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Two problems of Calderón-Zygmund theory on product-spaces


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Robert Fefferman introduced in [1] the notion of a rectangle atom on $\mathbb{R}^n \times \mathbb{R}^m$ and proved the following theorem.

**Theorem A.** - Let $T$ be a bounded linear operator on $L^2(\mathbb{R}^n \times \mathbb{R}^m)$. Suppose that for any $H^p(\mathbb{R}^n \times \mathbb{R}^m)(0 < p \leq 1)$ rectangle atom $a$ supported on the rectangle $R$ we have

$$\int_{(R)\epsilon} |Ta|^p dx_1 dx_2 \leq c\gamma^{-\delta}$$

for some fixed $\delta > 0$ and all $\gamma \geq 2$. Then $T$ is a bounded operator from $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ to $L^p(\mathbb{R}^n \times \mathbb{R}^m)$.

The definitions and tools involved in this theorem and its proof have been generalized to product spaces with an arbitrary number of factors [2], [3], but the question of whether Theorem A extends for three or more factors or not, raised implicitly in [4] and explicitly in [2] was open. Our purpose is to show that in the case $p = 1$, and of the space $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$, Theorem A does not extend without any further assumptions on the nature of $T$. If, however, one supposes that $T$ is

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a convolution operator and if \( \delta > (1/8) \), then Theorem A extends. As will be apparent from the proof $1/8$ is probably not sharp and it is reasonable to conjecture that $\delta > 0$ should suffice.

The second question which we shall answer has been raised by Raphy Coifman and concerns the $L^2$-boundedness of the operator $c_a$ defined for $a \in L^\infty_0(\mathbb{R}^2)$ and $\|a\|_\infty < 1$, by the kernel

$$c_a(x, y) = \frac{1}{(x_1 - y_1)(x_2 - y_2) + \int_{x_1}^{y_1} \int_{x_2}^{y_2} a(u_1, u_2) du_1 du_2}.$$  

The case $\|a\|_\infty < \varepsilon$ was handled in [3] and was a consequence of the estimate

$$\|L_{k,a}\|_{2,2} \leq c^k \|a\|_\infty^k$$

where $L_{k,a}$ is the operator defined by the kernel

$$\frac{1}{(x_1 - u_1)(x_2 - y_2)} \left[ \int_{x_1}^{y_1} \int_{x_2}^{y_2} a(u_1, u_2) du_1 du_2 \right]^k \left[ \frac{(x_1 - y_1)(x_2 - y_2)}{(x_1 - u_1)(x_2 - y_2)} \right].$$

Here we improve this estimate and obtain $\|L_{k,a}\|_{2,2} \leq c_\delta (1 + k)^{2+\delta}$ for all $\delta > 0$, which yields the general case $\|a\|_\infty < 1$.

In Section 1 we recall some facts about bounded mean oscillation over rectangles, and state Theorem A, restricted to $p = 1$, in this dual setting. In Section 2 we present the counterexample to the extension of Theorem A for $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and $p = 1$. In Section 3 we show how the positive result for convolution operators can be reduced to a problem on finite families of convolution operators, which is handled in Section 4. In Section 5 we treat the operators $L_{k,a}$. This section essentially combines ideas already contained elsewhere, and for this reason, is rather sketchy.

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1. Bounded mean oscillation over rectangles.

The space $B(R \times R)$, introduced in [5], is the dual space of the atomic $H^1$-space, constructed from rectangle atoms. In other words, let $b \in L^2_{loc}(R^2)$. For every rectangle $R = I \times J$ in $R^2$ let

$$\text{Osc}_R b = \inf_{b_1, b_2} \left( \frac{1}{|R|} \int_R |b(x_1, x_2) - b_1(x_1) - b_2(x_2)|^2 dx_1 dx_2 \right)^{1/2},$$

where the inf is taken over all $b_1, b_2$ respectively in $L^2(I)$ and $L^2(J)$. Then $b \in B$ if and only if

$$\sup_{R} \text{Osc}_R b < +\infty.$$  

The left hand side of 1.1 is denoted $\|b\|_B$.

An equivalent definition can be given in terms of Carleson measures over rectangles. Let $\psi$ be a real even $C^\infty(R)$ function such that $\int \psi \, dx = 0$. For $t > 0$ and $i \in \{1, 2\}$, let $Q_{ti}$ be the convolution operator on $R^2$ of symbol $\hat{\psi}(t\xi_i)$. We normalize $\psi$ so that $\int_0^{+\infty} Q_{ti}^2 \frac{dt_i}{t_i} = I$. For each rectangle $R$, the set $S(R)$ denotes the subset of $R_+^2 \times R_+^2$ of those $(x_1, t_1, x_2, t_2) = (x, t)$ such that $|x_1 - t_1, x_1 + t_1[\times|x_2 - t_2, x_2 + t_2[ \subset R$.

**Lemma 1.** - A function $b \in L^2_{loc}(R^2)$ is in $B$ if and only if for some constant $c_b$

$$\int_{S(R)} |Q_{t_1} Q_{b_2} b|^2 dx_1 \frac{dt_1}{t_1} dx_2 \frac{dt_2}{t_2} \leq c_b |R|.$$  

Moreover if $c_b$ is optimal, $c_b \approx \|b\|^2_B$.

Notice that 1.1 $\Rightarrow$ 1.2 is clear since $\psi$ has compact support. We shall prove the converse in the non-product setting but the proof we give extends easily.

It is enough to show that if $b \in B(R) = \text{BMO}(R)$, and for all interval $I \subseteq R$

$$\int_{(x,t), |x-\epsilon,x+\epsilon| \subseteq I} |Q_t b|^2 \frac{dt}{t} \leq |I|.$$
then \[ \|b\|_B = \|b\|_{BMO} \leq c \] where \( c \) depends only on \( \psi \). For all \( t > 0 \), let \( P_t \) be the operator \( I - \int_0^t Q_s^2 \frac{ds}{s} \). Then for all \( t > 0 \), \( P_t b \) is \( C^\infty \) and \( \|(P_t b)'\|_\infty \leq c \|b\|_B t^{-1} \). It follows that if \( I \) is an interval of center \( x_0 \) and \( t_I = K|I| \) for \( K \) fixed large enough, \[
\left( \frac{1}{|I|} \int_I |P_t b(x) - P_t b(x_0)|^2 \, dx \right)^{1/2} \leq \frac{1}{2} \|b\|_B.
\]
Therefore \[
\left( \frac{1}{|I|} \int_I |b(x) - P_t b(x_0)|^2 \, dx \right)^{1/2} \leq \left( \frac{1}{|I|} \int_I |b(x) - P_t b(x)|^2 \, dx \right)^{1/2} + \frac{1}{2} \|b\|_B.
\]
By taking the sup over \( I \), we see that \[ \|b\|_B \leq 2 \sup_I \left( \frac{1}{|I|} \int_I |b(x) - P_t b(x)|^2 \, dx \right)^{1/2} \]. To estimate the right hand side we let \( g \) be in \( L^2(I) \), with \( \|g\|_2 = 1 \) and dominate \( <g, b - P_t b> \) by \[ \frac{1}{2} \int_{s \leq t_f} |Q_s g(x)| \cdot |Q_s b(x)| \frac{ds}{s} \, dx. \]

