UNIVERSAL TRANSITIVITY OF SIMPLE AND 2-SIMPLE PREHOMOGENEOUS VECTOR SPACES

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Introduction.

We denote by $k$ a field of characteristic zero. Let $\tilde{G}$ be a connected $k$-split linear algebraic group acting on $X = \text{Aff}^n$ rationally by $\rho$ which is defined over $k$. If there exists a Zariski-dense $\tilde{G}$-orbit $Y$, we say that $(\tilde{G}, \rho, X)$ is a prehomogeneous vector space (abbrev. P.V.). When $\rho$ is irreducible or $[G, G]$ is a simple algebraic group, or a product of two simple algebraic groups, they are completely classified over $\mathbb{C}$ (see [3] ~ [6]). Put $G = \rho(\tilde{G})$. Let $\ell$ be the number of $G(k)$-orbits in $Y(k)$, i.e., $\ell = \ell_k(G, X) = |G(k) \backslash Y(k)|$. In this paper, we shall assume that there exists a nonsplit quaternion $k$-algebra. In other words, $H^1(k, \text{Aut}(SL_2)) \neq 0$. This condition is satisfied by every local field $k$ other than $\mathbb{C}$. We say that $Y$ is a universally transitive open orbit if $\ell = \ell_k(G, X) = 1$ for all such fields $k$, i.e., $Y(k)$ is a $G(k)$-orbit. Note that $G(k) \neq \rho(\tilde{G}(k))$ in general. Professor J.-I. Igusa classified all irreducible regular P.V.’s with universally transitive open orbits ([1], [2]). He also proved in [2] that $\ell$ is invariant under castling transformations.

In this paper, we shall classify simple or 2-simple P.V.’s with universally transitive open orbits. We shall also prove that $\ell$ is invariant under some P.V.-equivalences such as (1) $(Sp_{2n} \times G, \Lambda_1 \otimes \rho)$ (deg $\rho \leq 2n$) $\leftrightarrow (G, \Lambda^2(\rho))$ (see Proposition 3.7) (2) $(G \times GL_n, \rho_1 \otimes \Lambda_1 + \rho_2 \otimes \Lambda^\dagger)$ ($n \geq \deg \rho_1 \geq \deg \rho_2$) $\leftrightarrow (G, \rho_1 \otimes \rho_2)$ (see Proposition 4.1), and

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others (cf. Lemma 4.3-Proposition 4.7). This paper consists of the following four sections:

1. Preliminaries.
2. Simple P.V.'s with Universally Transitive Open Orbits.
3. 2-Simple P.V.'s of Type I with Universally Transitive Open Orbits.
4. 2-Simple P.V.'s of Type II with Universally Transitive Open Orbits.

The results are given in Theorems 2.19; 3.20; 4.2; 4.18; 4.25; 4.26; and Corollaries 2.20; 3.21. Also we shall check universal transitivity for non-regular irreducible P.V.'s (see Corollary 3.22). The first author would like to express his hearty thanks to Professor J.-I. Igusa and other members at The Johns Hopkins University in U.S.A. for their mathematical stimulation and hospitality while he stayed there in 1986. The idea for this work was first obtained that time.

1. Preliminaries.

We shall use the same notations as in [2]. For $\xi \in Y(k)$, put $\tilde{G}_\xi = \{g \in \tilde{G}; p(g)\xi = \xi\}$ and $\tilde{G}_\xi = \rho(\tilde{G}_\xi)$. Let $\ell$ be a number of $G(k)$-orbits in $Y(k)$, i.e., $\ell = |G(k)\backslash Y(k)|$.

**Proposition 1.1.** We have $G(k)\backslash Y(k) = \alpha^{-1}(1)$, where $\alpha : H^1(k,G) \to H^1(k,G)$.

**Corollary 1.2.** Assume that (1) $H^1(k,\tilde{G}) = \{1\}$, (2) $H^1(k,\tilde{G}) \to H^1(k,G)$ is surjective. Then we have $G(k)\backslash Y(k) = \tilde{G}(k,G)$.

**Proof.** See [2].

**Corollary 1.3.** Assume that (1) $H^1(k,\tilde{G}) = \{1\}$, (2) Ker $\rho = \{1\}$. Then we have $G(k)\backslash Y(k) = H^1(k,\tilde{G})$.

**Proof.** If Ker $\rho = \{1\}$, then we have $\tilde{G}_\xi \simeq G_\xi$ and hence $H^1(k,\tilde{G}_\xi) \to H^1(k,G_\xi)$ is bijective. Q.E.D.

**Corollary 1.4.** If $\tilde{G}_\xi = \{1\}$, then we have $\ell = 1$, i.e., $Y(k)$ is a $G(k)$-orbit.
Proof. - We have \( G_\xi = \rho(\tilde{G}_\xi) = \{1\} \) and hence \( G(k)\backslash Y(k) = \alpha^{-1}(1) = \{1\} \) for \( \alpha: H^1(k,G_\xi) = \{1\} \to H^1(k,G) \). Q.E.D.

**Proposition 1.5.** - We have \( \ell = 1 \) for \((\tilde{G}, \rho_1 \oplus \rho_2, X_1 \oplus X_2)\) if and only if \( \ell = 1 \) for \((\tilde{G}, \rho_1, X_1)\) and \( \ell = 1 \) for \((H, \rho_2|_H, X_2)\) where \( H \) is a generic isotropy subgroup of \((\tilde{G}, \rho_1, X_1)\).

Proof. - Let \( Y \) (resp. \( Y_1, Y_2 \)) be the open orbit of \((\tilde{G}, \rho_1 \oplus \rho_2, X_1 \oplus X_2)\) (resp. \((\tilde{G}, \rho_1, X_1), (H, \rho_2|_H, X_2)\)) \((\Rightarrow)\) For any \( \xi_1 \in Y_1(k) \) and \( H = \tilde{G}_{\xi_1} \), take \( \xi_2 \in Y_2(k) \). Then we have \((\xi_1, \xi_2) \in Y(k)\) and hence the projection \( Y(k) \to Y_1(k) \) is a \( \tilde{G} \)-equivariant surjective map. Since \( Y(k) \) is a \( G(k) \)-orbit, \( Y_1(k) \) must be a \( G(k) \)-orbit, i.e., \( \ell = 1 \) for \((\tilde{G}, \rho_1, X_1)\). Now take any two points \( \xi_2, \xi_2' \in Y_2(k) \) for \( H = \tilde{G}_{\xi_1} \). Since \((\xi_1, \xi_2)\) and \((\xi_1, \xi_2')\) are elements of \( Y(k) \), there exists \( g \in G(k) \) satisfying \((g\xi_1, g\xi_2) = (\xi_1, \xi_2)\). This implies that \( g \in G_{\xi_1}(k) = H(k) \) satisfying \( g\xi_2 = \xi_2' \), i.e., \( \ell = 1 \) for \((H, \rho_2|_H, X_2)\). \((\Leftarrow)\) Take any two points \((\xi_1, \xi_2)\) and \((\xi_1', \xi_2')\) of \( Y(k) \). Then there exists \( g \in G(k) \) such that \( g\xi_1 = \xi_1' \). We have \( g(\xi_1, \xi_2) = (\xi_1, g\xi_2) \), and two points \( \xi_2 \) and \( g\xi_2 \) belong to \( Y_2(k) \) for \( H = \tilde{G}_{\xi_1} \). Hence there exists \( h \in H(k) \) satisfying \( hg\xi_2 = \xi_2 \), i.e., \( hg(\xi_1, \xi_2) = (\xi_1, \xi_2) \), with \( hg \in G(k) \) Q.E.D.

**Corollary 1.6.** - Assume that \( \ell = 1 \) for \((\tilde{G}, \rho_1, X_1)\) and \((H^*, \rho_2|_{H^*}, X_2)\) where \( H^* \) is the connected component of a generic isotropy subgroup \( H \) of \((\tilde{G}, \rho_1, X_1)\). Then we have \( \ell = 1 \) for \((\tilde{G}, \rho_1 \oplus \rho_2, X_1 \oplus X_2)\).

**Remark 1.7.** - Assume that \( \ell = 1 \) for \((G, \rho, X)\). Then \( \ell = 1 \) for \((\tilde{G}, \rho, X)\) with \( \rho(\tilde{G}) \Rightarrow \rho(G) \).

**Theorem 1.8** (J.-I. Igusa [1], [2]). - A regular irreducible P.V. has a universally transitive open orbit (i.e., \( \ell = 1 \)) if and only if it is castling-equivalent to one of the following P.V.'s:

1. \((G \times GL_m, \rho \otimes \Lambda_1)\) where \( \rho \) is an \( m \)-dimensional irreducible representation of \( G \).
2. \((GL_{2m}, \Lambda_2)\).
3. \((Sp_n \times GL_{2m}, \Lambda_1 \otimes \Lambda_1)\).
4. \((GL_1 \times SO_n, \Lambda_1 \otimes \Lambda_1)\) where \( n \) is even, and \( n \geq 4 \).
5. \((GL_1 \times Spin_2, \Lambda_1 \otimes the\ spin\ rep.)\).
6. \((GL_1 \times Spin_9, \Lambda_1 \otimes the\ spin\ rep.)\).
7. \((Spin_{10} \times GL_2, a\ half-spin\ rep. \otimes \Lambda_1)\).
8. \((GL_1 \times E_6, \Lambda_1 \otimes \Lambda_1)\) with \( \deg(\Lambda_1) = 27 \) for \( E_6 \).
2. Simple P.V.'s with Universally Transitive Open Orbits.

**Theorem 2.1 ([4] with a correction [5]).** — All non-irreductible simple P.V.'s with scalar multiplications are given as follows:

1. \((GL_1^n \times SL_n, \Lambda_1 \oplus \cdots \oplus \Lambda_1 \oplus \Lambda_1^{(*)}) (1 \leq k \leq n, n \geq 2)\).
2. \((GL_1^{n+1} \times SL_n, \Lambda_2 \oplus \Lambda_1^{(*)} \oplus \cdots \oplus \Lambda_1^{(*)}) (1 \leq k \leq 3, n \geq 4)\) except \((GL_1^4 \times SL_n, \Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1^{(*)})\) with \(n = \text{odd}\).
3. \((GL_1^3 \times SL_{2m+1}, \Lambda_2 \oplus \Lambda_2)\) for \(m \geq 2\).
4. \((GL_1^2 \times SL_n, 2\Lambda_1 \oplus \Lambda_1^{(*)})\).
5. \((GL_1^3 \times SL_5, \Lambda_2 \oplus \Lambda_2 \oplus \Lambda_1^{(*)})\).
6. \((GL_1^2 \times SL_n, \Lambda_3 \oplus \Lambda_1^{(*)}) (n = 6, 7)\).
7. \((GL_1^2 \times SL_6, \Lambda_3 \oplus \Lambda_1 \oplus \Lambda_1)\).
8. \((GL_1^4 \times Sp_n, \Lambda_1 \oplus \cdots \oplus \Lambda_1^{(*)} (k = 2, 3)\).
9. \((GL_1^3 \times Sp_2, \Lambda_2 \oplus \Lambda_1)\).
10. \((GL_1^3 \times Sp_3, \Lambda_3 \oplus \Lambda_1)\).
11. \((GL_1^2 \times Spin_n, (\text{half-})\text{spin rep.} \oplus \text{the vector rep.})\) with \(n = 7, 8, 10\).
12. \((GL_1^2 \times Spin_{10}, \Lambda \oplus \Lambda)\) where \(\Lambda = \text{the even half-spin representation}\).

Here \(\Lambda^{(*)}\) stands for \(\Lambda\) or its dual \(\Lambda^*\). Note that \((G, \rho, X) \simeq (G, \rho^*, X^*)\) as triplets if \(G\) is reductive.

**Lemma 2.2.** — We have \(\ell = 1\) for \((GL_n, \Lambda_1 \oplus \cdots \oplus \Lambda_1, M(n))\).

**Proof.** — Clearly the isotropy subgroup at \(I_n \in M(n)\) is \(\{I_n\}\), and hence \(\ell = 1\) by Corollary 1.4. Q.E.D.

**Lemma 2.3.** — We have \(\ell = 1\) for \((GL_1^n \times GL_1^n, (\Lambda_1 \oplus \cdots \oplus \Lambda_1) \oplus \Lambda_1^{(*)})\) where \(GL_1^n\) acts independently on each irreducible component of \((\Lambda_1 \oplus \cdots \oplus \Lambda_1)\) and it acts on \(\Lambda_1^{(*)}\) trivially.
Proof. — By Remark 1.7 and Lemma 2.2, we have \( \ell = 1 \) for
\[(GL^n \times GL_n, \Lambda_1 \oplus \cdots \oplus \Lambda_1).\]
Its isotropy subgroup at \( I_n \) is
\[H = \{ (\alpha_1, \cdots, \alpha_n, \begin{pmatrix} \alpha_1^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_n^{-1} \end{pmatrix}); \alpha_1, \cdots, \alpha_n \in GL_1 \}.
\]
By Proposition 1.5 and Lemma 2.2 for \( n = 1 \), we have \( \ell = 1 \) for \((H, \Lambda_1^{(*)})\). Again by Proposition 1.5, we have \( \ell = 1 \) for our P.V.

Q.E.D.

Proposition 2.4. — We have \( \ell = 1 \) for
\[(GL_1^{k+1} \times SL_n, \Lambda_1 \oplus \cdots \oplus \Lambda_1^{(*)}) \quad (1 \leq k \leq n, n \geq 2).\]

Proof. — By Proposition 1.5, Lemma 2.2 and Lemma 2.3, we have our result. Q.E.D.

