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AN F. AND M. RIESZ THEOREM FOR BOUNDED SYMMETRIC DOMAINS

by R. G. M. BRUMMELHUIS (*)

to the memory of my mother

1. Introduction.

In [11] J. H. Shapiro has given new proofs of the classical F. and M. Riesz theorem for the circle group $\mathbf{T} = \{z \in \mathbf{C} : |z|=1\}$ and of Bochner's generalization of F. and M. Riesz to the torus $\mathbf{T} \times \mathbf{T}$. These proofs were based on a study of the duals of certain subspaces of $L^p(\mathbf{T})$, respectively $L^p(\mathbf{T} \times \mathbf{T})$ for p 's between 0 and 1.

In this paper Shapiro's methods are generalized to arbitrary compact groups. As a result, we obtain in section 3 a general F. and M. Riesz theorem for compact groups whose center contains a circle group.

A typical special case of our F. and M. Riesz theorem is the unit sphere \mathbf{S} in \mathbf{C}^n : $\mathbf{S} = \mathbf{S}_{2n-1} = U(n)/U(n-1)$, where $U(n)$ is the unitary group. For the formulation we have to recall some definitions from harmonic analysis on \mathbf{S} , cf. [7], chapter 12. Let $\mathbf{H}(p,q)$ be the set of restrictions to \mathbf{S} of harmonic polynomials in z and \bar{z} which are homogeneous of degree p in z and of degree q in \bar{z} . Let σ denote the $U(n)$ -invariant measure on \mathbf{S} with total mass 1. The spaces $\mathbf{H}(p,q)$ span $L^2(\mathbf{S},\sigma)$ and are pairwise orthogonal. Let π_{pq} denote the orthogonal projection of $L^2(\mathbf{S},\sigma)$ onto $\mathbf{H}(p,q)$. The map $f \rightarrow (\pi_{pq}f)(z) (z \in \mathbf{S})$ can be represented as the inner product in L^2 of f with an element \mathbf{K}_z in $\mathbf{H}(p,q)$. Hence we can define $\pi_{pq}\mu \in \mathbf{H}(p,q)$ for any finite Borel measure μ on \mathbf{S} . Let $\text{spec } \mu = \{(p,q) \in \mathbf{N} \times \mathbf{N} : \pi_{pq}\mu \neq 0\}$.

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1.1. THEOREM. — *Let $\Delta \subseteq \mathbf{N} \times \mathbf{N}$ satisfy the following two conditions :*

- (i) *For each $m \in \mathbf{Z}$ the set $\{(p, q) \in \Delta : p - q = m\}$ is finite.*
- (ii) *The set $\{p - q : (p, q) \in \Delta\}$ is bounded from below (or above).*

Let μ be a finite Borel measure on S such that $\text{spec } \mu \subseteq \Delta$. Then μ is absolutely continuous with respect to σ . \square

Examples of sets Δ which satisfy conditions (i) and (ii) of 1.1 are the sets $\Delta_\alpha = \{(p, q) \in \mathbf{N} \times \mathbf{N} : q \leq \alpha p\}$ for $\alpha < 1$. The singular measures τ_m defined by

$$\int_S f d\tau_m = \int_{-\pi}^{\pi} f(e^{i\theta}\zeta) e^{im\theta} d\theta, \quad f \in C(S),$$

($\zeta \in S$ fixed) show that condition 1.1 (ii) by itself is not sufficient. Similarly, the existence of a singular pluriharmonic measure μ (that is,

$$\text{spec } \mu \subseteq \mathbf{N} \times \{0\} \cup \{0\} \times \mathbf{N},$$

cf. Aleksandrov [1] or Rudin [8]) shows that some condition on the set $\{p - q : (p, q) \in \text{spec } \mu\}$ is necessary. Cf. also remark 3.4 below.