The conditions \( Q_s g \neq 0 \), \( s \leq t_f \), and \( g \in L^2(I) \), imply \( x K' I \) for some \( K' \) fixed. Using Cauchy-Schwarz, 1.3 and \( \|g\|_2 = 1 \), we can dominate 1.5 by an absolute constant, which proves the lemma.

In the following lemma the notations and definitions are those of [3].

**Lemma 2.** – Let \( T \) be a translation invariant \( \delta - CZO \) on \( \mathbb{R} \times \mathbb{R} \). Then \( T \) is bounded on \( B \).

This lemma is an easy consequence of lemma 1. For simplicity we shall consider the non-product situation, but give a proof which extends trivially.
Let $T$ be a translation invariant $\delta - \text{CZO}$ on $\mathbb{R}$. The kernel $(Q_t TQ_t')(x - y)$ of $T$ is easily seen to satisfy

$$|(Q_t TQ_t')(x - y)| \leq c \, w_{\delta', t, t'}(x - y) \left( \frac{t \wedge t'}{t \vee t'} \right)^{\delta'}$$

for some $\delta' < \delta$, where $w_{\delta', t}(z) = \frac{t^{\delta'}}{t^{1+\delta'} + |z|^{1+\delta'}}$. For $(x, t) \in \mathbb{R}^2_+$

and $b \in B \,
\left| Q_t T b(x) \right| = \left| \int_{\mathbb{R}^2_+} (Q_t TQ_t') (x, y) (Q_t'b)(y) \, dy \, \frac{dt'}{t'} \right|$.

By Cauchy-Schwarz and because of 1.6 this is less than

$$c \left[ \int_{\mathbb{R}^2_+} w_{\delta', t, t'}(x - y) \left( \frac{t \wedge t'}{t \vee t'} \right)^{\delta'} |Q_t'b(y)|^2 \, dy \, \frac{dt'}{t'} \right]^{1/2}.$$ 

It follows that if $|Q_t'b(y)|^2 \, dy \frac{dt'}{t'}$ is a Carleson measure, then

$$|Q_tTb(x)|^2 \, dx \frac{dt}{t}$$

is a Carleson measure. The same proof using Carleson measures over rectangles yields the result in the product case. Lemma 2 is proved, by Lemma 1.

We conclude this section by stating Theorem A, restricted to $p = 1$, in dual form [4].

\textbf{Theorem A′.} – \textit{Let $T$ be a linear operator bounded on $L^2(\mathbb{R} \times \mathbb{R})$. Suppose that for any rectangle $R$, and any $L^\infty$-function $a$ supported out of $\gamma R$,

$$\text{Osc}_R Ta \leq c \gamma^{-\delta}$$

for some $\delta > 0$ and all $\gamma > 2$. Then $T$ maps $L^\infty$ to $\text{BMO}(\mathbb{R} \times \mathbb{R})$.}

\textbf{2. A Counterexample.}

Any counterexample in this kind of question has to be related to the counterexample of Carleson [6] showing that rectangular Carleson measures are not, on the bidisc, a good substitute for classical Carleson-measures. As shown by R. Fefferman in [5], this...
counterexample implies that there can be no a priori estimate $||b||_{\text{BMO}} \leq c||b||_{B}$ on the bidisc. We shall denote for each $k \geq 0$ by $b_k$ a function on $\mathbb{R} \times \mathbb{R}$ such that $||b_k||_{\text{BMO}} = 1$ and $||b||_{B} \leq 2^{-k}$.

Using $b_k$ we form a $\delta$--CZO on $T_k$ on $\mathbb{R} \times \mathbb{R}$ by letting $T_k f = \int \int Q_{t_1}Q_{t_2} \{ (Q_{t_1}Q_{t_2}b_k) (P_{t_1}P_{t_2}f) \} \frac{dt_1}{t_1} \frac{dt_2}{t_2}$ for $f \in C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R})$.

The fact that $T_k$ is a $\delta$--CZO is clear, and proved in [3]. Let also $S_k$ be defined on $\mathbb{R}$ as the operator of convolution by $\frac{1}{x} \{-\phi(x) + \phi(2^{-k}x)\}$, where $\phi$ is a $C_0^\infty(\mathbb{R})$ function equal to 1 near 0, followed by the multiplication by $e^{i\xi}$. Finally let $U_k = T_k \otimes S_k$.

We claim that $U_k$ satisfies uniformly the assumptions of Theorem A', adapted to the case of three factors. Clearly $||U_k||_{2,2} \leq c$.

To check 1.7, only the oscillation of $b_k$ over rectangles is used, introducing a gain of $2^{-k}$. On the other hand the kernel $s_k(x,y)$ of $S_k$ satisfies $|\nabla_x s_k(x,y)| \leq \frac{c}{|x-y|}$, but since on the support of $s_k(.,.)$, $|x-y| \leq c2^k$, one also has $|\nabla_x s_k(x,y)| \leq \frac{c2^k}{|x-y|^2}$, and $|\nabla_y s_k(x,y)| \leq \frac{c}{|x-y|^2}$. Therefore, writing $U_k = (2^kT_k) \otimes (2^{-k}S_k)$, we see that $U_k$ satisfies 1.7 as any tensor product of a $\delta$--CZO on $\mathbb{R} \times \mathbb{R}$ with a $1$--CZO on $\mathbb{R}$. The contradiction comes from the fact that the functions $u_k = U_k (1 \otimes \text{sgn } x) = b_k \otimes S_k \text{sgn } x$, are not a bounded sequence in $\text{BMO}(\mathbb{R} \times \mathbb{R} \times \mathbb{R})$. Indeed $||u_k||_{\text{BMO}} \approx ||S_k \text{sgn } (x)||_{\text{BMO}} \approx k$.

3. Extension of Theorem A in the convolution case.

We wish to prove the following.

