EDWARD BIERSTONE
P. D. MILMAN

Relations among analytic functions. II


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RELATIONS AMONG ANALYTIC FUNCTIONS II

by E. BIERSTONE (¹) and P. D. MILMAN (²)

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CHAPTER II
DIFFERENTIABLE FUNCTIONS

10. Ideals generated by analytic functions.

We give an elementary proof of the theorem of Malgrange [27, Ch. VI]. Let $N$ be a real analytic manifold. Put $\mathcal{O} = \mathcal{O}_N$. Let $A$ be a $p \times q$ matrix of real analytic functions on $N$, and let $A^*: \mathcal{C}^\infty(N)^q \to \mathcal{C}^\infty(N)^p$ denote the $\mathcal{C}^\infty(N)$-homomorphism defined by multiplication by $A$.

**Theorem 10.1.** $A^* \mathcal{C}^\infty(N)^q = (A^* \mathcal{C}^\infty(N)^q)^{\simeq}$.

**Remark 10.2.** Let $Z \subset Y$ be closed subanalytic subsets of $N$. Suppose that $f \in \mathcal{C}^\infty(N;Z)^p$ and, for all $a \in Y$, there exists $G_a \in \hat{\mathcal{O}}_a$ such that $f_a = A_a \cdot G_a$. The following proof shows, moreover, that there exists $g \in \mathcal{C}^\infty(N;Y)^q$ such that $f - A \cdot g \in \mathcal{C}^\infty(N;Y)^q$ (cf. [7, Thm. 0.1.1]).

**Proof of Theorem 10.1.** Let $\mathcal{A}$ denote the sheaf of submodules of $\mathcal{C}^p$ generated by the columns $\varphi^1, \ldots, \varphi^q$ of $A$. Let $\mathcal{B}$ be the subsheaf of $\mathcal{C}^q$ of (germs of) relations among the columns of $A$. Then $\mathcal{B}$ is coherent.

We can assume that $N$ is an open subset of $\mathbb{R}^n$. If $a \in N$, we identify $\hat{\mathcal{O}}_a$ with $\mathbb{R}[\![y]\!]$, $y = (y_1, \ldots, y_n)$. By Lemma 7.2 and Remark 7.3, we can suppose there is a filtration of $N$ by closed analytic subsets,

$$N = X_0 \supset X_1 \supset \cdots \supset X_{r+1} = \emptyset,$$

such that, for each $k = 0, \ldots, r$:

1. $X_k - X_{k+1}$ is smooth.
2. $\mathfrak{M}(\mathcal{A}_a)$ and $\mathfrak{M}(\mathcal{B}_a)$ are constant on $X_k - X_{k+1}$. We write $\mathfrak{M}_k(\mathcal{A}) = \mathfrak{M}(\mathcal{A}_a)$ and $\mathfrak{M}_k(\mathcal{B}) = \mathfrak{M}(\mathcal{B}_a)$, $a \in X_k - X_{k+1}$.
3. Let $(\beta_i, j_i)$, $i = 1, \ldots, t$, denote the vertices of $\mathfrak{M}_k(\mathcal{A})$. Then, for each $i$, there exists $\psi^i$ in the submodule of $\mathcal{C}^\infty(X_a)[[y]]^p$ generated by
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(the elements induced by) the $\varphi'$ (cf. Remark 7.3), such that, for all $a \in X_k - X_{k+1}$, $v(\varphi'(a; \cdot)) = (\beta_i, j_i)$ and $\psi'_a \in \mathcal{A}_a$, where $\psi'_a(y) = \psi'(a; y)$.

(4) There exist $\sigma'$ in the submodule of $\mathcal{O}(X_k) [[y]]^t$ induced by $\mathcal{B}(N)$ such that the $v(\sigma'(a; \cdot))$ are the vertices of $\mathcal{R}_k(\mathcal{A})$, for all $a \in X_k - X_{k+1}$.

Fix $k$. Let $\{\Delta_i, \Delta\}$ denote the decomposition of $N^n \times \{1, \ldots, p\}$ determined by the vertices $(\beta_i, j_i)$ of $\mathcal{R}_k(\mathcal{A})$, as in § 6. Let $a \in X_k - X_{k+1}$. By the formal division algorithm (Theorem 6.2) and Remark 6.7, there exist unique $r_i^a \in \mathcal{O}_a^{\sigma}$ and $q_{i, \alpha}^a \in \mathcal{O}_a$, $\ell = 1, \ldots, t$, such that $\text{supp } r_i^a \subset \Delta$, $(\beta_i, j_i) + \text{supp } q_{i, \alpha}^a \subset \Delta_i$, and

\begin{equation}
(10.3) \quad y^{\beta_i, j_i} = \sum_{\ell=1}^{t} q_{i, \alpha}^a(y) \psi'_a(y) + r_i^a(y).
\end{equation}

Put $\theta_i^a(y) = y^{\beta_i, j_i} - r_i^a(y)$, $i = 1, \ldots, t$; then the $\theta_i^a \in \mathcal{A}_a$ (cf. Corollary 7.7). The coefficients $\theta_i^a(a)$ of $\theta_i^a(y) = \sum_{\beta} \theta_{\beta, j_i}^a(a) y^{\beta, j_i}$, as well as the coefficients of the $q_{i, \alpha}^a$, are analytic on $X_k - X_{k+1}$, and extend to $X_k$ as quotients of analytic functions by products of powers of the $\psi_{\beta, j_i}^a(a)$, where $\psi'_a(y) = \sum_{\beta} \psi_{\beta, j_i}^a(a) y^{\beta, j_i}$. There exist analytic functions $\theta^i$ defined in a neighborhood of $X_k - X_{k+1}$, whose power series expansions at each $a \in X_k - X_{k+1}$ are the $\theta_i^a$ (cf. Corollary 7.7(3)).

Suppose that $f \in (A \cdot \mathcal{C}^\infty(N)^q)^-$ and that $f$ is flat on $X_{k+1}$. It suffices to find $h \in \mathcal{C}^1(N ; X_{k+1})^q$ such that $f - A \cdot h \in \mathcal{C}^1(N ; X_k)^q$.

Let $a \in X_k - X_{k+1}$. Then $\hat{f}_a \in \mathcal{A}_a$. By the formal division algorithm, there are unique $G_{i, \alpha} \in \mathcal{O}_a$, $i = 1, \ldots, t$, such that $(\beta_i, j_i) + \text{supp } G_{i, \alpha} \subset \Delta_i$ and

\begin{equation}
(10.4) \quad \hat{f}_a = \sum_{i=1}^{t} G_{i, \alpha} \theta_i^a.
\end{equation}

Put $G_{i, \alpha} = 0$ if $a \in X_{k+1}$.

We claim there exist $g_i \in \mathcal{C}^1(N ; X_{k+1})$ such that $G_{i, \alpha} = g_i a$ for all $a \in X_k$. Write $G_{i, \alpha} = \sum_{\beta} G_{i, \beta} a y^{\beta}$. By the formal division algorithm and Łojasiewicz's inequality [27, IV.4.1], each $G_{i, \beta}$ is the restriction to $X_k$ of a $\mathcal{C}^\infty$ function which is flat on $X_{k+1}$. Let $a \in X_k - X_{k+1}$. Since $f$ is $\mathcal{C}^\infty$ and the $\theta^i$ are analytic, then, regarding both $a$ and $y$ as variables
in \( N \), we have

\[
\frac{\partial f_\lambda(y)}{\partial a_j} = \frac{\partial f_\lambda(y)}{\partial y_j},
\]

(10.5)

\[
\frac{\partial \theta_\lambda^i(y)}{\partial a_j} = \frac{\partial \theta_\lambda^i(y)}{\partial y_j},
\]

\( j = 1, \ldots, n \) ("Taylor expansion commutes with differentiation"). If \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \), write \( D_{\lambda,a} = \sum \lambda_i \partial \partial a_j \); \( D_{\lambda,a} \) is the directional derivative with respect to the \( a \) variables in the direction \( \lambda \). If \( D_{\lambda,a} \) is tangent to \( X_k - X_{k+1} \) at \( a \), then \( D_{\lambda,a} G_{i,a}(y) \) is well-defined, and, by (10.4) and (10.5), \( \sum_{i=1}^{r} (D_{\lambda,a} G_{i,a} - D_{\lambda,y} G_{i,a}) \cdot \theta_\lambda^i = 0 \). For each \( i \), \( (\beta_0, j) + \text{supp} (D_{\lambda,a} G_{i,a} - D_{\lambda,y} G_{i,a}) \subseteq \Delta_i \) (where \( \text{supp} \) is with respect to \( y \)). Therefore, by the uniqueness of formal division, for each \( i = 1, \ldots, t \),

\[
D_{\lambda,a} G_{i,a} = D_{\lambda,y} G_{i,a}.
\]

Choose local coordinates \((u, v) = (u_1, \ldots, u_m, v_1, \ldots, v_{n-m})\) near \( a \in X_k - X_{k+1} \) such that \( X_k - X_{k+1} \) is given by \( v = 0 \). Write \( G_{i,a} \) as

\[
G_{i,a}(u, v) = \sum_{\beta \in \mathbb{N}^{n-m}} \left( \sum_{\alpha \in \mathbb{N}^m} G_{i,a}^{\alpha, \beta}(u) \frac{u^\alpha}{\alpha!} \right) \frac{v^\beta}{\beta!}.
\]

Then (10.6) implies that \( \sum_{\alpha} G_{i,a}^{\alpha, \beta}(a) u^\alpha / \alpha! \) is the formal Taylor series of \( G^0, \beta \) at \( a \). By Whitney's extension theorem [27, I.4.1] and Hestenes's lemma [37, IV.4.3], there exists \( g_i \in \mathscr{R}(N; X_{k+1}) \) such that \( G_{i,a} = \hat{g}_i(a) \), for all \( a \in X_k \), as claimed.

To finish the proof, we must express \( f \) in terms of the columns \( \varphi^j \) of \( A \). By (3) and (10.3), \( \theta^{i}(\lambda) = \sum y_{j,i,a}(y) \varphi^{i}(y), \ i = 1, \ldots, t \), where \( \psi^i(y) = \varphi^i(a + y), \ \xi_{j,i,a} \in \mathcal{O}_a \), and the coefficients \( \xi_{j,i,a}(y) = \sum_{\beta} \xi_{j,i,a}^{\beta}(y) y^{\beta} \) are quotients of analytic functions by products of powers of the \( \psi_{j,i,a}(a) \). Put \( \xi_{i,a} = (\xi_{i1,a}, \ldots, \xi_{iq,a}) \). By the formal division algorithm and Remark 6.7, there exist unique \( \eta_{i,a}(y) \in \mathcal{O}_a^q \) such that \( \xi_{i,a} - \eta_{i,a} \in \mathcal{B}_a \) and \( \text{supp} \eta_{i,a} \cap \mathcal{B}_k(\mathcal{A}) = \emptyset \). Write \( \eta_{i,a} = (\eta_{i1,a}, \ldots, \eta_{iq,a}) \) and \( \eta_{j,i,a}(y) = \sum_{\beta} \eta_{j,i,a}^{\beta}(a) y^{\beta}, \ j = 1, \ldots, q \). By (4), the \( \eta_{j,i,a}(a) \) extend to \( X_k \) as
quotients of analytic functions. By the uniqueness of formal division, 
\[ \eta_{\gamma, a}(b - a + \gamma) = \eta_{\gamma, a}(\gamma), \]
for \( b \) in some neighborhood of \( a \) in \( X_k - X_{k+1} \)
(cf. the proof of Corollary 7.7 (3)). Thus the \( \eta_{\gamma, a} \) are the formal power
series expansions at \( a \) of analytic functions \( \eta_{\gamma} \) defined in a neighborhood
of \( X_k - X_{k+1} \).