Proposition 2.5. — We have \( \ell \geq 2 \) for following P.V.'s:
1. \( (GL_2^2 \times SL_n, 2 \Lambda_1 \oplus \Lambda_1^{(*)}) \).
2. \( (GL_2^2 \times SL_n, \Lambda_3 \oplus \Lambda_1^{(*)}) \). \((n = 6, 7)\).
3. \( (GL_2^2 \times SL_5, \Lambda_3 \oplus \Lambda_1 \oplus \Lambda_1) \).
4. \( (GL_2^2 \times Sp_2, \Lambda_2 \oplus \Lambda_1) \).
5. \( (GL_2^2 \times Sp_3, \Lambda_3 \oplus \Lambda_1) \).
6. \( (GL_2^2 \times Spin_n, \text{(half-)spin rep.} \oplus \text{the vector rep.}) \), with \( n = 7 \) and 12.

Proof. — By Theorem 1.8, we have \( \ell \geq 2 \) for \((GL_1 \times SL_2, 2 \Lambda_1)\), \((GL_n, \Lambda_3)\), \((n = 6, 7)\) \((GL_1 \times Sp_2, \Lambda_2) \simeq (GL_1 \times SO_5, \Lambda_1)\), \((GL_1 \times Sp_3, \Lambda_3)\), \((GL_1 \times Spin_7, \text{the vector rep.}) \simeq (GL_1 \times SO_7, \Lambda_1)\), and \((GL_1 \times Spin_{12}, \text{a half-spin rep.})\). By Proposition 1.5, we have our result. Q.E.D.

Remarks 2.6. — In [2], it is proved that, for \((GL_7, \Lambda_3)\), \( Y(k) \) is \( G(k) \)-transitive for any local field \( k \) other than \( \mathbb{R} \). However, for \((GL_7^2 \times SL_7, \Lambda_3 \oplus \Lambda_1^{(*)})\), \( Y(k) \) is not \( G(k) \)-transitive even when \( k \) is a \( p \)-adic field. Because its generic isotropy subgroup \( H \) is \( (G_2) \times \{ cI_7; c^3 = 1 \} \) (see, p. 86 in [3]) and \((GL_1 \times (G_2)_1, \Lambda_2) \subseteq (GL_1 \times SO_7, \Lambda_1)\), we have our result by Proposition 1.5 and [2].
LEMMA 2.7. - We have \( t = 1 \) for \((GL_1 \times Sp_n, 1 \otimes \Lambda_1 + \Lambda_1 \otimes \Lambda_1)\).

Proof. - We have \( X = M(2n,2) \) and \( \rho(g)x = \tilde{A}x \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} \) for \( g = (\alpha, \tilde{A}) \in GL_1 \times Sp_n, x \in X, \rho = 1 \otimes \Lambda_1 + \Lambda_1 \otimes \Lambda_1 \). For \( \tilde{A} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in GL_{2n} \) with \( A, B, C, D \in M_n \), we have \( \tilde{A} \in Sp_n \) if and only if (1) \( A' B \) and \( C' D \) are symmetric matrices, (2) \( A' D - B' C = I_n \). We shall calculate the isotropy subgroup \( G_x \) at

\[
\xi = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} = (e_1, e_{n+1}).
\]

Put

\[
A = \begin{bmatrix} a_1 & a_2 \\ a_3 & A_4 \end{bmatrix}, \ldots, D = \begin{bmatrix} d_1 & d_2 \\ d_3 & D_4 \end{bmatrix}.
\]

Then

\[
\tilde{A} \xi \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \alpha \\ a_3 & b_3 \alpha \\ c_1 & d_1 \alpha \\ c_3 & d_3 \alpha \end{bmatrix} = \xi
\]

implies that \( a_1 = 1, \ d_1 = \alpha^{-1}, \ b_1 = c_1 = 0, \) and \( a_3 = b_3 = c_3 = d_3 = 0 \). By the condition \( A \in Sp_n \), we get

\[
\begin{align*}
(1) \quad & \begin{bmatrix} A_4 & B_4 \\ C_4 & D_4 \end{bmatrix} \in Sp_{n-1} \\
(2) \quad & \begin{bmatrix} A_4 & B_4 \\ C_4 & D_4 \end{bmatrix} \begin{bmatrix} b_2 \\ -a_2 \end{bmatrix} = 0, \\
(3) \quad & \begin{bmatrix} A_4 & B_4 \\ C_4 & D_4 \end{bmatrix} \begin{bmatrix} d_2 \\ -c_2 \end{bmatrix} = 0, \\
(4) \quad & \alpha^{-1} + a_2' d_2 - b_2' c_2 = 1.
\end{align*}
\]
Thus we have

\[ \tilde{G}_\xi = \{(\alpha, \tilde{A}) \in GL_1 \times SP_n, \alpha = 1, \tilde{A} = \begin{bmatrix} 1 & 0 \\ A_4 & B_4 \\ 0 & 1 \\ C_4 & D_4 \end{bmatrix} \} \cong SP_{n-1}. \]

On the other hand, \( \text{Ker} \rho = \{1\} \) and \( H^1(k, GL_1 \times SP_n) = \{1\} \), we have \( G(k) \backslash Y(k) = H^1(k, \tilde{G}_\xi) = H^1(k, \text{Sp}^n) = \{1\} \) by Corollary 1.3. Q.E.D.

**Proposition 2.8.** — We have \( \ell = 1 \) for \( (GL^2_1 \times SP_n, \Lambda_1 \oplus \Lambda_1) \).

**Proof.** — By Remark 1.7 and Lemma 2.7, we have our result.

Q.E.D.

**Proposition 2.9.** — We have \( \ell = 1 \) for \( (GL^3_1 \times SP_n, \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1) \).

**Proof.** — Similar calculation as Lemma 2.7 shows that a generic isotropy subgroup \( H \) of \( (GL^2_1 \times SP_n, \Lambda_1 \oplus \Lambda_1) \) is isomorphic to

\[ \{(\alpha, \alpha^{-1}, A) \}; A \in SP_{n-1}, \alpha \in GL_1 \} . \]

By Propositions 1.5 and 2.8, it is enough to show \( \ell = 1 \) for \( (GL_1 \times H, \Lambda_1) \cong (GL_1 \times GL_1 \times SP_n, (\Lambda_1 \otimes \Lambda_1^* + \Lambda_1 \otimes \Lambda_1) \otimes 1 + \Lambda_1 \otimes 1 \otimes \Lambda_1) \). Since \( \ell = 1 \) for \( (SP_n, \Lambda_1) \) by Lemma 2.7, it is enough to show \( \ell = 1 \) for \( (GL_1 \times GL_1, \Lambda_1 \otimes \Lambda_1^* + \Lambda_1 \otimes \Lambda_1) \). Put

\[ \tilde{G} = GL_1 \times GL_1, \rho = \Lambda_1 \otimes \Lambda_1^* + \Lambda_1 \otimes \Lambda_1, \]

i.e., \( \rho(\alpha, \beta) = (\alpha \beta^{-1}, \alpha \beta) \) and \( G = \rho(\tilde{G}) \). Since \( G = GL_1 \times GL_1 \), we have \( G(k) = GL_1(k) \times GL_1(k) \cong \rho(\tilde{G}(k)) \) and

\[ Y(k) = \{(\alpha, \beta) \in k^2; \alpha \beta \neq 0\} = G(k). (1, 1), \] i.e., \( \ell = 1 \). Q.E.D.

**Proposition 2.10.** — We have \( \ell = 1 \) for

\[ (GL^{k+1}_1 \times SL_{2m}, \Lambda_2 \oplus \Lambda_1^* \oplus \cdots \oplus \Lambda_1^*) \] (\( 1 \leq k \leq m > 2 \)).

**Proof.** — By Proposition 1.5, it is enough to show \( \ell = 1 \) when \( k = 3 \), i.e., \( (GL^3_1 \times GL_{2m}, \Lambda_2 \oplus \Lambda_1^* \oplus \Lambda_1^* \oplus \Lambda_1^*) \) where \( GL^3_1 \) acts on \( \Lambda_1^* \oplus \Lambda_1^* \oplus \Lambda_1^* \) as independent scalar multiplications. Since the isotropy subgroup of \( (GL_{2m}, \Lambda_2) \) is \( SP_m \), we have result by Proposition 2.9.

Q.E.D.
Lemma 2.11. - We have $\ell = 1$ for $(GL_{2m+1}, A_2 \oplus A_1)$.

Proof. - The isotropy subgroup

$$H = \{ A \in GL_{2m+1} ; (AJ^t A, Ae_1) = (J', e_1) \}$$

at

$$\xi = (J' = \begin{pmatrix} 0 & 1 \\ 0 & J \end{pmatrix}, \quad e_1 = \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix}, \quad J = \begin{pmatrix} -I_m & 0 \\ 0 & -I_m \end{pmatrix},$$

is given by

$$H = \{ \begin{pmatrix} 1 & 0 \\ 0 & A' \end{pmatrix} ; \quad A' \in Sp_m \} \cong Sp_m.$$

Since $\text{Ker} \rho = \{1\}$ and $H^1(k, GL_{2m+1}) = \{1\}$, we have $G(\mathbb{k})/\mathcal{Y}(\mathbb{k}) = H^1(k, Sp_m) = \{1\}$ by Corollary 1.3 Q.E.D.

Proposition 2.12. - We have $\ell = 1$ for $(GL_1^4 \times SL_{2m+1}, A_2 \oplus A_1 \oplus (A_1 \oplus A_1)^{*})$.

Proof. - It is enough to show $\ell = 1$ when

$$\bar{G} = GL_1 \times GL_1 \times GL_1 \times GL_{2m+1},$$

$$\rho = (1 \otimes 1 \otimes 1) \otimes A_2 + A_1 \otimes 1 \otimes 1 \otimes A_1$$

$$+ (1 \otimes A_1 \otimes 1 + 1 \otimes 1 \otimes A_1) \otimes A_1^{(*)} .$$

A generic isotropy subgroup of $(1 \otimes 1 \otimes 1) \otimes A_2 + A_1 \otimes 1 \otimes 1 \otimes A_1$ is

$$\bar{G} = \{(\alpha, \beta, \gamma, \zeta) \in G ; \quad A \in Sp_m \}$$

(cf. Lemma 2.11). Hence, by Proposition 1.5 and Lemma 2.11, it is enough to show $\ell = 1$ for $\bar{G} = GL_1 \times GL_1 \times GL_1 \times Sp_m$, $\rho = A_1^{(*)} \otimes (A_1 \otimes 1 + 1 \otimes A_1) \otimes 1 + 1 \otimes (A_1 \otimes 1 + 1 \otimes A_1)$. One can prove that $\ell = 1$ for $(GL_1 \times Sp_m, A_1 \otimes (A_1 + A_1))$ similarly as Lemma 2.7. Note that $\bar{G} \cong Sp_{m-1} \times \text{Ker} \rho$ in this case. Then our assertion is clear.

Q.E.D.

Proposition 2.13. - We have $\ell = 1$ for $(GL_1^4 \times SL_{2m+1}, A_2 \oplus A_1^{*} \oplus A_1^{*} \oplus A_1^{*})$. 
**Proof.** It is enough to show $\ell = 1$ when

$$\tilde{G} = GL_1 \times GL_1 \times GL_1 \times GL_{2m+1},$$

$\rho = (1 \otimes 1 \otimes 1) \otimes \Lambda_2 + (\Lambda_1 \otimes 1 \otimes 1 + 1 \otimes \Lambda_1 \otimes 1 + 1 \otimes 1 \otimes \Lambda_1) \otimes \Lambda_2^\iota$. The isotropy subgroup of $(1 \otimes 1 \otimes 1) \otimes \Lambda_2$ at

$$J' = \begin{pmatrix} 0 & 0 \\ 0 & J \end{pmatrix}$$

is

$$H = \left\{ \begin{pmatrix} \alpha & 0 \\ A' & A \end{pmatrix} \in GL_{2m+1}; A \in Sp_m \right\}.$$

By Proposition 1.5 and Lemma 2.11, it is enough to show $\ell = 1$ for a P.V. given by

$$X \mapsto \begin{pmatrix} \alpha^{-1} & B \\ 0 & A \end{pmatrix} X \begin{pmatrix} \beta & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \delta \end{pmatrix} = \begin{pmatrix} (\alpha^{-1}x_1, \alpha^{-1}x_2, \alpha^{-1}x_3) + BZ \\ AZ \begin{pmatrix} \beta \\ \gamma \\ \delta \end{pmatrix} \end{pmatrix}$$

for $X = \left[ x_1, x_2, x_3 \right] \in M(2m+1, 3), A \in Sp_m$. Now by Proposition 2.9, any point $X = \left[ x_1, x_2, x_3 \right] / Z$ of $Y(k)$ is $G(k)$-equivalent to

$$X_0 = \left[ z_1, z_2, z_3 \right] / Z_0,$$

where

$$Z_0 = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \ldots & 0 \end{pmatrix} (= (e_1, e_{m+1}, e_1 + e_2 + e_{m+1}),$$

cf. p. 81 in [4]) and $(z_1, z_2, z_3) \in k^3$. Put $B = (b_1, \ldots, b_{2m})$ with $b_1 = -z_1, b_2 = z_1 + z_2 - z_3, b_{m+1} = -z_2, b_j = 0$ for all $j \neq 1, 2, m+1$. Then we have

$$\begin{pmatrix} 1 & B \\ 0 & I_{2m} \end{pmatrix} X_0 I_3 = \begin{pmatrix} 0 \\ Z_0 \end{pmatrix}.$$

This implies that $G(k)$ acts on $Y(k)$ transitively. Q.E.D.

**Proposition 2.14.** We have $\ell = 1$ for $(GL_1^2 \times SL_{2m+1}, \Lambda_2 \oplus \Lambda_2^\iota \oplus \Lambda_1^\iota)$ and $(GL_1^2 \times SL_{2m+1}, \Lambda_2 \oplus \Lambda_1^\iota)$.