**Theorem 1.** Let $T$ be a bounded convolution operator on $L^2(\mathbb{R}^3)$. Suppose that for any rectangle $R$, and any $L^\infty$-function supported out of $\gamma R$,

3.1 $\text{Osc}_R Ta \leq c \gamma^{-\delta}$

for some $\delta > 1/8$ and all $\gamma > 2$. Then $T$ maps $L^\infty(\mathbb{R}^3)$ to $\text{BMO} (\mathbb{R} \times \mathbb{R} \times \mathbb{R})$. 
In [3] the $L^\infty - \text{BMO}$ boundedness of operators bounded on $L^2$, whose kernels satisfy some assumptions of Calderón-Zygmund type, is proved using the characterization of BMO in terms of Carleson measures [7] and some geometrical lemmas. Assumption 3.1 can be thought of as a very weak Calderón-Zygmund type of assumption and is reminiscent of the weak vector-valued standard estimates satisfied by the kernel of the square function operator of J.-L. Rubio de Francia [8]. It turns out that the technique used in [3] to extend Rubio de Francia's theorem in several parameters also applies here. Indeed this technique can be summarized in the following lemma.

**Lemma 3.** Let $T$ be a bounded operator on $L^2(\mathbb{R}^n)$. Let $i \in \{1, 2, \ldots, n\}$ and let $(l_1, \ldots, l_i) \in \mathbb{Z}^i$. Let $(x_1, \ldots, x_i) \in \mathbb{R}^i$. Let $b \in L^2_{\text{loc}}(\mathbb{R}^n)$ be supported in \{$(z_1, \ldots, z_i) \in \mathbb{R}^n, 2^{t_k} \leq |z_k - x_k| \leq 2^{t_k+1}$ for $1 \leq k \leq i$\}, such that for all $(z_1, \ldots, z_i)$,

\[
3.2 \quad \int_{z_{i+1}, \ldots, z_n} |b(z_1, \ldots, z_n)|^2 dz_{i+1}, \ldots, dz_n \leq 1.
\]

Suppose that for $(t_1, \ldots, t_i)$ such that $t_k \leq 2^{t_k-1}$ for $1 \leq k \leq i$,

\[
3.3 \quad \int_{x_{i+1}, \ldots, x_n} |Q_{t_1} \ldots Q_{t_i} Tb(x_1, \ldots, x_n)|^2 dx_{i+1}, \ldots, dx_n \leq c \prod_{1 \leq k \leq i} \left( \frac{t_k}{2^{t_k}} \right)^\varepsilon
\]

for some $\varepsilon > 0$ and that all of this remains true if the set $\{1, \ldots, i\}$ is replaced by any other non-empty subset of $\{1, \ldots, n\}$.

Then $T$ maps $L^\infty(\mathbb{R}^n)$ to $\text{BMO}(\mathbb{R} \times \ldots \times \mathbb{R})$.

Of course when $i = n$, then 3.2 and 3.3 simply mean that when

\[
||b||_\infty \leq 1, |Q_{t_1} \ldots Q_{t_n} Tb(x_1, \ldots, x_n)| \leq c \prod_{1 \leq i \leq n} \left( \frac{t_k}{2^{t_k}} \right)^\varepsilon,
\]

which follows trivially from 3.1.

Now we are going to see how to reduce the proof of Theorem 1 to a problem on finite families of convolution operators. In this reduction we shall suppose that $n = 2$ and $i = 1$. We want to prove...
that if $T$ satisfies the assumptions of Theorem 1, then it satisfies the assumptions of Lemma 3.

We shall assume that the function $\psi$ defining $Q_{t_1}$, $i \in \{1,2\}$, is of the form $\tilde{\psi} \ast \tilde{\psi}$, where $\tilde{\psi}$ is real even $C^\infty$, supported in $[-\frac{1}{2}, \frac{1}{2}]$ and with mean-value 0. Then if $|x_1 - z_1| \geq 2t_1$ we can define an operator $(Q_{t_1}T)_{x_1-z_1}$ acting on functions of the second variable by letting, for

$$f, g \in C^\infty_0(\mathbb{R}) < g, (Q_{t_1}T)_{x_1-z_1}f > = < \tilde{\psi}_{t_1}^{z_1} \otimes g, T\tilde{\psi}_{t_1}^{z_1} \otimes f >$$

where $\tilde{\psi}_{t_1}^{z_1}(u)$ is defined as $\frac{1}{t_1} \tilde{\psi} \left( \frac{u-x_1}{t_1} \right)$ and similarly for $\tilde{\psi}_{t_1}^{z_1}(u)$.

Let $(x_1, t_1) \in \mathbb{R}^2$; $\ell_1 \in \mathbb{Z}$ such that $2\ell_1 \geq 2t_1$ and $b \in L^2_{\text{loc}}(\mathbb{R}^2)$ such that $\text{supp } b \subseteq \{(z_1, z_2) \in \mathbb{R}^2, 2\ell_1 \leq |x_1 - z_1| \leq 2\ell_1 + 1\}$ and for all $z_1 \in \mathbb{R}, \int |b(z_1, z_2)|^2 dz_2 \leq 1$. Then $Q_{t_1}T b(x_1, \cdot) = \int (Q_{t_1}T)_{x_1-z_1} b(z_1, \cdot) dz_1$. In order to prove that $||Q_{t_1}T b(x_1, \cdot)||_2 \leq c \left( \frac{t_1}{2t_1} \right)^\varepsilon$ for some $\varepsilon > 0$, it suffices to prove that for all finite sequence $(z_{1,k} - z_{1,k+1}) = 2t_1$ and $2\ell_1 \leq |x_1 - z_{1,k}| \leq 2\ell_1 + 1$ for all $1 \leq k \leq N$,

$$3.4 \quad \left\| \sum_{k=1}^N (Q_{t_1}T)_{x_1-z_{1,k}} b(z_{1,k}, \cdot) \right\|_2 \leq \frac{c}{t_1} \left( \frac{t_1}{2t_1} \right)^\varepsilon.$$

On the other hand we are going to see that if $||b||_{\infty} \leq 1$, 3.1 implies

$$3.5 \quad \left\| \sum_{k=1}^N (Q_{t_1}T)_{x_1-z_{1,k}} b(z_{1,k}, \cdot) \right\|_{L^2(\mathbb{R})} \leq \frac{c}{t_1} \left( \frac{t_1}{2t_1} \right)^\delta.$$

Indeed, using the factorization of $Q_{t_1}$ as $\tilde{Q}_{t_1}^2$, we can rewrite the sum

$$\sum_{k=1}^N (Q_{t_1}T)_{x_1-z_{1,k}} b(z_{1,k}, \cdot)$$

as

$$\int (\tilde{Q}_{t_1}T)_{x_1-y_1} \left[ \sum_k \tilde{\psi}_{t_1}(y_1 - z_{1,k}) b(z_{1,k}, \cdot) \right] dy_1.$$
As a function of \((y_1, \cdot), \sum_k \tilde{\psi}_k(y_1 - z_{1,k})b(z_{1,k}, \cdot)\) is bounded of norm \(\frac{f}{t_1}\) and supported in a strip \(\{(y_1, \cdot) \in \mathbb{R}^2; |x_1 - y_1| \approx 2^{t_1}\}\). It follows that 3.1 implies 3.5.