If \( a \in X_k - X_{k+1} \), then 
\[ \hat{f}_a = \sum_i G_{i,a} \theta_a^i = \sum_i \eta_{\gamma, a} G_{i,a} \varphi_a. \]
Put \( H_{j,a} = \sum_i \eta_{\gamma, a} G_{i,a} \) if \( a \in X_k - X_{k+1} \), and \( H_{j,a} = 0 \) if \( a \in X_{k+1} \), \( j = 1, \ldots, q \). Then
there exist \( h_j \in \mathcal{O}(N ; X_{k+1}) \) such that \( H_{j,a} = h_j \) for all \( a \in X_k \), \( j = 1, \ldots, q \). Thus, 
\[ f - A \cdot h \in \mathcal{O}(N ; X_k)^p, \]
where \( h = (h_1, \ldots, h_q) \).

11. Modules over a ring
of composite differentiable functions.

Let \( K = \mathbb{R} \) or \( \mathbb{C} \). Let \( M \) and \( N \) denote analytic manifolds (over \( K \)),
and let \( \varphi : M \to N \) be an analytic mapping. Let \( A \) and \( B \) be \( p \times q \)
and \( p \times r \) matrices of analytic functions on \( M \), respectively. We use
the notation of 8.2. If \( a \in M \), let \( \mathcal{R}_a = \{ G \in \mathcal{O}_{\varphi(a)} : \varphi_a(G) \in \text{Im } \hat{B}_a \} \).

Let \( \mathcal{B} \subset \mathcal{O}_M^q \) denote the sheaf of \( \mathcal{O}_M \)-modules generated by the
columns of \( B \). Let \( U \) be a coordinate neighborhood of some point in
\( M \), with coordinates \( x_1, \ldots, x_m \), say. By Theorem 7.4, the diagram of
initial exponents \( \mathcal{R}(\mathcal{B}_a) \in \mathbb{N}^m \times \{1, \ldots, p\} \) is Zariski semicontinuous on
\( U \). Thus, after perhaps shrinking \( U \), there is a filtration by closed
analytic subsets, \( U = X_0 \supset X_1 \supset \ldots \supset X_{r+1} = \emptyset \), such that \( \mathcal{R}(\mathcal{B}_a) \)
is constant on each \( X_\lambda - X_{\lambda+1} \). Let \( b \in N \). The following proposition
shows that \( \mathcal{R}_a \) is constant on every connected component of
\( (X_\lambda - X_{\lambda+1}) \cap \varphi^{-1}(b), \lambda = 0, \ldots, t \).

**Proposition 11.1.** — Let \( U \) be a local coordinate chart in \( M \). Let \( b \in N \) and let \( S \) be a locally closed semianalytic subset of \( U \) such that 
\( S \subset \varphi^{-1}(b) \). Suppose that \( \mathcal{R}(\mathcal{B}_a) \) is constant on \( S \). Let \( f \in \mathcal{O}(U)^p \) and let 
\( G \in \mathcal{O}_a \). Then
\[ \mathcal{H} = \{ a \in S : \hat{f}_a - \varphi_a(G) \in \text{Im } \hat{B}_a \} \]
is open and closed in \( S \).

**Proof.** — We can assume that \( U \) (respectively, \( N \)) is an open
neighborhood of the origin in \( K^m \) (respectively, \( K^n \)), and that \( \varphi(0) = 0 \)
and \( b = 0 \). We identify (the components of) \( \varphi \) and \( f \) and (the entries
of A and B with their convergent power series expansions at 0. If \( x = (x_1, \ldots, x_m) \) and \( y = (y_1, \ldots, y_n) \), then

\[
f(x + y) - A(x + y) \cdot G(\varphi(x + y) - \varphi(x)) = \sum_{\alpha \in \mathbb{N}^m} \frac{D^\alpha f(x)}{\alpha!} y^\alpha - A(x + y) \cdot \sum_{\beta \in \mathbb{N}^n} \frac{D^\beta G(0)}{\beta!} \left( \sum_{\alpha > 0} \frac{D^\alpha \varphi(x)}{\alpha!} y^\alpha \right)^\beta,
\]

where \( \alpha \) (respectively, \( \beta \)) denotes a multiindex in \( \mathbb{N}^m \) (respectively, \( \mathbb{N}^n \)).

Thus

\[
f(x + y) - A(x + y) \cdot G(\varphi(x + y) - \varphi(x)) = \sum_{\alpha \in \mathbb{N}^m} \frac{H_\alpha(x)}{\alpha!} y^\alpha,
\]

where the \( H_\alpha \) converge in a common neighborhood of 0 (which we can take to be \( U \)). (For all \( \alpha \in \mathbb{N}^m \), each component of \( H_\alpha(x) - D^\alpha f(x) \) is a finite linear combination of certain products of derivatives of the components of \( \varphi \) times derivatives of the entries of \( A \).)

Let \( \mathcal{R} = \mathcal{R}(\mathcal{M}_a) \), \( a \in S \), and let \( (\alpha, j_i), i = 1, \ldots, k \), denote the vertices of \( \mathcal{R} \). For each \( a \in S \), let \( g_\alpha'(y) \in \hat{\mathcal{B}}_a = K[[y]]^p \), \( i = 1, \ldots, k \), denote the standard basis of \( \hat{\mathcal{B}}_a \), where in \( g_\alpha' = y^{j_1} \). Then each \( g_\alpha'(y) = \sum_{\alpha, j} g_{\alpha, j}(a) y^{j} \) is convergent, and each \( g_{\alpha, j}(a) \) is analytic on \( S \) (Corollary 6.8).

Let \( a \in S \) and let \( h_a(y) = \sum_a H_\alpha(a) y^\alpha/\alpha! \). By Theorem 6.2, there exist unique \( q_{i, a}(y) \in \hat{\mathcal{B}}_a \) and \( r_a(y) \in \hat{\mathcal{B}}_a^p \) such that \( (\alpha, j_i) + \text{supp } q_{i, a} \subset \Delta_i \), \( \text{supp } r_a \subset \Delta \) (where \( \Delta_i, \Delta \) are as in § 6), and

\[
(11.2) \quad h_a(y) = \sum_{i=1}^{k} q_{i, a}(y) g_{\alpha, j_i} + r_a(y).
\]

Write \( r_a(y) = \sum_{\alpha, j} r_{\alpha, j}(a) y^{j} \). Then each \( r_{\alpha, j}(a) \) is analytic on \( S \) (cf. Remark 6.5). By (11.2), \( h_a \in \text{Im } \hat{\mathcal{B}}_a \) if and only if each \( r_{\alpha, j}(a) = 0 \); i.e., \( \mathcal{H} \) is closed.

Since \( f(y) - A(y) \cdot G(\varphi(y)) \in \hat{\mathcal{B}}_0 = K[[y]]^p \), there exist unique \( q_i(y) \in \hat{\mathcal{B}}_0 \) such that \( (\alpha, j_i) + \text{supp } q_i \subset \Delta_i \) and \( f(y) - A(y) \cdot G(\varphi(y)) = \sum_{i=1}^{k} q_i(y) g_\alpha(y) \). Consider the identity

\[
(11.3) \quad f(x + y) - A(x + y) \cdot G(\varphi(x + y)) = \sum_{i=1}^{k} q_i(x + y) g_\alpha(x + y).
\]
Suppose that $0 \in S$. Let $\mathcal{I} \subseteq \mathcal{O}_0 = K(x)$ denote the ideal of germs of analytic functions at 0 which vanish on $S$. Write $\mathcal{O}_{S,0} = \mathcal{O}_0 / \mathcal{I}$ and $\hat{\mathcal{O}}_{S,0} = \hat{\mathcal{O}}_0 / \mathcal{I} \cdot \hat{\mathcal{O}}_0$. We expand each term of (11.3) as a power series in $y$ with coefficients in $\hat{\mathcal{O}}_0 = K[[x]]$, and take the induced power series in $y$ with coefficients in $\hat{\mathcal{O}}_{S,0}$. Since each component of $\phi$ vanishes on $S$, the left-hand side of (11.3) gives the same result as reducing the coefficients of $\sum H_\alpha(x)y^\alpha/\alpha!$ modulo $\mathcal{I}$; write $h_\alpha(y)$ for the resulting element of $\mathcal{O}_{S,0}[[y]]^p$. Likewise, write $q_\alpha(x,y)$ and $g_\alpha(x,y)$ for the elements of $\hat{\mathcal{O}}_{S,0}[[y]]$ and $\hat{\mathcal{O}}_{S,0}[[y]]^p$ induced by $q_\alpha(x+y)$ and $g_\alpha(x+y)$, respectively. Thus,

$$(11.4) \quad h_\alpha(y) = \sum_{i=1}^k q_{\alpha,i}(y)g_\alpha^i(y).$$

Since $(\alpha_i,j_i) + \text{supp } q_i \subseteq \Delta_i$, then $(\alpha_i,j_i) + \text{supp } q_{i,x} \subseteq \Delta_i$. Clearly, in $g_\alpha^i(y) = y^{\alpha_i/j_i}$.

On the other hand, by the formal division algorithm, there are unique $Q_{\alpha,i}(y) \in \hat{\mathcal{O}}_{S,0}[[y]]$ and $R_\alpha(y) \in \hat{\mathcal{O}}_{S,0}[[y]]^p$ such that $(\alpha_i,j_i) + \text{supp } Q_{i,x} \subseteq \Delta_i$, $\text{supp } R_x \subseteq \Delta$, and

$$(11.5) \quad h_\alpha(y) = \sum_{i=1}^k Q_{\alpha,i}(y)g_\alpha^i(y) + R_\alpha(y).$$

Since the coefficients of $h_\alpha(y)$ belong to $\mathcal{O}_{S,0}$, so do those of $Q_{i,x}(y)$ and $R_\alpha(y)$ (cf. Remark 6.5); moreover, all coefficients can be evaluated in a common neighborhood of 0 in $S$.

Comparing (11.4) and (11.5), we get $R_\alpha(y) = 0$. But from (11.2) and (11.5), $R_a(y) = r_a(y)$ for $a \in S$ sufficiently close to 0. Therefore, all $r_a(a)$ vanish on $S$ near 0; i.e., $\mathcal{H}$ is open. 

**Corollary 11.6.** — If $\phi$ is proper, then (locally in $N$), there is a bound $s$ on the number of distinct submodules $\mathcal{R}_a$ of $\hat{\mathcal{O}}_b^q$, where $a \in \phi^{-1}(b)$.

**Proof.** — Let $U, X_0, \ldots, X_{t+1}$ be as above. Suppose that $U$ is relatively compact and each $X_\lambda$ is semianalytic in $M$. Then, for each $\lambda = 0, \ldots, t$, there is a bound on the number of connected components of $(X_\lambda - X_{\lambda+1}) \cap \phi^{-1}(b)$ [11], [12], [20, Thm. 2.5]. The result follows from Proposition 11.1. 

**Remark 11.7.** — Suppose $\phi$ is proper. Then (locally in $N$), there is a bound $s'$ on the number of connected components of a fiber $\phi^{-1}(b)$. If $B = 0$, then Corollary 11.6 is satisfied with $s = s'$. 
In the remainder of this section, we assume that $K = \mathbb{R}$. Let $\varphi^* : \mathcal{C}^\infty(N) \to \mathcal{C}^\infty(M)$ denote the ring homomorphism induced by $\varphi$, and let $\Phi : \mathcal{C}^\infty(N)^a \to \mathcal{C}^\infty(M)^p$ denote the module homomorphism over $\varphi^*$ defined by $\Phi(g) = A \cdot (g \circ \varphi)$, where $g \in \mathcal{C}^\infty(N)^a$. Let $B : \mathcal{C}^\infty(M)^p \to \mathcal{C}^\infty(M)^p$ denote the $\mathcal{C}^\infty(M)$-homomorphism induced by multiplication by the matrix $B$.