Another application of our F. and M. Riesz theorem is made to the Bergman-Shilov boundary S of a bounded symmetric domain Ω : we get another proof of the known result that an H^1 function on Ω can be written as the Poisson integral of an L^1 function on S . Finally, our F. and M. Riesz theorem contains the classical results of the Riesz brothers and of Bochner as special cases.

Kanjin [5] has proved an F. and M. Riesz theorem for zonal (i.e. $U(n-1)$ -invariant) measures on S : such a measure μ is absolutely continuous with respect to σ if $\text{spec } \mu \subseteq \{(p, q) \in \mathbf{N} \times \mathbf{N} : \min(p, q) \leq N\}$ for some $N \in \mathbf{N}$. I do not know if Kanjin's result can be proved (and extended) by the methods in this paper.

As in the classical case, if μ is a measure such that $\text{spec } \mu$ satisfies 1.1 (i) and (ii) then not only is μ absolutely continuous with respect to σ but σ is absolutely continuous with respect to μ as well. This will be shown in the final section of this paper.

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2. Absolute continuity and the existence of L^p continuous linear functionals, $p < 1$.

2.1. Notations. — If X is a compact topological space, let $C(X)$ denote the space of complex valued continuous functions on X , with the sup norm. $M(X)$ denotes the dual of $C(X)$, the space of finite Borel measures on X .

Throughout this paper, K will denote a compact group with a countable basis of neighborhoods at e . By dk we denote the Haar measure on K , normalized to total mass 1; $L^p(K, dk) = L^p(K)$ and $\|f\|_p$ ($0 < p < \infty$) have their usual meaning. If $\mu \in M(K)$ we write (as usual) $\mu \ll dk$, $\mu \perp dk$ for « μ is absolutely continuous with respect to dk », respectively, « μ is singular with respect to dk ».

2.2 Fourier transform on K . — Let \hat{K} be the unitary dual of K , i.e. \hat{K} is the set of (equivalence classes of) irreducible unitary continuous representations of K . For $\tau \in \hat{K}$, let $H(\tau)$ denote the representation space of τ , and d_τ the complex dimension of $H(\tau)$, the degree of τ . The Fourier transform $\hat{\mu}$ of $\mu \in M(K)$ is defined as the following (operator valued) function on K :

$$\hat{\mu}(\tau) = \int_K \tau(x^{-1}) d\mu(x).$$

Let $T(K)$ be the space of trigonometric polynomials on K ; i.e. $T(K)$ is the set of finite linear combinations of functions $k \rightarrow (\tau(k)v, w)$ where $v, w \in H(\tau)$, $\tau \in \hat{K}$ and $(., .)$ is the inner product of $H(\tau)$. If χ_τ denotes the character of τ then for $F \in T(K)$

$$F(k) = \sum_{\tau} d_{\tau}(\chi_{\tau} * F)(k) = \sum_{\tau} d_{\tau} \text{Tr} \{ \hat{F}(\tau) \tau(k) \}$$

where $*$ denotes convolution on K and Tr means trace.

For $\tau \in \hat{K}$ let $T_{\tau}(K)$ denote the linear span of all functions $k \rightarrow (\tau(k)v, w)$, where $v, w \in H(\tau)$. The map $f \rightarrow d_{\tau} \chi_{\tau} * f$ is the L^2 -orthogonal projection of $L^2(K)$ onto $T_{\tau}(K)$. For $\mu \in M(K)$, the Fourier-Stieltjes series of μ is defined as the formal series

$$\sum_{\tau \in \hat{K}} d_{\tau}(\chi_{\tau} * \mu)(k) = \sum d_{\tau} \text{Tr} \{ \hat{\mu}(\tau) \tau(k) \}.$$

2.3. The spectrum of a measure. — For $\mu \in M(K)$, let $\text{spec } \mu$ be the support of $\hat{\mu} : \text{spec } \mu = \{\tau \in \hat{K} : \hat{\mu}(\tau) \neq 0\}$.

Clearly, $\text{spec } \mu = \{\tau \in \hat{K} : \chi_\tau * \mu \neq 0\}$.