Observe that now we just have to show that 3.5 for \(\delta > 1/8\) implies 3.4 for some \(\epsilon > 0\). Since \(N \leq \frac{2t_1}{t_1}\) it will be a consequence of the following.

**Proposition 1.** Let \(N\) be an integer and \((T_j)_{1 \leq j \leq N}\) be a family of convolution operators on \(L^2(\mathbb{R})\). Suppose that \((b_j)_{1 \leq j \leq N}\) is a sequence of bounded functions

\[|\sum_j T_j b_j|_B \leq \sup_j |b_j|_{\infty}.\]

Then if \((f_j)_{1 \leq j \leq N}\) is a sequence of \(L^2\)-functions

\[|\sum_j T_j f_j|_2 \leq CN^{1+\eta} \sup_j |f_j|_2\]

for all \(\eta > 0\).

Of course to prove Theorem 1 we need an analogue of Proposition 1 in a higher number of parameters. The extension of the proof of Proposition 1 which we shall give in the next section relies only on Lemma 2, and on the characterization of BMO in terms of \(L^\infty\) and partial Hilbert transforms [7]. Therefore we shall leave it to the reader. We shall however use in our proof the symbols \(\| \cdot \|_B\) and \(\| \cdot \|_{BMO}\), even though the norms they denote coincide in the one-parameter case, in order to indicate which one should be used in several parameters, at which place.


**Lemma 4.** Let \((\xi_m)_{1 \leq m \leq M}\) be a finite collection of distinct real numbers and let \((c_m)_{1 \leq m \leq M}\) be a finite collection of complex numbers. The function \(b = \sum c_m e^{i\xi_m} \cdot \cdot \cdot \) is in \(B\) with a norm at
least

\[ \left( \sum_{m=1}^{M} |c_m|^2 \right)^{1/2}. \]

To see this, it suffices to test the oscillation of \( b \) on intervals whose length tends to \( \infty \). We omit the details.

From this lemma it follows that, under the assumption \( 3.6 \) applied when the \( b_j \)'s are characters,

4.1 \[
\sum_{i\leq j\leq N} ||T_j||_{2,2}^2 \leq 1.
\]

Let us prove that 4.1 allows us to make the assumption that \( ||T_j||_{2,2} \leq \frac{1}{\sqrt{N}} \) for all \( j [1,N] \), without loss of generality. Indeed let \( c(N) \) be the best constant for which 3.7 holds with \( c(N) \) instead of \( cN^{1+\eta} \), and similarly for \( c'(N) \) but assuming \( ||T_j||_{2,2} \leq \frac{1}{\sqrt{N}} \) for \( j \in [1, N] \). Let \( \alpha > 0 \). Let \((T_j)_{1\leq j \leq N}\) be a collection of operators satisfying to 3.6. By 4.1, the number of \( j \)'s for which \( ||T_j||_{2,2} \geq N^{\alpha-\frac{1}{2}} \) is less than \([N^{1-2\alpha}]\), where \([\ ]\) denotes the integer part. It follows, by considering the set of \( j \)'s for which \( ||T_j||_{2,2} \leq N^{\alpha-\frac{1}{2}} \), and those for which \( ||T_j||_{2,2} \geq N^{\alpha-\frac{1}{2}} \), that

4.2 \[
c(N) \leq c'(N)N^{\alpha} + c([N^{1-2\alpha}]).
\]

Hence if \( c'(N) \) grows at most like \( N^{1+\eta} \) for all \( \eta > 0 \), so does \( c(N) \).

We now suppose that for all \( 1 \leq j \leq N \), \( ||T_j||_{2,2} \leq \frac{1}{\sqrt{N}} \). Since the Hilbert transform is bounded on \( B \) and any BMO function \( b \) can be written as \( a_1 + Ha_2 \), where \( a_1 \) and \( a_2 \) are in \( L^\infty \) and satisfy \( ||a_1||_\infty + ||a_2||_\infty \leq c||b||_{\text{BMO}} \), we see that 3.6 implies

4.3 \[
||\sum_j T_j b_j||_B \leq c \sup_j ||b_j||_{\text{BMO}}.
\]

We can also assume that the symbol of each \( T_j \) vanishes on \( \cup_{k \in \mathbb{Z}} [2^k, \frac{5}{4}, 2^k] \cup [3, \infty] \). We let \( \Delta_k \) be the multiplier of symbol \( X[\frac{5}{4}, 2^{k+1}] \) and let \( T_{j,k} \) be \( T_j \Delta_k \).
LEMMA 5. - For all \( k \in \mathbb{Z} \), \( \sum \| T_{j,k} \|_{2,2} \leq c \).

In order to prove this lemma it suffices to show that if \( (S_j)_{1 \leq j \leq N} \) is a family of convolution operators whose symbols \( \sigma_j \) are supported in \( \{ 1 \leq \xi \leq 1 + \beta \} \) for some \( \beta > 0 \) and if for all sequences \( (\xi_j)_{1 \leq j \leq N} \) of real numbers and all sequence \( (c_j)_{1 \leq j \leq N} \) of complex numbers

\[
4.4 \quad \left\| \sum_j S_j c_j e^{i\xi_j \cdot \cdot} \right\|_B \leq \sup_j |c_j|
\]

then

\[
4.5 \quad \sum_j \| S_j \|_{2,2} = \sum_j \| \sigma_j \|_{\infty} \leq c.
\]

To prove 4.5 it suffices to show that for any sequence \( (\xi_j)_{1 \leq j \leq N} \),

\[
4.6 \quad \sum_j |\sigma_j(\xi_j)| < c.
\]

We may assume that the \( \xi_j \)'s take their values in \([1, 1 + \beta]\) and that the \( \sigma_j(\xi_j) \)'s are valued in \([0, 1]\). Then we just have to show that

\[
4.7 \quad \int_0^1 \left| \sum_j \sigma_j(\xi_j) e^{i\xi_j \cdot \cdot} - m_{[0,1]} \left( \sum_j \sigma_j(\xi_j) e^{i\xi_j \cdot \cdot} \right) \right|^2 \, dx \geq c \left( \sum_j \sigma_j(\xi_j) \right)^2.
\]

