). Since \(v\) is one to one, we have \(v(I-e) \cap v(e) = \phi\); thus \(w \in \Sigma_+ + \Sigma_-\). Hence there exists \(y \in e - e_0\) such that either \(w = v(y+)\) or \(w = v(y-)\). Therefore \(x = \mu \circ v(x) = \mu(w) = \mu \circ v(y \pm) = y\). This contradiction implies that \(\Lambda \cap v(I-e) = \phi\); that is, \(v(I-e) \subset \Sigma - \Lambda\). Thus it suffices to show that \(\Sigma - \Lambda \subset v(I-e)\).

Suppose now that there exists a word \(w = (w_n) \in \Sigma - \Lambda\) such that \(w \notin v(I-e)\). Put \(z = (z_n) \equiv v \circ \mu(w)\). Then it follows that \(w \neq z\). For otherwise, we have \(\mu(w) \in e\); thus, \(w \in v(e) \subset \Lambda\), contrary to \(w \in \Sigma - \Lambda\). Let \(N \geq 1\) be the smallest integer such that \(w_N \neq z_N\). Since \(\mu(w) = \mu \circ v \circ \mu(w) = \mu(z)\), it follows that

\[
h_{w_N} \circ h_{w_{N+1}} \circ \ldots = h_{z_N} \circ h_{z_{N+1}} \circ \ldots, \text{ say } p.
\]

Then we have \(p \in e_1\) and \(w, z \in \Lambda_N\), contrary to \(w \in \Sigma - \Lambda\). This completes the proof.


Let \(f_j : I \to I\) be a contraction for \(1 \leq j \leq m\). We assume that \(c_0 = 0\) and \(c_m = 1\) are unique fixed points of \(f_1(x)\) and \(f_m(x)\) respectively. The following lemma is a special case of the general theorem obtained by the author [4, p. 397], but we include the proof for completeness.
LEMMA 4.1. — The functional equations

\[(4.1) \quad \psi(x) = f_j(\psi(g_j(x))) \quad \text{for} \quad x \in I, \quad 1 \leq j \leq m\]

possess a unique continuous solution \(\psi(x)\) if and only if

\[(4.2) \quad f_j\left(\frac{1+e_j}{2}\right) = f_{j+1}\left(\frac{1-e_{j+1}}{2}\right) \quad \text{for} \quad 1 \leq j \leq m - 1.\]

Remark. — This is a generalization of the theorem obtained by G. de Rham [10]; indeed he has shown that the equations

\[M\left(\frac{x}{2}\right) = F_0(M(x)), \quad M\left(\frac{1+x}{2}\right) = F_1(M(x)) \quad \text{for} \quad x \in I\]

possess a unique continuous solution \(M(x)\) if and only if \(F_1(p_0) = F_0(p_1)\) where \(p_0, p_1\) are unique fixed points of the contractions \(F_0, F_1\) respectively. Lebesgue's singular functions and Pólya's space-filling curves satisfy the above equations for certain affine contractions \(F_0\) and \(F_1\).

Proof. — The conditions (4.2) are obviously necessary; thus it suffices to show the sufficiency. Let \(\mathcal{F}\) be the set of continuous functions \(u(x)\) defined on \(I\) satisfying \(u(0) = 0\) and \(u(1) = 1\); obviously \(\mathcal{F}\) is a closed subset of the Banach space \(C([0,1])\) with the usual uniform norm. We now consider the following operator:

\[Tu(x) = f_{A(x)}(u(G(x))).\]

Then it is easily seen that the conditions (4.2) imply that \(T(\mathcal{F}) \subset \mathcal{F}\); moreover \(T\) is a contraction, since

\[\|Tu - Tv\| \leq \lambda \max_{x \in I} |u(G(x)) - v(G(x))| \leq \lambda \|u - v\|,\]

where \(\lambda = \max_{1 \leq j \leq m} \text{Lip}(f_j) \in [1/m,1]\), for any \(u, v \in \mathcal{F}\). Hence \(T\) has a unique fixed point \(\psi\) in \(\mathcal{F}\); namely

\[\psi(x) = f_j(\psi(g_j(x))) \quad \text{for} \quad c_{j-1} \leq x < c_j, \quad 1 \leq j \leq m.\]

Obviously this equality holds also true for \(x = c_j\). This completes the proof.

For \(n \geq 1\) and \(x, y \in I\), we define

\[F_{n,x}(y) = f_{A_0(x)} \circ f_{A_1(x)} \circ \cdots \circ f_{A_{n-1}(x)}(y)\].
The function $F_{n,x}$ is a contraction satisfying $\text{Lip}(F_{n,x}) \leq \lambda^n$. Put 
\[ \beta = \max_{1 \leq j \leq m} \text{Lip}(g_j) \in [m, \infty]. \] Then we have 

**Lemma 4.2.** Suppose that \{\(f_j\)\} satisfy the conditions (4.2). If 
\[ \beta < \infty, \] \[ \text{then the continuous solution } \psi(x) \text{ satisfies H"older's condition of order } \log(1/\lambda)/\log \beta. \]