Let X_μ be the subspace of $T(K)$ defined by

$$X_\mu = \{F * \mu : F \in T(K)\}.$$

X_μ determines $\text{spec } \mu$ completely, since $\text{spec } \mu = \{\tau \in \hat{K} : X_\mu \cap T_\tau(K) \neq \emptyset\}$. Conversely, $\text{spec } \mu$ does not determine X_μ in general: note that X_μ is spanned by the functions

$$k \rightarrow (\tau(k)v, w), \quad v \in \text{Range } \hat{\mu}(\tau), \quad w \in H(\tau):$$

a short computation shows that for $w_1, w_2 \in H(\tau)$ one has

$$((\tau(\cdot)w_1, w_2) * \mu)(k) = (\tau(k)\hat{\mu}(\tau)w_1, w_2).$$

If K is abelian, \hat{K} can be identified with the character group of K and then X_μ is the linear span of $\text{spec } \mu$.

It is expedient to use X_μ instead of $\text{spec } \mu$ when generalizing Shapiro's results to non-abelian K .

Recall that a space of functions Y on K is called invariant under left translation if $f \in Y$ implies ${}^k f \in Y$, where ${}^k f(x) := f(kx)$. Note that X_μ is invariant under left translation.

If Y is a subspace of $T(K)$, let Y^p denote the closure of Y in $L^p(K)$.

2.4. THEOREM. — *Let μ in $M(K)$ be singular with respect to dk . Then X_μ^p has no nonzero continuous linear functionals if $0 < p < 1$.*

Compare [11], theorem 2.1. For the proof we need some lemmas.

2.5. LEMMA. — *There exists a sequence $\{F_n\}$ of trigonometric polynomials, with $\{\|F_n\|_1\}$ bounded such that*

- (i) *if $\mu \in M(K)$, $\mu \perp dk$, then $F_n * \mu \rightarrow 0$ in Haar measure as $n \rightarrow \infty$;*
- (ii) *if $f \in L^1(K)$, then $F_n * f \rightarrow f$ in $L^1(K)$ as $n \rightarrow \infty$.*

Proof. — Let $\{V_n : n \in \mathbf{N}\}$, $V_{n+1} \subseteq V_n$, be a countable basis of neighborhoods of e (the identity element) in K . Let h_n be the characteristic function of V_n , divided by the Haar measure of V_n . A rather straightforward argument shows that (i) and (ii) hold with F_n replaced by h_n , cf. [11], proof of lemma 1.1.

Since each h_n is in $L^2(K)$, there exists $F_n \in T(K)$ such that $\|F_n - h_n\|_1 < 2^{-n}$. Hence (ii) holds. Furthermore, $(F_n - h_n) * \mu \rightarrow 0$ in Haar measure as $n \rightarrow \infty$ for every $\mu \in M(K)$ since $\|(F_n - h_n) * \mu\|_1 \rightarrow 0$. Hence (i) follows. □

2.6. LEMMA. — *Let $\{f_n\}$ be a sequence of functions in $L^1(K)$ which converges to 0 in Haar measure. Suppose there exists a $C > 0$ such that $\|f_n\|_1 \leq C$ for all n . Then $\|f_n\|_p \rightarrow 0$ as $n \rightarrow \infty$ for all $p \in (0,1)$.*

Proof. — It is enough to observe the following: if $E \subseteq K$ is Borel measurable then if $|E|$ denotes the Haar measure of E ,

$$\begin{aligned} \int_E |f_n|^p dk &= \int |f_n|^p \chi_E dk \leq \|f_n\|_1^p |E|^{1-p} \\ &\leq C^p \cdot |E|^{1-p} \end{aligned}$$

by Hölder's inequality with exponent $1/p$. Now take for $E = E_n$ the set where $|f_n| > \varepsilon$; for large n , it will have small measure. □

Proof of 2.4. — Fix $p, p \in (0,1)$ and write X for X_μ^p . Let $\{F_n\}$ be as in lemma 2.5.