Equivalently it suffices to prove for all \( j \)

\[
4.8 \quad \int_0^1 |e^{i\xi_j \cdot \cdot} - m_{[0,1]} e^{i\xi_j \cdot \cdot}|^2 \, dx \geq c
\]

and for \( j, \ell, j \neq \ell \),

\[
4.9 \quad \text{Re} \int_0^1 (e^{i\xi_j \cdot \cdot} - m_{[0,1]} e^{i\xi_j \cdot \cdot}) (e^{i\xi_{\ell} \cdot \cdot} - m_{[0,1]} e^{i\xi_{\ell} \cdot \cdot}) \, dx \geq c.
\]

This is clear if \( \beta \) is small enough, which proves Lemma 5. We may therefore assume that \( \sum_j \| T_{j,k} \|_{2,2} \leq 1 \) for all \( k \in \mathbb{Z} \).
The last lemma we shall need is classical.

**Lemma 6.** Let \((c_k)_{k \in \mathbb{Z}} \in \ell^2(O(Z))\) and \((\xi_k)_{k \in \mathbb{Z}}\) be a sequence of real numbers such that \(\xi_k \in \left[\frac{5}{4} 2^k, 2^{k+1}\right]\) for all \(k \in \mathbb{Z}\). Then \(\sum_k c_k e^{i\xi_k}\) is in BMO and \(\left\| \sum c_k e^{i\xi_k}\right\|_{\text{BMO}} \leq c \left(\sum |c_k|^2\right)^{1/2}\).

Let us now indicate the strategy to go from 3.6 to 3.7, assuming that \(\left\| T_j \right\|_{2,2} \leq \frac{1}{\sqrt{N}}\) and \(T_j = T_j \sum \Delta_k\), for all \(j\), and that \(\sum_j \left\| T_{j,k} \right\|_{2,2} \leq 1\) for all \(k \in \mathbb{Z}\).

Suppose there exists for each \(k\) a number \(\xi_k\) in \(\left[\frac{5}{4} 2^k, 2^{k+1}\right]\), such that for all \(j\),

4.10 \(\left\| T_{j,k} \right\|_{2,2} \leq c |m_j(\xi_k)|\).

For all \(j \in [1, N]\), let \((c_{j,k})_k\) be such that \(\sum_j |c_{j,k}|^2 \leq 1\). Then by Lemmas 4 and 6 and by 4.3 we obtain \(\sum_k \left(\sum_j m_j(\xi_k) c_{j,k}\right)^2 \leq c\) and even

4.11 \(\sum_k \left(\sum_j |m_j(\xi_k)| |c_{j,k}|\right)^2 \leq c\).

Now if \((f_j)_{1 \leq j \leq N}\) are \(L^2\)-functions with norm 1, by 4.10 and 4.11,

\[
\left\| \sum T_j f_j \right\|_2^2 = \sum_k \left\| \sum_j T_{j,k} \Delta_k f_j \right\|_2^2 
\leq \sum_k \left(\sum_j \left\| T_{j,k} \right\|_{2,2} \left\| \Delta_k f_j \right\|_2\right)^2 
\leq c \sum_k \left(\sum_j |m_j(\xi_k)| \left\| \Delta_k f_j \right\|_2\right)^2 \leq c,
\]

since for all \(j\), \(\sum \left\| \Delta_k f_j \right\|_2 \leq 1\).
Unfortunately the existence of these miraculous $\xi_k$'s is not guaranteed in general and matters are slightly more complicated. The point will be to select a small number of $\xi_{k,\ell}$'s for each $k$, in such a way that $\sup_{j,k} \left( \frac{||T_{j,k}||_{2,2}}{\sup_{\ell} m_j(\xi_{k,\ell})} \right)$ be not too large, and then apply essentially the previous argument. We are now going to describe how to select these $\xi_{k,\ell}$'s.

Let $k$ be fixed $p$ be a large fixed integer, and $\mu = \frac{3}{4p}$. Let $r$ and $s$ be two integers such that $0 \leq s \leq r \leq p - 1$. We pick up a $\xi$, denoted $\xi_{k,1}$ such that

$$\sum_j |m_j(\xi)|^2 \geq \frac{1}{p^2} n^{-\frac{3}{4}}$$

where the sum runs over those $j$'s such that

$$N^{-\frac{3}{4} + \mu r} \leq ||T_{j,k}||_{2,2} \leq N^{-\frac{3}{4} + \mu (r+1)}$$

and

$$N^{-\frac{3}{4} + \mu s} \leq |m_j(\xi)| \leq N^{-\frac{3}{4} + \mu (s+1)}.$$

We take off all the $j$'s satisfying 4.13 and 4.14 and select another $\xi$, denoted $\xi_{k,2}$, in the same fashion. When we cannot go on for a fixed $r, s$, we choose another couple $r', s'$, and obtain a collection of $(\xi_{k,\ell}')_{\ell}$. Finally when the process stops we can conclude that for all $\xi$'s in $[\frac{5}{4} 2^k, 2^{k+1}]$,

$$\sum_j |m_j(\xi)|^2 \leq N^{-\frac{3}{4}},$$

where the sum is restricted to those $j$'s which have not been taken off during the selection process. We call this set $E_k$. So we have a decomposition of $[1,N]$ as $E_k \cup \bigcup_{r, \ell} E_{k,r}^{r',s',\ell}$ where $E_{k,r}^{r',s',\ell}$ is the set of $j$'s which have been taken off after selecting $\xi_{k,\ell}'. Notice that all these sets are pairwise disjoints. We define $T_j^{r',s'} = T_j \left( \sum_k \Delta_k \right)$, where the
sum is extended to those $k$'s such that $j$ belongs to $U^E$. Notice that for each $(r, s)$, the collection $(T_j^{r,s})_{1 \leq j \leq N}$ satisfy 4.3 uniformly.