**Proof.** Consider arbitrary two points \(x < y\) in \(I\). Let \(N \geq 0\) be the smallest integer satisfying \# \(\{S_{N+1} \cap (x,y)\} \geq 2\). We now distinguish two cases: (a) \(S_N \cap (x,y) = \emptyset\); (b) \(S_N \cap (x,y)\) consists of a single point, say \(p\). In case (a), it follows that 
\[ |\psi(x) - \psi(y)| = \lim_{\varepsilon \to 0^+} |\psi(x+\varepsilon) - \psi(y-\varepsilon)| 
= \lim_{\varepsilon \to 0^+} |F_{N,x+\varepsilon}(\psi(G^N(x+\varepsilon))) - F_{N,x+\varepsilon}(\psi(G^N(y-\varepsilon)))| \leq \lambda^N. \]

Similarly we have \(|\psi(x) - \psi(y)| \leq 2\lambda^N\) in case (b), since \((x,p) \cap S_N = (p,y) \cap S_N = \emptyset\). Now let \(s < t\) be any two consecutive points of \(e_{N+1}\) contained in \((x,y)\). Then it follows that \(|x-y| > |s-t| \geq \beta^{-N-1}\); thus 
\[ |\psi(x) - \psi(y)| \leq 2\lambda^N = \frac{2}{\lambda} \beta^{-\xi(N+1)} \leq \frac{2}{\lambda} |x - y|^\xi \]
where \(\xi = \log(1/\lambda)/\log \beta\), which obviously completes the proof. \(\square\)

**5. Some Properties.**

The continuous solution \(\psi(x)\) of the equations (4.1) is not necessarily singular in general; for example, if we take 
\[ g_j(x) = mx - j + 1 \quad \text{and} \quad f_j(x) = \frac{x}{m} + \frac{j - 1}{m} \]
for \(1 \leq j \leq m\), then obviously \(\psi(x) \equiv x\) is a smooth solution of (4.1). In this paper, to discuss the singularities of \(\psi(x)\), we shall restrict ourselves to the following case:

\[ (5.1) \]
\[ \varepsilon_j = 1 + 2 \left[ \frac{j}{4} \right] - 2 \left[ \frac{j+1}{4} \right] \]

and
\[ f_j(x) = \frac{1}{2k} \left\{ (-1)^{j/2} x + \left[ \frac{j}{2} \right] - \left[ \frac{j}{4} \right] + \left[ \frac{j-1}{4} \right] \right\} \]
for $1 \leq j \leq m = 4k$, where $k$ is a positive integer; so $\lambda = 1/2k$. Then it is easily seen that the functions $\{f_j\}$ satisfy the conditions (4.2); therefore the equations (4.1) possess a unique continuous solution $\psi(x)$, which depends only on the functions $\{g_j\}$ satisfying the conditions (5.1). Let $\eta_j$ be the sign of the function $f_j$; namely $\eta_j = (-1)^{[j/2]}$, for $1 \leq j \leq 4k$. For brevity, put

$$
\varepsilon_{n,x} = \prod_{j=0}^{n-1} \varepsilon_{A_j(x)} \quad \text{and} \quad \eta_{n,x} = \prod_{j=0}^{n-1} \eta_{A_j(x)}
$$

for $n \geq 1$, $x \in I$.

Consider now an arbitrary point $x \in I - \varepsilon$. We define

$$
p_{j,n,x} = H_{n,x}(c_j) \quad \text{for} \quad n \geq 1, \ 0 \leq j \leq 4k.
$$

Obviously $p_{j,n,x} \neq x$. Since $p_{j,n,x} \in G^{-n}(c_j) \subset e_{n+1}$ for $1 \leq j \leq 4k - 1$, we have

$$
G^n(p_{j,n,x}) = c_j \quad \text{for} \quad 1 \leq j \leq 4k - 1.
$$

The points $p_{0,n,x}$ and $p_{4k,n,x}$ are two end points of $K_{n,x}$ and do not satisfy the above equality in general; however,

$$
\lim_{y \to p_{j,n,x}} G^n(y) = c_j \quad \text{for} \quad j = 0, 4k.
$$

Note that $0 < |x - p_{j,n,x}| < \gamma^n$ for any $n \geq 1$. Then we have

**Lemma 5.1.**— Suppose that $x \in I - \varepsilon$. Then the points $\{p_{j,n,x}\}$ satisfy the following properties:

1. $\text{sign} \ (x - p_{j,n,x}) = \varepsilon_{n,x} \text{sign} \left\{A_n(x) - j - \frac{1}{2}\right\}$,

2. $\psi(x) - \psi(p_{j,n,x}) = \frac{\eta_{n,x}}{(2k)^n} \left\{\psi(G^n(x)) - \frac{1 - (-1)^j}{4k} - \frac{1}{k \left\lfloor \frac{j}{4} \right\rfloor}\right\}$

for $n \geq 1$ and $0 \leq j \leq 4k$.

**Proof.**— Since $p_{j,n,x} = H_{n,x}(c_j)$, we have

$$
\text{sign} \ (x - p_{j,n,x}) = \text{sign} \ \{H_{n,x}(G^n(x)) - H_{n,x}(c_j)\} = \varepsilon_{n,x} \text{sign} \ \{G^n(x) - c_j\};
$$
thus the property (1) follows immediately. Since $K_{n,x} \cap S_n = \emptyset$,

$$
\psi(p_{j,n,x}) = \lim_{y \to p_{j,n,x}} \psi(y) = \lim_{y \to p_{j,n,x}} F_{n,x}(\psi(G^n(y))) = F_{n,x}(\psi(c_j))
$$

for $0 \leq j \leq 4k$; hence

$$
\psi(x) - \psi(p_{j,n,x}) = F_{n,x}(\psi(G^n(x))) - F_{n,x}(\psi(c_j)) \frac{\eta_{n,x}}{(2k)^n} \{\psi(G^n(x)) - \psi(c_j)\},
$$

