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Graded morphisms of $G$-modules


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1. Introduction.

During the 1987 meeting in honor of J. K. Koszul, Steve Halperin explained to us the following conjecture (motivated by the study of the spectral sequence associated to a homogeneous space).

1.1. Conjecture. — If \( f_1, f_2, \ldots, f_n \) is a regular sequence in the polynomial ring \( \mathbb{C}[x_1, x_2, \ldots, x_n] \), the connected component of the automorphism group of the (finite dimensional) algebra \( \mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_n) \) is solvable.

In this paper we prove a weak form of this (Corollary 4.3) which implies the conjecture at least when the \( f_i \)'s are homogeneous (Remark 4.4).

2. Preliminaries.

Our base field is \( \mathbb{C} \), the field of complex numbers, or any other algebraically closed field of characteristic zero.

2.1. Definition. — A morphism \( \varphi : V \to W \) between finite dimensional vector spaces \( V \) and \( W \) is called graded if there is a basis of \( W \) such that the components of \( \varphi \) are all homogeneous polynomials.

Let us denote by \( \mathcal{O}(V) \), \( \mathcal{O}(W) \) the ring of regular functions on \( V \) and \( W \). These \( \mathbb{C} \)-algebras are naturally graded by degree: \( \mathcal{O}(V) = \bigoplus \mathcal{O}(V)_i \). A subspace \( S \subset \mathcal{O}(V) \) is called graded if \( S = \bigoplus_i S \cap \mathcal{O}(V)_i \).

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If \( \varphi : V \to W \) is a morphism and \( \varphi^* : \mathcal{O}(W) \to \mathcal{O}(V) \) the corresponding comorphism we have the following equivalence:

\[ \varphi \text{ is graded } \iff \varphi^*(W^*) \text{ is a graded subspace of } \mathcal{O}(V). \]

2.2. **Lemma.** — For any graded morphism \( \varphi : V \to W \) there is a unique decomposition \( W = \bigoplus W_v \) and homogeneous morphisms \( \varphi_v : V \to W_v \) of degree \( v \) such that

\[ \varphi = (\varphi_0, \varphi_1, \varphi_2, \ldots) : V \to W_0 \oplus W_1 \oplus W_2 \oplus \cdots. \]

(This is clear from the definitions.)

2.3. **Remark.** — Let \( G \) be an algebraic group. Assume that \( V \) and \( W \) are \( G \)-modules and that \( \varphi : V \to W \) is graded and \( G \)-equivariant. Then in the notations of lemma 2.2 all \( W_v \) are submodules and all components \( \varphi_v \) are \( G \)-equivariant.

2.4. **Remark.** — If \( \varphi : V \to W \) is graded and dominant with \( \varphi^{-1}(0) = \{0\} \), then \( \varphi \) is a finite surjective morphism. In fact given a finitely generated graded algebra \( A = \bigoplus A_i \) with \( A_0 = \mathbb{C} \) and a graded subspace \( S \subset A \) such that the radical \( \text{rad}(S) \) of the ideal generated by \( S \) is the homogeneous maximal ideal \( \bigoplus A_i \) of \( A \), then \( A \) is a finitely generated module over the subalgebra \( \mathbb{C}[S] \) generated by \( S \) (see [1, II.4.3 Satz 8]).

3. The Main Theorem.

3.1. **Theorem.** — Let \( G \) be a connected reductive algebraic group and let \( V, W \) be two \( G \)-modules. Assume that \( V \) and \( W \) do not contain 1-dimensional submodules. Then any graded \( G \)-equivariant dominant morphism with finite fibres is a linear isomorphism.

We first prove this for \( G = SL_2 \) and then reduce to this situation.

For any \( C^* \)-module \( V \) we have the weight decomposition

\[ V = \bigoplus_j V_j, \quad V_j := \{ v \in V | t(v) = t^j \cdot v \}. \]

We say that \( V \) has only positive weights if \( V = \bigoplus_{j>0} V_j \).
3.2. Lemma. — Let $V$, $W$ be two $\mathbb{C}^*$-modules with only positive weights, and let $\varphi : V \to W$ be a $\mathbb{C}^*$-equivariant graded morphism with finite fibres. For all $k \geq 0$ we have

$$\varphi^{-1} \left( \bigoplus_{j \leq k} W_j \right) \subseteq \bigoplus_{j \leq k} V_j,$$

and the inclusion is strict for at least one $k$ in case $\varphi$ is not linear.