**Proof.** By Propositions 1.5 and 2.12, we have our result. Q.E.D.
PROPOSITION 2.15. - We have \( \ell = 1 \) for \((GL_1^2 \times SL_2m+1, \Lambda_2 \oplus \Lambda_2)\).

Proof. - A direct calculation shows that the isotropy subgroup \( G_{\xi} \) of \((SL_2m+1, \Lambda_2 \oplus \Lambda_2)\) at

\[
\xi = \begin{pmatrix}
0 & I_m & 0 \\
-I_m & 0 & I_m \\
0 & I_m & 0
\end{pmatrix}
\]

is given by

\[
G_{\xi} = \begin{pmatrix}
I_{m+1} & 0 \\
\begin{array}{cccc}
a_1 & a_2 & \cdots & a_{m+1} \\
a_2 & a_3 & \cdots & a_{m+2} \\
\vdots & \vdots & & \vdots \\
a_m & a_{m+1} & \cdots & a_{2m}
\end{array} & I_m
\end{pmatrix} \sim G_{a}^{2m}.
\]

Since \( H^1(k, SL_2m+1) = \{1\} \), \( \text{Ker } \rho = \{1\} \), and \( H^1(k, G_2^{2m}) = \{1\} \), we have \( \ell = 1 \) for \((SL_2m+1, \Lambda_2 \oplus \Lambda_2)\) by Corollary 1.3. Hence we obtain our result. Q.E.D.

PROPOSITION 2.16. - We have \( \ell = 1 \) for \((GL_1^2 \times SL_5, \Lambda_2 \oplus \Lambda_2 \oplus \Lambda_2^\flat)\).

Proof. - Let \( H \) be the generic isotropy subgroup of \((GL_1^2 \times SL_5, \Lambda_2 \oplus \Lambda_2)\) at \( \xi = (e_2 \wedge e_3 + e_1 \wedge e_4, e_1 \wedge e_3 + e_2 \wedge e_5)\). Clearly \( H \) contains \( \{(e_1, e_2, \text{diag}(e_1^{-1}e_2^{-2}, e_1^{-1}e_1^{-2}e_2^{-1}, e_1e_2, e_2^2, e_3^2)) \in GL_1^2 \times SL_5\} \) and

\[
\{(1,1, \begin{pmatrix} I_2 & A \\ 0 & I_3 \end{pmatrix}); A = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_2 & \gamma_3 & \gamma_4 \end{pmatrix}, (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \in G_{a}^4\}.
\]

By Corollary 1.6 and Proposition 2.15, it is enough to show that \( \ell = 1 \) for \((GL_1 \times H, \Lambda_1 \otimes \Lambda_1^\flat)\). An element \( x = (x_1, x_2, x_3, x_4, x_5) \in \text{Aff}^5 \) is a generic point of \((GL_1 \times H, \Lambda_1 \otimes \Lambda_1^\flat)\) if and only if \( x_1x_2 \neq 0 \) (cf. Proposition 1.1 in [5]). Assume that \( x \) is in \( Y(k) \), then by the action of \( g_1 = (e, \text{diag}(e_1^{-1}e_2^{-2}, \ldots, e_5^2)) \in H(k) \) with \( e = x_1/x_2^2, e_1 = 1, e_2 = x_2/x_1 \), we may assume that \( x_1 = x_2 = 1 \). Now it is transformed to \( x_0 = ((1,1,0,0,0) \) by the action of

\[
g_2 = (1, \begin{pmatrix} I_3 & A \\ 0 & I_2 \end{pmatrix}) \in H(k).
\]
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with $A = \begin{bmatrix} x_3, & x_4 - x_3, & 0 \\ 0, & x_3, & x_2 \end{bmatrix}$. Thus $GL_1(k) \times H(k)$ acts on $Y(k)$ transitively.

Q.E.D.

**Proposition 2.17.** We have $\ell = 1$ for $(GL^2_1 \times \text{Spin}_n, \Lambda \oplus \Lambda)$ with $n = 8$ and 10.

**Proof.** Let $n$ be 8 or 10. Then by Theorem 1.8, we have $l = 1$ for $(GL_1 \times \text{Spin}_n, \text{the vector rep.})$ and $(GL_1 \times \text{Spin}_{n-1}, \text{the spin rep.})$. Since the restriction of a half-spin representation of $\text{Spin}_n$ to a generic isotropy subgroup of $(GL_1 \times \text{Spin}_n, \text{the vector rep.})$ gives $(GL_1 \times \text{Spin}_{n-1}, \text{the spin rep.})$, we have our result by Corollary 1.6.

Q.E.D.

**Proposition 2.18.** We have $\ell = 1$ for $(GL^2_1 \times \text{Spin}_{10}, \Lambda \oplus \Lambda)$ where $\Lambda = \text{the even half-spin representation}.$

**Proof.** Prof. J.-I. Igusa proved that $\ell = 1$ for $(GL_1 \times \text{Spin}_{10}, \Lambda \oplus (\Lambda \oplus \Lambda))$ (See p. 14 in [1]) and our assertion is obvious by Remark 1.7.

Q.E.D.

**Theorem 2.19.** All non-irreducible simple P.V.'s with universally transitive open orbits are given as follows:

1. $(GL_1^{k+1} \times SL_n, \Lambda_1 \oplus \cdots \oplus \Lambda_1 \oplus \Lambda_1^{(\ast)})$ ($1 \leq k \leq n, n \geq 2$),

2. $(GL_1^{k+1} \times SL_n, \Lambda_2 \oplus \Lambda_1^{(\ast)} \oplus \cdots \oplus \Lambda^{(\ast 1)})$ ($1 \leq k \leq 3, n \geq 4$),

except $(GL_1^2 \times SL_n, \Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1^{(\ast)})$ with $n = \text{odd}$.

3. $(GL_1^2 \times SL_{2m + 1}, \Lambda_2 \oplus \Lambda_2)$ for $m \geq 2$.

4. $(GL_1^3 \times SL_5, \Lambda_2 \oplus \Lambda_2 \oplus \Lambda_1^{(\ast)})$.

5. $(GL_1^k \times Sp_n, \Lambda_1 \oplus \cdots \oplus \Lambda_1)$ ($k = 2, 3$).

6. $(GL_1^2 \times \text{Spin}_n, \text{a half-spin rep.} \oplus \text{the vector rep.})$ with $n = 8, 10$.

7. $(GL_1^2 \times \text{Spin}_{10}, \Lambda \oplus \Lambda)$ where $\Lambda = \text{the even half-spin representation}$.

**Proof.** By Proposition 2.4, 2.5, 2.8-2.10; 2.12-2.18, we have our result.

Q.E.D.
COROLLARY 2.20. — All non-irreducible regular simple P.V.'s with universally transitive open orbits are given as follows:

(1) \((GL_1^n \times SL_n, \Lambda_1 \oplus \Lambda_1^*)\).

(2) \((GL_1^n \times SL_n, \Lambda_1 \oplus \Lambda_1^*)\).

(3) \((GL_1^{n+1} \times SL_n, \Lambda_1 \oplus \Lambda_1^*)\).

(4) \((GL_1^3 \times SL_{2m}, \Lambda_2 \oplus \Lambda_1^* \oplus \Lambda_1^*)\).

(5) \((GL_1^3 \times SL_{2m+1}, \Lambda_2 \oplus \Lambda_1)\).

(6) \((GL_1^4 \times SL_{2m+1}, \Lambda_2 \oplus \Lambda_1 \oplus (\Lambda_1 \oplus \Lambda_1)^*)\).

(7) \((GL_1^4 \times SP_n, \Lambda_1 \oplus \Lambda_1^*)\).

(8) \((GL_1^4 \times \text{Spin}_n, \text{a half-spin rep.} \oplus \text{the vector rep.})\) with \(n = 8, 10\).

(9) \((GL_1^4 \times \text{Spin}_{10}, \Lambda \oplus \Lambda)\) where \(\Lambda = \text{the even half-spin representation}\).

3. 2-Simple P.V.'s of Type I with Universally Transitive Open Orbits.

THEOREM 3.1. ([5]). — All non-irreducible 2-simple P.V.'s \((GL_1^k \times G (= G_1 \times G_2), \rho (= \rho_1 \oplus \ldots \oplus \rho_k))\) of Type I, which do not contain a regular irreducible P.V.'s with \(\varepsilon \geq 2\), are castling-equivalent to the following P.V.'s:

(I) (1) \(G = SL_{2m+1} \times SL_2, \quad \rho = \Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1 (+ T)\) with \(T = 1 \otimes \Lambda_1 (+ 1 \otimes \Lambda_1)\).

(2) \(G = \text{Spin}_{10} \times SL_2, \quad \rho = \text{a half-spin rep.} \otimes \Lambda_1 + 1 \otimes \Lambda_1 (+ T)\) with \(T = 1 \otimes \Lambda_1 (+ 1 \otimes \Lambda_1)\).

(II) (3) \(G = SO_n \times SL_{n-1}, \quad \rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_1^* (n = \text{even})\).

(4) \(G = SL_4 \times SL_5, \quad \rho = \Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^*\).

(5) \(G = \text{Spin}_7 \times SL_7, \quad \rho = \text{the spin rep.} \otimes \Lambda_1 + 1 \otimes \Lambda_1^*\).

(6) \(G = \text{Spin}_8 \times SL_7, \quad \rho = \text{the vector rep.} \otimes \Lambda_1 + \text{a half spin rep.} \otimes 1 + 1 \otimes \Lambda_1^*\).

(III) (7) \(G = Sp_n \times SL_m, \quad \rho = \Lambda_1 \otimes \Lambda_1 + T, \quad \text{with} \quad T = 1 \otimes \Lambda_1^* \quad (\Lambda_1^* + \Lambda_1^* + \Lambda_1^*) (1 \leq k \leq 3) \text{ except } 1 \otimes (\Lambda_1 + \Lambda_1 + \Lambda_1^*)\) with \(m = \text{odd} \quad \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^* + \Lambda_1^* (0 \leq k \leq 2) \text{ except } \Lambda_1 \otimes 1 + 1 \otimes (\Lambda_1 + \Lambda_1^*)\) with \(m = \text{odd}, \ 1 \otimes \Lambda_2 (m = \text{odd}), \ 1 \otimes (\Lambda_2 + \Lambda_1^*) (m = 5)\).

(8) \(G = Sp_n \times SL_{2m+1}, \quad \rho = \Lambda_1 \otimes \Lambda_1 + (\Lambda_1 + \Lambda_1 \otimes 1)\).

(9) \(G = Sp_n \times SL_2, \quad \rho = \Lambda_1 \otimes 2 \Lambda_1 + 1 \otimes \Lambda_1\).
(10) $G = SL_5 \times SL_2, \quad \rho = \Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1 (\Lambda_1 + \Lambda_1)$.
(11) $G = SL_5 \times SL_2, \quad \rho = \Lambda_2 \otimes \Lambda_1 + \Lambda_1^* \otimes 1 (\Lambda_1 + \Lambda_1)$.
(12) $G = SL_5 \times SL_2, \quad \rho = \Lambda_2 \otimes \Lambda_1 + (\Lambda_1 + \Lambda_1) \otimes 1$.
(13) $G = SL_5 \times SL_8, \quad \rho = \Lambda_2 \otimes \Lambda_1 + (\Lambda_1^* + \Lambda_1) \otimes 1$.
(14) $G = SL_5 \times SL_9, \quad \rho = \Lambda_2 \otimes \Lambda_1 + (\Lambda_1^* + \Lambda_1) \otimes 1$.
(15) $G = SL_7 \times SL_2, \quad \rho = \Lambda_2 \otimes \Lambda_1 + (\Lambda_1 + \Lambda_1) \otimes 1$.
(16) $G = SL_9 \times SL_2, \quad \rho = \Lambda_2 \otimes \Lambda_1 + \Lambda_1^* \otimes 1$.
(17) $G = Spin_{10} \times SL_n, \quad (n = 14, 15), \quad \rho = a \text{ half-spin rep.}$

**Proposition 3.2.** - We have $\ell = 1$ for $P.V's$ in (I), i.e., (1) and (2) in Theorem 3.1.

**Proof.** - For (1), it is enough to show $\ell = 1$ when $\rho = \Lambda_2 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1)$.
Since we have $\ell = 1$ for $(GL_1 \times SL_2, \Lambda_1 + \Lambda_1 + \Lambda_1)$ and $(SL_{2m} \times (I_2, \Lambda_2 \otimes \Lambda_1) = (SL_{2m+1}, \Lambda_2 \otimes \Lambda_2)$, we have our result by Corollary 1.6 and the proof of Proposition 2.15. For (2), one can prove similarly as above by the proof of Proposition 2.18.

**Q.E.D.**

**Proposition 3.3.** - We have $\ell \geq 2$ for $P.V's$ in (II), i.e., (3)-(6) in Theorem 3.1.

**Proof.** - For (3), the $GL_{n-1}$ part of a generic isotropy subgroup $H$ of $(SO_n \times GL_{n-1}, \Lambda_1 \otimes \Lambda_1)$ is $O_{n-1}$ (cf. p. 109 in [3]). Since $\ell \geq 2$ for $(GL_1 \times O_{n-1}, \Lambda_2 \otimes \Lambda_1) (n-1 = \text{odd})$, we have our result by Proposition 1.5. For remaining $P.V.'s$, since $(Spin_7, \text{the spin rep.}) \subset (SO_8, \Lambda_1) \simeq (Spin_8, \text{the vector rep.})$ and $(SL_4, \Lambda_2) \simeq (SO_6, \Lambda_1)$, we have our result.