which obviously completes the proof. \qed

We now consider an arbitrary point $x \in e_N$, $N \geq 1$. Then it is easily seen that, for $1 \leq j \leq 4k - 1$, each of the sets $K_{j,n,x}$ contains exactly one point of $G^{-n}(c_j) \subset e_{n+1}$, say $q_{j,n,x}^\pm$ respectively. Obviously $q_{j,n,x}^\pm \neq x$. Similarly we can define $\{q_{j,n,0}\}$ and $\{q_{j,n,1}\}$ for $n \geq 0$, $1 \leq j \leq 4k - 1$. Note that $0 < |x - q_{j,n,x}^\pm| < \gamma^n$ for any $n \geq N$. It also follows that

$$
\lim_{\varepsilon \to 0 \pm} G^n(x + \varepsilon) = \frac{1}{2} \left(1 \mp \varepsilon_{N,x} \mp\right)
$$

for every $n \geq N$, respectively. We, of course, adopt the rule:

$\varepsilon_{0,0^+} = \varepsilon_{0,1^-} = \eta_{0,0^+} = \eta_{0,1^-} = 1$. Then we have

**LEMMA 5.2. — Suppose that $x \in e_N$, $N \geq 0$. Then the points $\{q_{j,n,x}^\pm\}$ satisfy the following :**

$$
\psi(x) - \psi(q_{j,n,x}^\pm) = \frac{\eta_{N,x}^\pm}{(2k)^n} \left\{\frac{1}{2} \left(1 \mp \varepsilon_{N,x} \mp\right) - \frac{1}{4k} - \frac{1}{k} \left[\frac{j}{4}\right]\right\}
$$

for $n \geq N$ and $1 \leq j \leq 4k - 1$, respectively.

**Proof. —** Since $K_{n,x}^\pm \cap S_n = \emptyset$, we have

$$
\psi(x) - \psi(q_{j,n,x}^\pm) = \lim_{\varepsilon \to 0 \pm} \{\psi(x + \varepsilon) - \psi(q_{j,n,x}^\pm)\} =

\lim_{\varepsilon \to 0 \pm} \left\{F_{n,x+\varepsilon}(\psi(G^n(x + \varepsilon))) - F_{n,x}(\psi(c_j))\right\} = \frac{\eta_{N,x}^\pm}{(2k)^n} \left\{\frac{1}{2} \left(1 \mp \varepsilon_{N,x} \mp\right) - \psi(c_j)\right\}
$$

for every $n \geq N$, respectively. This completes the proof. \qed

For any \( x \neq y \in I \), we define \( \Delta \psi(x, y) = (\psi(x) - \psi(y))/(x - y) \). Let \( W \) be the set of points \( x \in I \) at which \( A_n(x) \equiv 2 \) or \( 3 \) \((\text{mod } 4)\) for infinitely many \( n \)'s. Obviously \( W \subset I - e \). First of all, we have

**Theorem 6.1.** Suppose that \( \gamma \leq 1/2k \). Then we have

\[
D^+ \psi(x) \geq 0 \geq D_- \psi(x) \quad \text{and} \quad D^+ \psi(x) - D_- \psi(x) \geq 1/4k
\]

respectively, for every \( x \in W \).

**Proof.** We distinguish two cases (not exclusive) as follows:

**Case A.** \( A_n(x) \equiv 3 \) \((\text{mod } 4)\) for infinitely many \( n \)'s.

Let \( 0 < n_1 < n_2 < \cdots \) be the subsequence of integers such that \( A_n(x) = 4N_i + 3 \), where \( 0 \leq N_i < k \). From the functional equations (4.1), we have

\[
\frac{N_i}{k} \leq \psi(G^{n_i}(x)) \leq \frac{2N_i + 1}{k};
\]

therefore \( \{\psi(x) - \psi(P_{i, 1})\}/\{\psi(x) - \psi(P_{i, 2})\} \leq 0 \) by (2) of Lemma 5.1, where \( p_{i,j} = p_{4N_i+j,n_i,x} \) for \( 0 \leq j \leq 4 \). On the other hand, we have

\[
\text{sign}(x - P_{i, 1}) = \text{sign}(x - P_{i, 2}) = \varepsilon_{n_i,x} \quad \text{by (1) of Lemma 5.1.}
\]

Since \( \varepsilon_{n_i,x} \) changes the sign infinitely many times as \( i \) increases, it follows that

\[
D^\pm \psi(x) \geq 0 \geq D_- \psi(x).
\]

It also follows that

\[
|\Delta \psi(x, P_{i, 1})| + |\Delta \psi(x, P_{i, 2})| \geq \frac{(2k)^{-n_i-1}}{|x - P_{i, 1}|} > \frac{1}{2k} (2k\gamma)^{-n_i} \geq \frac{1}{2k};
\]

therefore \( D^+ \psi(x) - D_- \psi(x) \geq 1/4k \) respectively, as required.