Let $(\Phi \mathcal{C}^\infty(N)^a + B \cdot \mathcal{C}^\infty(M))^p = \{ f \in \mathcal{C}^\infty(M)^p : \text{for all } b \in \varphi(M), \text{there exists } G_b \in \mathcal{C}^\infty(N)^a \text{ such that } f_a - \hat{G}_b(G_a) \in \text{Im } \hat{B}_a, \text{ for all } a \in \varphi^{-1}(b) \}$.

**Theorem 11.8.** — Suppose that $\varphi$ is proper. Then each of the equivalent conditions of Theorem 8.2.5 implies that

$$\Phi \mathcal{C}^\infty(N)^a + B \cdot \mathcal{C}^\infty(M)^p = (\Phi \mathcal{C}^\infty(N)^a + B \cdot \mathcal{C}^\infty(M)^p)^p.$$

**Remark 11.9.** — Let $Z$ be a closed subanalytic subset of $N$. Our proof of Theorem 11.8 will show that each of the equivalent conditions of Theorem 8.2.5 implies the following stronger result: If $f \in (\Phi \mathcal{C}^\infty(N)^a + B \cdot \mathcal{C}^\infty(M)^p)^p$ and $f_a \in \text{Im } \hat{B}_a$ for all $a \in \varphi^{-1}(Z)$, then there exists $g \in \mathcal{I}(N;Z)^a$ and $h \in \mathcal{C}^\infty(M)^p$ such that $f = \Phi(g) + B \cdot h$.

**Remark 11.10.** — In the case that $A = I$ and $B = 0$, it is enough to assume that $\varphi$ is semiproper [5, Rmk. 3.5]. The following example shows that «semiproper» is not sufficient in general: Let $M = M_1 \cup M_2$ be the disjoint union of $M_1 = \mathbb{R}^2$ and $M_2 = \mathbb{R}^2$. Let $N = \mathbb{R}^2$. Define $\varphi : M \to N$ by $\varphi(x,y) = (x,y)$ if $(x,y) \in M_1$, $\varphi(x,y) = (x,xy)$ if $(x,y) \in M_2$. Let $p = q = 1$ and let $A(x,y) = 0$ on $M_1$, $A(x,y) = 1$ on $M_2$. Take $B = 0$. Define $f \in \mathcal{C}^\infty(M)$ by $f(x,y) = 0$ on $M_1$ and $f(x,y) = ye^{-1/2y^2}$ on $M_2$. Let $(u,v)$ denote the coordinates of $N$. Then $f$ is flat on $\varphi^{-1}(\{u = 0\})$, and outside $\varphi^{-1}(\{u = 0\})$, $f = \Phi(g)$, where $g(u,v) = (v/u)e^{-1/2v^2}$. Hence $f \in (\Phi \mathcal{C}^\infty(N))^p$. Clearly, $f \notin \Phi \mathcal{C}^\infty(N)$. This example satisfies the conditions of Theorem 8.2.5 because $\varphi|M_2$ is generically a submersion (cf. § 13).

**Remark 11.11.** — The assertion that $\Phi \mathcal{C}^\infty(N)^a + B \cdot \mathcal{C}^\infty(M)^p = (\Phi \mathcal{C}^\infty(N)^a + B \cdot \mathcal{C}^\infty(M)^p)^p$ is local in $N$. Hence we can assume that $N$ is an open subset of $\mathbb{R}^n$ and, by Corollary 11.6, that there is a bound $s$ on the number of distinct submodules $\mathcal{R}_a \subset \mathcal{C}^\infty$, where $a \in \varphi^{-1}(b)$, $b \in N$. We will prove Theorem 11.8 using the conditions of Theorem 8.2.5 with this $s$.

We will also use the following:
Remark 11.12. - Let X be a germ at the origin of a closed analytic subset of $\mathbb{R}^n$. Let $X^C$ denote the complexification of X, and let Sing $X^C$ denote (the germ of) the singular points of $X^C$. The real part $\Sigma$ of Sing $X^C$ is (a germ of) a proper analytic subset of X. There exist $f_i(x) \in \mathbb{R}\{x\} = \mathbb{R}\{x_1, \ldots, x_m\}, 1 \leq i \leq k$, such that the complexifications $f_i(z)$ of the $f_i(x)$ generate the ideal in $C[z] = C[z_1, \ldots, z_m]$ of convergent power series which vanish on $X^C$. Then, for all $a \in X - \Sigma$, $I_{X,a}$ is generated by the $f_i$ (where we have used the same symbol for a germ at the origin and a representative of the germ in a suitable neighborhood, and where $I_X$ denotes the sheaf of germs of real analytic functions vanishing on X).

Proof of Theorem 11.8. - We make the assumptions of Remark 11.11. If $b \in \phi(M)$, then there exist $a^1, \ldots, a^e \in \phi^{-1}(b)$ such that $\bigcap_{a^i \in \phi^{-1}(b)} R_a = \bigcap_{i=1}^s R_d$. If $a \in M^\varphi$, $a = (a^1, \ldots, a^e)$, we put $R_a = \bigcap_{i=1}^s R_d$. Since the

...
flat on $\varphi^{-1}(Y_{\mu+1})$. By induction, we can assume that $f$ is flat on $\varphi^{-1}(Y_\mu)$.

Let $X = X_\mu - \varphi^{-1}(Y_\mu)$. If $X = \emptyset$, we can take $g = 0$ and $h = 0$. Suppose $X \neq \emptyset$. Then $\varphi|_X : X \to \mathbb{N} - Y_\mu$ is proper. Let $a \in X$, $a = (a_1, \ldots, a_s)$, and let $b = \varphi(a)$. By (3) and the formal division algorithm (Theorem 6.2), there is a unique $G_b \in \partial^\beta$ such that

$$\text{supp } G_b \cap \mathcal{R}_\mu = \emptyset,$$

and $\tilde{f}_a - \tilde{g}_a(G_b) \in \text{Im } \tilde{B}_d$, $i = 1, \ldots, s$. Then, by (4), for all $a \in \varphi^{-1}(b)$, $\tilde{f}_a - \tilde{g}_a(G_b) \in \text{Im } \tilde{B}_d$.

Write $G_b = (G_{1,b}, \ldots, G_{q,b})$. $G_{j,b} = \sum_{\beta \in \mathbb{N}^n} G^\beta_{j,b} y^\beta \in \partial^\beta = \mathbb{R}[[y]]$, where $y = (y_1, \ldots, y_n)$. Then (11.13) is equivalent to: $D^\beta G_{j,b} = 0$ for all $(\beta, j) \in \mathcal{R}_\mu$.

**Lemma 11.14.** - For each $(\beta, j) \in \mathbb{N}^n \times \{1, \ldots, q\}$, there exists $g^\beta_j \in C^\infty(X)$ such that:

(i) $g^\beta_j$ extends continuously to zero on $\bar{X} - X$.

(ii) For all $a \in X$, $g^\beta_{j,a} = \iota^*_a \Phi^*_a(D^\beta G_{j,\varphi(a)})$, where $\iota^*_a : \partial^\beta_{M^\phi,a} \to \partial^\beta_{X,a}$ is induced by the inclusion $\iota : X \to M^\phi$.

It follows from (ii) and an estimate of Glaeser [16, §§4, 5] (or [37, pp. 180-181]) that, for each $j = 1, \ldots, q$, there exists $g^\beta_j \in C^\infty(N - Y_\mu)$ such that $\hat{g}^\beta_{j,b} = G_{j,b}$ for all $b \in \varphi(X) = Y_{\mu+1} - Y_\mu$. By (i), for all $(\beta, j) \in \mathbb{N}^n \times \{1, \ldots, q\}$, $D^\beta g^\beta_{j,\varphi(X)}$ extends continuously to zero on $Y_\mu$. Since $Y_{\mu+1}$ is subanalytic, it follows that there exist $g_j \in C^\infty(N)$ such that $g_j$ is flat on $Y_\mu$ and $\hat{g}^\beta_{j,b} = G_{j,b}$, for all $b \in \varphi(X)$. Put $g = (g_1, \ldots, g_q)$. Then $(f - \Phi(g))_a \in \text{Im } \tilde{B}_a$ for all $a \in \varphi^{-1}(Y_{\mu+1})$. By Theorem 10.1 (and Remark 10.2), there exists $h \in C^\infty(M^\phi)$ such that $f_\varphi - \Phi(g) - B \cdot h$ is flat on $\varphi^{-1}(Y_{\mu+1})$, as required.

**Proof of Lemma 11.14.** - If $(\beta, j) \in \mathcal{R}_\mu$, then $D^\beta G_{j,b} = 0$, for all $b \in \varphi(X)$. Hence it is enough to prove the assertion for $(\beta, j) \notin \mathcal{R}_\mu$. Let $a \in X$, $a = (a_1, \ldots, a_s)$. We have $\tilde{f}_{a,i} - \tilde{A}_{a,i}(G_{\varphi(a)} \Phi^*_a) \in \text{Im } \tilde{B}_{d,i}$, $i = 1, \ldots, s$; i.e., $(\tilde{f}_{a,i})_{1 \leq i \leq s} = \Phi^*_a(G_{\varphi(a)}) \in \text{Im } \tilde{B}_{a,i}$.

For each $\ell \in \mathbb{N}$, let $'F_a$ (respectively, $'G_a$) denote the image of $(\tilde{f}_{a,i})_{1 \leq i \leq s}$ (respectively, of $G_{\varphi(a)}$) by the lower (respectively, upper) horizontal arrow in the completion of the left-hand diagram (8.2.6); thus,

$$'F_a - \tilde{A}_{a,i}.'G_a \in \text{Im } \tilde{B}_{a,i}.$$
Recall that \( \hat{G}_a \) is the element of \( \bigoplus_{\beta \in \mathbb{N}^n} \hat{\mathcal{O}}_{X,a} \) induced by \( (D^\theta G_{\psi(a)} \hat{\phi}_a)_{\beta \in \ell} \). Write \( \hat{G}_a = (G^\beta_a)_{\beta \in \ell, 1 \leq j \leq q} \), where each \( G^\beta_{j,a} \in \hat{\mathcal{O}}_{X,a} \) and \( G^\beta_a = (G^\beta_{j,a})_{1 \leq j \leq q} \). Then \( G^\beta_{j,a} = 0 \) for all \( (\beta,j) \in \mathfrak{N}_a \).

We use the notation of 8.2, 8.3. Let \( k \in \mathbb{N} \). According to Theorem 8.2.5. (1), there exists \( \ell = \ell(k) \in \mathbb{N} \) such that \( \ell(k,a) \leq \ell \) for all \( a \in X \). Let \( \rho_{r,k}(X) = \max_{a \in X} \rho_{r,k}(a) \) and let \( \sigma_{r,k}(X) = \max_{a \in X} \sigma^\ell_{r,k}(a) \).