**Q.E.D.**

**Sublemma 3.4.** - Let $V = K^{2n}$ with $\langle v, v' \rangle = \langle v J v' \rangle$ where $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. Assume that $\{v_1, \ldots, v_r\}$ and $\{u_1, \ldots, u_r\}$ are linearly independent subsets of $V$ satisfying $\langle v_i, v_j \rangle = \langle u_i, u_j \rangle$ for $i, j = 1, \ldots, r$ with $r < 2n$. Then there exist $v_{r+1}$ and $u_{r+1}$ such that (1) $\{v_1, \ldots, v_{r+1}\}$ and $\{u_1, \ldots, u_{r+1}\}$ are linearly independent, (2) $\langle v_i, v_j \rangle = \langle u_i, u_j \rangle$ for all $i, j = 1, \ldots, r + 1$. 
Proof. — (I) The case when \( \langle v_1, \ldots, v_r \rangle \not\subset \langle v_1, \ldots, v_r \rangle \). Take \( u_r \) such that \( u_{r+1} \notin \langle u_1, \ldots, u_r \rangle \). Since \( \{v_1, \ldots, v_r\} \) is linearly independent, the linear equation \( t(v_1, \ldots, v_r)Ju = t(u_1, \ldots, u_r)Ju_{r+1} \) (i.e. \( \langle v_i, v \rangle = \langle u_i, u_{r+1} \rangle \) for \( i = 1, \ldots, r \)) has a solution \( v_0 \), and the set \( v \) solution is given by \( v_0 + \langle v_1, \ldots, v_r \rangle \) (\( \not\subset \langle v_1, \ldots, v_r \rangle \)). Hence there exists \( v_{r+1} \notin \langle v_1, \ldots, v_r \rangle \) such that \( \langle v_i, v_{r+1} \rangle = \langle u_i, u_{r+1} \rangle \) for \( i = 1, \ldots, r \).

(II) The case when \( \langle v_1, \ldots, v_r \rangle \not\subset \langle v_1, \ldots, v_r \rangle \). Take \( u_{r+1} \notin \langle v_1, \ldots, v_r \rangle \). Assume that any solution \( u \) of \( t(u_1, \ldots, u_r)Ju = t(v_1, \ldots, v_r)Ju_{r+1} \) belongs to \( \langle u_1, \ldots, u_r \rangle \). Let \( u = a_1u_1 + \cdots + a_ru_r \) be a solution. Since \( t(u_iJu_j = t(v_iJu_j \) for \( i, j = 1, \ldots, r \), we have \( t(v_1, \ldots, v_r)Ju_{r+1} = t(u_1, \ldots, u_r)Ju_{r+1} \), i.e., \( v_{r+1} = a_1v_1 + \cdots + a_rv_r \in \langle v_1, \ldots, v_r \rangle \) and hence \( v_{r+1} = \langle v_1, \ldots, v_r \rangle \) a contradiction. Hence there exists \( u_{r+1} \in V \) satisfying \( u_{r+1} \notin \langle u_1, \ldots, u_r \rangle \) and \( t(u_1, \ldots, u_r)Ju_{r+1} = t(v_1, \ldots, v_r)Ju_{r+1} \). Q.E.D

**Lemma 3.5.** — Let \( \{v_1, \ldots, v_r\} \) and \( \{u_1, \ldots, u_r\} \) are linearly independent subsets of \( V = K^{2n} \) satisfying \( \langle v_i, v_j \rangle = \langle u_i, u_j \rangle \) for \( i, j = 1, \ldots, r \). Then there exists an element \( g \) of the symplectic group \( Sp_n(K) \) such that \( gv_i = u_i \) for \( i = 1, \ldots, r \).

Proof. — By Sublemma 3.4, there exist basis \( \{v_1, \ldots, v_r, \ldots, v_{2n}\} \) and \( \{u_1, \ldots, u_r, \ldots, u_{2n}\} \) satisfying \( \langle v_i, v_j \rangle = \langle u_i, u_j \rangle \). Define an element \( g \) of \( GL_{2n} \) by \( (v_1, \ldots, v_{2n})g = (u_1, \ldots, u_{2n}) \). Then it is clear that \( g \in Sp_n(K) \). Q.E.D

**Lemma 3.6.** — Let \( \Omega \) be the universal domain and \( K \) a subfield. For \( 2n \geq m \), put \( W = \{v \in M_{2n,m}(\Omega) ; \text{rank } v = m\} \) and \( W' = \{w \in Alt_m(\Omega) ; \text{rank } w \text{ is maximal}\} \). Define a map \( \psi : W \to Alt_m(\Omega) \) by \( \psi(v) = \langle \langle v, v \rangle \rangle \) for \( v = (v_1, \ldots, v_{2m}) \in W \). Then \( \psi(W) = W' \) and \( \psi(W(K)) = W'(K) \).

Proof. — Note that \( W \) (resp. \( W' \)) is the Zariski dense orbit of \( (Sp_n \times GL_m, \Lambda_1 \otimes \Lambda_1, M_{2n,m}(\Omega)) \) (resp. \( (GL_m, \Lambda_2, Alt_m(\Omega)) \)). Since \( \psi(Av'B) = B\psi(v)'B \) for any \( (A, B) \in Sp_n \times GL_m \), \( \psi(W) \) is an orbit of \( (GL_m, \Lambda_2) \). Let \( X_0 \) be the generic point of \( (Sp_n \times GL_m, \Lambda_1 \otimes \Lambda_1) \) given in p. 101 in [3]. Then we have \( \psi(X_0) = J(m=\text{even}) \) or \( \psi(X_0) = \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} \) (\( m=\text{odd} \)), i.e., \( \psi(X_0) \) is a generic point of \( (GL_m, \Lambda_2) \). Hence \( \psi(W) = W' \). Since \( \psi \) is defined over the prime field, we have
\[ \psi(W(K)) \subset W'(K). \] Since \( \ell' = 1 \) for \( (GL_m, \Lambda_2) \), \( W'(K) \) is a single \( \Lambda_2(GL_m)(K) \)-orbit. Since \( \psi(W(K)) \) is \( \Lambda_2(GL_m)(K) \)-admissible, we have \( \psi(W(K)) = W'(K) \). Q.E.D.

**Proposition 3.7.** We have \( \ell' = 1 \) for \( (Sp_n \times G, \Lambda_1 \otimes \rho) \) (\( m = \deg \rho \leq 2n \)) if and only if \( \ell' = 1 \) for \( (G, \Lambda^2(\rho)) \).

**Proof.** Let \( Y(\subset W \subset M_{2n,m}(\Omega)) \) and \( Y'(\subset W' \subset Alt_{m}(\Omega)) \) be the Zariski-dense orbits of \( (Sp_n \times G, \Lambda_1 \otimes \rho) \) and \( (G, \Lambda^2(\rho)) \) respectively. Then the map \( \psi : W \rightarrow W' \) in Lemma 3.6 gives the surjective \( Sp_n \times G \)-equivariant map \( \psi : Y \rightarrow Y' \). Clearly we have \( \psi(Y(K)) \subset Y'(K) \). Take any element \( x \) of \( Y'(K) \). Since \( \psi(W(K)) = W'(K) \Rightarrow Y'(K) \), there exists \( v = (v_1, \ldots, v_m) \in W(K) \) such that \( \psi(v) = x \). On the other hand, we have \( \psi(Y) = Y' \Rightarrow Y'(K) \) there exists \( u = (u_1, \ldots, u_m) \in Y \) such that \( \psi(u) = x \). By Lemma 3.5, there exists \( g \in Sp_n \) satisfying \( v = gu \in Y \), i.e., \( v \in Y \cap W(K) = Y(K) \). Hence \( \psi : Y(K) \rightarrow Y'(K) \) is surjective. By Lemma 3.5, each fibre is \( Sp_n(K) \)-homogeneous. Thus the orbits in \( Y(K) \) and \( Y'(K) \) correspond bijectively. Q.E.D.

**Corollary 3.8.** (1) We have \( \ell' = 1 \) for \( (Sp_n \times G, \Lambda_1 \otimes \rho + 1 \otimes \sigma) \) (\( \deg \rho \leq 2n \)) if and only if \( \ell' = 1 \) for \( (G, \Lambda^2(\rho) + \sigma) \).

(2) We have \( \ell' = 1 \) for \( (Sp_n \times G \times GL_1, \Lambda_1 \otimes \rho \otimes 1 + \Lambda_1 \otimes 1 \otimes \Lambda_1 + 1 \otimes \sigma \otimes 1) \) (\( \deg \rho \leq 2n - 1 \)) if and only if \( \ell' = 1 \) for \( (G \times GL_1, \Lambda^2(\rho) \otimes 1 + \rho \otimes \Lambda_1 + \sigma \otimes 1) \).

(3) We have \( \ell' = 1 \) for \( (GL_2^2 \times Sp_n \times GL_{2m+1}, 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1 + (\Lambda_1 \otimes 1 + 1 \otimes \Lambda_1) \otimes \Lambda_1 \otimes 1) (2m + 3 \leq 2n) \) if and only if \( \ell' = 1 \) for \( (GL_1 \times GL_{2m+1}, 1 \otimes \Lambda_2 + (\Lambda_1 + \Lambda^2) \otimes \Lambda_1) \).

**Proof.** (1) is obvious. Since \( \Lambda^2(\rho \otimes 1 + 1 \otimes \Lambda_1) = \Lambda^2(\rho) \otimes 1 + \rho \otimes \Lambda_1 \), we have (2). Since \( \Lambda^2(1 \otimes \Lambda_1 + \Lambda_1 \otimes 1 \otimes 1 + 1 \otimes \Lambda_1) \) for \( GL_1^2 \times GL_{2m+1} \), is \( GL_2^2 \times GL_{2m+1} \), \( 1 \otimes 1 \otimes \Lambda_2 + (\Lambda_1 \otimes 1 + 1 \otimes \Lambda_1) \otimes \Lambda_1 \) (\( \Lambda_1 \otimes \Lambda_1 \otimes 1) \), we have (3) by Proposition 1.5. Q.E.D.

**Proposition 3.9.** For P.V.'s in (III) in Theorem 3.1, we have \( \ell' = 1 \) for (7), (8) and \( \ell' \geq 2 \) for (9).

**Proof.** By Theorem 2.19 and Corollary 3.8, we have \( \ell' = 1 \) for (7). By Lemma 2.7, the proof of Proposition 2.12, and (3) of Corollary 3.8, we have \( \ell' = 1 \) for (8). Since \( (SL_3, \Lambda^2(\Lambda_1) = (SL_3, \Lambda_2) = (SL_3, \Lambda^2) \), we have \( (SO_3, \Lambda^2(\Lambda_1)) = (SO_3, \Lambda_1) \). Hence we have \( \ell' \geq 2 \) for \( (Sp_n \times GL_2, \Lambda_1 \otimes 2 \Lambda_1) = (Sp_n \times GO_3, \Lambda_1 \otimes \Lambda_1) \). Thus \( \ell' \geq 2 \) for (9). Q.E.D.
Lemma 3.10. - We have \( \ell \geq 2 \) for \((GL_{2m+1} \times GL_2, \Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1)(m=2,3)\).

Proof. - Assume that \( \ell = 1 \). Then, by Proposition 1.5, we have \( \ell = 1 \) for \((H \times GL_2, \Lambda_2 \otimes \Lambda_1)\) where

\[
H = \begin{bmatrix} 1 & A' \\ 0 & A \end{bmatrix} ; A \in GL_{2m}.
\]

Since \( \begin{bmatrix} 1 & A' \\ 0 & A \end{bmatrix} = \begin{bmatrix} 1 & -y \\ 0 & X \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -A' & A \end{bmatrix} = \begin{bmatrix} 0 & * \\ * & AXA \end{bmatrix} \), this implies \( \ell = 1 \) for \((GL_{2m} \times GL_2, \Lambda_2 \otimes \Lambda_1)\), which is a contradiction by Theorem 1.8.

Proposition 3.11. - We have \( \ell \leq 2 \) for any P.V. in (10) in Theorem 3.1.

Proof. - By Proposition 1.5 and Lemma 3.10, we have our result.

Q.E.D.

Proposition 3.12. - We have \( \ell = 1 \) for any P.V. in (11) in Theorem 3.1.

Proof. - It is enough to show \( \ell = 1 \) for \((GL_2^1 \times SL_5 \times SL_2, \Lambda_2 \otimes \Lambda_1 + \Lambda_1^* \otimes 1 + 1 \otimes (\Lambda_1 + \Lambda_1))\). Since \( \ell = 1 \) for \((GL_2^1 \times SL_2, \Lambda_1 + \Lambda_1)\), it is enough to show \( \ell = 1 \) for

\[
(GL_2^1 \times SL_5 \times SL_2, \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix}, \Lambda_2 \otimes \Lambda_1 + \Lambda_1^* \otimes 1) \approx (GL_2^1 \times SL_5, \Lambda_2 \otimes \Lambda_2 \otimes \Lambda_1^*).
\]

Thus we have our result by Theorem 2.19.

Q.E.D.

Proposition 3.13. - We have \( \ell = 1 \) for a P.V. (12) in Theorem 3.1.