**Case B.** \( A_n(x) \equiv 2 \) \((\text{mod } 4)\) for infinitely many \( n \)'s.

Let \( 0 < n_1 < n_2 < \cdots \) be the subsequence of integers such that \( A_n(x) = 4N_i + 2 \), where \( 0 \leq N_i < k \). Since

\[
\frac{N_i}{k} \leq \psi(G^{n_i}(x)) \leq \frac{2N_i + 1}{k},
\]

it is easily seen that \( \{\psi(x) - \psi(P_{i, 0})\}/\{\psi(x) - \psi(P_{i, 1})\} \leq 0 \) and \( \{\psi(x) - \psi(P_{i, 2})\}/\{\psi(x) - \psi(P_{i, 3})\} \leq 0 \). On the other hand, we have
\[
\text{sign}(x - P_{i,0}) = \text{sign}(x - P_{i,1}) = \text{sign}(P_{i,2} - x) = \text{sign}(P_{i,3} - x) \quad \text{therefore} \quad D^\psi(x) \geq 0 \geq D_\psi(x). \quad \text{Moreover,}
\]
\[
|\Delta \psi(x, P_{i,0})| + |\Delta \psi(x, P_{i,1})| \geq \frac{(2k)^{-n-1}}{|x - P_{i,0}|} > \frac{1}{2k} \left(\frac{2k\gamma}{n}\right)^{-n} \geq \frac{1}{2k}.
\]

The same estimate holds true if we replace \( P_{i,0}, P_{i,1} \) by \( P_{i,2}, P_{i,3} \), respectively; thus \( D^\pm \psi(x) - D_\pm \psi(x) \geq 1/4k \) respectively. This completes the proof. \( \square \)

Let \( W_0 \subset W \) be the set of points \( x \in I \) at which \( A_n(x) \equiv 2 \) or \( 3 \) (mod 4) and \( A_{n+1}(x) \equiv 2 \) or \( 3 \) (mod 4) for infinitely many \( n \)’s. Then we have

**THEOREM 6.2.** - Suppose that \( \gamma \leq 1/2k \). Then \( W_0 \) is contained in the set \( \text{Knot}(\psi) \) except for a set of measure zero.

**Proof.** - We consider an arbitrary point \( x \) of \( W_0 \). Let \( 0 \leq n_1 < n_2 < \cdots \) be the subsequence of integers such that \( A_{n_i}(x) = 4N_i + \delta_i \) and \( A_{n+1}(x) = 4L_i + \omega_i \), where \( 0 \leq N_i, L_i < k \) and \( 2 \leq \delta_i, \omega_i \leq 3 \). Then it is easily seen that
\[
\frac{2N_i + 1}{2k} - \frac{2L_i + 1}{(2k)^2} \leq \psi(G^n(x)) \leq \frac{2N_i + 1}{k} - \frac{L_i}{2k^2};
\]
therefore by (2) of Lemma 5.1,
\[
\eta_{n_i, x}(2k)^n(\psi(x) - \psi(P_{i,0})) = \psi(G^n(x)) - \frac{N_i}{k} \geq \frac{1}{2k} - \frac{2L_i + 1}{(2k)^2} \geq (2k)^{-2}.
\]
Similarly we have
\[
\eta_{n_i, x}(2k)^n(\psi(P_{i,4}) - \psi(x)) = \frac{N_i + 1}{k} - \psi(G^n(x)) \geq \frac{1}{2k} + \frac{L_i}{2k^2} \geq \frac{1}{2k},
\]
Therefore, since \( \text{sign}(x - P_{i,0}) = \text{sign}(P_{i,4} - x) \), it follows that
\[
\text{sign}(\Delta \psi(x, P_{i,0})) = \text{sign}(\Delta \psi(x, P_{i,4}))
\]
and
\[
|\Delta \psi(x, P_{i,0})| > (2k)^{-2}, \quad |\Delta \psi(x, P_{i,4})| > \frac{1}{2k}.
\]
Hence the set \([D_+ \psi(x), D^+ \psi(x)] \cap [D_- \psi(x), D^- \psi(x)]\) contains an interval of length \((2k)^{-2}\) by Theorem 6.1. Thus it follows from Denjoy's theorem
[2, p. 105] that except for a set of measure zero, every point of $W_0$ is a knot point of $\psi(x)$. This completes the proof.

For $N \geq 0$, let $Y_N$ be the set of points $x \in I$ at which $A_n(x) \equiv 0$ or $1 \pmod{4}$ for all $n \geq N$ and $A_{N-1}(x) \equiv 2$ or $3 \pmod{4}$. Obviously $I - W = \bigcup_{n \geq 0} Y_n$. For brevity, put $Y_n^* = Y_n \cap (I - e)$ for $n \geq 0$. Then the unit interval $I$ is decomposed as follows:

$$I = W + e + \bigcup_{n \geq 0} Y_n^*.$$

For $n \geq 1$, let $\Xi_n$ be the set of finite words $(w_1, \ldots, w_n)$ of length $n$ such that $1 \leq w_j \leq 4k$ and $w_j \equiv 0$ or $1 \pmod{4}$ for $1 \leq j \leq n$. Then we have

**Theorem 6.3.** Suppose that there exists a positive constant $C_0$, independent of $n$, satisfying

$$\min_{(w_1, \ldots, w_n) \in \Xi_n} |h_{w_1} \circ \ldots \circ h_{w_n}(I)| \geq C_0 (2k)^{-n}$$

for all $n \geq 1$. Suppose further that $\beta < \infty$. Then we have

$$D^+ \psi(x) - D^- \psi(x) \geq \frac{1}{2k}$$

respectively, for every $x \in I - W$.