Proof. — By lemma 2.2 and remark 2.3 we have $\varphi = \sum_{v \geq 1} \varphi_v$ where $\varphi_v : V \to W_v$ is homogeneous of degree $v$ and $\mathbb{C}^*$-equivariant. Let $v = \sum_{j=1}^{\infty} v_j \in \bigoplus_{j>0} V_j = V$ with $v_k \neq 0$. Then

$$\lim_{\lambda \to 0} \lambda^k t_k^{-1}(v) = v_k.$$

(Here $t_k$ denotes the action of $\mathbb{C}^*$.) Since $\varphi_v$ is homogeneous of degree $v$ and $\mathbb{C}^*$-equivariant we obtain

$$\lim_{\lambda \to 0} \lambda^{v_k} t_k^{-1}(\varphi_v(v)) = \varphi_v(v_k).$$

This implies that $\varphi_v(v) \in \bigoplus_{j \leq v_k} W_j$ for all $v$, proving the first claim.

If $\varphi$ is not linear, i.e. $\varphi \neq \varphi_1$, then there is a $v > 1$, an index $k$ and an element $v \in V_k$ such that $\varphi_v(v) \neq 0$. But $\varphi_v(v) \in W_{v_k}$ by (1) and so $v \notin \varphi^{-1} \left( \sum_{j \leq k} W_j \right)$.

3.3. Corollary. — Under the assumptions of lemma 3.2 suppose that $\varphi$ is surjective. Put $\lambda_j := \dim V_j$ and $\mu_j := \dim W_j$. Then for all $k \geq 1$ we have

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k \geq \mu_1 + \mu_2 + \cdots + \mu_k.$$

If $\varphi$ is not linear the inequality is strict for at least one $k$.

(This is clear.)

3.4. Proposition. — Let $V$, $W$ be two $\text{SL}_2$-modules containing no fixed lines. Let $\varphi : V \to W$ be a graded $\text{SL}_2$-equivariant morphism, which is dominant and has finite fibres. Then $\varphi$ is a linear isomorphism.
Proof. — Consider the maximal unipotent subgroup
\[ U := \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \subseteq \text{SL}_2 \]
and the maximal torus
\[ T := \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \mid \lambda \in \mathbb{C}^* \right\} \cong \mathbb{C}^*. \]
By assumption \( \varphi \) is finite and surjective (Remark 2.4), and \( \varphi^{-1}(W^U) = V^U \). Hence the induced morphism
\[ \varphi|_{V^U} : V^U \to W^U \]
is graded, \( T \)-equivariant, finite and surjective too. Furthermore all weights \( \lambda_j \) of \( V^U \) and \( \mu_j \) of \( W^U \) are positive. It follows from (2) that
\[ \lambda_k + \lambda_{k+1} + \cdots \leq \mu_k + \mu_{k+1} + \cdots \]
for all \( k \), because \( \sum_j \lambda_j = \dim V^U = \dim W^U = \sum_j \mu_j \). From this we get
\[ \dim V = 2\lambda_1 + 3\lambda_2 + \cdots + (n+1)\lambda_n \leq 2\mu_1 + 3\mu_2 + \cdots + (n+1)\mu_n = \dim W \]
for all \( n \) which are big enough. (Remember that an irreducible \( \text{SL}_2 \)-module of highest weight \( j \) is of dimension \( j + 1 \)). If \( \varphi \) is not linear this inequality is strict (Corollary 3.3), contradicting the fact that \( \varphi \) is finite and surjective. \( \Box \)

3.5. Proof of the Theorem. — Assume that \( \varphi : V \to W \) is not linear, i.e. there is a \( v_0 > 1 \) such that the component \( \varphi_{v_0} : V \to V_{v_0} \) is non-zero. Then there is a homomorphism \( \text{SL}_2 \to G \) and a non-trivial irreducible \( \text{SL}_2 \)-submodule \( M \subseteq V \) such that \( \varphi_j|_M \neq 0 \). (In fact the intersection of the fixed point sets \( V^{(\text{SL}_2)} \) for all homomorphisms \( \iota : \text{SL}_2 \to G \) is zero.) Now consider the \( G \)-stable decompositions \( V = V^{\text{SL}_2} \oplus V' \) and \( W = W^{\text{SL}_2} \oplus W' \) and the following morphism:
\[ \varphi' : V' \hookrightarrow V \xrightarrow{\varphi} W \xrightarrow{\text{Pr}} W'. \]
Since \( V' \) and \( W' \) are sums of isotypic components the morphism \( \varphi' \) is again graded. Furthermore \( \varphi^{-1}(W^{\text{SL}_2}) = V^{\text{SL}_2} \), hence \( \varphi^{-1}(0) = V^{\text{SL}_2} \cap V' = \{0\} \). This implies that \( \varphi' : V' \to W' \) is dominant.
with finite fibres and satisfies therefore the assumptions of proposition 3.4. As a consequence \( \varphi' \) is linear. Since \( \varphi|_V : V' \to W \) is graded too we have \( \varphi|_V = 0 \) for all \( v > 1 \). This contradicts the facts that \( M \subseteq V' \) and \( \varphi_{v_0}|_M \neq 0 \) (see the construction above).