**Lemma 7.** Let $(f_j)_{1 \leq j \leq N}$ be a sequence of $L^2$-functions of norm less than 1. Then
\[ \left\| \sum_j T_j^{r,s} f_j \right\|_2 \leq c \, p \, N^{\mu\left(\frac{r}{2}+2\right)}. \]

To prove this lemma we first observe that for each $\ell$, the set $E_{k,\ell}^{r,s}$ has at least $p^{-2} N^{\frac{3}{4} - 2\mu(s+1)}$ elements because of 4.12 and 4.14. On the other hand, since $\sum_j \|T_{j,k}\|_2 \leq 1$, the set $U_{\ell} E_{k,\ell}^{r,s}$ has at most $N^{\frac{3}{4} - \mu r}$ elements by 4.13. Hence there is at most $p^2 N^{\mu(2s+2-r)}$ distinct values of $\xi_{k,\ell}^{r,s}$, for $r, s$ and $k$ fixed. For each $j$ we consider a sequence $(c_{j,k})_{k \in \mathbb{Z}}$ in $\ell^2(\mathbb{Z})$ such that $\sum_k |c_{j,k}|^2 \leq 1$, and $c_{j,k} = 0$ if $T_j^{r,s} \Delta_k = 0$. Let $b_j$ be the BMO function $\sum_k c_{j,k} e^{i \xi_{k,\ell}^{r,s}}$, where $\xi_{k,\ell}^{r,s}$ is the element of $\{\xi_{k,\ell}^{r,s}, 1 \leq \ell \leq p^2 N^{\mu(2s+2-r)}\}$ for which $j \in E_{k,\ell}^{r,s}$. By Lemmas 4 and 6 and by 4.3 and 4.14

\[ \sum_{k,\ell} \left( \sum_j |c_{j,k}| \right)^2 \leq c \, N^{\frac{3}{2} - 2\mu s}, \]

where the sum in $j$ runs over $E_{k,\ell}^{r,s}$. If $(f_j)_{1 \leq j \leq N}$ are $L^2$-functions of norms less than 1,\n\[ \left\| \sum_j T_j^{r,s} f_j \right\|_2 \leq \sum_k \left( \sum_j \|T_j^{r,s}\|_2 \|\Delta_k f_j\|_2 \right)^2, \]

which is, by 4.13, dominated by $\sum_k \left( \sum_j \|\Delta_k f_j\|_2 \right)^2 N^{-\frac{3}{2} + 2\mu(r+1)}$, the sum in $j$ running over $\bigcup_{\ell} E_{k,\ell}^{r,s}$. Hence, for each $k$,

\[ \left( \sum_{j \in \bigcup_{\ell} E_{k,\ell}^{r,s}} \|\Delta_k f_j\|_2 \right)^2 \leq p^2 N^{\mu(2s+2-r)} \sum_{\ell} \left( \sum_{j \in E_{k,\ell}^{r,s}} \|\Delta_k f_j\|_2 \right)^2. \]

Summing over $k$ and using 4.16 we obtain
\[ \left\| \sum_j T_j^{r,s} f_j \right\|_2^2 \]
\[ \leq c \, p^2 N^{\mu(r+4)} \] and Lemma 7 is proved.

By the assumption \( \|T_j\|_{2,2} \leq \frac{1}{\sqrt{N}} \) for all \( j \), we see that \( \mu(r+1) \leq \frac{1}{4} \). We then deduce from Lemma 7 that \( \left\| \sum \sum T_j^{r,s} f_j \right\|_2 \)

\[ \leq c \, p^{3} N^{\frac{3}{2} \mu + \frac{1}{8}} \sup_{j} \|f_j\|_2. \]

Since \( \mu \) can be made arbitrarily small, we just have to estimate the remainder term \( \left\| \sum_{j} T_j f_j - \sum_{r,s} \left( \sum_{j} T_j^{r,s} f_j \right) \right\|_2 \).

Using 4.15, Plancherel and Cauchy-Schwarz we easily obtain a domination by \( c \, N^{1/8} \). This concludes the proof of Proposition 1.

5. Tensor-products of multilinear singular integral operators.

We wish to prove the following.

Theorem 2. - For all \( \delta > 0 \), there exists \( c_{\delta} > 0 \), such that for all \( a \in L^{2}(\mathbb{R}^2) \) and \( k \in \mathbb{N} \)

\[ |||L_{k,a}|||_{2,2} \leq c_{\delta}(1 + k)^{2+\delta}||a||_{\infty}^{k}. \]

This theorem will essentially follow from a general result on multilinear singular integral operators.

Let \( T_1 \) and \( T_2 \) be two bounded operators on \( L^2(\mathbb{R}) \). Then it is a simple consequence of Fubini's Theorem that \( T_1 \otimes T_2 \), defined on \( L^2(\mathbb{R} \times \mathbb{R}) \), extends boundedly to all of \( L^2(\mathbb{R} \times \mathbb{R}) \). If however one considers two bilinear operators bounded from \( L^{\infty}(\mathbb{R}) \times L^2(\mathbb{R}) \) to \( L^2(\mathbb{R}) \), then their tensor-product, defined on \( [L^{\infty}(\mathbb{R}) \otimes L^{\infty}(\mathbb{R})] \times [L^2(\mathbb{R}) \otimes L^2(\mathbb{R})] \), is not in general bounded from \( L^{\infty}(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \) to \( L^2(\mathbb{R}^2) \) \([9]\). It is a surprising fact that it is bounded when the bilinear operators are of Coifman-Meyer type.

We shall see that this is also true for multilinear singular operators.
As in [10] we shall deal with multilinear singular integral forms instead of multilinear operators. We refer to [10] for the precise definition of a multilinear singular integral form, $\delta - n\text{SIF}$, and for a boundedness criterion concerning them. We shall denote by $U$ a $\delta - n\text{SIF}$ on $\mathbb{R}$ and refer to [10] for the notations $U_{i1}$, $i \in [1,n]$, $|U|_{\delta}$, $|U|_{W}$, $|U|_{i,j}$ and $|U_{ij}|_{\delta}$. We recall that $U$ is bounded if for $1 \leq i \leq j \leq n$,

$$5.1 \quad |U(h_1, h_2, \ldots, h_n)| \leq c_{ij} \left( \prod_{k \neq i,j} \|h_k\|_{\infty} \right) \|h_i\|_2 \|h_j\|_2$$

for all $h_{\ell}$, $1 \leq \ell \leq n$, in $C_0^\infty(\mathbb{R})$.

If $U$ and $U'$ are two bounded $\delta - n\text{SIF}'s$, then their tensor-product $U \otimes U'$ is well defined on $[C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R})]^n$. We then say that $U$ is bounded if for $1 \leq i \leq j \leq n$,

$$5.2 \quad |U \otimes U'(h_1, h_2, \ldots, h_n)| \leq c_{ij} \left( \prod_{k \neq i,j} \|h_k\|_{\infty} \right) \|h_i\|_2 \|h_j\|_2$$

for all $h_{\ell}$, $1 \leq \ell \leq n$, $C_0^\infty(\mathbb{R})$, and we denote by $|||U \otimes U'||||$, $\sup_{i,j} \tilde{c}_{ij}$, where $\tilde{c}_{ij}$ is the best constant in 5.2.