Then $f_n := F_n * \mu \rightarrow 0$ in $L^p(K)$ by 2.5 (i), 2.6 and the fact that $\|F_n * \mu\|_1 \leq \|F_n\|_1 \|\mu\| \leq C \|\mu\|$ for all n .

Suppose Φ is an L^p continuous linear functional on X . Then Φ is L^1 continuous on $L^1(K) \cap X$, since $\|\cdot\|_p \leq \|\cdot\|_1$. By Hahn-Banach and the fact that the dual of $L^1(K)$ is $L^\infty(K)$, there exists a φ in $L^\infty(K)$ such that

$$\Phi(f) = \int_K f(x)\varphi(x^{-1}) dx, \quad f \in X \cap L^1(K).$$

By the left translation invariance of X_μ , ${}^k f_n \in X_\mu$ for all $k \in K$ and ${}^k f_n \rightarrow 0$ in $L^p(K)$ as $n \rightarrow \infty$ since dk is left invariant. Therefore

$$\Phi({}^k f_n) = (f_n * \varphi)(k) = ((F_n * \mu) * \varphi)(k) \rightarrow 0, \quad n \rightarrow \infty.$$

Since $\mu * \varphi \in L^1(K)$, $F_n * (\mu * \varphi) \rightarrow (\mu * \varphi)$ in $L^1(K)$ (2.5(ii)). Hence $\mu * \varphi = 0$ a.e. . Since

$$\Phi(F * \mu) = ((F * \mu) * \varphi)(e), \quad F \in T(K),$$

$\Phi = 0$ on X . □

2.7. THEOREM. — *Let $0 < p < 1$ and let $Y \subseteq L^p(K)$ be a closed subspace, invariant under left translation. Suppose that $Y \cap T(K)$ has sufficiently many L^p continuous linear functionals to separate points. Let μ in $M(K)$ be such that $X_\mu \subseteq Y$. Then μ is absolutely continuous with respect to dk .*

Compare [11], corollary 5.2.

Proof. — Let $\mu = fdk + \nu$ be the Lebesgue decomposition of μ , $f \in L^1(K)$, $\nu \perp dk$.

Choose $\{F_n\}$ as in lemma 2.5. Then by 2.5 (i) and (ii) and 2.6,

$$F_n * \mu \rightarrow f \quad \text{in } L^p \quad \text{as } n \rightarrow \infty \quad (0 < p < 1).$$

This implies that $f \in Y$, since $X_\mu \subseteq Y$.

Let V be the closed subspace of $L^1(K)$ spanned by the left translates of f . Then $V \subseteq Y$ since Y is closed under left translation. Hence $F * f \in V \subseteq Y$ for all $F \in T(K)$. Also $F * \mu \in Y$ for all $F \in T(K)$. Hence $X_\nu \subseteq Y \cap T(K)$. But this implies $\nu = 0$, by theorem 2.4. \square

3. A general F. and M. Riesz theorem.

3.1. Our main theorem concerns compact groups K (with a countable neighborhood basis at e) whose center $Z(K)$ contains a circle group T . Throughout this section, let K be such a group, and fix an identification $T \rightarrow Z(K)$, so that $e^{i\theta}$ denotes an element of K as well as of T . By Schur's lemma, there exists for each $\tau \in \hat{K}$ an $n(\tau) \in \mathbf{Z}$ such that

$$\tau(e^{i\theta}) = e^{in(\tau)\theta} \cdot \text{Id}, \quad \theta \in \mathbf{R}.$$

We can now formulate our main result.

3.2. THEOREM. — *Let $\Delta \subseteq \hat{K}$ satisfy the following two conditions :*

- (i) *For each $m \in \mathbf{Z}$ the set $\{\tau \in \Delta : n(\tau) = m\}$ is finite.*
- (ii) *The set $\{n(\tau) : \tau \in \Delta\}$ is bounded from below.*

Let $\mu \in M(K)$ be such that $\text{spec } \mu \subseteq \Delta$. Then μ is absolutely continuous with respect to dk .