**Proof.** We distinguish two cases as follows:

**Case A.** $x \in Y_n^*$ for some $N \geq 0$.

By Lemma 3.1, we have $A_n(x) \neq 1$ for infinitely many $n$'s. Let $N \leq n_1 < n_2 < \cdots$ be the subsequence of integers such that $A_{n_i}(x) \geq 4$. Put $Q_{i,j} = p_{j,n_i,x}$ for $0 \leq j \leq 2$. Since

$$\psi(G^n(x)) \geq \frac{1}{2k}$$

and $\text{sign}(x - Q_{i,1}) = \text{sign}(x - Q_{i,2}) = \text{sign}(Q_{i,2} - Q_{i,1}) = \epsilon_{n,x}$, we have

$$|\Delta \psi(x, Q_{i,1}) - \Delta \psi(x, Q_{i,2})| = (2k)^{-n} \left| \psi(G^n(x)) \left\{ \frac{1}{x - Q_{i,2}} - \frac{1}{x - Q_{i,1}} \right\} + \frac{1}{2k(x - Q_{i,1})} \right| \geq (2k)^{-n_i-1} \frac{1}{|x - Q_{i,1}|} > \frac{1}{2k}.$$
On the other hand, it follows that
\[ |x - Q_{i,0}| > |Q_{i,1} - Q_{i,0}| \geq \beta^{-N} \left| h_{A_N(x)} \circ \ldots \circ h_{A_{N_i}(x)} \circ h_1(I) \right| \]
and
\[ C_0 \beta^{-N} (2k)^{-n_1 + N - 1}. \]

therefore
\[ |\Delta \psi(x, Q_{i,0})| = (2k)^{-n_1} \left| \frac{\psi(G^{n_i}(x))}{x - Q_{i,0}} \right| \leq \frac{2k}{C_0} \left( \frac{\beta}{2k} \right)^N. \]

Since \( \text{sign} (x - Q_{i,0}) = \varepsilon_{N,x} \), we conclude that either \([D_+ \psi(x), D_+ \psi(x)]\) or \([D_- \psi(x), D_- \psi(x)]\) contains an interval of length \(1/2k\) according as \( \varepsilon_{N,x} = -1 \) or \(+1\).

It also follows from Lemma 3.1 that \( A_n(x) \neq 4k \) for infinitely many \( n \)'s. Let \( N \leq n_1 < n_2 < \ldots \) be the subsequence of integers such that \( A_{n_i}(x) \leq 4k - 3 \). Put \( R_{i,j} = p_{4k-j,n_i,x} \) for \( 0 \leq j \leq 3 \). Since
\[ \psi(G^{n_i}(x)) \leq \frac{2k - 1}{2k} \]
and \( \text{sign} (x - R_{i,2}) = \text{sign} (x - R_{i,3}) = \text{sign} (R_{i,3} - R_{i,2}) = -\varepsilon_{N,x} \), we have
\[ |\Delta \psi(x, R_{i,2}) - \Delta \psi(x, R_{i,3})| = \]
\[ (2k)^{-n_1} \left| \left( \frac{2k - 1}{2k} - \psi(G^{n_i}(x)) \right) \left( \frac{1}{x - R_{i,3}} - \frac{1}{x - R_{i,2}} \right) + \frac{1}{2k(x - R_{i,2})} \right| \geq \]
\[ (2k)^{-n_1 - 1} \frac{1}{|x - R_{i,2}|} > \frac{1}{2k}. \]

On the other hand, \( |x - R_{i,0}| > |R_{i,1} - R_{i,0}| \geq C_0 \beta^{-N} (2k)^{-n_1 + N - 1} \); thus
\[ |\Delta \psi(x, R_{i,0})| = (2k)^{-n_1} \left| \frac{\psi(G^{n_i}(x)) - 1}{x - R_{i,0}} \right| \leq \frac{2k}{C_0} \left( \frac{\beta}{2k} \right)^N. \]

Since \( \text{sign} (x - R_{i,0}) = -\varepsilon_{N,x} \), it follows that either \([D_+ \psi(x), D_+ \psi(x)]\) or \([D_- \psi(x), D_- \psi(x)]\) contains an interval of length \(1/2k\) according as \( \varepsilon_{N,x} = +1 \) or \(-1\). Hence \( D_+ \psi(x) - D_- \psi(x) \geq 1/2k \) respectively.

Case B. \( x \in \varepsilon_N \) for some \( N \geq 0 \).

For \( n \geq N \), let \( Q_n^+ = \max \{ q_{1,n,x}^+, q_{3,n,x}^+ \} \), \( Q_n^- = \min \{ q_{1,n,x}^-, q_{3,n,x}^- \} \) and let \( R_n^+ = q_{2,n,x}^+ \) respectively. Then \( Q_n^- < R_n^- < x < R_n^+ < Q_n^+ \).
Since \( \text{sign} (x - Q_n^+) = \text{sign} (Q_n^+ - R_n^+) = \pm 1 \) respectively, it follows from Lemma 5.2 that
\[
|\Delta \psi(x, R_n^+) - \Delta \psi(x, Q_n^+)| =
(2k)^{-n} \left( \frac{1}{2} (1 + \varepsilon_{n,x}) \left\{ \frac{1}{x - R_n^+} - \frac{1}{x - Q_n^+} + \frac{1}{2k(x - Q_n^+)} \right\} \right) \geq \frac{(2k)^{-n-1}}{|x - Q_n^+|} > \frac{1}{2k},
\]
respectively. On the other hand, we have
\[
|x - R_n^+| > |K_n^+| \geq \beta^{-N}|h_{A_n(x \pm)} \circ \cdots \circ h_{A_n(x \pm)}(I)| \geq C_0 \beta^{-N}(2k)^{-n+N-1};
\]
therefore
\[
|\Delta \psi(x, R_n^+)| \leq \frac{(2k)^{-n}}{|x - R_n^+|} < \frac{2k(\beta)^N}{C_0 (2k)}.
\]
Hence \( D^+ \psi(x) - D^- \psi(x) \geq 1/2k \) respectively. This completes the proof.