Proof. - We shall prove that a generic isotropy subgroup of \((GL_1 \times GL_2 \times GL_2, 1 \otimes \Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1^* \otimes 1 + \Lambda_1 \otimes \Lambda_1^* \otimes 1)\) is \( \{1\} \). Then we have \( \ell = 1 \) by Corollary 1.4. The representation space \( V \) is given by \( V = \{(X,Y), Z; X, Y \in M_5, 'X = -X, 'Y = -Y, Z \in M_{5,2}\} \). Then the action is given by \( \rho(g)x = \{(AXA, AYA)^{t}B, A^{-1}Z(\alpha)\} \) for \( g = (\alpha, A, B) \in GL_1 \times GL_5 \times GL_2 \) and \( x = \{(X, Y), Z\} \in V \). Put \( x_0 = \{(X_0, Y_0), Z_0\} \) with \( X_0 = (-e_4, -e_5, 0, e_1 + e_5, e_2 - e_4) \), \( Y_0 = (0, e_4, e_5, -e_2 + e_5, -e_3 - e_4) \), \( Z_0 = (e_4, e_5) \) where \( e_1 = (0 \ldots 1 \ldots 0) \in \Omega^5 \). We shall calculate the isotropy subgroup...
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\[ H = \{ g \in GL_1 \times GL_5 \times GL_2 : \rho(g)x_0 = x_0 \} \]. One can easily check that 
\[ \rho^{-1}(X_0) = Z_0 \] if and only if \( A \) is of the form

\[
\begin{pmatrix}
A_1 & A_2 \\
0 & (0)
\end{pmatrix}
\]

We shall determine \((A, B)\) satisfying \((AX_0, AY_0 A')B = (X_0, Y_0)\) where \( A \) is of the above form. By comparing the components of \((1,4), (1,5), (2,4), (2,5), (3,4), (3,5), (4,5)\), we obtain
\[ b_{12} = \alpha^{-1} - b_{11}, \]
\[ b_{21} = \alpha^{-1} - b_{22}, \]
\[ a_{12} = c - a_{11}, \quad a_{13} = a_{11} - c, \quad a_{14} = c - a_{11}, \]
\[ a_{15} = \alpha b_{22} c - a_{11}, \quad a_{21} = c - a_{22}, \quad a_{23} = \alpha c^{-1} - a_{22}, \]
\[ a_{24} = a_{22} - \alpha b_{22} c, \quad a_{25} = a_{22} - \alpha b_{11} c, \quad a_{31} = a_{33} - \alpha c^{-1}, \]
\[ a_{32} = \alpha c^{-1} - a_{33}, \quad a_{34} = b_{11} c - a_{33}, \quad a_{35} = \alpha c^{-1} - a_{33}, \]
where \( c(b_{11} + b_{22} - \alpha^{-1}) = 1, A = (a_{ij}) \) and \( B = (b_{ij}) \). Then, by comparing the \((1,2), (1,3), (2,3)\) components, we obtain
\[ a_{11} = a_{22} = a_{33} = b_{11} = c = a = 1 \] Thus we have \( H = \{1\} \).

Q.E.D.

**Proposition 3.14.** — We have \( \ell \geq 2 \) for a P.V. (13) in Theorem 3.1.

**Proof.** — Let \( \mathfrak{s} \) be the \( sl_8 \)-part of the generic isotropy subalgebra of \((GL_1 \times SL_5 \times SL_8, \Lambda_1 \otimes \Lambda_2 \otimes \Lambda_1)\) at \( x_0 = (\omega_1, 2\omega_3, 2\omega_2, \omega_{10}, \omega_5 - \omega_8, \omega_4 - \omega_9, \omega_6, \omega_7)\) (see P.95 in [3]). Then its image by \( \Lambda^*_1 \) is given by

\[
\Lambda^*_1(\mathfrak{s}) = \left\{ \begin{pmatrix} A & B & 0 \\ 0 & A' & B' \\ 0 & 0 & a \end{pmatrix} \in M_8 \mid B = \begin{pmatrix} -2d_3 & 4d_2 & -2d_1 & 0 \\ -d_4 & d_3 & d_2 & -d_1 \\ 0 & -d_4 & 2d_3 & -d_2 \end{pmatrix} \right\},
\]

\[ B' = 2'(d_1, d_2, d_3, d_4), \quad a = -25t, \quad A = 15tI_3 + (2\Lambda_1)(C), \]

\[ A' = -5tI_4 + (3\Lambda_1)(C) \quad \text{for} \quad C \in sl_2. \]

Let \( H \) be any algebraic subgroup of \( GL_8 \) with \( \text{Lie}(H) = \Lambda^*_1(\mathfrak{s}) \). It is enough to show \( \ell \geq 2 \) for \((GL_1 \times H, \Lambda_1 \otimes \Lambda_1, \Omega^8)\). Since \( hAh^{-1} \in \Lambda^*_1(\mathfrak{s}) \), for any \( h \in H \) and \( A \in \Lambda^*_1(\mathfrak{s}) \), we have

\[ H \subset \left\{ \begin{pmatrix} h_1 & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\}
\]

by Schur's lemma. Since the normalizer of \( GO_3 \) is \( GO_3 \), we may assume that \( h_1 \in GO_3 \). Let \( x = (x_1, \ldots, x_8) \) be a point of \( Y(k) \) for
(GL_1 \times H, \Lambda_1 \otimes \Lambda_1, \Omega^8). Clearly we may assume that x_8 = 1. By the action of one parameter subgroups obtained from B and B' in \Lambda^*_1(\mathfrak{f}), we may also assume that x_4 = x_5 = x_6 = x_7 = 0. Let H_1 be the subgroup of H fixing x_4 = x_5 = x_6 = x_7 = 0 and x_8 = 1. Then the corresponding Lie subalgebra \mathfrak{s}_1 of \Lambda^*_1(\mathfrak{f}) consists of A of \Lambda^*_1(\mathfrak{f}) satisfying B = B' = 0. Since H_1 normalizes \mathfrak{s}_1, we have

\[ H_1 = \{ \begin{pmatrix} A & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}; \quad A \in GO_3 \} \]

and hence the action on (x_1, x_2, x_3)-space is \((GO_3, \Lambda_1)\) which \(l \geq 2\) by Theorem 1.8.

**Q.E.D.**

**Proposition 3.15.** — We have \(l \geq 2\) for any P.V. in (14) in Theorem 3.1.

**Proof.** — The generic isotropy subgroup of \((GL_5, \Lambda_2)\) is connected (see P.76 in [3]). Hence the generic isotropy subgroup H of its castling transform \((SL_5 \times GL_9, \Lambda_2 \otimes \Lambda_1)\) is connected and it is contained in

\[ \{ \begin{pmatrix} A & * \\ 0 & * \end{pmatrix}; \quad A \in GO_5 \} \]

(see the proof of Lemma 2.6 in [5]). Since \(l \geq 2\) for \((GO_5, \Lambda_1)\), we have \(l \geq 2\) for \((GL_1 \times H, \Lambda_1 \otimes \Lambda^*_1, \Omega^8)\). This proves our assertion.

**Q.E.D.**

**Proposition 3.16.** — We have \(l \geq 2\) for any P.V. in (15) in Theorem 3.1.

**Proof.** — For the first P.V. in (15), we have \(l \geq 2\) by Lemma 3.10. Now let H be the \(SL_7\) part of a generic isotropy subgroup of \((GL_1 \times SL_7 \times SL_2, \Lambda_2 \otimes \Lambda_1)\). Then we have

\[
\text{Lie}(H) = \{ \begin{pmatrix} A_1 & 0 \\ * & A_2 \end{pmatrix}; \quad A_1 = 3\Lambda^*_1(C) + 3tI_4, \\
A_2 = 2\Lambda_1(C) - 4tI_3 \text{ for } C \in \mathfrak{sl}_2 \}
\]
(see Lemma 1.4 in [5]). By the fact that the normalizer of $GO_3$ is $GO_3$ and by Schur's lemma, we have

$$H \subset \{ \begin{pmatrix} \ast & 0 \\ \ast & A \end{pmatrix}; A \in GO_3 \}. $$

Since $\ell \geq 2$ for $(GO_3, \Lambda_1)$, we have $\ell \geq 2$ for $(GL_1 \times H, \Lambda_1 \otimes \Lambda_1^*, \Omega^7)$ and hence $\ell \geq 2$ for the latter P.V.'s in (15).

Q.E.D.

**Proposition 3.17.** - We have $\ell \geq 2$ for a P.V. (16) in Theorem 3.1.

**Proof.** - Let $H$ be the $SL_9$-part of a generic isotropy subgroup of $(GL_1 \times SL_9 \times SL_2, \Lambda_1 \otimes \Lambda_2 \otimes \Lambda_1)$. Then, similarly as the proof of Proposition 3.16, we have

$$H \subset \{ \begin{pmatrix} \ast & 0 \\ \ast & A \end{pmatrix}; A \in 3\Lambda_1(GL_2) \}. $$

Since $\ell \geq 2$ for $(GL_2, 3\Lambda_1)$, we have $\ell \geq 2$ for $(GL_1 \times H, \Lambda_1 \otimes \Lambda_1^*, \Omega^7)$ and hence we obtain our result.

Q.E.D.

**Proposition 3.18.** - We have $\ell \geq 2$ for $(GL_2^2 \times Spin_{10} \times SL_{15}$, a half-spin rep. $\otimes \Lambda_1 + 1 \otimes \Lambda_1^*)$.

**Proof.** - Let $H$ be the $SL_{15}$-part of the generic isotropy subgroup of $(GL_1 \times Spin_{10} \times SL_{15}$, a half-spin rep. $\otimes \Lambda_1$) at $X_0 = (e_1 e_5, e_2 e_5, e_3 e_5, e_4 e_5, e_2 e_3 e_4 e_5, -e_1 e_3 e_4 e_5, e_1 e_2 e_4 e_5, -e_1 e_2 e_3 e_4, e_1 e_2, e_1 e_3, e_1 e_4, -e_3 e_4, e_2 e_3, -e_2 e_3)$. Then we have

$$\text{Lie}(H) = \{ \begin{pmatrix} A_1 & 0 \\ \ast & A_2 \end{pmatrix} \in M_{15}, A_1 = \Lambda(B), A_2 = \Lambda'(B) \text{ for } B \in o_{7} \}$$

where $\Lambda$(resp. $\Lambda'$) is the spin (resp. the vector) representation of $o_{7}$. By the fact that the normalizer of $GO_7$ is $GO_7$ and by Schur's lemma, we have $H \subset \{ \begin{pmatrix} \ast & 0 \\ \ast & A \end{pmatrix}; A \in GO_7 \}$. Since $\ell \geq 2$ for $(GO_7, \Lambda_1)$, we have $\ell \geq 2$ for $(GL_1 \times H, \Lambda_1 \otimes \Lambda_1^*)$. This implies our assertion.

Q.E.D.

**Proposition 3.19.** - We have $\ell \geq 2$ for $(GL_2^2 \times Spin_{10} \times SL_{14}$, a half-spin rep. $\otimes \Lambda_1 + 1 \otimes \Lambda_1^*)$.

**Proof.** - Let $H$ be the $SL_{14}$-part of a generic isotropy subgroup of $(GL_1 \times Spin_{10} \times SL_{14}, \Lambda_1 \otimes$ a half-spin rep. $\otimes \Lambda_1$). By checking the
weights, one obtains $\text{Lie}(H) = \text{Lie}(G_2 \otimes SL_2)$. Let $G$ be the image of $(GL_1 \times H, \Lambda_1 \otimes \Lambda_\ell)$. Then we have $1 \rightarrow GL_1 \rightarrow G \rightarrow \text{Aut}(G_2 \otimes SL_2) \rightarrow 1$ (exact) and hence $G$ is connected. Since $G \cong G_2 \otimes GL_2$ and $\dim G = \dim G_2 \otimes GL_2$, we have $(GL_1 \times H, \Lambda_1 \otimes \Lambda_\ell) \cong (G_2 \times GL_2, \Lambda_2 \otimes \Lambda_1, \Omega^7 \otimes \Omega^2)$ which has $\ell \geq 2$ by Theorem 1.8. This completes the proof. Q.E.D.

**Theorem 3.20.** All non-irreducible 2-simple P.V.'s $(GL_1^k \times G, \rho (= \rho_1 \oplus \cdots \oplus \rho_k))$ of type I with universally transitive open orbits are given as follows:

1. $G = SL_{2m+1} \times SL_2, \quad \rho = \Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1(+)T$ with $T = 1 \otimes \Lambda_1(+)T_1$.
2. $G = SL_5 \times SL_2, \quad \rho = \Lambda_2 \otimes \Lambda_1 + (\Lambda_1^* + \Lambda_1^* \otimes 1)$.
3. $G = SL_5 \times SL_2, \quad \rho = \Lambda_2 \otimes \Lambda_1 + (\Lambda_1^* + \Lambda_1^* \otimes 1)$.
4. $G = Sp_n \times SL_m, \quad \rho = \Lambda_1 \otimes \Lambda_1 + T, \quad$ with $T = 1 \otimes (\Lambda_1^* + \cdots + \Lambda_1^*)$ except $1 \otimes (\Lambda_1^* + \Lambda_1^* \otimes 1)$ with $m$ odd, $\Lambda_1 \otimes 1 + 1 \otimes (\Lambda_1^* + \cdots + \Lambda_1^*)$ except $\Lambda_1 \otimes 1 + 1 \otimes (\Lambda_1^* + \Lambda_1^* \otimes 1)$ with $m = odd, 1 \otimes \Lambda_2(m=odd), 1 \otimes (\Lambda_2 + \Lambda_1^*)$.
5. $G = Sp_n \times SL_{2m+1}, \quad \rho = \Lambda_1 \otimes \Lambda_1 + (\Lambda_1 + \Lambda_1) \otimes 1$.
6. $G = Spin_{10} \times SL_2, \quad \rho = a$ half-spin rep. $\otimes \Lambda_1 + 1 \otimes \Lambda_1(+)T$ with $T = 1 \otimes \Lambda_1(+)T_1$.

**Corollary 3.21.** All non-irreducible regular 2-simple P.V.'s of type I with universally transitive orbits are given as follows:

1. $(GL_1^3 \times SL_5 \times SL_2, \Lambda_2 \otimes \Lambda_1 + (\Lambda_1^* + \Lambda_1^* \otimes 1))$.
2. $(GL_1^3 \times Sp_n \times SL_{2m}, \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1^* + \Lambda_1^*))$.
3. $(GL_1^3 \times Sp_n \times SL_{2m+1}, \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1)$.
4. $(GL_1^3 \times Sp_n \times SL_{2m+1}, \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1)^*)$.
5. $(GL_1^3 \times Spin_{10} \times SL_2, a$ half-spin rep $\otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1))$.
6. $(GL_1^3 \times Spin_{10} \times SL_2, a$ half-spin rep $\otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1))$.