Put \( Y_{r,k} = \{ a \in X : \rho_{r,k}(a) < \rho_{r,k}(X) \} \) and \( Z_{r,k} = \{ a \in X : \sigma^\ell_{r,k}(a) < \sigma_{r,k}(X) \} \). Then \( Y_{r,k} \) and \( Z_{r,k} \) are proper analytic subsets of \( X \). Let \( a \in X \). Define \( T^X_{r,k}(a) \) and \( \hat{T}_{r,k,a} \) as in 8.3. From (11.15):

\[
\text{ad} \sigma_{r,k}(X) \hat{S}_{r,k,a} \circ \text{Ad}^\rho_{r,k}(X) \hat{D}_{r,k,a} \cdot \hat{F}_a = \hat{T}_{r,k,a} \cdot \hat{G}_a,
\]

where \( \hat{S}_{r,k,a} = \text{Ad}^\rho_{r,k}(X) \hat{D}_{r,k,a} \circ \hat{B}_{r,k,a} \).

Let \( e(k) \) denote the number of exponents \( (\beta,j) \in \mathbb{N}^n \times \{1, \ldots, q\} \) such that \( (\beta,j) \notin \mathfrak{N}_a \) and \( |\beta| \leq k \). Suppose \( a \in X - (Y_{r,k} \cup Z_{r,k}) \). By the formal division algorithm (Theorem 6.2) and Remarks 8.2.4 and 8.3.1, rank \( T^X_{r,k}(a) = e(k) \); moreover, if \( V_a(k) \) denotes the subspace

\[
\{ G = (G^\beta_{j,a})_{\beta \in \mathbb{N}^n, 1 \leq j \leq q} \in \bigoplus_{\beta \in \mathbb{N}^n} (\mathcal{O}_{X,a}/m_{X,a} \cdot \hat{\mathcal{O}}_{X,a})^q : G^\beta_{j,a} = 0 \text{ if } (\beta,j) \notin \mathfrak{N}_a \},
\]

then rank \( T^X_{r,k}(a) \mid V_a(k) = e(k) \).

By the induction hypothesis and Cramer’s rule, there is a minor \( \delta = \delta_a \) of order \( e(k) \) of \( T^X_{r,k} \) such that \( \delta \) is not identically zero on \( X \) and such that, for all \( a \in X \) and \( (\beta,j) \notin \mathfrak{N}_a, |\beta| \leq k \),

\[
(11.16) \quad \delta_a \cdot G^\beta_{j,a} = (\xi^\beta_{j,a})^\wedge_a,
\]

where \( \xi^\beta_{j,a} \in \mathcal{C}^\infty(X) \) is the restriction to \( X = X_a - \varphi^{-1}(Y_a) \) of a \( \mathcal{C}^\infty \) function on \( U_a \) which is flat on \( \varphi^{-1}(Y_a) \). The minor \( \delta \) is the restriction to \( X \) of an analytic function defined on \( U_a \) (which we also denote \( \delta \)).

Suppose \( (\beta,j) \notin \mathfrak{N}_a, |\beta| \leq k \). By Whitney’s extension theorem [27, I.4.1], there exists \( \eta^\beta_{j,a} \in \mathcal{C}^\infty(U_a) \) such that \( \eta^\beta_{j,a} \) is flat on \( W_a - X \) and \( \eta^\beta_{j,a} \mid X = \xi^\beta_{j,a} \). Then, by (11.16) and (5) above, for all \( a \in U_a \), \( \eta^\beta_{j,a} \) belongs to the ideal in \( \mathcal{O}_{U_a} \) generated by \( \delta_a \) and the \( \delta_{\mu,a} \). By Theorem 10.1, there exists \( h^\beta_{j,a} \in \mathcal{C}^\infty(U_a) \) such that \( \eta^\beta_{j,a} - \delta \cdot h^\beta_{j,a} \) belongs to the ideal generated by the \( \delta_{\mu,a} \) in \( \mathcal{C}^\infty(U_a) \). Then \( h^\beta_{j,a} \) vanishes on \( X - X \) and, if \( g^\beta_{j,a} = h^\beta_{j,a} \mid X \), then \( g^\beta_{j,a} = G^\beta_{j,a} \) for all \( a \in X \), as required.
CHAPTER III

SEMICONTINUITY RESULTS

12. Algebraic morphisms.

Let $K = \mathbb{R}$ or $\mathbb{C}$. Let $K[x]$ (respectively, $K[[x]]$) denote the ring of polynomials (respectively, formal power series) in $x = (x_1, \ldots, x_m)$.

**Definition 12.1.** Let $U$ be an open subset of $K^m$. An analytic function $f \in \mathcal{O}(U)$ is Nash if it is algebraic over the ring $K[x]$ of polynomials in the coordinates $x = (x_1, \ldots, x_m)$ of $K^m$; i.e., there is a nonzero polynomial $P(x, y) \in K[x, y]$ such that $P(x, f(x)) = 0$ for all $x \in U$. Let $N(U)$ denote the ring of Nash functions on $U$.

We can define a category of Nash manifolds and Nash mappings using, as local models, open subsets $U$ of $K^m$, $m \in \mathbb{N}$, together with the rings $N(U)$.

**Theorem 12.2.** Let $M$ and $N$ denote Nash manifolds, and let $\varphi : M \to N$ be a Nash mapping. Let $A$ and $B$ be $p \times q$ and $p \times r$ matrices, respectively, whose entries are Nash functions on $M$. We use the notation of 8.2, 8.4. Let $s \in \mathbb{N}$. Assume that $N$ is an open subset of $K^s$. Then the diagram of initial exponents $\mathcal{N}_s = \mathcal{N}(\mathcal{R}_s)$ is Zariski semicontinuous on $M^p$.

**Remarks 12.3.** (1) Our proof of Theorem 12.2 together with Proposition 9.6 in fact establishes 12.2 under the following more general hypothesis: Let $M$ and $N$ denote analytic manifolds. Let $\varphi : M \to N$ be an analytic mapping, and $A, B$ matrices of analytic functions on $M$, satisfying the following condition: For every $a \in M$, there are (analytic) coordinate neighborhoods $U$ of $a$ in $M$ and $V$ of $\varphi(a)$ in $N$, such that $\varphi(U) \subseteq V$ and both the components of $\varphi|U$ and the entries of $A|U$ and $B|U$ belong to $N(U)$.

(2) In the special case that $M$ and $N$ are algebraic manifolds, $\varphi$ is a regular (rational) mapping, and $A, B$ are matrices of regular functions on $M$, our proofs actually show that $\mathcal{N}_s$ is Zariski semicontinuous in the algebraic sense; i.e., for each $a \in M^p$, $\{x \in M^p : \mathcal{N}_s \geq \mathcal{N}_s\}$ is a closed algebraic subset of $M^p$. 

To prove Theorem 12.2, we will use a version of « Artin approximation with respect to nested subrings » (cf. [2], [3], [33]) :

**Definition 12.4.** — A formal power series $f(x) \in K[[x]]$ is algebraic if it is algebraic over $K[x]$. The algebraic elements of $K[[x]]$ form a subring which we denote $K\langle x \rangle$.

Clearly, $K\langle x \rangle \subset K\{x\}$, the ring of convergent power series. Let $(x) = (x_1, \ldots, x_m)$ denote the ideal in $K[[x]]$ generated by $x_1, \ldots, x_m$.

**Remark 12.5** [3]. — Let $f_i(x) \in K[[x]]$. Then $f_i(x)$ is algebraic if and only if there exist $r \in \mathbb{N}$, $f_i(x) \in K[[x]]$, $i = 2, \ldots, r$, and $F_j(x,y) \in K[x,y]$, $j = 1, \ldots, r$ where $y = (y_1, \ldots, y_r)$, such that :

1. $F(x,f(x)) = 0$, where $f = (f_1, \ldots, f_r)$ and $F = (F_1, \ldots, F_r)$;

2. $\det \left( \frac{\partial F_i}{\partial y_j} \right)(0,f(0)) \neq 0$.

**Theorem 12.6.** — Let

$$f(x,y,u,v) = 0$$

be a system of equations in $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_n)$, $u = (u_1, \ldots, u_p)$ and $v = (v_1, \ldots, v_q)$, where $f = (f_1, \ldots, f_r)$ and each $f_j \in K\langle x, y, u, v \rangle$. Assume that $f$ is linear with respect to $v$; i.e.,

$$f(x,y,u,v) = \sum_{i=0}^{q} v_i g_i(x,y,u),$$

where $v_0 = 1$ and each $g_i \in K\langle x, y, u \rangle$. Suppose that (12.7) admits a solution $u = \hat{u}(x) \in K[[x]]^p$, $v = \hat{v}(x,y) \in K[[x,y]]^q$, where $\hat{u}(0) = 0$. Then, for all $t \in \mathbb{N}$, (12.7) has a solution $u = u(x) \in K\langle x \rangle^p$, $v = v(x,y) \in K\langle x, y \rangle^q$ such that $u(x) - \hat{u}(x) \in (x)^t K[[x]]^p$ and $v(x,y) - \hat{v}(x,y) \in (x,y)^t K[[x,y]]^q$.

**Remark 12.8.** — The analogue of Theorem 12.6 for convergent power series is false : Let $f(x) = f(x_1, x_2)$ and $\varphi_i(x)$, $i = 1, 2, 3$, be as in Example 2.8. Then the equation $f(x) - g(y) = \sum_{i=1}^{3} h_i(x,y)(y_i - \varphi_i(x))$ admits a formal solution $g(y)$, $h_i(x,y)$, $i = 1, 2, 3$, but no such convergent solution.
LEMMA 12.9. — Theorem 12.6 holds under the stronger assumption that each $f_i(x,y,u,v) \in K[x,y,u,v]$. (In this case, it is unnecessary to assume $u(0) = 0$.)

Proof. — For convenience, we make the following change of notation: $v$ will mean $(v_0, v_1, \ldots, v_q)$, where $v_0 = 1$. We also put $\hat{v}(x,y) = (\hat{v}_0(x,y), \ldots, \hat{v}_q(x,y))$, where $\hat{v}_0(x,y) = 1$. Let $A$ denote the localization of the ring $K[[x,y]]$ at the ideal generated by $x$ and $y$. Let $\hat{A}$ denote the completion of $A$; of course, $\hat{A} = K[[x,y]]$.

Each $g_i(x,y,\hat{u}(x)) \in A$. Since $v = \hat{v}(x,y)$ is a solution of the system

$$\sum_{i=0}^{q} v_i g_i(x,y,\hat{u}(x)) = 0,$$

then, by Krull's theorem, there is a solution $v = \hat{v}(x,y)$, where $\hat{v}_0 = 1$ and each $\hat{v}_i(x,y) \in A$. Clearly, $\hat{v}$ can be chosen to approximate $v$ to any given order.

We can write $\hat{v}(x,y) = \hat{w}(x,y)/\hat{w}_0(x,y)$, where $\hat{w} = (\hat{w}_0, \ldots, \hat{w}_q)$, each $\hat{w}_i \in K[[x,y]]$ and $\hat{w}_0(0,0) \neq 0$. Then $\sum_i \hat{w}_i(x,y)g_i(x,y,\hat{u}(x)) = 0$. Write each $\hat{w}_i$ and $g_i$ as a polynomial in $y_1, \ldots, y_n$:

$$\hat{w}_i(x,y) = \sum_{\alpha} \hat{w}_i(\alpha)(x)y^\alpha \in K[[x,y]]$$

and

$$g_i(x,y,u) = \sum_{\alpha} g_{i\alpha}(x,u)y^\alpha \in K[x,u][y]^r,$$

where $\alpha$ denotes a multiindex in $N^n$. Then $u = \hat{u}(x)$, $w_{i\alpha} = \hat{w}_{i\alpha}(x)$ is a formal solution of the system of polynomial equations

$$\sum_{i=0}^{q} \sum_{\alpha+\beta=\gamma} \hat{w}_{i\alpha}g_{i\beta}(x,u) = 0, \quad \gamma \in N^n.$$ 

By Artin's theorem [2, Thm. 1.10], there is an algebraic solution $u = u(x)$, $w_{i\alpha} = w_{i\alpha}(x)$ which approximates the given formal solution to any specified order.