In condition (ii) of 3.2 «from below» may be replaced by «from above»: just replace μ by $\bar{\mu}$.

Proof. — Let Y be the linear span of the $T_\tau(K)$'s with $\tau \in \Delta$. By theorem 2.7 it is sufficient to show that for $p < 1$

$$(3.1a) \quad Y^p \cap T(K) = Y,$$

(3.1b) Y has sufficiently many L^p continuous linear functionals to separate points.

In the proof of (3.1a) and (3.1b) we will use the following lemma :

3.3. LEMMA. — For $m \in \mathbf{Z}$ define the projection $\Pi_m : T(K) \rightarrow T(K)$ by

$$\begin{aligned} \Pi_m f(k) &= \int_{-\pi}^{\pi} f(e^{i\theta}k) e^{-im\theta} d\theta/2\pi \\ &= \sum_{n(\tau)=m} d_\tau(\chi_\tau * f)(k). \end{aligned}$$

If Y is a subspace of $T(K)$ such that the set $\{n(\tau) : \exists f \in Y : \chi_\tau * f \neq 0\}$ is bounded from below, then Π_m is L^p continuous on Y for all $p > 0$. (The interesting case is of course $0 < p < 1$.)

Proof. — For $k \in K$, $f \in T(K)$ define the « slice function » f_k on T by $f_k(e^{i\theta}) := f(e^{i\theta}k)$. Obviously

$$\begin{aligned} f_k(e^{i\theta}) &= \sum_{m \in \mathbf{Z}} \left(\sum_{n(\tau)=m} d_\tau(\chi_\tau * f)(k) \right) e^{im\theta} \\ &= \sum_{m \in \mathbf{Z}} \Pi_m f(k) e^{im\theta}. \end{aligned}$$

Let $N \in \mathbf{Z}$ be such that $n(\tau) \geq N$ for all τ for which $d_\tau \chi_\tau * f \neq 0$ for some $f \in Y$. Suppose first that $N \geq 0$. Then $f_k(e^{i\theta})$ is an analytic trigonometric polynomial for each f in Y . Hence, by a result from one variable H^p theory due to Hardy and Littlewood (cf. [3], theorem 6.4; cf. also [2], p. 68, for a short proof) for each $p > 0$ and each $m \in \mathbf{Z}$ there exists a constant $C = C(p, m)$ such that

$$|\Pi_m f(k)|^p \leq C \int_{-\pi}^{\pi} |f(e^{i\theta}k)|^p d\theta/2\pi.$$

Integration over K yields the lemma when $N \geq 0$.

If $N < 0$ then for each $f \in Y$, $f_k(e^{i\theta}) = e^{iN\theta} \cdot F(e^{i\theta})$ where F is again an analytic trigonometric polynomial on T . Apply the one variable result mentioned above to F and note that $|F| = |f_k|$ on T . □

We now return to the proof of theorem 3.2. To prove (3.1a) suppose that $f_n \in Y$, $f \in T(\mathbf{K})$ such that $f_n \rightarrow f$ in $L^p(\mathbf{K})$ as $n \rightarrow \infty$. By lemma 3.3 applied to $Y' = \text{span}\{Y, f\}$, $\Pi_m(f_n) \rightarrow \Pi_m(f)$ in $L^p(\mathbf{K})$ for all $m \in \mathbf{Z}$. Since Y is invariant under left translation, $\Pi_m(f_n)$ belongs to $Y \cap \bigoplus \{T_\tau(\mathbf{K}) : n(\tau) = m\}$. The latter is a finite dimensional subspace of $T(\mathbf{K})$ by condition 3.2(i). Since all vector space topologies on a finite dimensional vector space are complete, $\Pi_m(f) \in Y$ for all $m \in \mathbf{Z}$, which implies that $f \in Y$.