**Corollary 3.22.** Any non-regular irreducible P.V., which is not castling-equivalent to $(Sp_n \times GO_3, \Lambda_1 \otimes \Lambda_1)$, has the universally transitive open orbit.

**Proof.** By Theorem 2.19 and the proof of Proposition 3.9, we have our result. Note that $\ell = 1$ for any trivial P.V.
(G × GLₙ, ρ⊗Λ₁, Ω_m⊗Ω) (deg ρ= m≤ n) since we have ℓ = 1 for
((I_m) × GLₙ, ρ⊗Λ₁) ≃ (GLₙ, Λ₁ ⊕ · · · ⊕ Λ₁)(m≤ n) by Proposition 1.5 and
Lemma 2.2. Q.E.D.

4. 2-Simple P.V.'s of Type II with Universally Transitive Open orbits.

PROPOSITION 4.1. — For n ≥ m₁ ≥ m₂, we have ℓ = 1 for a P.V.
(G × GLₙ, ρ₁⊗Λ₁ + ρ₂⊗Λ₁*, M_m₁,n ⊕ M_m₂,n) if and only if ℓ = 1 for a
P.V. (G, ρ₁⊗ρ₂, M_m₁,m₂).

Proof. — Define a map ψ : M_m₁,n ⊕ M_m₂,n → M_m₁,m₂ by
ψ(X, Y) = X'Y for (X, Y) ∈ M_m₁,n ⊕ M_m₂,n. Since ψ((A, Y) ∈ M_m₁,n ⊕ M_m₂,n.
ρ₂(A)YB⁻¹) = ρ₂(A)ψ(X, Y) for (A, B) ∈ G × GLₙ, it is G × GLₙ-
equivariant. Let W(resp. W') be the Zariski-dense orbit of the first P.V.
(resp. the latter P.V.). By Theorems 1.4 and 1.6 in [6], we have
ψ(W) = W'. It is enough to show that ψ : W'(k) → W'(k) is surjective
with GL_n(k)-homogeneous fibres. Clearly we have W ⊂ U =
{(X, Y) ∈ M_m₁,n ⊕ M_m₂,n; rank X = m₁, rank Y = rank X'Y = m₂} and
W' ⊂ U' = {Z ∈ M_m₁,m₂n; rank Z = m₂}. Since ψ((I_mₐ, 0), (Z, 0)) = Z, the
maps ψ : U → U' and ψ : U'(k) → W'(k) are surjective. For any
(X, Y) ∈ ψ⁻¹(Z) ∩ U, there exists B ∈ GLₙ satisfying X'B = (I_m₁, 0) and
YZB⁻¹ = (Z', Z'). Since rank 'Z = m₂, we have 'ZC' = Z' for some
C' ∈ M_m₂,n. Put C = \begin{pmatrix} I & C' \\ 0 & I \end{pmatrix} ∈ GLₙ. Then we obtain X'B'C =
(I_m₁, 0) and YZB⁻¹ = (Z, 'ZC' + Z') = (Z, 0), i.e., (X, Y) ~
((I_m₁, 0), (Z, 0)). This implies that each fibre of ψ : U → U' (resp.
ψ : U(k) → U'(k)) is GL_n (resp. GL_n(k))-homogeneous. Hence GL_n(k) acts
on each fibre of ψ : W(k) → W'(k) transitively. For any
Z ∈ W'(k) = U'(k) ∩ W', there exists (X, Y) in U(k) satisfying
ψ(X, Y) = Z. Since ψ(W) = W' ∩ Z, there exists (X', Y') in W satisfying
ψ(X', Y') = Z. Hence (X, Y) = (X'B, Y'B⁻¹) ∈ U(k) ∩ W = W(k) for
some B ∈ GLₙ, i.e., ψ(W(k)) = W'(k). Q.E.D.

THEOREM 4.2. — We have ℓ = 1 for the following 2-simple P.V.'s
(4.a)-(4.c) of type II if and only if ℓ = 1 for a simple P.V.
(GL¹ × G, ρ₁+· · ·+ρᵣ) (deg ρᵢ ≥ 2 for i = 1, · · ·, r) (see Theorem 2.19).

(4.a) (GL¹ × G × SLₙ, (σ₁ + · · · + σₙ)⊗Λ₁ + (ρ₁ + · · · + ρᵣ)⊗1) for
any representation σ₁ + · · · + σₙ of G and any natural number n
satisfying n ≥ deg σ₁ + · · · + deg σₙ.
(4.b) \((GL_1^{t+r} \times G \times SL(\Sigma \deg p_i + r - 1), (\rho_1 + \cdots + \rho_k) \otimes \Lambda_1 + (\rho_{k+1}^\ast + \cdots + \rho_r^\ast) \otimes 1 + 1 \otimes (\Lambda_1 + \cdots + \Lambda_1))(1 \leq k \leq r)\) for any \(t \geq 0\).

(4.c) \((GL_1^{t+r} \times G \times SL_n, (\rho_1 + \cdots + \rho_k) \otimes \Lambda_1 + (\rho_{k+1}^\ast + \cdots + \rho_r^\ast) \otimes 1 + 1 \otimes (\Lambda_1 + \cdots + \Lambda_1^{t-1} + \Lambda_1 + \Lambda_1^\ast))(1 \leq k \leq r)\) for any pair of natural number \((t,n)\) satisfying \(t \geq 1\) and \(n \geq t - 1 + \deg \rho_1 + \cdots + \deg \rho_r\).

**Proof.** - For (4.a), we have our result by Proposition 1.5 and the remark in the proof of Corollary 3.22. A P.V. (4.b) is a castling transform of \((GL_1^{t+r} \times G, \rho_1^\ast + \cdots + \rho_r^\ast + 1 + \cdots + 1)\). Clearly it has \(\ell = 1\) if and only if \(\ell = 1\) for \((GL_1^t \times G, \rho_1 + \cdots + \rho_r)\) (see §2 in [2]). By proposition 4.1, we have \(\ell = 1\) for (4.c) if and only if \(\ell = 1\) for \((GL_1^{t+r+1} \times G, \rho_1 + \cdots + \rho_r + 1 + \cdots + 1)\), i.e., \(\ell = 1\) for \((GL_1^t \times G, \rho_1 + \cdots + \rho_r)\).

Q.E.D.

From now on, for simplicity, we shall write \((G, \rho)\) instead of \((GL_1^k \times G, \rho(=\rho_1 \oplus \cdots \oplus \rho_k))\) where \(GL_1^k\) acts on each irreducible component \(\rho_i(1 \leq i \leq k)\) independently.

**Lemma 4.3.** - We have \(\ell = 1\) for \((GL_2^{m+1} \times H, \Lambda_2 \otimes 1 + \rho \otimes \rho'(\text{resp. } \Lambda_2^\ast \otimes 1 + \rho \otimes \rho'\ast))\) if and only if \(\ell = 1\) for \((Sp_m \times GL_2^{m+1} \times H, \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \rho \otimes \rho'(\text{resp. } \Lambda_1 \otimes \Lambda_1^\ast \otimes 1 + 1 \otimes \rho \otimes \rho'\ast))\).

**Proof.** - Let \(H'\) be a generic isotropy subgroup of \((GL_2^{m+1}, \Lambda_2)\) (resp. \(\Lambda_2^\ast\)). Then the \(GL_2^{m+1}\)-part of a generic isotropy subgroup of \((Sp_m \times GL_2^{m+1}, \Lambda_1 \otimes \Lambda_1'\) (resp. \(\Lambda_1 \otimes \Lambda_1^\ast\)) is \(H'\). Since \(\ell = 1\) for \((GL_2^{m+1}, \Lambda_2)\) and \((Sp_m \times GL_2^{m+1}, \Lambda_1 \otimes \Lambda_1^\ast)\), by Proposition 1.5, both of \(\ell\) coincide with \(\ell\) for \((H \times H', \rho \otimes \rho')\).

Q.E.D.

**Proposition 4.4.** - We have \(\ell = 1\) for \((G \times GL_2^{m+1}, \rho \otimes \Lambda_1 + 1 \otimes \Lambda_2 + \sigma \otimes 1)'\) with \(\deg \rho \leq 2m + 1\), if and only if \(\ell = 1\) for \((G \times GL(\deg \rho - 1), \rho^\ast \otimes \Lambda_1 + 1 \otimes \Lambda_2 + \sigma \otimes 1)'\).

**Proof.** - By Lemma 4.3, \(\ell\) for the first P.V. coincides with \(\ell\) for \((G \times Sp_m \times GL_2^{m+1}, \sigma \otimes 1 \otimes 1 + (\rho \otimes 1 + 1 \otimes \Lambda_1) \otimes \Lambda_1)\), which is castling-equivalent to \((G \times Sp_m \times GL(\deg \rho - 1), \sigma \otimes 1 \otimes 1 + (\rho^\ast \otimes 1 + 1 \otimes \Lambda_1) \otimes \Lambda_1)\). Then, by Proposition 3.7, we have our result.

Q.E.D.

**Proposition 4.5.** - We have \(\ell = 1\) for \((G \times GL_2^{m+1}, \rho \otimes \Lambda_1 + 1 \otimes \Lambda_2 + \sigma \otimes 1)'\) with \(\deg \rho \leq 2m + 1\), if and only if \(\ell = 1\) for \((G \times Sp_m, \rho \otimes \Lambda_1 + 1 \otimes \Lambda_2 + \sigma \otimes 1)'\).
Proof. — By Lemma 4.3, the number \( \ell \) for the first P.V. coincides with \( \ell' \) for \( (G \times Sp_n \times GL_{2m+1}, \sigma \otimes 1 + \rho \otimes 1 \otimes 1 + 1 \otimes 1 \otimes 1, \sigma') \). By Proposition 4.1, it has the same \( \ell \) as \( (G \times Sp_m, \sigma \otimes 1 + (\rho \otimes 1) \otimes (1 \otimes 1), = \sigma \otimes 1 + \rho \otimes 1) \)'.

PROPOSITION 4.6. — The following P.V.'s (1), (2), (3) have \( \ell = 1 \) if and only if \( \ell = 1 \) for 

1. \( (G \times GL_{2n'+1}, \rho \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_1) + \sigma \otimes 1)' \)
2. \( (G \times GL_{2n'+1}, \rho \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_1) + \sigma \otimes 1)' \)
3. \( (G \times GL_{2n'+1}, \rho \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_1) + \sigma \otimes 1)' \)

Proof. — By Proposition 4.4, (1) is equivalent to 
\( (G \times GL_{2n'+1}, \rho \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_1) + \sigma \otimes 1)' \), which has a generic isotropy subgroup 
\( \{ (g, \rho(g)); g \in G \} \), we have our result for (1). Since \( \ell = 1 \) for 
\( (GL_{2n'+1}, (\Lambda_2 + \Lambda_1)', \) and their generic isotropy subgroups coincide, we have our result for (3). By Proposition 4.5, (2) is equivalent to 
\( (G \times Sp_{2m'}, (\rho + 1) \otimes \Lambda_1 + \sigma \otimes 1)' \), which is equivalent to 
\( (G, \Lambda^2 (\rho + 1) + \sigma)' = (G, \Lambda^2 (\rho + \sigma)' \) by Proposition 3.7. Q.E.D.

PROPOSITION 4.7. — Assume that \( \deg \rho = \text{odd} < 2n' + 1 \). Then we have \( \ell = 1 \) for 
\( (G \times GL_{2n'+1}, \rho \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_1) + \sigma \otimes 1)' \) if and only if 
\( \ell = 1 \) for 
\( (G, \Lambda^2 (\rho + \sigma)' \)

Proof. — Let \( (W, \begin{bmatrix} X & Y \\ -Y & Z \end{bmatrix}) \) be a \( k \)-rational generic point of 
\( (G \times GL_{2n'+1}, \rho \otimes \Lambda_1 + 1 \otimes \Lambda_2, M_{2m'+1, 2n'+1} \otimes \text{Alt}_{2n'+1}) \)' (\( \deg \rho = 2m'+1 \)). Since \( \ell = 1 \) for a trivial P.V. 
\( (G \times GL_{2n'+1}, \rho \otimes \Lambda_1) \), we may assume that 
\( W = (I_{2m'+1}, 0) \). Then the fixer at \( W \) acts on \( Z \)-spaces as 
\( (GL_{2(n'-m')}, \Lambda_2, \text{Alt}_{2(n'-m')}) \) which has \( \ell = 1 \). Hence we may take 
\( Z = J = \begin{bmatrix} 0 & I_{n'-m'} \\ -I_{n'-m'} & 0 \end{bmatrix} \).

By the action of 
\( \begin{bmatrix} I_{2m'+1} & \begin{bmatrix} YJ \\ 0 \end{bmatrix} \\ \begin{bmatrix} I_{2(n'-m')} \\ I_{2(n'-m')} \end{bmatrix} \end{bmatrix} (\in GL_{2n'+1}) \), 
we may assume that \( Y = 0 \). The generic isotropy subgroup of 
\( (GL_1 \times G \times \)
\( GL_{2n+1}, 1 \otimes \rho \otimes \Lambda_1 + \Lambda_1 \otimes \otimes \Lambda_2 \) at this point is given by

\[
\{(\alpha, A) \mid \begin{vmatrix} \alpha \rho(A)^{-1} \overset{0}{\alpha} \overset{0}{B} \end{vmatrix} \in GL_1 \times G \times GL_{2n+1} ; \alpha \rho(A)^{-1} X = \rho(A)^{-1} = X, \alpha BJ B = J \}.
\]

Since

\[
\begin{vmatrix} \alpha \rho(A)^{-1} & 0 \\ 0 & B \end{vmatrix} = \begin{vmatrix} \rho(A) & 0 \\ 0 & \alpha B^{-1} \end{vmatrix}
\]

and \( \ell = 1 \) for \((Sp_{m-n}, \Lambda_1^*)\), our P.V. has \( \ell = 1 \) if and only if \((G, \Lambda^2(p) + \rho + \sigma)p^2\) has \( \ell = 1 \).