**Theorem 3.** - If $U$ and $U'$ are two bounded $\delta - n\text{SIF}'s$, then $U \otimes U'$ is bounded and

$$5.3 \quad |||U \otimes U'|||| \leq c \left\{ \sum_i ||U_{i1}||_{\text{BMO}} + n^2(|U|_{\delta} + |U|_{W}) \right\}$$

Notice that the constant $c$ appearing in 5.3 is independent of $n$.

An application of Theorem 3 in the case where $U = U'$ is the form determined by the $(n - 2) - nd$ Calderón-commutator (see [10] section 4) yields $||L_{k,a}||_{2,2} \leq c_\delta(1 + n)^{4+\delta}$ for all $n \in \mathbb{N}$ and $\delta > 0$. As in [10] the antisymmetry of the kernel $\frac{1}{x - y}$ permits to improve this estimate and to obtain Theorem 2. Since this will be clear from the proof of Theorem 3 which we shall now outline, we omit the details.
The proof of Theorem 3 is along the same lines as the proof of Theorem 2 in [10], which we shall assume familiar to the reader. Let us recall however that the main ingredients of this proof are Carleson measures, quadratic estimates that have been developed in [5], [7], and [11] in the context of product-spaces. Another important element is the equivalence between $H^1 \rightarrow L^1$ and $L^\infty \rightarrow \text{BMO}$ boundedness for a singular integral operator. The proof of this equivalence given in [12, p. 49] relies on the atomic decomposition of $H^1$ in $H^{1,\infty}$-atoms. Such a decomposition does not seem to exist on product-spaces where the atoms are only in $L^2$. However this equivalence is still true on product-spaces.

**Proposition 2.** Let $T$ be a $\delta$—SIO on $\mathbb{R} \times \mathbb{R}$. Then $T$ is bounded on $L^2$ if and only if $T$ maps $L^\infty$ to BMO.

We refer to [3] for the definition of a $\delta$—SIO on $\mathbb{R} \times \mathbb{R}$. Notice that by applying Proposition 2 simultaneously to $T$ and $T^*$ we see that a $\delta$—SIO maps $L^\infty$ to BMO if and only if it maps $H^1$ to $L^1$.

The fact that the $L^2$-boundedness of $T$ implies its $L^\infty$—BMO boundedness is already known [3]. The converse is then an easy consequence. Suppose that $T$ is a $\delta$—SIO bounded from $L^\infty$ to BMO, and let us also assume that $||T||_2,2 < +\infty$. Then, by the direct part of Proposition 2 applied to $T^*$ we obtain $||T||_{H^1,L^1} \leq c||T||_2,2 + c(T)$, where $c(T)$ depends only on the constants for the standard estimates of the kernel of $T$. By interpolation [13] $||T||_2,2 \leq c(||T||_{L^\infty,\text{BMO}},||T||_{H^1,L^1})^{1/2}$. It follows that, $||T||_2,2 \leq c(||T||_{L^\infty,\text{BMO}} + c(T))$, which easily implies Proposition 2.

The connection between $\delta$—SIO's on $\mathbb{R} \times \mathbb{R}$ and tensor products of $\delta$—SIF's on $\mathbb{R}$ is provided by the following lemma.

**Lemma 8.** Let $U$ and $U'$ be two bounded $\delta$—SIF's on $\mathbb{R}$. For all $1 \leq i < j \leq n$ and all $h_k \in C_0^\infty(\mathbb{R}^d) \otimes C_0^\infty(\mathbb{R}^d)$, $k \neq i,j$, the operator $T = (U \otimes U')_{i,j}(h_1,\ldots,h_k,\ldots,h_n)$ defined by $< h_1, Th_j > = (U \otimes U')(h_1,\ldots,h_n)$, is a $\delta$—SIO on $\mathbb{R} \times \mathbb{R}$, of norm less than $c(||U||_{i,j} + |U_{ij}|)U'_{ij}|\delta + (||U'||_{i,j} + |U'_{ij}|)U_{ij}|\delta \prod_{k \neq i,j} ||h_k||_{\infty}$. 


The proof is routine and we omit it.

We turn to the proof of Theorem 3. Let \( \phi \) be a non-negative function in \( C_0^\infty(\mathbb{R}) \) such that \( \int \phi \, dx = 1 \). For all \( t > 0 \), we denote by \( P_t \) the convolution operator on \( \mathbb{R}^2 \) of symbol \( \sigma(\xi, \xi') = \hat{\phi}(t\xi) \). Similarly \( P_t' \) is the operator of symbol \( \hat{\phi}(t\xi') \). Finally \( Q_t = -t \frac{\partial}{\partial t} P_t \) and \( Q_t' = -t' \frac{\partial}{\partial t'} P_t' \).

As in [10] we choose \( h_1, \ldots, h_n \) in \( C_0^\infty(\mathbb{R}^d) \otimes C_0^\infty(\mathbb{R}^d) \) and express \( U \otimes U'(h_1, \ldots, h_n) \) as the sum of \( n^2 \) double integrals of two different types:

\[
\begin{align*}
I & = \iint U \otimes U'(Q_t, Q_t', h_1, P_t P_t', h_2, \ldots, P_t P_t', h_n) \frac{dt}{t} \frac{dt'}{t'} \\
II & = \iint U \otimes U'(Q_t, P_t', h_1, P_t Q_t', h_2, P_t P_t', h_3, \ldots, P_t P_t', h_n) \frac{dt}{t} \frac{dt'}{t'}.
\end{align*}
\]

It is clear that the estimates of the \( n \) integrals of type I can be reduced to Carleson measure estimates. To see that this is also true for the \( n^2 - n \) integrals of type II we need the following.

**Lemma 9.** Let \( T \) be a \( \delta - \text{SIO} \) on \( \mathbb{R} \times \mathbb{R} \). If \( T_1 \) and \( T_1^* \) vanish (or are in BMO), and the partial adjoints of \( T \) are bounded on \( L^2 \), then \( T \) is a \( \delta - \text{CZO} \).

This lemma follows immediately from the T1-Theorem and Theorem 3 of [3].

Let \( V' \) be the \( \delta - \text{nSIF} \) obtained from \( U' \) by letting \( V'(f_1, f_2, \ldots, f_n) = U'(f_2, f_1, f_3, \ldots, f_n) \). By lemmas 8 and 9 we see that

\[
\left| \iint U \otimes V'(Q_t, Q_t', h_1, P_t P_t', h_2, \ldots) \frac{dt}{t} \frac{dt'}{t'} \right| \leq c_1 \|h_1\|_2 \|h_2\|_2 \prod_{k \geq 3} \|h_k\|_\infty
\]

implies

\[
\left| \iint U \otimes U'(Q_t, P_t', h_1, P_t Q_t', h_2, \ldots) \frac{dt}{t} \frac{dt'}{t'} \right|
\]
Since $V'$ has the same properties as $U'$, we are reduced to estimating integrals of type I.