4. Some Consequences.

We add some corollaries of the theorem. Let \( G \) be a connected reductive group. For every \( G \)-module \( V \) we have the canonical \( G \)-stable decomposition \( V = V^0 \oplus V' \) where \( V^0 \) is the sum of all 1-dimensional representations (i.e. \( V^0 = V^{(G,G)} \)) and \( V' \) the sum of all others. The proof of the theorem above easily generalizes to obtain the following result:

**4.1. Theorem.** Let \( \varphi : V \to W \) be a graded \( G \)-equivariant dominant morphism with finite fibres. Then \( \varphi \) induces a linear isomorphism \( \varphi|_V : V' \cong W'. \)

**4.2. Corollary.** Let \( \mathcal{O}(V) \) be the ring of regular functions on a \( G \)-module \( V \), and let \( f_1, \ldots, f_n \) be a regular sequence of homogenous elements of \( \mathcal{O}(V) \) such that the linear span \( \langle f_1, \ldots, f_n \rangle \) is \( G \)-stable. Then \( \langle f_1, \ldots, f_n \rangle \) contains all non-trivial representations of \((G,G)\) in \( \mathcal{O}(V)_1 \), the linear part of \( \mathcal{O}(V) \).

*Proof.* The regular sequence \( f_1, \ldots, f_n \) defines a \( G \)-equivariant finite morphism \( \varphi : V \to W, W := \langle f_1, \ldots, f_n \rangle^* \). By the theorem above the restriction \( \varphi'|_V : V' \to W' \) is a linear isomorphism which means that every non-trivial \((G,G)\)-submodule of \( \langle f_1, \ldots, f_n \rangle \) is contained in the linear part \( \mathcal{O}(V_1) \) of \( \mathcal{O}(V) \). \( \square \)

**4.3.** Recall that a finite dimensional \( C \)-algebra is called a complete intersection if it is of the form \( C[x_1, \ldots, x_n]/(f_1, \ldots, f_n) \) with a regular sequence \( f_1, \ldots, f_n \).

**Corollary.** Let \( A \) be a finite dimensional local \( C \)-algebra with maximal ideal \( m \) and let \( \text{gr}_mA \) be the associated graded algebra (with respect to the \( m \)-adic filtration). If \( \text{gr}_mA \) is a complete intersection then the connected component of the automorphism group of \( A \) is solvable.
Proof. — Let $G$ and $\overline{G}$ be the connected components of the automorphism groups of $A$ and of $\text{gr}_m A$ respectively. Since the $m$-adic filtration of $A$ is $G$-stable we have a canonical homomorphism $\rho : G \rightarrow \overline{G}$. It is easy to see that $\ker \rho$ is unipotent, so it remains to show that $G$ is solvable.

Assume that $G$ is not solvable. Then $G$ contains a (non-trivial) semisimple subgroup $H$. By assumption we have an isomorphism

$$\text{gr}_m A \cong C[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$$

with a regular sequence $f_1, \ldots, f_n$ where all $f_i$ are homogeneous of degree $\geq 2$. Clearly the action of $G$ on $\text{gr}_m A$ is induced from a (faithful) linear representation on $C[x_1, \ldots, x_n]$. Hence it follows from corollary 4.2 that $\langle f_1, \ldots, f_n \rangle$ contains all non-trivial $H$-submodules of $C[x_1, \ldots, x_n]_1$, contradicting the fact that all $f_i$ have degree $\geq 2$. □

4.4. Remark. — The corollary above implies that conjecture 1.1 is true in case all $f_i$ are homogeneous, i.e. if the algebra

$$A = C[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$$

is finite dimensional and graded with all $x_i$ of degree 1.

4.5. Remark. — Another formulation of our result is the following: Let $V$ be a representation of a connected algebraic group $G$ and $Z \subseteq V$ a $G$-stable graded subscheme, which is a complete intersection supported in $\{0\}$. Then $(G, G)$ acts trivially on $Z$.

BIBLIOGRAPHY


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