For (3.1b) it is sufficient to show that for each $\sigma \in \Delta$ and each $k \in \mathbf{K}$ the linear functional

$$(3.2) \quad f \rightarrow d_\sigma(\chi_\sigma * f)(k)$$

is L^p continuous on Y . Take a σ in Δ . Clearly, the linear functional (3.2) is equal to the composition of the projection $\Pi_{n(\sigma)}$ with the restriction of (3.2) to $\bigoplus \{T_\tau(\mathbf{K}) : \tau \in \Delta, n(\tau) = n(\sigma)\}$. Since this subspace is finite dimensional, the L^p continuity of (3.2) follows from the L^p continuity of $\Pi_{n(\sigma)}$. This proves the theorem. \square

3.4. Remark. — Recall that a subset Σ of $\hat{\mathbf{K}}$ is called a $\Lambda(1)$ set (Rudin [10]) if there exists a $p < 1$ and a constant C such that for all f in $\bigoplus \{T_\tau(\mathbf{K}) : \tau \in \Sigma\}$,

$$\|f\|_1 \leq C \|f\|_p,$$

i.e. if the L^1 and L^p topologies coincide on $\text{span}\{T_\tau(\mathbf{K}) : \tau \in \Sigma\}$.

We can replace condition (i) of 3.2 by the following weaker condition :

(i)' For each $m \in \mathbf{Z}$ the set $\{\tau \in \Delta : n(\tau) = m\}$ is a $\Lambda(1)$ subset of $\hat{\mathbf{K}}$.

The proof remains essentially the same: instead of the finite dimensionality of the subspaces $\bigoplus \{T_\tau(\mathbf{K}) : \tau \in \Delta, n(\tau) = m\}$ we now use the equivalence, for some $p < 1$, of the L^1 and L^p topologies on these subspaces, and the L^1 continuity of the linear functionals (3.2) (for all $\sigma \in \mathbf{K}$).

Similarly, we may also replace condition (ii) by

(ii)' The set $\{n(\tau) : \tau \in \Delta\}$ is a $\Lambda(1)$ subset of \mathbf{Z} (considered as the dual of \mathbf{T}).

In this case the analogue of lemma 3.3 becomes trivial.

Note, by the way, that for arbitrary compact K the conclusion of the F. and M. Riesz theorem holds for all $\Lambda(1)$ subsets of \hat{K} : this follows immediately from theorem 2.7 and the definition of a $\Lambda(1)$ set.

3.5. Example. — Let $K = \mathbf{T} \times \mathbf{T}$ and identify \mathbf{T} with a subgroup of K via the map $e^{i\theta} \rightarrow (e^{i\theta}, 1)$, $\theta \in (-\pi, \pi]$. The irreducible unitary representations of K are the characters $\chi_{p,q}: (e^{i\theta}, e^{i\psi}) \rightarrow e^{i(p\theta + q\psi)}$ and $n(\chi_{p,q}) = p$. In this case theorem 3.2 contains Bochner's theorem where the spectrum is required to lie in an angle to the right of opening less than π (cf. [9], theorem 8.2.5 for the precise formulation).

According to the remarks made in 3.4 it suffices to require in condition (i) that for each p the set $\{q \in \mathbf{Z} : (p, q) \in \text{spec } \mu\}$ is a $\Lambda(1)$ set. We refer to the appendix of [2] for another strengthening of Bochner's theorem which only requires that for each p these sets are either bounded from above or from below. This can easily be proved by the method of proof of theorem 3.2 if we note that $H^p(\mathbf{T}) \cap \mathbf{T}(\mathbf{T})$ consists of analytic polynomials.