Put $w_i(x,y) = \sum_{\alpha} w_{i\alpha}(x)y^\alpha$ and $v(x,y) = w(x,y)/w_0(x,y)$, where $w = (w_0, \ldots, w_q)$. Then $u = u(x)$, $v = v(x,y)$ is an algebraic solution of (12.7). Clearly, the solution can be chosen to approximate $\hat{u}(x)$, $\hat{v}(x,y)$ to any specified order. 

Proof of Theorem 12.6. — We make the same notational changes as in Lemma 12.9: $v$ will mean $v = (v_0, v_1, \ldots, v_q)$, where $v_0 = 1$, etc. Write $g_i = (g_{i1}, \ldots, g_{iq})$, $i = 0, \ldots, q$, where each $g_{iy} \in K[x,y,u]$. By Remark 12.5, there exist $s \in N$, $s > q$, as well as $g_{iy}(x,y,u) \in K[x,y,u]$, $i = 0, \ldots, q$, $j = 1, \ldots, r$, and $G_{k\ell}(x,y,u,z) \in K[x,y,u,z]$. 


RELATIONS AMONG ANALYTIC FUNCTIONS

$k = 0, \ldots, s$, $\ell = 1, \ldots, r$, where $z = (z_y)$, $i = 0, \ldots, s$, $j = 1, \ldots, r$, such that:

1. $G(x, y, u, g(x, y, u)) = 0$, where $g = (g_{ij})$, $G = (G_{kr})$;
2. $\det \left( \frac{\partial G}{\partial z} \right)(0, g(0)) \neq 0$.

By the implicit function theorem,

$$z - g(x, y, u) + g(0) = c(x, y, u, z)G(x, y, u, g(0) + z),$$

where $c(x, y, u, z) = (c_{ijk}(x, y, u, z))$ is a matrix whose rows are indexed by $(i, j)$ and whose columns are indexed by $(k, \ell)$. Each entry $c_{ijk}(x, y, u, z) \in K \langle x, y, u, z \rangle$. Then, for each $j = 1, \ldots, r$,

$$\sum_{i=0}^{q} v_i g_{ij}(x, y, u)$$

$$= \sum_{i=0}^{q} v_i (g_{ij}(0) + z_{ij}) - \sum_{i=0}^{q} \sum_{k=1}^{r} v_i c_{ijk}(x, y, u, z)G_{kr}(x, y, u, g(0) + z).$$

Consider the system of polynomial equations

$$(12.10) \quad \sum_{i=0}^{q} v_i (g_{ij}(0) + z_{ij}) = \sum_{k=1}^{r} w_{jk} g_{kj}(x, y, u, g(0) + z),$$

$j = 1, \ldots, r$, where $u, v$ and $w = (w_{jk})$ are the unknowns. Then (12.10) admits a formal solution $u = \hat{u}(x)$, $v = \hat{v}(x, y)$ and $w_{jk} = \tilde{w}_{jk}(x, y, z) = \sum_{i=0}^{q} \hat{v}_i (x, y) c_{ijk}(x, y, \hat{u}(x), z)$. Let $t \in \mathbb{N}$. By Lemma 12.9, there exist $u = u(x) \in K \langle x \rangle^p$, $v = v'(x, y, z) \in K \langle x, y, z \rangle^{q+1}$ and $w_{jk} = w_{jk}(x, y, z) \in K \langle x, y, z \rangle$ such that $v_0'(x, y, z) = 1$, $u(x) - \hat{u}(x) \in (x)^t K[[x]]^p$, $v'(x, y, z) - \hat{v}(x, y) \in (x, y, z)^t K[[x, y, z]]^{q+1}$, and

$$(12.11) \quad \sum_{i=0}^{q} v_i'(x, y, z) (g_{ij}(0) + z_{ij}) = \sum_{k=1}^{r} w_{ij} g_{kj}(x, y, u(x), g(0) + z),$$

$j = 1, \ldots, r$. Substitute $z_{ij} = g_{ij}(x, y, u(x)) - g_{ij}(0)$ into (12.11), to get

$$\sum_{i=0}^{q} v_i(x, y) g_i(x, y, u(x)) = 0,$$

where $v_i(x, y) = v_i'(x, y, g(x, y, u(x)) - g(0))$, $i = 0, \ldots, q$. ⊓⊔
Remark 12.12. — Let \( f_i(x) \in \mathbb{C}\langle x \rangle = \mathbb{C}\langle x_1, \ldots, x_m \rangle \). Let \( f_i(x) \), \( i = 2, \ldots, r \), and \( F_j(x, y) \), \( j = 1, \ldots, r \), \( y = (y_1, \ldots, y_r) \), be as in Remark 12.5. Put \( Z = \{(x, y) \in \mathbb{C}^{m+r} : F(x, y) = 0\} \). We can assume that the projection \( \pi(x, y) = x \) of \( Z \) onto \( \mathbb{C}^m \) is finite. The smooth points of \( Z \) which are not critical points of \( \pi \) project onto the complement of a proper algebraic subset \( V \) of \( \mathbb{C}^m \). Clearly, \( f_i \) extends to \( \mathbb{C}^m - V \) as a multivalued holomorphic function, whose various determinations are algebraic at every point of \( \mathbb{C}^m - V \). By differentiating the system of equations \( F(x, f(x)) = 0 \) with respect to \( x_j \), we can see that the partial derivative \( \partial f_i / \partial x_j \) also extends to \( \mathbb{C}^m - V \) as a multivalued holomorphic function whose various determinations are algebraic at every point.

Proof of Theorem 12.2. — By Lemma 9.5, we can assume that \( M \) is connected. Let \( a_0 \in M \cap U \), \( a = (a^1, \ldots, a^s) \). Let \( \Phi_\ast : \mathbb{C}_d(\Phi_{\ast}(a)) \to \mathbb{C}_d(\Phi_{\ast}(a)) \) and \( B_\ast : \mathbb{C}_d(\Phi_{\ast}(a)) \to \mathbb{C}_d(\Phi_{\ast}(a)) \), as well as \( \hat{\Phi}_\ast \) and \( \hat{B}_\ast \), be as in 8.2. Let \( G \in \mathbb{C}_d(\Phi_{\ast}(a)) \) and \( H \in \mathbb{C}_d(\Phi_{\ast}(a)) \). Put \( f = \Phi_\ast(G) + \hat{B}_\ast(H) \in \mathbb{C}_d(\Phi_{\ast}(a)) \), \( f = (f^1, \ldots, f^r) \). Suppose each \( f^i \in \mathbb{C}_d(\Phi_{\ast}(a)) \) is algebraic. Let \( t \in \mathbb{N} \). Then there exist \( g \in \mathbb{C}_d(\Phi_{\ast}(a)) \) and \( h \in \mathbb{C}_d(\Phi_{\ast}(a)) \) such that \( g \) and \( h \) are algebraic, \( f = \Phi_\ast(g) + \hat{B}_\ast(h) \), and \( g - G \in m_{\Phi_{\ast}(a)}^{1} \cdot \mathbb{C}_d(\Phi_{\ast}(a)) \), \( h - H \in m_{\Phi_{\ast}(a)}^{1} \cdot \mathbb{C}_d(\Phi_{\ast}(a)) \).

Proof. — Write \( H = (H^1, \ldots, H^r) \). Then

\[
(12.14) \quad f^i(x) = \hat{A}_{d}(x) \cdot G(\Phi_{\ast}(x) - \varphi(a^i)) + \hat{B}_{d}(x) \cdot H^i(x),
\]

\( i = 1, \ldots, s \). In other words, for each \( i = 1, \ldots, s \), there is a \( p \times n \) matrix \( Q^i(x, y) \) with entries in \( \mathbb{K}[[x, y]] \) such that

\[
(12.15) \quad f^i(x) - \hat{A}_{d}(x) \cdot G(y) - \hat{B}_{d}(x) \cdot H^i(x) = Q^i(x, y) \cdot (y - \Phi_{\ast}(x) + \varphi(a^i)).
\]
In this system of equations, $G(y)$ and the $H_i(x), Q^x(y)$ are the «unknowns». Since $A, B$ and $\varphi$ are algebraic, then, by Theorem 12.6, there is an algebraic solution $g(y), h^i(x,y), q^x(y)$ of (12.15); i.e.,

\begin{equation}
(12.16) \quad f^i(x) - \hat{A}_{d}(x) \cdot g(y) - \hat{B}_{d}(x) \cdot h^i(x,y) = q^x(x,y) \cdot (y - \hat{\varphi}_{d}(x) + \varphi(a^i)),
\end{equation}

\[ i = 1, \ldots, s, \] such that $g(y) - G(y) \in (y)^r \cdot K[[y]]^q$ and each $h^i(x,y) - H^i(x) \in (x,y)^r \cdot K[[x,y]]^r$. Substitute $y = \hat{\varphi}_{d}(x) - \varphi(a^i)$ back into (12.16), for each $i$, to see that $g(y), h^i(x) = \hat{h}^i(x,\hat{\varphi}_{d}(x) - \varphi(a^i))$ is a solution of (12.14); clearly $h^i(x) - H^i(x) \in (x,y)^r \cdot K[[x,y]]^r$.

\begin{corollary}
Corollary 12.17. \quad \mathcal{R}_a = \{ G \in \hat{\Phi}_{a}(G) : \Phi_a(G) \in \text{Im} \hat{B}_a \} is generated by algebraic elements.
\end{corollary}

Proof. \quad Let $(\beta,j)$ be a vertex of $\mathfrak{N}_a = \mathfrak{N}(\mathcal{R}_a)$. By Lemma 12.13, there exists $g \in \mathcal{R}_a$ such that $g$ is algebraic and in $g = j^{\beta,j}$.

We now complete the proof of Theorem 12.2. We can assume that $K = C$. Let $X$ denote an irreducible germ at $a_0$ of a closed analytic subset of $M'_a$. We can assume that $X$ is a closed analytic subset of $U$ and that its smooth points are connected. Let $\mathfrak{N}_X$ denote the generic diagram of initial exponents (Definition 8.4.3). By Proposition 8.4.6(1), it suffices to find a proper closed analytic subset $W$ of $X$ such that $\mathfrak{N}_a = \mathfrak{N}_X$ for all $a \in X - W$.

Let $(\beta_\ell,k_\ell), \ell = 1, \ldots, t,$ denote the vertices of $\mathfrak{N}_X$. Let $k = k(X)$ as in Definition 4.4.1, so that each $|\beta_\ell| \leq k$. Let $D_k$ be as in (3.3.2) and let $Z \subset X$ be as in Remark 8.4.4. By Lemma 8.4.5, $\mathfrak{N}_a = \mathfrak{N}_X$ for all $a \in D_k \cap (X - Z)$.