We are going to prove the following estimate:

$$\left| \int \int U \otimes U'(Q_t Q'_t h_1, P_t P'_t h_2, \ldots) \frac{dt}{t} \frac{dt'}{t'} \right|$$

5.4

$$\leq c ||U_1||_{BMO} + n(||U||_{\delta} + ||U||_W)$$

$$\left( ||U'_1||_{BMO} + n(||U'||_{\delta} + ||U'||_W) \right) \left( \prod_{k \geq 3} ||h_k||_{\infty} \right) ||h_1||_2 ||h_2||_2.$$

It is easy to see from the previous remarks that 5.4 implies Theorem 3. In turn 5.4 is itself an immediate consequence, after reduction to a Carleson-measure estimate, of an extension of Theorem 1 of [10] to the setting of product spaces, which we now describe. We first need to recall the notion of an $\epsilon$-family introduced in [10].

**Definition 1.** - A family $S = (s_t)_{t \geq 0}$ of operators given by kernels satisfying

5.5

$$|s_t(x, y)| \leq c \frac{t\epsilon}{t^{1+\epsilon} + |x - y|^{1+\epsilon}}$$

5.6

$$|s_t(x, y) - s_t(x, z)| \leq c \frac{t\epsilon}{t^{1+\epsilon} + |x - y|^{1+\epsilon}} \left( \frac{y - z}{t + |x - y|} \right)^\epsilon,$$

for all $x, y, z$ such that $|y - z| \leq \frac{1}{2}(t + |x - y|)$, is an $\epsilon$-family. It is bounded if for all $f \in L^2$,

5.7

$$\left[ \int_0^{+\infty} ||s_t f||_2^2 \frac{dt}{t} \right]^{1/2} \leq c ||f||_2.$$

Following the procedure of [3], to extend this notion to product spaces, we first put a norm on the space of $\epsilon$-families by letting $||S||_\epsilon = ||S||_2 + ||S||_\epsilon$, where $||S||_2$ is the best constant in 5.7 and
|S|<\epsilon in 5.5 and 5.6. An \epsilon-family on \mathbb{R} \times \mathbb{R} will then be a two-parameter family \((T_t, t'), t', > 0\) of operators given by integrable kernels \(T_t, t'(x, x, y, y')\). For \(t, x, y\) fixed, we shall denote by \((T_t, [x, y])_t', > 0\), the one-parameter family of operators acting on the second variable, and of kernels \((T_t, [x, y])(x', y') = T_t, t'(x, x', y, y')\), and similarly for \((T_t, [x', y'])_t', > 0\). Then \((T_t, t'), t', > 0\) is an \epsilon-family if

\[
||| (T_t, t') [x, y]_t', > 0 ||| \leq c \frac{t^\epsilon}{|x - y|^{1+\epsilon} + t^{1+\epsilon}}
\]

\[
||| (T_t, t') [x, z]_t', > 0 ||| \leq c \left( \frac{|y - z|}{t + |x - y|} \right)^\epsilon \frac{t^\epsilon}{t^{1+\epsilon} + |x - y|^{1+\epsilon}}
\]

when \(|y - z| \leq \frac{1}{2} (t + |x - y|)\) and similarly for \((T_t, t') [x', y']_t', > 0\). We denote by \(|T^{(t', t')}|_c\) the best constant in 5.8 and 5.9. The family \((T_t, t')_t', > 0\) is bounded if for all \(f \in L^2\)

\[
\left[ \iint |T_t, t' f|^2 \frac{dt}{t} \frac{dt'}{t'} \right]^{1/2} \leq c ||f||_2.
\]

We also introduce a "Carleson norm" on functions \(w\) from \(\mathbb{R}^2_+ \times \mathbb{R}^2_+\) into \(c\), by letting

\[
|w|_c = \sup_{\Omega \subseteq \mathbb{R}^2} \left[ \frac{1}{|\Omega|} \iint_{S(\Omega)} |w(x, x', t, t')|^2 dx dx' \left[ \frac{dt}{t} \frac{dt'}{t'} \right]^{1/2} \right],
\]

where \(\Omega\) is an arbitrary bounded open subset of \(\mathbb{R}^2\), and \(S(\Omega)\) consists of these \((x, x', t, t')\) such that \(|x - t, x + t[x] x' - t', x' + t'| \subseteq \Omega\).

**Theorem 4.** - Let \((T_t, t')_t', > 0\) be an \epsilon-family. It is bounded if and only if \(||(T_t, 1)(\cdot, \cdot)||_c < +\infty\). In this case for all \(a \in L^\infty(\mathbb{R}^2)\)

\[
|(T_t, a)(\cdot, \cdot)||_c \leq ||a||_\infty ||(T_t, 1)(\cdot, \cdot)|| + c_\epsilon ||a||_\epsilon |T_t, t'|_c.
\]

By the same argument as for Theorem 1 for [10], we need to consider only the case where \(T_t, t' = 0\) for all \(t, t' > 0\). We then decompose \(T_t, t'\) as \(X_t, t' + Y_t, t'\) where \(X_t, t' f(x, x') = \)
Notice that $(X_{t',u'})_{t',u'>0}$ is itself an $\varepsilon$-family, as well as $(Y_{t',u'})_{t',u'>0}$. Furthermore if $f$ does not depend on the first variable $X_{t',u'}f = 0$, while if it depends only on the first variable $Y_{t',u'}f = 0$. Therefore we are reduced to the case where not only $T_{t',u'}1 = 0$ but also $T_{t',u'}f = 0$ for all functions $f$ of the first variable. To show that $(T_{t',u'})_{t',u'>0}$ is bounded in this case it suffices to show that

$$Z = \iint T_{t',u'}^{*} \frac{dt}{t} \frac{dt'}{t'}$$

is bounded on $L^2$. But $Z$ is an SIO on $\mathbb{R} \times \mathbb{R}$, to which it is easy to see that the $T1$-Theorem of [3] applies. To deduce 5.11 from the boundedness of $(T_{t',u'})_{t',u'>0}$ one proceeds exactly as in the proof of Theorem 3 on [3]. This proves Theorem 4. Routine arguments, which we shall omit, now yield 5.4 and then Theorem 3.

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