**Q.E.D.**

**THEOREM 4.8.** - We have \( \ell = 1 \) for \((GL_1^4 \times SL_m \times SL_n, \rho) \) \( m < n \Rightarrow odd \)
for the following \( p's \):

\[
(4.1) \ p = \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes (\Lambda_1 + \Lambda_1 + \Lambda_1) \quad (m = odd, \ or \ m = 2n', \ n = 2n' + 1).
\]

\[
(4.2) \ p = \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1 + \Lambda_1) \quad (m = odd).
\]

\[
(4.3) \ p = \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1 + \Lambda_1) \quad (m = odd).
\]

\[
(4.4) \ p = \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1 + \Lambda_1) \quad (m = even).
\]

\[
(4.5) \ p = \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1 + \Lambda_1) \quad (m = even).
\]

\[
(4.6) \ p = \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1 + \Lambda_1) \quad (m = even).
\]

**Proof.** - For (4.1) with \( m = 2n', \ n = 2n' + 1 \), it is castling-equivalent to \((GL_1^4 \times SL_n, \Lambda_1^2 + \Lambda_1 + \Lambda_1 + \Lambda_1^*)\), which has \( \ell = 1 \). When \( m = odd \), by Lemma 4.3 and Proposition 4.1, it is equivalent to \((GL_1^4 \times SL_m \times Sp_{m'}, \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1))\), which has \( \ell = 1 \) by (5) in Theorem 3.2. By Proposition 4.4 (with \( \rho = \Lambda_1 + 1 + 1 \)) and by a castling transformation, (4.2) is equivalent to \((GL_1^4 \times SL_m, \Lambda_2 + \Lambda_1 + \Lambda_1 + \Lambda_1^*)\), which has \( \ell = 1 \). Since the generic isotropy subgroups of \((GL_{2n'+1}, \Lambda_2 + \Lambda_1^2)^{(\omega)}\) coincide, we have (4.3) from (4.2). Now (4.4) (resp. (4.5), (4.6)) is a castling transform of (4.1) (resp. (4.2), 4.3)).

**Q.E.D.**

**LEMMA 4.9.** - We have \( \ell = 1 \) for \((GL_1 \times GL_{2m'} \times GL_{2n'+1}, \)

\[
1 \otimes (\Lambda_1^2 \otimes 1 + \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2 + \Lambda_1 \otimes 1 \otimes \Lambda_1^*) \) with \( 2m' < 2n' + 1 \).

**Proof.** - Since \( \ell = 1 \) for \((GL_{2m'}, \Lambda_1^*)\) and \((GL_{2n'+1}, \Lambda_2)\), we may assume that a \( k \)-rational generic point of

\[
(GL_{2m'} \times GL_{2n'+1}, \Lambda_1^2 \otimes 1 + \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2, \Omega^{2m'} \otimes M_{2m'}, \Lambda_1 \otimes \Lambda_2, \Omega^{2n'+1})
\]
is $(1,0,\ldots,0)$, \[ \begin{bmatrix} x & Y \\ Z & W \end{bmatrix} \] $(1,0,\ldots,0))$. By the action of

\[ \begin{bmatrix} x^{-1} & 0 \\ x^{-1}Z & I \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ x^{-1}Y & I \end{bmatrix} (\in (GL_{2m} \times GL_{2n+1})(k)), \]

we may assume that $x = 1$ and $Y = Z = 0$. The isotropy subgroup at this point is

\[ \{ \begin{bmatrix} \alpha & 0 \\ 0 & A \end{bmatrix}, \begin{bmatrix} \alpha^{-1} & 0 \\ 0 & B \end{bmatrix} \in GL_{2m} \times GL_{2n+1}; \alpha \in GL_1, A \in GL_{2m-1}, B \in Sp_n, AW^B = W \}. \]

Since

\[ \Lambda^*_1 \begin{bmatrix} \alpha^{-1} & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & B \end{bmatrix} \]

and $\ell = 1$ for $(GL^2 \times Sp_n \times SL_{2m-1}, \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1)$ by Theorem 3.20, we have our result.

**Theorem 4.10.** We have $\ell' = 1$ for

\[ (GL^2 \times SL_m \times SL_n, \rho(=\rho_1 \oplus \ldots \oplus \rho_k))(m < n = \text{odd}) \]

where $\rho$ is one of (4.7) $\sim$ (4.13). Here $T$ stands for any one of $\Lambda_2 \oplus \Lambda_1$, $\Lambda_2^* \oplus \Lambda_1^*(\cdot)$:

(4.7) $\rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes T + (\Lambda_1 + \Lambda_1^{(*)}) \otimes 1$.

(4.8) $\rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes T + (\Lambda_1 + \Lambda_1^*) \otimes 1$ (m = even).

(4.9) $\rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_2^*) + (\Lambda_1 + \Lambda_1^*) \otimes 1$ (m = odd).

(4.10) $\rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_2^*) + (\Lambda_1 + \Lambda_1) \otimes 1$.

(4.11) $\rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2^* + \Lambda_1^{(*)})(+\Lambda_1^{(*)}\otimes 1)$.

(4.12) $\rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_2^*) + \Lambda_2^* \otimes 1$ (m = 5).

(4.13) $\rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_2^*) + \Lambda_2 \otimes 1$ (m = 4).

**Proof.** By Theorem 2.19 and Proposition 4.6, we have (4.7) and (4.8). By Proposition 4.7, we have (4.9) and (4.12). Now (4.10) is a castling transform of (one of) (4.7). From (4.7), (4.9), (4.10) and Lemma 4.8, we have (4.11). By (4.12) and Lemma 4.3, we have $\ell' = 1$ for $(SL_4 \times SL_{2n'+1} \times SL_5, \Lambda_2 \otimes 1 \otimes 1 + 1 \otimes (\Lambda_2 + \Lambda_2^*) \otimes 1 + (1 \otimes \Lambda_1) \otimes \Lambda_1 + (\Lambda_1 \otimes 1) \otimes \Lambda_1^*)'$. Now the proof of Proposition 4.1 shows that if $\ell' = 1$ for $(G \times GL_n, \rho_1 \otimes \Lambda_1 + \rho_2 \otimes \Lambda_1^*)$ with $m_1 > n \geq m_2$, then we have $\ell' = 1$ for $(G, \rho_1 \otimes \rho_2)$. In our case, we have $\ell' = 1$ for (4.13). Q.E.D.
THEOREM 4.11. – We have \( \ell = 1 \) for the following \( \text{P.V.}'s \):

\[(4.14) \quad \rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2 + \sigma \otimes 1 \quad (m = \text{odd}) \quad \text{with} \quad \sigma = \Lambda_2^*, \]
\[(4.15) \quad \rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2 + \sigma \otimes 1 \quad (m = \text{even}) \quad \text{with} \quad \sigma = \Lambda_2^*, \]
\[(4.16) \quad \rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2^* + \sigma \otimes 1 \quad \text{with} \quad \sigma = \Lambda_2 \quad (m = \text{odd}), \]
\[(4.17) \quad \rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2^* + (\Lambda_1^* \otimes 1 + (\Lambda_1^* \otimes 1))^\prime. \]

Proof. – By Proposition 4.4, (4.14) is equivalent to \((SL_{2m'} \times SL_{2m'}) \otimes \Lambda_1 + \sigma \otimes 1)^\prime \) \((m = 2m' + 1)\), which is equivalent to \((SL_{2m' + 1}, \Lambda_2^* \otimes \Lambda_1 + \sigma \otimes 1)^\prime \) by Lemma 4.3. Hence, by Theorem 2.19, we have \( l = 1 \) for (4.14). For (4.15) with \( \sigma = \Lambda_2^* \), it is equivalent to \((SL_{2m' + 1}, \Lambda_2^* \otimes \Lambda_1 + 1 \otimes \Lambda_2)^\prime \) since \( \ell = 1 \) for \((GL_{2m'}, \Lambda_2^*) \) \((m = 2m')\). Then, by Propositions 4.4 and 3.7, it is equivalent to \((SL_{2m' + 1}, \Lambda_2^* \otimes \Lambda_2)^\prime \) which has \( \ell = 1 \) by Theorem 2.19. For (4.15) with \( \sigma = (\Lambda_1 + \Lambda_1 + \Lambda_1)^\prime \), by Proposition 4.4, it is equivalent to \((SL_{2m' + 1}, \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2 + (\Lambda_1 + \Lambda_1 + \Lambda_1)^\prime \otimes 1)^\prime \). When \( \sigma = \Lambda_1 + \Lambda_1 + \Lambda_1 \), it is castling-equivalent to \((SL_{2m' + 1}, \Lambda_1 + \Lambda_1 + \Lambda_1 \otimes 1 + \Lambda_1 \otimes \Lambda_2^* + 1 \otimes \Lambda_2)^\prime \). Since \( \ell = 1 \) for \((GL_{2m'} \times SL_{2m'}, \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2 + (\Lambda_1 + \Lambda_1 + \Lambda_1)^\prime \otimes 1)^\prime \), it is equivalent to \((GL_{2m'}, \Lambda_1 \otimes \Lambda_1 + \sigma \otimes 1)^\prime \) \((m = 4)\), it is equivalent to \((SL_{4m'}, \Lambda_1 \otimes \Lambda_1 + \sigma \otimes 1)^\prime \) by Proposition 4.4. Clearly it is also equivalent to \((SL_{4m'}, \Lambda_1 \otimes \Lambda_1 + \sigma \otimes 1)^\prime \) by Theorem 3.20. For (4.16), by Proposition 4.5, it is equivalent to \((SL_{m'}, \Lambda_1 \otimes \Lambda_1 + \sigma \otimes 1)^\prime \), which is again equivalent to \((SL_{m'}, \Lambda_2 + \sigma)^\prime \) by Proposition 3.7. Hence we have our result by Theorem 2.19. By above results and Theorem 4.10, we have (4.17).

Q.E.D.

THEOREM 4.12. – We have \( l = 1 \) for the following \( \text{P.V.'s} \):

\[(4.18) \quad (GL_{2m'} \times SL_{2m'}, \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2) \quad \text{with} \quad 2m' \leq 2n' + 1. \]
\[(4.19) \quad (GL_{2m'} \times SL_{2m'}, \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2 + \Lambda_1 \otimes 1). \]
\[(4.20) \quad (GL_{2m'} \times SL_{2m'}, \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2 + 1 \otimes \Lambda_2^*). \]

Proof. – We have \( l = 1 \) for (4.18) \((\text{resp.} \quad 4.19, \quad 4.20)\) by (4.15) with \( \sigma = \Lambda_2^* \) \((\text{resp.} \quad 4.15) \quad \text{with} \quad \sigma = (\Lambda_2 + \Lambda_1^* \otimes 1 + (\Lambda_1^* \otimes 1))^\prime \) \((m = 4)\), (4.13)) Q.E.D.
**Theorem 4.13.** We have $\ell = 1$ for the following P.V.'s:

(4.21) $(GL_1^3 \times SL_2 \times SL_5, \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2^* + \Lambda_4^*))$.

(4.22) $(GL_1^3 \times SL_3 \times SL_5, \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_2^*))$.

(4.23) $(GL_1^3 \times SL_4 \times SL_5, \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_2^*))$.

**Proof.** Since (4.23) is castling-equivalent to $(GL_1^3 \times SL_5, \Lambda_2 \oplus \Lambda_2 \oplus \Lambda_4^*)$, we have $\ell = 1$ for (4.23). Since (4.21) is a castling transform of (4.22), it is enough to show $\ell = 1$ for (4.22), namely, for $(GL_1 \times GL_2 \times GL_3, (1 \otimes \Lambda_2 + \Lambda_1 \otimes \Lambda_2) \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_1)$. The isotropy subalgebra $\mathfrak{H}$ of $(GL_1 \times GL_2, 1 \otimes \Lambda_2 + \Lambda_1 \otimes \Lambda_2)$ at $\xi$ with $m = 2$ in the proof of Proposition 2.15 is given by

$$\{(\alpha, A) \in gl_1 \oplus gl_5; A = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}; A_1 = \text{diag}(a, a - \alpha, a - 2\alpha), A_2 = \text{diag}(-a, a - \alpha), A_3 = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{bmatrix} \}.$$ 

Therefore the $GL_5$-part $H$ of the isotropy subgroup at $\xi$ contains $\{\text{diag}(\varepsilon, \varepsilon^{-1}, \varepsilon^{-2}, \varepsilon^{-1}, \varepsilon^{-1})\}$, with $\varepsilon, \eta \in GL_1$ and $G_\xi$ with $m = 2$ in the proof of Proposition 2.15. We shall show $\ell = 1$ for $(H \times GL_3, \Lambda_1 \otimes \Lambda_1, M_{5,3})$.