Let $a_1 \in D_k \cap (X - Z), \ a_i = (a^1_i, \ldots, a^t_i)$. Put $b_1 = \varphi(a_i)$. Let $G'(y) = y^{\beta_\ell,k_\ell} - r'(y), \ell = 1, \ldots, t,$ denote the standard basis of $\mathcal{R}_a$, so that $\text{supp} r' \cap \mathfrak{N}_X = \emptyset$; for each $\ell$. By Corollaries 6.8 and 12.17, each $G'(y)$ is convergent. Thus, for $b$ in some neighborhood of $b_1$, we can substitute $b - b_1 + y$ into $G'$, and expand in powers of $y$:

\[ G'(b - b_1 + y) = (b - b_1 + y)^{\beta_\ell,k_\ell} - r'(b - b_1 + y) = y^{\beta_\ell,k_\ell} - r'(y), \]
where \( \text{supp } r_i(y) \cap \mathfrak{M}_X = \emptyset \). For \( a \) in a sufficiently small neighborhood of \( a_i \) in \( M_v \), put \( G'_i(y) = G'(\phi(a) - b_1 + y) \). Then \( G'_i(y) = y^{b_{\ell},k_\ell} - r'_i(y) \), where \( r'_i = r'_{\phi(a)} \). Clearly, \( G'_i \in \mathfrak{R}_a \). If \( a \in X - Z \), then \( \mathfrak{N}_a \subset \mathfrak{M}_X \) by Proposition 8.4.6.(2), and it follows that in \( G'_i = y^{b_{\ell},k_\ell} \). In particular, \( \mathfrak{N}_a = \mathfrak{M}_X \) in a neighborhood of \( a_i \) in \( X \).

By Lemma 12.13, for each \( \ell = 1, \ldots, t \), there exist \( g' \in \hat{\mathcal{O}}^q_{\phi(a_i)} \), \( h_\ell \in \bigoplus \hat{\mathcal{O}}'_{a_i} \), \( h_\ell = (h_{1,\ell}, \ldots, h_{s,\ell}) \), such that \( g' \) and each \( h_\ell \) are algebraic, in \( g' = y^{b_{\ell},k_\ell} \), and \( \Phi_{a_i}(g') = B_{a_i}(h_\ell) \). In particular, \( g' \in \mathfrak{R}_a \). For each \( \ell = 1, \ldots, t \), put

\[
G'(v; y) = \sum_{\beta \in \mathbb{N}^n} (D^\beta g'(v)) \frac{y^\beta}{\beta!} \in \hat{\mathcal{O}}_{h_1}[[y]]^q,
\]

\[
H^i_\ell(u; x) = \sum_{x \in \mathbb{N}^m} (D^x h_\ell(x)) \frac{x^u}{u!} \in \hat{\mathcal{O}}_{a_i}[[x]]', \quad i = 1, \ldots, s,
\]

where \( u = (u_1, \ldots, u_m) \) and \( v = (v_1, \ldots, v_n) \). By the formal division algorithm (cf. Remark 6.5),

\[
(12.18) \quad y^{b_{\ell},k_\ell} = \sum_{j=1}^{t} Q_j(v; y) G'(v; y) + R'(v; y),
\]

\( \ell = 1, \ldots, t \), where, for each \( \ell \),

\( Q_\ell(v; y) \in \hat{\mathcal{O}}_{h_\ell}[[y]] \), \quad \( R'(v; y) \in \hat{\mathcal{O}}_{h_1}[[y]]^q \), \quad \text{supp } R'(v; y) \cap \mathfrak{M}_X = \emptyset,

and the coefficients of \( Q_\ell \) and \( R' \) (as elements of \( \hat{\mathcal{O}}_{h_\ell} \)) are algebraic. (They are linear combinations of the coefficients of the \( G'(v; y) \) divided by products of powers of the \( D^b g'_{\ell}(v) \), where \( g' = (g'_1, \ldots, g'_t) \).)

For each \( \ell = 1, \ldots, t \), write

\[
R'(v; y) = \sum_{\beta \neq \mathfrak{R}_X} \bar{R}_{\beta,j}(v) y^{\beta,j}.
\]

It follows from Remark 12.12 that there exist :

(1) A proper algebraic subset \( V \) of \( \mathbb{N} \) such that \( b_1 \notin V \), and, for each \( i = 1, \ldots, s \), a proper algebraic subset \( W^i \) of \( U^i \) such that \( a_i^j \notin W^i \).
(2) For each $\ell = 1, \ldots, t$ and $(\beta, j) \notin N_X$, an (a priori, multivalued) analytic function $p_{\beta, j}^\ell$ defined on $N - V$, such that $\hat{R}_{\beta, j}^\ell(v)$ is the formal Taylor expansion $(\hat{R}_{\beta, j}^\ell)^\prime\left(v\right)$ of some branch $R_{\beta, j}^\ell$ of $p_{\beta, j}^\ell$ at $b_1$. Likewise, for each $\ell = 1, \ldots, t$, multivalued analytic functions defined on $N - V$ (respectively, multivalued analytic functions defined on $U^i - W^i$, $i = 1, \ldots, s$) which extend the coefficients of $Q_\ell$ (respectively, the coefficients of $H_i$, $i = 1, \ldots, s$).

For each $\ell = 1, \ldots, t$, write $r^\ell_\alpha(y) = \sum_{(\beta, j) \notin N_\alpha} r^\ell_{\beta, j}(a) y^{\beta, j}$. We claim that, for $a$ in a sufficiently small neighborhood of $a_1$ in $X - Z$,

$$r^\ell_{\beta, j}(a) = R^\ell_{\beta, j}(\phi(a)),$$

for all $\ell, \beta, j$. Indeed, if $a$ belongs to a suitable neighborhood of $a_1$, then $R^\ell_{\beta, j}(\phi(a)) = \hat{R}_{\beta, j}^\ell(\phi(a) - b_1)$ and $G^\ell(\phi(a) - b_1; y) = g^\ell(\phi(a) - b_1 + y) \in R_*$. Thus $y^{\beta, \ell} - R^\ell(\phi(a) - b_1; y) \in R_*$. Moreover, $\text{supp} R^\ell(\phi(a) - b_1; y) \cap N_X = \emptyset$.

For $a$ close enough to $a_1$ in $X - Z$, $N_* = N_X$, so that

$$G^\ell(y) = y^{\beta, \ell} - R^\ell(\phi(a) - b_1; y),$$

by uniqueness of the standard basis; hence (12.19).

Let $W = X \cap (\phi^{-1}(V) \cup \bigcup_{i=1}^s (\mu^i)^{-1}(W^i))$, where $\mu^i : M^i_0 \to M$ denotes the projection $\mu^i(x) = x^i$, $x = (x^1, \ldots, x^s)$. Then $W$ is a closed analytic subset of $X$, and $a_1 \notin W$. By (12.19) and (2) above, the coefficients $r^\ell_{\beta, j}(a)$ of each $G^\ell_\alpha(y) = y^{\beta, \ell} - r^\ell_\alpha(y)$, as well as the coefficients of the $Q_\ell$ composed with $\phi$, and the coefficients of the $H_i$, can be analytically continued (as multivalued functions) throughout $X - W$. By continuity and (12.18), if $a \in W$, then any analytic continuation of (the coefficients of) $G^\ell_\alpha(y)$ to $a$ results in an element of $R_*$. If $a \in X - (Z \cup W)$, then $N_* \subset N_X$; it follows from uniqueness of the standard basis that any analytic continuation of $G^\ell_\alpha(y)$ to $a$ gives the same result, and that $N_* = N_X$. \qed
13. Regular mappings.

Let $K = \mathbb{R}$ or $\mathbb{C}$.

**Theorem 13.1.** Let $M$ and $N$ be analytic manifolds (over $K$) and let $\varphi : M \to N$ be an analytic mapping. Suppose that $\varphi$ is regular (as in 2.7). Let $s \in N$. For each $a \in M_s$, let $H_a$ denote the Hilbert-Samuel function of the ring $\mathcal{O}_{\varphi(a)}/\mathcal{R}_a$, where $\mathcal{R}_a = \bigcap_{i=1}^s \ker \varphi_a^*$, $a = (a^1, \ldots, a^s)$. Then $H_a$ is Zariski semicontinuous on $M_s$.

**Remark 13.2 (Tougeron).** If $s = 1$, the uniform Chevalley estimate (8.2.5(1)) can be proved using results of [39].

**Remark 13.3.** Let $V$ be an analytic manifold, and let $Z$ be a closed analytic subset of $V$. We denote by $\mathcal{I}_Z$ the subsheaf of ideals of $\mathcal{O}_V$ of germs of analytic functions which vanish on $Z$. Suppose that $\dim V = n$ and that $Z$ has pure dimension $n - 1$. Let $b \in V$. Then $\mathcal{I}_{Z,b}$ is a principal ideal. Let $\mu$ be as in Remark 6.10(2); we call $\mu_Z(b) = \mu$ the multiplicity of $Z$ at $b$. Thus $\mu_Z(b)$ is the largest $\mu \in \mathbb{N}$ such that $\mathcal{I}_{Z,b} \subseteq m_b^\mu$, where $m_b$ is the maximal ideal of $\mathcal{O}_{V,b}$.

**Proof of Theorem 13.1.** By Lemma 9.5, we can assume that the generic rank $r_1(a)$ of $\varphi$ near $a$ is constant on $M$; say $r_1(a) = n - k$, $a \in M$. Let $a_0 \in M_0$, $a_0 = (a^1_0, \ldots, a^s_0)$. Put $b_0 = \varphi(a_0)$. We can assume that $N$ is an open subset of $K^n$ and $b_0 = 0$. Since $\varphi$ is regular, then, after replacing $M$ and $N$ by suitable neighborhoods of $\{a^1_0, \ldots, a^s_0\}$ and $b_0$ (respectively) if necessary, there is a closed analytic subset $Z$ of $N$ of dimension $n = k$, such that $\varphi(M) \subseteq Z$ and $\mathcal{I}_{Z,0} = \bigcap_{i=1}^s \ker \varphi_{a^i_0}$.

The result is trivial if $k = 0$. Suppose that $k = 1$. We can assume that $K = \mathbb{C}$ and that $Z$ has pure dimension $n - 1$. Since $Z$ is coherent, the multiplicity of $Z$ is Zariski semicontinuous, by Theorem 7.4 and Remark 6.10. Let $\eta : Z' \to Z$ denote the normalization of $Z$. Since $\eta$ is finite, it follows that (after shrinking $N$ if necessary) there is a filtration of $Z$ by closed analytic subsets,

$$Z = Z_0 \supseteq Z_1 \supseteq \ldots \supseteq Z_{t+1} = \emptyset,$$
such that, for each \( i = 0, \ldots, t \):

1. \( Z_i - Z_{i+1} \) is smooth and connected.

2. Let \( Z'_i = \eta^{-1}(Z_i) \). Then \( \eta'(Z'_i - Z'_{i+1}) : Z'_i - Z'_{i+1} \to Z_i - Z_{i+1} \) is a smooth covering projection.

3. The multiplicity of \( Z \) is constant on \( Z_i - Z_{i+1} \).

It follows from (2) that, for each \( i \), there are finitely many analytic sets \( Z_y \) defined in a neighborhood of \( Z_i - Z_{i+1} \), such that, for all \( b \in Z_i - Z_{i+1} \), the germs \( Z_{y,b} \) of the \( Z_y \) at \( b \) are the distinct irreducible components of \( Z_b \). Then, by (3), for each \( i \) and \( j \), the multiplicity of \( Z_{y,b} \) is constant on \( Z_i - Z_{i+1} \).

Let \( X_i = \varphi^{-1}(Z_i) \), \( i = 0, \ldots, t \). Suppose that \( a = (a^1, \ldots, a^{t}) \in X_i - X_{i+1} \). Then, for each \( \ell = 1, \ldots, s \), there is a \( j \) such that \( \text{Ker } \varphi^*_i = \mathcal{I}_{Z_{y,b}(a)} \). It follows that \( \text{Ker } \varphi^*_x = \mathcal{I}_{Z_{y,b}(a)} \) for \( x = (x^1, \ldots, x^t) \) in some neighborhood of \( a \) in \( X_i - X_{i+1} \). Therefore, by Remark 6.10, the Hilbert-Samuel function \( H_\bullet \) is constant on each connected component of \( X_i - X_{i+1} \). By Proposition 8.3.7, \( H_\bullet \) is Zariski semicontinuous on \( M^t \). This completes the proof in the case \( k = 1 \).