Let \( Y^* = \bigcup_{n \geq 0} Y_n^* \) for brevity. Then we have

**Theorem 6.4.** - Knot \( (\psi) \cap Y^* = \emptyset \).

**Proof.** - We consider an arbitrary point \( x \) of \( Y_n^* \) for some \( N \geq 0 \). Let \( s_n = p_{0,n,x} \) for \( n \geq N \). Since \( \text{sign} (x - s_n) = \varepsilon_{n,x} \) is independent of \( n \geq N \), the sequence \( \{s_n\} \) is monotone, either increasing or decreasing, and converges to \( x \). Note that \( s_n = s_{n+1} \) if and only if \( A_n(x) = 1 \). Put \( J_n = [s_n, s_{n+1}] \subset K_{n,x} \) for \( n \geq N \). Then it is easily seen that
\[
(x, s_N] = \bigcup_{n \geq N} J_n.
\]
Since the function \( G^*(x) \) maps \( K_{n,x} \) homeomorphically onto \( (0,1) \), we have \( A_n(x) > A_n(y) \) for all \( y \in J_n \). Therefore
\[
\psi(G^*(x)) \geq f_{A_n(x)}(0) \geq \max_{j < A_n(x)} \|f_j\| \geq \psi(G^*(y));
\]
thus
\[
\eta_{N,x} \text{ sign } \{\psi(x) - \psi(y)\} = \eta_{N,x} \text{ sign } \{F_{n,x}(\psi(G^*(x)) - F_{n,x}(\psi(G^*(y))))\} =
\text{ sign } \{\psi(G^*(x)) - \psi(G^*(y))\} \geq 0.
\]
By the continuity of \( \psi \), we conclude that
\[
\eta_{N,x} \text{ sign } \{\psi(x) - \psi(y)\} \geq 0 \quad \text{for every} \quad y \in [x, s_N].
\]
This means that \( x \) is not a knot point of \( \psi(x) \).
7. Proof of Theorem 2.1.

First of all, for any integer \( k \geq 1 \) and positive numbers \( \sigma, \tau, \rho \) satisfying

\[
2k(\sigma + \tau) < 1 \quad \text{and} \quad \sigma \geq \rho,
\]

we shall construct two Cantor sets \( E_0 \equiv E_0(k, \sigma, \tau) \) and \( E_1 \equiv E_1(k, \sigma, \rho) \). The set \( E_0(k, \sigma, \tau) \) is obtained from the unit interval \( I \) by a sequence of deletions of open intervals known as middle thirds, as follows: First divide \( I \) into \( k \) equal parts, say

\[
I_{1,1} = \left[ 0, \frac{1}{k} \right], \quad I_{1,2} = \left[ \frac{1}{k}, \frac{2}{k} \right], \quad \ldots, \quad I_{1,k} = \left[ \frac{k-1}{k}, 1 \right],
\]

and remove from each closed interval \( I_{1,j} \) the open interval \( U_{1,j} \) centered at \( (2j-1)/2k \) and of length \( 2\sigma \). We subdivide each of the \( 2k \) remaining closed intervals into \( k \) equal parts, say \( I_{2,j}, 1 \leq j \leq 2k^2 \), ordered from left to right, each of length \( (1-2k\sigma)/(2k^2) \). Then remove from each closed interval \( I_{2,j} \) the middle open interval \( U_{2,j} \) of length \( 2\sigma \tau \), leaving the \( 4k^2 \) closed intervals, each of length \( (1-2k\sigma-4k^2\sigma\tau)/(4k^2) \). This process is permitted to continue indefinitely. At the \( n \)th stage of deletion, each length of the \( 2^{n-1}k^n \) open intervals \( U_{n,j} \) is \( 2\sigma \tau^{n-1} \), and therefore the measure of the union of the open intervals removed in the entire sequence of removal operations is \( 2k\sigma/(1-2k\tau) \). The set \( E_0 \) is defined to be the closed set remaining; thus

\[
|E_0| = \frac{1 - 2k(\sigma + \tau)}{1 - 2k\tau}.
\]

We next define the set \( E_1(k, \sigma, \rho) \), which is slightly different from \( E_0 \) defined above, as follows: First divide the unit interval \( I \) into \( k \) equal parts, say

\[
J_{1,1} = \left[ 0, \frac{1}{k} \right], \quad J_{1,2} = \left[ \frac{1}{k}, \frac{2}{k} \right], \quad \ldots, \quad J_{1,k} = \left[ \frac{k-1}{k}, 1 \right],
\]