3.6. F. and M. Riesz for homogeneous spaces. — Let H be a closed subgroup of K . Functions and measures on K/H can be identified with functions and measures on K which are right H -invariant. If $\mu \in M(K/H)$ is a right H -invariant measure on K , then $\pi_\tau \mu := d_\tau \chi_\tau * \mu$ is again right H -invariant. Let σ be the K -invariant measure on K/H , normalized to 1. The map $\pi_\tau: f \rightarrow d_\tau \chi_\tau * f$ ($\tau \in \hat{K}$) is an L^2 orthogonal projection of $L^2(K/H, \sigma) = L^2(\sigma)$ which is different from zero iff τ occurs in the left regular representation of K on $L^2(\sigma)$. As in the case of the unit sphere, π_τ can be represented by an integral operator with continuous kernel.

Theorem 3.2 can now be formulated for measures on K/H entirely in terms of π_τ and σ :

3.7. THEOREM. — Let $\Delta \subseteq \hat{K}$ be such that all $\tau \in \Delta$ occur in the left regular representation of K on $L^2(\sigma)$ and suppose Δ satisfies conditions (i) and (ii) of 3.2. Let $\mu \in M(K/H)$ be such that $\pi_\tau \mu = 0$ if $\tau \notin \Delta$. Then μ is absolutely continuous with respect to σ .

If we take $K = U(n)$, $H = U(n-1)$ then $K/H = S$ and we get theorem 1.1: $Z(K)$ contains the multiplications by $e^{i\theta}$. If τ_{pq} denotes the restriction of the left regular representation of $U(n)$ on $L^2(\sigma)$ to $H(p, q)$, i.e. $\tau_{pq}(U)f(\zeta) = f(U^{-1}\zeta)$, $f \in H(p, q)$, $U \in U(n)$, $\zeta \in S$, then τ_{pq} is irreducible, $n(\tau_{pq}) = q - p$, the τ_{pq} are pairwise inequivalent and they represent all irreducible representations of $U(n)$ which occur in $L^2(\sigma)$ (cf. for example [7], chapter 12).

In case $H = T$ theorem 3.7 becomes trivial: $n(\tau) = 0$ for all τ which occur in K/T and (i) then implies that Δ is finite.

3.8. Application to bounded symmetric domains. — Let $\Omega \subseteq \mathbb{C}^n$ be a bounded symmetric domain. (Cf. [6], [4] for the relevant facts.) We may assume that Ω is convex and circular (i.e. $z \in \Omega$ implies $e^{i\theta} \cdot z \in \Omega$ for all $\theta \in \mathbb{R}$). Let K be the stabilizer of 0 in the component of the identity of the group of holomorphic automorphisms of Ω . The action of K on Ω incorporates multiplication by $e^{i\theta}$; in particular, $T \subseteq Z(K)$. Let S denote the Bergman-Shilov boundary of Ω . Then K acts transitively on S and we can apply the principal theorems 3.2, 3.7 to S . As above, let σ be the normalized K -invariant measure on S .

Let $H^2(S)$ be the closure in $L^2(S, \sigma) = L^2(\sigma)$ of the holomorphic polynomials, restricted to S . Obviously $H^2(S)$ is K -invariant under the left regular representation of K on $L^2(\sigma)$. Let \hat{K}_{Hol} be the set of irreducible representations of K which occur in $H^2(S)$; for a description of \hat{K}_{Hol} , cf. [12].

We claim that \hat{K}_{Hol} satisfies conditions (i) and (ii) of 3.2. For let $H(p)$ be the space of holomorphic polynomials which are homogeneous of degree p , restricted to S . By a well known theorem of H. Cartan (cf. [7], theorem 2.1.3) K acts on Ω by complex linear transformations. Hence each $H(p)$ is K -invariant and decomposes as a finite sum of representations in \hat{K}_{Hol} . Obviously, $n(\tau) \leq 0$ if τ is in \hat{K}_{Hol} and $n(\tau) = -p$ if τ occurs in $H(p)$. This proves the claim.