In general, by the representation theorem for germs of analytic sets [32, Ch. III], we can assume:

1. There is a neighborhood \( V' \) of \( 0 \) in \( K^{n-k} \) such that \( N = V' \times K \subset K^{n-k} \times K^k \).

2. Let \( y = (y_1, \ldots, y_n) \) denote the coordinates in \( K^n \). Then, for each \( i = 1, \ldots, k \), there is a monic polynomial \( P_i \in \mathcal{O}(V')[Y_{n-i+1}] \) such that \( P_i \) vanishes on \( Z \).

3. Let \( d_i = \text{degree } P_i \), \( i = 1, \ldots, k \). Put \( P = P_k \) and \( d = d_k \). Let \( \Delta(y_1, \ldots, y_{n-k}) \) denote the discriminant of \( P \). Then \( \Delta \) is not identically zero and, for all \( j = 1, \ldots, d \) and all \( \alpha = (\alpha_1, \ldots, \alpha_d) \in N^k \) with \( 0 \leq \alpha_i < d_i, i = 1, \ldots, k \), there exists \( v_{\alpha} \in \mathcal{O}(V') \) such that

\[
Q_\alpha = \Delta \cdot y_{n-k+1}^{\alpha_{n-k+1}} \cdots y_n^{\alpha_n} - \sum_{j=1}^{d} v_{\alpha_j} \cdot y_{n-k+1}^{d-j}
\]

vanishes on \( Z \).

Suppose \( a = (a^1, \ldots, a^t) \in M^t \) and \( b = \varphi(a) \), \( b = (b_1, \ldots, b_n) \). Set \( b' = (b_1, \ldots, b_{n-k}) \). Suppose \( G \in \mathcal{O}_{b'} = K[[y]] \). Then, by the formal Weierstrass division theorem, there exist \( G_{\alpha} \in \mathcal{O}_{b'}, 0 \leq \alpha_i < d_i, \)
\[ G = \sum_{0 < n_i < d_i} G_i \cdot y^2_{n-k+1} \cdots y^2_{n} \in (P_i) \cdot \mathcal{O}_b, \]

where \((P_i)\) denotes the ideal of \(\mathcal{O}_b\) generated by the \(P_i\). By (3), there exist \(H_j \in \mathcal{O}_{b^j}, j = 1, \ldots, d\), such that

\[ \Delta_p \cdot G - \sum_{j=1}^{d} H_j \cdot y^{d-1}_{n-k+1} \in (P_i, Q_i) \cdot \mathcal{O}_b. \]

Let \(\pi : N \to V = V' \times K\) denote the projection \(\pi(y_1, \ldots, y_n) = (y_1, \ldots, y_{n-k+1})\). Put \(\psi = \pi \circ \phi\). Then \(\psi\) is regular and has generic rank \(n-k\). If \(G \in \bigcap \ker \mathcal{O}_{a^*}\), then \(H = \sum_{j=1}^{d} H_j \cdot y^{d-1}_{n-k+1} \in \bigcap \ker \mathcal{O}_{a^*}\). It follows from the case \(k = 1\) and Theorems 8.2.5 and 9.1, that there is a neighborhood \(U'\) of \(a_0\) in \(M_{\phi}\) and a filtration of \(U'\) by closed analytic sets, \(U' = Y_0 \supset Y_1 \supset \ldots \supset Y_{t+1} = \emptyset\), such that, for each \(\lambda = 0, \ldots, t\), there exist finitely many \(h_{\lambda \mu} \in \mathcal{M}(Y_{\lambda}; Y_{\lambda+1})[[y_1, \ldots, y_{n-k+1}]]\) such that the \(h_{\lambda \mu}(a; y_1, \ldots, y_{n-k+1})\) generate \(\bigcap_{\lambda = 1}^{s} \ker \mathcal{O}_{a^*}\), \(a = (a^1, \ldots, a^r) \in Y_{\lambda} - Y_{\lambda+1}\).

Then by Proposition 9.4, there is a neighborhood \(U\) of \(a_0\) in \(M_{\phi}\) and a filtration of \(U\) by closed analytic sets, \(U = X_0 \supset X_1 \supset \ldots \supset X_{r+1} = \emptyset\), such that, for each \(\lambda = 0, \ldots, r\), there exist finitely many elements \(g_{\lambda \mu} \in \mathcal{M}(X_{\lambda}; X_{\lambda+1})[[y]]\) such that the \(g_{\lambda \mu}(a; y)\) generate \(\bigcap_{\lambda = 1}^{s} \ker \mathcal{O}_{a^*}\), for all \(a = (a^1, \ldots, a^r) \in X_{\lambda} - X_{\lambda+1}\).

Therefore, by Lemma 7.2 (2) and Proposition 8.3.7, the Hilbert-Samuel function \(H_a\) is Zariski semi-continuous on \(M_{\phi}\). \(\square\)

14. The finite case.

Let \(K = R\) or \(C\). Let \(M\) and \(N\) denote analytic manifolds (over \(K\)) and let \(\phi : M \to N\) be an analytic mapping. If \(a \in M\), then \(\mathcal{O}_a\) is an \(\mathcal{O}_{\phi(a)}\)-module via the homomorphism \(\phi_a^* : \mathcal{O}_{\phi(a)} \to \mathcal{O}_a\).
DEFINITION 14.1. - We say that \( \varphi \) is locally finite if, for every \( a \in M \), \( \mathcal{O}_a \) is a finitely generated \( \mathcal{O}_{\varphi(a)} \)-module. (This definition extends to morphisms of (possibly singular) analytic spaces.)

THEOREM 14.2. - Let \( M \) and \( N \) be analytic manifolds, and let \( \varphi : M \to N \) be a locally finite analytic mapping. Let \( A \) and \( B \) be \( p \times q \) and \( p \times r \) matrices of analytic functions on \( M \), respectively. We use the notation of 8.2. Let \( s \in N \). Then there is a uniform Chevalley estimate (8.2.5(1)) on \( M_s \).

Theorem 14.2 extends to the case that \( M \) is a (possibly singular) analytic space which is Cohen-Macauley: see Remark 14.13 after the proof.

Proof of Theorem 14.2. - We can assume that \( K = \mathbb{C} \) and that \( N \) is an open neighborhood of 0 in \( \mathbb{C}^n \). By Lemma 9.5, we can assume that \( M \) has pure dimension \( m \). Let \( a_0 = (a_1, \ldots, a_0) \in M_r \). Shrinking \( N \) and replacing \( M \) by an appropriate neighborhood of \( \{a_0, \ldots, a_0\} \), we can assume that \( \varphi \) is proper and that \( Z = \varphi(M) \) is a closed analytic subset of \( N \), each irreducible component of which contains \( \varphi(a_0) \).

Suppose that \( \varphi(a_0) = 0 \) in \( N \subseteq \mathbb{C}^n \). Since \( \dim Z = m \), we can assume that \( N = N' \times N'' \subseteq \mathbb{C}^m \times \mathbb{C}^{n-m} \) and that the projection \( \pi : N \to N' \) induces a finite (i.e., proper and locally finite) mapping of \( Z \) onto \( N' \). Let \( \theta = \pi \circ \varphi \), \( \theta = (\theta_1, \ldots, \theta_m) \). Let \( a \in M \) and let \( m_{\theta(a)} \cdot \mathcal{O}_a \) denote the ideal in \( \mathcal{O}_a \) generated by \( m_{\theta(a)} \) (via the homomorphism \( \theta_a^* \)). Since \( \theta \) is finite, \( \dim \mathcal{O}_a/m_{\theta(a)} \cdot \mathcal{O}_a < \infty \).

LEMMA 14.3. - Let \( \ell = \dim \mathcal{O}_a/m_{\theta(a)} \cdot \mathcal{O}_a \). Then \( m_a^{\ell+1} \subset m_{\theta(a)} \cdot \mathcal{O}_a \).

Proof. - If \( j \geq 1 \) and \( m_{\theta(a)} \cdot \mathcal{O}_a + m_j = m_{\theta(a)} \cdot \mathcal{O}_a + m_j^{-1} \), then, by Nakayama’s lemma, \( m_{\theta(a)} \cdot \mathcal{O}_a = m_{\theta(a)} \cdot \mathcal{O}_a + m_j \), so that \( m_j = m_{\theta(a)} \cdot \mathcal{O}_a \). Suppose \( m_j^{\ell+1} \not\subset m_{\theta(a)} \cdot \mathcal{O}_a \). Then, for all \( j \leq \ell + 1 \),

\[
\dim \mathcal{O}_a/(m_{\theta(a)} \cdot \mathcal{O}_a + m_j^{\ell+1}) > \dim \mathcal{O}_a/(m_{\theta(a)} \cdot \mathcal{O}_a + m_j^\ell).
\]

Therefore, \( \dim \mathcal{O}_a/m_{\theta(a)} \cdot \mathcal{O}_a \geq \dim \mathcal{O}_a/(m_{\theta(a)} \cdot \mathcal{O}_a + m_j^{\ell+2}) > \ell \); a contradiction.

Remark 14.4. - We define the multiplicity \( \text{mult}_a \theta \) of \( \theta \) at \( a \) by

\[
\text{mult}_a \theta = \dim_{K_{\theta(a)}} \mathcal{O}_a \otimes_{\mathcal{O}_{\theta(a)}} K_{\theta(a)}.
\]
where \( K_{\theta(a)} \) denotes the field of fractions of \( \mathcal{O}_{\theta(a)} \). Then \( \text{mult}_a \theta = \dim c \mathcal{O}_a / \mathfrak{m}_{\theta(a)} \cdot \mathcal{O}_a \) (by \([31, \text{Ch. 6, Thm. A.10}]\) and \([40, \text{App. 6, Thm. 3}]\)). Let \( d \) denote the number of points in a generic fiber of \( \theta \). Then, for all \( b \in N' \), \( \sum_{a \in \theta^{-1}(b)} \text{mult}_a \theta = d \) (Weil's formula \([31, \text{Ch. 6, (A.8)]})

**Corollary 14.5.** - For all \( a \in M \), \( m_a^{d+1} \subset \mathfrak{m}_{\theta(a)} \cdot \mathcal{O}_a \).

Let \( X \) be an irreducible germ at \( a_0 \) of a closed analytic subset of \( M_{\theta} \). In order to prove Theorem 14.2, it suffices to find (a germ at \( a_0 \) of) a proper closed analytic subset \( Y \) of \( X \), and a function \( \ell = \ell(k) \) from \( N \) to itself, such that, for \( a \in X - Y \) in some neighborhood of \( a_0 \), \( \ell(k, a) \leq \ell(k) \) for all \( k \in N \). (We use the same symbol for a germ at \( a_0 \) and a suitable representative of the germ in some neighborhood.)

Put \( \theta = \pi \circ \phi : M_{\theta} \to N' \). (Clearly, \( M_{\theta}^i \subset M_\theta^i \subset M^i \); \( \theta \) is the restriction to \( M_{\theta}^i \) of the mapping \( M_\theta^i \to N' \) induced by \( \theta \).) Then \( \theta \) is finite.