Let $X = \begin{bmatrix} Y \\ Z \end{bmatrix}$ be a $k$-rational generic point where $Y \in M_3(k)$ and $Z = \begin{pmatrix} u_1, u_2, u_3, z_1, z_2, u_4 \end{pmatrix} \in M_{1,3}(k)$. Since $\det Y \neq 0$, we may assume that $Y = I_3$ by the action of $GL_3$. Similarly we have $u_i = 0(1 \leq i \leq 4)$ by the action of $G_\xi$. In this case, we have $z_1 z_2 \neq 0$ since otherwise it cannot be a generic point. For example, one can check this by calculation of the isotropy subalgebra. By the action of

$$g = \text{diag}(\varepsilon, \varepsilon \eta^{-1}, \varepsilon \eta^{-2}, \varepsilon^{-1}, \varepsilon^{-1} \eta) \times \text{diag}(\varepsilon^{-1}, \varepsilon^{-1} \eta, \varepsilon^{-1} \eta^2) \in H \times GL_3$$

with $\varepsilon^2 = z_1^2 z_2^{-1}$ and $\eta = z_1 z_2^{-1}$, we have $z_1 = z_2 = 1$, i.e., $\ell = 1$. Note that $\Lambda_1 \otimes \Lambda_1(g)$ is $k$-rational even if $g \notin (H \times GL_3)(k)$. Q.E.D.

**Theorem 4.14.** We have $\ell = 1$ for the following P.V.'s $(GL_k^k \times SL_m \times SL_n, \rho(= \rho_1 \oplus \ldots \oplus \rho_k))$ where $n = 2n'$ (= even):

(4.24) $\rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2^* (+ \sigma \otimes 1)$ with

$\sigma = \Lambda_1^*(*)$, $\Lambda_1^*(*) + \Lambda_1^*(*)$, $\Lambda_1^* + \Lambda_2^* + \Lambda_4^*$,

$(\Lambda_1 + \Lambda_1 + \Lambda_1)(*)$, $\Lambda_1 + \Lambda_1 + \Lambda_2^*$ ($m$ = even), $\Lambda_2$ ($m$ = odd).
(4.25) \( \rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda^*_2 + \Lambda^*_1) (+ \sigma \otimes 1) \) with \( \sigma = \Lambda_1^{(*)} \),

(4.26) \( \rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda^*_2 + \Lambda^*_1) + \Lambda^*_1 \) (\( m = \text{odd} \)).

(4.27) \( \rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda^*_2 (+ \Lambda_2 + \Lambda^*_1) \otimes 1 \) (\( m = 5 \)).

Proof. - Since \( \ell = 1 \) for \((G^L, \Lambda^*_2)\) with a generic isotropy subgroup \(Sp_n^r\), (4.24) \sim (4.27) reduce to the case of type I, and we have our result by Theorem 3.20. Q.E.D.

**Proposition 4.15.** - We have \( \ell \geq 1 \) for \((SL_2 \times SL_n, 2\Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda^*_2) (+ \Lambda_1 \otimes 1)')\).

Proof. - If \( n = 2n' \), it is equivalent to \((SL_2 \times Sp_{n'}, 2\Lambda_1 \otimes \Lambda_1 (+ \Lambda_1 \otimes 1))'\) which has \( \ell \geq 2 \) by Corollary 3.22. If \( n = \text{odd} \), we have \( \ell \geq 2 \) by Propositions 4.4 and 4.5. Q.E.D.

**Theorem 4.16.** - We have \( \ell = 1 \) for the following P.V.'s:

\[(4.28) \quad (GL^{*+1}_1 \times SL_n \times SL_n, \Lambda_1 \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_k) \otimes 1 + 1 \otimes (\tau_1 + \cdots + \tau_l)) \]

where \((GL^{*+1}_1 \times SL_n, \sigma_1^* + \cdots + \sigma_k^* + \tau_1 + \cdots + \tau_l)\)

is a simple P.V. with \( l = 1 \) (See Theorem 2.19).

Proof. - It is obvious. Q.E.D.

**Theorem 4.17.** - A P.V. of the type

\[(4.29) \quad (GL^{*+1}_1 \times G \times SL_n, (\rho_1 + \cdots + \rho_k) \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_k) \otimes 1 + 1 \otimes (\tau_1 + \cdots + \tau_l)) \]

with \( 2 \leq \deg \rho_i \leq n \) (\( i = 1, \ldots, k \)) and \( (\tau_1 + \cdots + \tau_l) \neq (\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)})\)

has \( \ell = 1 \) if and only if it is one of \((4.1) \sim (4.28)\).

Proof. - We can find the table of all P.V.'s of this type in §§ 5-2 in [6]. From Lemma 4.3 to Theorem 4.16, we have investigated the number \( \ell \) for all P.V.'s in §§ 5-2 in [6] except P.V.'s which have an irreducible component with \( \ell \geq 2 \). Q.E.D.

**Proposition 4.18.** - We have \( \ell \geq 2 \) for \((GL^2_1 \times SL_4 \times SL_8, (\Lambda_2 + \Lambda_1) \otimes \Lambda_1)\).
Proof. - It is castling-equivalent to \((GL_1^* \times SL_4 \times SL_2, (\Lambda_2 + \Lambda_1 \otimes \Lambda_1))\) where \((SL_4 \times GL_2, \Lambda_2 \otimes \Lambda_1) = (SO_6 \times GL_2, \Lambda_1 \otimes \Lambda_1)\) has \(r' \geq 2\). Hence we have our result. Q.E.D.

**Lemma 4.19.** - We have \(r' = 1\) for the following P.V.'s (1) \((GL_1 \times S_{p_m}, \Lambda_1 \otimes (\Lambda_1 + \Lambda_1))\), (2) \((GL_1 \times GL_{2m}, 1 \otimes \Lambda_2^{(*)} + \Lambda_1 \otimes (\Lambda_1 + \Lambda_1))\), (3) \((GL_1 \times GL_{2m+1}, 1 \otimes \Lambda_2^{(*)} + \Lambda_1 \otimes (\Lambda_1 + \Lambda_1))\).

Proof. - Applying Proposition 3.7 for \((G, \Lambda^2(\rho)) = (GL_1, \Lambda_1 \oplus \Lambda_1)\), we have \(r' = 1\) for \((G, \Lambda^2(\rho)) = (GL_1, 2\Lambda_1)\), i.e. (1). Hence we have (2). By Proposition 4.5, (3) is equivalent to (1). Note that \((GL_1 \times GL_{2m+1}, 1 \otimes \Lambda_2 + \Lambda_1 \otimes (\Lambda_1 + \Lambda_1))\) is a non P.V. since it has a non-constant absolute invariant. Q.E.D.

**Theorem 4.20.** - We have \(r' = 1\) for the following P.V.'s \((GL_1^* \times SL_{m} \times SL_n, \rho(=\rho_1 \oplus \ldots \oplus \rho_d))(n \geq m+1)\).

\[(4.29) \rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1^* + \Lambda_2^*) + \sigma \otimes 1 \quad \text{with} \quad \sigma = \Lambda_2^{(*)}, \Lambda_2^{(*)} + \Lambda_1, \Lambda_2^* + \Lambda_1^*, \Lambda_2 + \Lambda_1^* \quad (m=\text{even}).\]

\[(4.30) \rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2^* + \Lambda_1^* + \Lambda_2^*) + \Lambda_2^{(*)} \otimes 1,\]

\[(4.31) \rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2^* + \Lambda_1^* + \Lambda_1) + \sigma \otimes 1 \quad \text{with} \quad \sigma = \Lambda_2^*, \Lambda_2 \quad (m=\text{even}).\]

Proof. - By Proposition 4.1, (4.29) (resp. (4.30)) is equivalent to \((SL_m, \Lambda_1 \oplus \Lambda_1)\)' (resp. \((GL_1^* \times SL_m, \Lambda_2^{(*)} + \Lambda_1 + \Lambda_1 + \Lambda_1)\)) and hence, by Theorem 2.19, we have our result. For (4.31), it is equivalent to (2) or (3) in Lemma 4.20 by Proposition 4.1 and hence \(r' = 1\). Q.E.D.

**Theorem 4.21.** - We have \(r' = 1\) for the following P.V.:

\[(4.32) (GL_1^* \times SL_m \times SL_{m+1}, \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1 + \Lambda_1) + \sigma \otimes 1) \quad \text{with} \quad \sigma = \Lambda_2, \Lambda_2^* \quad (m=\text{even}).\]

Proof. - It is castling-equivalent to \((GL_1^* \times SL_m \times SL_2, \Lambda_1^* \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1 + \Lambda_1 + \sigma \otimes 1))\). Since \(r' = 1\) for \((GL_1^* \times SL_2, \Lambda_1 + \Lambda_1 + \Lambda_1)\) with a generic isotropy subgroup \{1\}, it is equivalent to \((GL_1 \times GL_1 \times SL_m, \Lambda_1 \otimes 1 \otimes (\Lambda_1^* + \Lambda_1^*) + 1 \otimes \Lambda_1 \otimes \sigma)\). By Lemma 4.19, we have our result.

**Theorem 4.22.** - We have \(r' = 1\) for the following P.V.'s \((GL_1^* \times SL_m \times SL_n, \rho(=\rho_1 \oplus \ldots \oplus \rho_d))(m=\text{odd})\):

\[(4.33) \rho = \Lambda_2 \otimes \Lambda_1 + 1 \otimes (\Lambda_1^* + \Lambda_1^*) \quad \text{when} \quad m = 5\]

\((n \geq 1/2m(m-1))).
(4.34) \( p = \Lambda_2 \otimes \Lambda_1 + 1 \otimes (\Lambda_1^* + \Lambda_1^* + \Lambda_1) \) \((n \geq 1/2m(m-1) + 1)\).

(4.35) \( p = \Lambda_2 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1) (+ \Lambda_1 \otimes 1 \text{ when } m = 5) \)
\((n = 1/2m(m-1))\).

(4.36) \( p = \Lambda_2 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1 + \Lambda_1) \) \((n = 1/2m(m-1) + 1)\).

Proof. – Since \( \ell = 1 \) for \((SL_m, \Lambda_2 \oplus \Lambda_2)\) (see the proof of Proposition 2.15), we have \( \ell = 1 \) for (4.33) and (4.34) by Propositions 4.1 and 2.16. A castling transform of (4.35) and (4.36) has \( \ell = 1 \) by Theorem 3.20.

THEOREM 4.23. – We have \( \ell = 1 \) for the following P.V.’s \((GL_1^k \times Sp_m \times SL_n, \rho(= \rho_1 \oplus \ldots \oplus \rho_k)) \) \((n \geq 2m)\):

(4.37) \( p = \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1^* + \Lambda_1^* + \Lambda_1) \) with \( T = \Lambda_1 \otimes 1, 1 \otimes \Lambda_1^* \) \((n \geq 2m + 1)\).

(4.38) \( p = \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1^* + \Lambda_1^* + \Lambda_1^*) \) \((n \geq 2m)\) with \( T = \Lambda_1 \otimes 1, 1 \otimes \Lambda_1^* \).

(4.39) \( p = \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1 + \Lambda_1) \) \((n = 2m + 1)\).

Proof. – By Propositions 4.1; 2.9 and Lemma 4.19, we have (4.37). Since \( \Lambda_1 \otimes \Lambda_1 + \sigma \otimes 1 + 1 \otimes \tau(n = m) \) is equivalent to \((Sp_m, \sigma + \tau)'\), we have (4.38). A castling transform of (4.39) has \( \ell = 1 \) by Theorem 3.20.

THEOREM 4.24. – We have \( \ell = 1 \) for the following P.V.’s \((GL_1^k \times Spin_{10} \times SL_n, \text{ a half-spin rep.}) \otimes \Lambda_1 + \rho'(= \rho_2 \oplus \ldots \oplus \rho_k)) \) \((n \geq 16)\):

(4.40) \( p' = 1 \otimes (\Lambda_1^* + \Lambda_1^*), 1 \otimes (\Lambda_1^* + \Lambda_1^* + \Lambda_1) \) \((n \geq 17)\).

(4.41) \( p' = 1 \otimes (\Lambda_1 + \Lambda_1) \) \((n = 16)\), \(1 \otimes (\Lambda_1 + \Lambda_1 + \Lambda_1) \) \((n = 17)\).

Proof. – Since \( \ell = 1 \) for \((GL_1 \times Spin_{10}, \Lambda_1 \otimes (\Lambda + \Lambda))\) (see P.14 in [1]) where \( \Lambda \) is the even half-spin representation, we have (4.40) by Proposition 4.1. A castling transform of (4.41) has \( \ell = 1 \) by Theorem 3.20.

THEOREM 4.25. – A P.V. of the type \((GL_1^{k+s+t} \times G \times SL_n, \rho_1 + \ldots + \rho_k) \otimes \Lambda_1 + (\sigma_1 + \ldots + \sigma_s) \otimes 1 + 1 \otimes (\Lambda_1^* + \Lambda_1^*) \) \((2 \leq \deg \rho_i \leq n(i = 1, \ldots, k) \text{ and } (G, \rho_1 + \ldots + \rho_k; \sigma_1 + \ldots + \sigma_s) \neq (SL_m, \Lambda_1 + \ldots + \Lambda_1; \Lambda_1^* + \ldots + \Lambda_1^*)\) has \( \ell = 1 \) if and only if it is one of (4.29)-(4.41).
Proof. — By §§5-3 in [6] and Proposition 4.18—Theorem 4.24, we have our result.

Q.E.D.

THEOREM 4.26. — A P.V. of the type $(GL_1^{k} \times SL_m \times SL_n, 
\Lambda_1 + \cdots + \Lambda_1) \otimes \Lambda_1 + (\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}) \otimes 1 + \Lambda_1^{(*)} - \Lambda_1^{(*)})$ has always the universally transitive open orbit, i.e., $\varepsilon = 1$.

Proof. — P.V.'s of such type are completely classified in §4 in [6]. P.V.-equivalences used there keep $\varepsilon$ invariant (cf. Proposition 4.1, etc.). They are essentially reduced to trivial P.V.'s or simple P.V.'s of type $(GL_1^s \times SL_n, \Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)})$ which have $\varepsilon = 1$, and hence we obtain our result.

Q.E.D.

BIBLIOGRAPHY


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