Then remove from each closed interval \( J_{1,j} \) the two intervals

\[
V_{1,j}^- = \left[ \frac{j-1}{k}, \frac{2j-1-2k\sigma}{2k} \right], V_{1,j}^+ = \left( \frac{2j-1+2k\sigma}{2k}, \frac{j}{k} \right],
\]

each of length \( (1-2k\sigma)/2k \). We subdivide each of the \( k \) remaining closed intervals into \( 2k \) equal parts, say \( J_{2,j}, 1 \leq j \leq 2k^2 \), ordered
from left to right, each of length \( \sigma/k \). Then delete from each closed 
interval \( J_{2,j} \) the two intervals \( V_{1,j}^+ \) of length \( \rho(1-2k\sigma)/2k \), leaving the 
\( 2k^2 \) middle closed intervals, each of length \( (\sigma-\rho+2k\sigma\rho)/k \). At the 
\( n \)th stage of deletion, we have \( |V_{n,j}^+| = \rho^{n-1}(1-2k\sigma)/2k \); therefore the 
measure of the union of the removed intervals in the entire sequence of 
removal operations is \( (1-2k\sigma)/(1-2k\rho) \). The set \( E_1 \) is defined to 
be the closed set remaining; thus

\[
|E_1| = \frac{2k(\sigma-\rho)}{1-2k\rho}.
\]

Note that the set \( E_1 \) is contained in \([1-2k\sigma/2k(1-\rho), 1-1-2k\sigma/2k(1-\rho)]\).

We now define the continuous function \( \zeta_0(x) \equiv \zeta_0(k,\sigma,\tau; x) \) by setting

\[
\zeta_0(x) = \int_0^x d_0(s) \, ds \quad \text{for} \quad 0 \leq x \leq 1,
\]

where \( d_0(s) = 1/2k \) if \( s \in E_0(k,\sigma,\tau) \) and \( d_0(s) = \tau \) otherwise. We also 
define the continuous function \( \zeta_1(x) \equiv \zeta_1(k,\sigma,\rho; x) \) by setting

\[
\zeta_1(x) = \frac{1}{2k} - \sigma + \int_0^x d_1(s) \, ds \quad \text{for} \quad 0 \leq x \leq 1,
\]

where \( d_1(s) = 1/2k \) if \( s \in E_1(k,\sigma,\rho) \) and \( d_1(s) = \rho \) otherwise. Then it 
is easily seen that \( \zeta_0(I) = [0,(1-2k\sigma)/2k], \zeta_1(I) = [(1-2k\sigma)/2k,1/2k] \) and 
\( \zeta_i(E_i) = E_i \cap \zeta_i(I) \) for \( i = 0,1 \).

We next define, for \( 0 \leq i < k \),

\[
\begin{align*}
g_{4i+1}(x) &= \zeta_0^{-1} \left( x - \frac{i}{k} \right) \quad \text{for} \quad x \in I_{4i+1} = \left[ \frac{i}{k}, \frac{i+1}{2k} - \sigma \right], \\
g_{4i+2}(x) &= \zeta_1^{-1} \left( x - \frac{i}{k} \right) \quad \text{for} \quad x \in I_{4i+2} = \left[ \frac{i+1}{2k} - \sigma, \frac{i+1}{2k} \right], \\
g_{4i+3}(x) &= \zeta_1^{-1} \left( \frac{i+1}{k} - x \right) \quad \text{for} \quad x \in I_{4i+3} = \left[ \frac{2i+1}{2k}, \frac{2i+1}{2k} + \sigma \right], \\
g_{4i+4}(x) &= \zeta_0^{-1} \left( x - \frac{2i+1}{2k} - \sigma \right) \quad \text{for} \quad x \in I_{4i+4} = \left[ \frac{2i+1}{2k} + \sigma, \frac{i+1}{k} \right];
\end{align*}
\]

thus the unit interval \( I \) is divided into \( m = 4k \) subintervals \( I_j = [c_{j-1}, c_j] \).
We have \( |I_{4i+1}| = |I_{4i+4}| = (1-2k\sigma)/2k \) and \( |I_{4i+2}| = |I_{4i+3}| = \sigma \).
Obviously the functions \( g_j(x) \) satisfy the conditions (5.1) and we denote
by $\psi(k, \sigma, \tau, \rho; x)$ the corresponding continuous solution of the equations (4.1).

It follows from Theorems 6.1 and 6.3 that $\psi(k, \sigma, \tau, \rho; x)$ has nowhere a unilateral derivative finite or infinite for any integer $k$ and positive numbers $\sigma$, $\tau$, $\rho$ satisfying (7.1), since we have

$$\gamma = \frac{1}{2k}, \quad \beta = \max \left\{ \frac{1}{\rho}, \frac{1}{\tau} \right\}$$

and

$$|h_{w_1} \circ \cdots \circ h_{w_n}(I)| = \frac{1}{(2k)^n} - \frac{\sigma}{(2k)^{n-1}} - \frac{\sigma \tau}{(2k)^{n-2}} - \cdots - \sigma \tau^{n-1} > \frac{|E_0|}{(2k)^n},$$

for every finite word $(w_1 \ldots w_n) \in \Xi_n$.