By theorem 3.7 a measure μ in $M(K/H)$ for which $\text{spec } \mu \subseteq \hat{K}_{\text{Hol}}$ is absolutely continuous with respect to σ . By a familiar weak- $*$ compactness argument this implies the following result:

3.9. COROLLARY. — *If f is in the Hardy space $H^1(\Omega)$ of Ω then f can be written as the Poisson integral of a function in $L^1(\sigma)$. (Cf. [6], [14] for the definitions of Hardy space and Poisson kernel.)* \square

The analogue of 3.9 for generalized Siegel half-planes is due to E. M. Stein [13]; 3.9 can be deduced from his result by using the generalized Cayley transform, cf. [6], p. 189. Corollary 3.9 can also be proved more directly, by using Bochner's method of slicing and the Hardy-Littlewood inequality for the radial maximal function from one dimensional H^p theory.

4. A supplement to theorem 3.2.

4.1. THEOREM. — Let K be a compact Lie group whose center contains the circle group T and let H be a closed subgroup of K such that K/H is connected. Suppose f in $L^1(K/H, \sigma)$ is such that $\text{spec } f$ satisfies conditions (i) and (ii) of theorem 3.7. Then either $f = 0$ or $f(\xi) \neq 0$ a.e. $[\sigma]$.

In particular, if $\mu \in M(K/H)$ is as in theorem 3.7 and $\mu \neq 0$ then $\sigma \ll \mu$ as well as $\mu \ll \sigma$. The case $K = T, H = \{1\}$ of this theorem is classical (cf. for example [3], theorem 2.2) and will be used in the proof of 4.1.

Proof. — Let f in $L^1(K/H, \sigma)$ satisfy the conditions of the theorem and suppose that $f = 0$ on a (Borel) set of nonzero measure. Identify f with a right H -invariant L^1 function on K , also denoted by f .

For almost all $k \in K$ the slice function $f_k(e^{i\theta}) = f(e^{i\theta}k)$ is in $L^1(T)$. Set $c_m(k) := \hat{f}_k(m)$, the m -th Fourier coefficient of f_k . Then $c_m \in L^1(K)$ and a calculation of $d_\tau \chi_\tau * c_m$ shows that

$$(4.1) \quad c_m(k) = \sum_{n(\tau)=m} d_\tau(\chi_\tau * f)(k)$$

(note that the sum is finite). The case $K = T$ of 4.1 now shows that for almost all k in K

$$(4.2) \quad f_k = 0 \quad \text{or} \quad f_k(e^{i\theta}) \neq 0 \quad \text{a.e.} [d\theta].$$

Since $f = 0$ on a set of nonzero Haar measure there exists an $F \subseteq K$ of strictly positive Haar measure such that for all k in F , $f_k = 0$ on a subset of T of nonzero (one-dimensional) Lebesgue measure. By (4.2) $f_k = 0$ for almost all k in F and hence, by (4.1)

$$\sum_{n(\tau)=m} d_\tau(\chi_\tau * f)(k) = 0 \quad \text{for all } m \in \mathbf{Z}, \quad \text{a.a. } k \in F.$$

Each $\tau \in \hat{K}$ is an analytic function on K and therefore $d_\tau \chi_\tau * f$ is an analytic function on the analytic manifold K/H . It is not difficult to prove that the zero set of a nonzero analytic function on a connected analytic manifold has Lebesgue measure zero. Hence

$$\sum_{n(\tau)=m} d_\tau \chi_\tau * f = 0$$

for all m and therefore $d_\tau \chi_\tau * f = 0$ for all $\tau \in \hat{K}$, i.e. $f = 0$. □

It is easy to find a counterexample to 4.1 if K/H is not connected. Take $K = T \times F$ and $H = \{e\}$, with F a finite non-commutative group. Let τ be an irreducible representation of F and choose a matrix coefficient τ_{mn} of τ such that $\tau_{mn}(e) = 0$, $\tau_{mn}(x) \neq 0$ for some $x \in F$. Now take $f = (1 \otimes \tau)_{mn}$: $\text{spec } f$ consists of one point but $f = 0$ on the component of the unit element of K .

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