Since the Cantor set $E_0$ is a unique compact subset of $I$ satisfying

$$E_0 = h_1(E_0) \cup h_4(E_0) \cup h_5(E_0) \cup \cdots \cup h_{4k}(E_0)$$

and since the mapping $v$ maps $Y_0^*$ homeomorphically onto $v(Y_0^*)$, it follows that $Y_0^* = E_0$. On the other hand, for every $x \in W + \bigcup_{n \geq 1} Y_n^*$, there exist $n = n(x)$ and $j = j(x)$ such that $x \in U_{n,j}$; thus $E_0 \subset Y_n^* + e$. Therefore $|Y_n^*| = |E_0|$, since $e$ is countable. Let $\Omega_n$ be the set of finite words $(w_1 \ldots w_n)$ of length $n$ such that $1 \leq w_j \leq 4k$ for $1 \leq j \leq n$. Then for any $n \geq 0$, the set $Y_{n+1}^*$ is decomposed as follows:

$$Y_{n+1}^* = \bigcup_{(w_1 \ldots w_n) \in \Omega_n} h_{w_1} \circ \cdots \circ h_{w_n} \circ h_j(Y_0^*).$$

On each interval $V_{1,j}^+$, for any $(w_1 \ldots w_n) \in \Omega_n$ and $j \in \Omega_1 - \Xi_1$, the function $h_{w_1} \circ \cdots \circ h_{w_n} \circ h_j(x)$ is a linear contraction; more precisely we have

$$\left| \frac{d}{dx} (h_{w_1} \circ \cdots \circ h_{w_n} \circ h_j)(x) \right| = \rho^{n+1} - r^2(w) \tau^2(w) \quad \text{for} \quad x \in V_{1,j}^+,$$

where $r(w) \equiv r(w_1, \ldots, w_n) = \frac{1}{2} \sum_{j=1}^{n} (1 + \eta_{w_j})$. Since $Y_0^* \cap U_{1,j} = \emptyset$ for all $j$, we have

$$|Y_{n+1}^*| = 2k|Y_0^*| \sum_{(w_1 \ldots w_n) \in \Omega_n} \rho^{n+1} - r^2(w) \tau^2(w) = 2k \rho |E_0|(2k(\rho + \tau))^n.$$
Therefore it follows that
\[ |Y^*| = \sum_{n=0}^{\infty} |Y_n^*| = |E_0| + 2k\rho |E_0| \sum_{n=0}^{\infty} (2k(\rho + \tau))^n = \frac{1 - 2k(\rho + \tau)}{1 - 2k(\rho + \tau)}. \]

For \( N \geq 0 \), let \( Z_N \) be the set of points \( x \in I \) at which \( A_n(x) \equiv 2 \) or \( 3 \pmod{4} \) for all \( n \geq N \) and \( A_{N-1}(x) \equiv 0 \) or \( 1 \pmod{4} \). Put \( Z = \bigcup_{n \geq 0} Z_n \). Obviously \( Z \subset W_0 \subset I - e \). Then it is easily seen that the set \( Z_0 \) is a compact subset of \( I \) satisfying
\[ Z_0 = h_2(Z_0) \cup h_3(Z_0) \cup h_6(Z_0) \cup \cdots \cup h_{4k-1}(Z_0); \]
therefore \( Z_0 = E_1 \). For any \( n \geq 0 \), the set \( Z_{n+1} \) is decomposed as follows:
\[ Z_{n+1} = \bigcup_{(w_1, \ldots, w_n) \in \Omega_n} h_{w_1} \circ \cdots \circ h_{w_n} \circ h_j(Z_0). \]

On each open interval \( U_{1,j} \), for any \( (w_1, \ldots, w_n) \in \Omega_n \) and \( j \in \Xi_1 \), the function \( h_{w_1} \circ \cdots \circ h_{w_n} \circ h_j(x) \) is a linear contraction such that
\[ \left| \frac{d}{dx} (h_{w_1} \circ \cdots \circ h_{w_n} \circ h_j)(x) \right| = \rho^{n-r(w)} \tau^1 + r(w) \quad \text{for} \quad x \in U_{1,j}. \]

Since \( Z_0 \cap V_{1,j} = \emptyset \) for all \( j \), we have
\[ |Z_{n+1}| = 2k |Z_0| \sum_{(w_1, \ldots, w_n) \in \Omega_n} \rho^{n-r(w)} \tau^1 + r(w) = 2k \tau |E_1| (2k(\rho + \tau))^n; \]
therefore
\[ |Z| = \sum_{n=0}^{\infty} |Z_n| = |E_1| + 2k \tau |E_1| \sum_{n=0}^{\infty} (2k(\rho + \tau))^n = \frac{2k(\sigma - \rho)}{1 - 2k(\rho + \tau)} = 1 - |Y^*|. \]

Then it follows from Theorems 6.2 and 6.4 that
\[ |Z| \leq |W_0| \leq |\text{Knot}(\psi)| \leq 1 - |Y^*| = |Z|; \]
hence we obtain
\[ |\text{Knot}(\psi)| = \frac{2k(\sigma - \rho)}{1 - 2k(\rho + \tau)}. \]

Thus if we take, for a fixed number \( \alpha \in [0,1) \),
\[ \sigma_0 = \frac{1 + \alpha}{8k}, \quad \tau_0 = \frac{1}{4k} \quad \text{and} \quad \rho_0 = \frac{1}{8k}, \]
then the function \( \psi_0(x) \equiv \psi(k, \sigma_0, \tau_0, \rho_0; x) \) satisfies \(|Knot(\psi_0)| = \alpha \) and Hölder's condition of order \( \log(2k)/\log(8k) \) by Lemma 4.2, which obviously converges to 1 as \( k \) tends to infinity. This completes the proof of Theorem 2.1. □

**Remark.** — Besicovitch's function \( B(x) \) illustrated in Figure 1 is precisely equal to the function \( \psi(1,1/8,1/4,1/8; x) \); thus \( B(x) \) satisfies Hölder's condition of order 1/3.

**BIBLIOGRAPHY**