**Lemma 14.6.** - There exists (a germ at \( a_0 \) of) a proper analytic subset \( Y' \) of \( X \) and, for all \( i = 1, \ldots, s \), a positive integer \( d_i \), such that:

1. \( Y' = X \cap \theta^{-1}(\theta(Y')) \);
2. \( \text{mult}_a \theta = d_i \) for all \( a = (a_1, \ldots, a^i) \in X - Y' \).

**Proof.** - Let \( a \in M \). By Remark 14.4 and Corollary 14.5, \( \text{mult}_a \theta = \dim c \mathcal{O}_a / \mathfrak{m}_a^{d+1} - \dim c \mathfrak{m}_{\theta(a)} \cdot \mathcal{O}_a / \mathfrak{m}_a^{d+1} \). With respect to local coordinates \( x = (x_1, \ldots, x_m) \) in \( M \), the vector space \( \mathfrak{m}_{\theta(a)} \cdot \mathcal{O}_a / \mathfrak{m}_a^{d+1} \) is generated by the equivalence classes modulo \( \mathfrak{m}_a^{d+1} \) of \((x-a)^\alpha \cdot (\theta_j(x) - \theta_j(a)) \), where \( j = 1, \ldots, m \) and \( \alpha \in N^m \), \( |\alpha| \leq d \). Thus \( \dim c \mathfrak{m}_{\theta(a)} \cdot \mathcal{O}_a / \mathfrak{m}_a^{d+1} \) is the rank of a matrix whose entries are analytic functions in \( a \). (Its columns are the partial derivatives through order \( d \) of \((x-a)^\alpha \cdot (\theta_j(x) - \theta_j(a)) \) with respect to \( x \), evaluated at \( x = a \).) Therefore, \( \text{mult}_a \theta \) is (analytic) Zariski (upper-) semicontinuous. The result follows since \( \theta \) is finite.

**Remark 14.7.** - Let \( a_1 = (a_1^1, \ldots, a_1^i) \in M_{\theta}^i \). Suppose that \( \{a_1^1, \ldots, a_1^i\} \) contains \( r \) distinct elements \( c^1, \ldots, c^r \), where \( c^j \) is repeated \( \mu_j \) times, \( j = 1, \ldots, r \), and \( \sum \mu_j = s \). Choose connected open neighborhoods \( U_j \) of \( c^j \) in \( M \), \( j = 1, \ldots, r \), and \( V \) of \( \theta(a_1) \) in \( N' \), such that the \( U_j \) are mutually disjoint and \( \theta(U_j) = V \) for each \( j \). Put \( U = \cup U_j \).
Then:

1. Since $\theta|U$ is finite, $\sum_{a \in U \cap \theta^{-1}(b)} \text{mult}_a \theta$ is constant on $V$.

2. If $a = (a^1, \ldots, a^s)$ is sufficiently close to $a_i$ in $M^i$, then $\{a^1, \ldots, a^s\}$ contains $\mu_i$ elements of $U^i$, for each $j$.

**Corollary 14.8.** Let $Y'$ be as in Lemma 14.6. There exists $r \leq s$ and a surjection $\sigma$ of $\{1, \ldots, s\}$ onto $\{1, \ldots, r\}$ satisfying the following conditions: Let $M^i_\sigma \to M^i$ denote the embedding given by $(a^1, \ldots, a^s) \to (a^{\sigma(1)}, \ldots, a^{\sigma(s)})$. Then:

1. $X \subset M^i$.

2. If $a = (a^1, \ldots, a^s) \in X - Y'$ and $i \neq j$, then $a^i \neq a^j$.

**Proof.** It follows from Lemma 14.6 and Remark 14.7 that, for each $i$ and $j$, $\{a = (a^1, \ldots, a^s) \in X - Y' : a^i = a^j\}$ is open in $X - Y'$. Clearly, it is closed. Since $X - Y'$ is connected, the result follows.

Let $Y'$ be as in Lemma 14.6. According to Corollary 14.8, we can assume, in our proof of Theorem 14.2, that if $a = (a^1, \ldots, a^s) \in X - Y'$ and $i \neq j$, then $a^i \neq a^j$.

For each $a = (a^1, \ldots, a^s) \in X - Y'$, put $\mathcal{F}_a = \bigoplus_{i=1}^s \mathcal{O}_{a^i}$ and $E_a = \bigoplus_{i=1}^s \mathcal{O}_{a^i}/m_{\theta(a^i)} \cdot \mathcal{O}_{a^i}$. Then $\mathcal{F}_a$ is an $\mathcal{O}_{\theta(a)}$-module via the homomorphism $(\theta(a^i))_{1 \leq i \leq s} : \mathcal{O}_{\theta(a)} \to \bigoplus_{i=1}^s \mathcal{O}_{a^i}$, and $E_a$ is a vector space over $\mathbb{C}$. Clearly, $E_a$ identifies with $\mathcal{F}_a/m_{\theta(a)} \cdot \mathcal{F}_a$.

Replacing $M$, if necessary, by a smaller neighborhood of $\{a^1_0, \ldots, a^s_0\}$, we can assume there exist $\eta_1, \ldots, \eta_s \in \mathcal{O}(M)$ and $a_i \in X - Y'$ such that the $\eta_i$ induce a basis of $E_{a_1}$. (We can, for example, choose $\eta_1, \ldots, \eta_s$ to be polynomial with respect to local coordinates in a neighborhood of each $a^i_0$.) By Lemma 14.6, $\dim_{\mathbb{C}} E_a = \sum_{i=1}^s d_i$ is constant on $X - Y'$. Thus there is (a germ at $a_0$ of) a proper analytic subset $Y$ of $X$ such that $Y' \subset Y$ and the $\eta_i$ induce a basis of $E_a$, for all $a \in X - Y$. Since $\theta$ is finite, we can assume that $Y = X \cap \theta^{-1}(\theta(Y))$.

**Lemma 14.9.** For each $a \in X - Y$, $\eta_1, \ldots, \eta_s$ induce a free set of generators of the module $\mathcal{F}_a$ over $\mathcal{O}_{\theta(a)}$. 
Proof. — Let \( a = (a_1, \ldots, a^s) \in X - Y \). By Nakayama’s lemma, \( \eta_1, \ldots, \eta_\sigma \) induce a set of generators of \( \mathcal{F}_a \) over \( \mathcal{O}_{\theta(a)} \). By Remark 14.4, 
\[
\sigma = \dim \mathcal{E}_a = \sum_{i=1}^s \text{mult}_i \theta = \sum_{i=1}^s \dim \mathcal{K}_{\theta(a)} \mathcal{O}_{\theta(a)} \mathcal{E}_{\theta(a)}, \quad \text{where } \mathcal{K}_{\theta(a)} \text{ is the field of fractions of } \mathcal{O}_{\theta(a)}. \]
Thus \( \sigma = \dim \mathcal{K}_{\theta(a)} \mathcal{E}_{\theta(a)} \), as required.

COROLLARY 14.10. — Put \( \ell(k) = (d+1)(k+1) - 1 \), where \( k \in \mathbb{N} \). Let \( a = (a^1, \ldots, a^s) \in X - Y \) and let \( H_j \in \varnothing_{\theta(a)}, \ j = 1, \ldots, \sigma \). If 
\[
\sum_{j=1}^\sigma \theta_d^\ast (H_j) \cdot \hat{\eta}_{j,a} \in \mathfrak{m}_{d^s}^{\ell(k)+1} \cdot \varnothing_{d^s}, \quad i = 1, \ldots, s, \ \text{then each } H_j \in \mathfrak{m}_{d^s}^{\ell(k)+1} \cdot \varnothing_{\theta(a)}. 
\]

Proof. — If \( a \in M \), then, by Corollary 14.5, \( \mathfrak{m}_{d^s}^{\ell(k)+1} \subset \mathfrak{m}_{\theta(a)} \mathcal{O}_a \). Therefore, for all \( a = (a^1, \ldots, a^s) \in M^s \), \( \oplus \mathfrak{m}_{d^s}^{\ell(k)+1} \cdot \varnothing_{d^s} \subset \mathfrak{m}_{\theta(a)} \mathcal{F}_a \), where \( \mathcal{F}_a = \oplus \varnothing_{d^s} \). The result follows from Lemma 14.9.

LEMMA 14.11. — Let \( f \in \mathcal{O}(M) \). Then:

(1) If \( a = (a^1, \ldots, a^s) \in X - Y \), there exist unique \( h_{j,a} \in \mathcal{O}_{\theta(a)}, \ j = 1, \ldots, \sigma \), such that, for each \( i = 1, \ldots, s \), \( f_a = \sum_{j=1}^\sigma \theta_d^\ast (h_{j,a}) \cdot \hat{\eta}_{j,a} \).

(2) For each \( j = 1, \ldots, \sigma \) and \( \beta \in \mathbb{N}^m \), let \( h_\beta^j(a) = D_\beta h_{j,a}(\theta(a)) \), where \( a \in X - Y \). Then \( h_\beta^j \in \mathcal{M}(X;Y) \).

Proof. — (1) By Lemma 14.9.

(2) If \( a \in M \), let \( \Theta_a : \mathcal{O}_{\theta(a)} \to \mathcal{O}_a \) denote the module homomorphism over \( \theta_a^\ast \) defined by \( \Theta_a(g) = \sum_{j=1}^\sigma \theta_a^\ast (g_j) \cdot \hat{\eta}_{j,a} \), where \( g = (g_1, \ldots, g_\sigma) \in \mathcal{O}_{\theta(a)} \). If \( a = (a^1, \ldots, a^s) \in M^s \), let \( \Theta_a : \mathcal{O}_{\theta(a)} \to \bigoplus_{i=1}^s \mathcal{O}_{d^s} \) denote the composition of \( \bigoplus \Theta_{d^s} \) with the diagonal injection \( \mathcal{O}_{\theta(a)} \to \bigoplus_{i=1}^s \mathcal{O}_{\theta(a)} \).

Suppose that \( a \in X - Y \). According to (1), \( (\hat{f}_{d^s})_{1 \leq i \leq s} = \Theta_a(h_a) \), where \( h_a = (h_{1,a}, \ldots, h_{\sigma,a}) \). We use the formalism of 8.2 and 8.3, where \( p = 1 \), \( q = \sigma \), \( B = 0 \), \( \Phi_a \) is replaced by \( \Theta_a \), etc. For each \( \ell \in \mathbb{N} \), let \( \mathcal{F}_a \) (respectively, \( \mathcal{H}_a \)) denote the image of \( (\hat{f}_{d^s})_{1 \leq i \leq s} \) (respectively, of \( h_a \)) by
the lower (respectively, upper) horizontal arrow in the left-hand diagram
of (8.2.6); thus, \( {\mathcal F}_s = A_{,s} \cdot {\mathcal H}_s \). Recall that \( {\mathcal H}_s \) is the element of
\( \oplus_{\beta \in \mathbb{C}^\infty} \mathcal{C}_\alpha^\infty \) induced by \( (D^\beta_{\mathcal{H}_s} \circ \partial_{\mathcal{H}_s})_{\beta \in \mathcal{C}_\alpha^\infty} \). Write \( {\mathcal H}_s = (H_{\beta,i,a})_{\beta \in \mathcal{C}_\alpha^\infty, 1 \leq i \leq \sigma} \), where
each \( H_{\beta,i,a} \in \mathcal{O}_{X,a} \).

Let \( k \in \mathbb{N} \) and let \( \ell = \ell_1(k) \). Then
\[
\text{Ad}^{\mathcal{P},k(x)}D_{\ell,k,a} \cdot {\mathcal F}_s = C_{\ell,k,a} \cdot {\mathcal H}_s.
\]
Let \( e(k) \) denote the number of pairs \((\beta, j) \in \mathbb{N}^m \times \{1, \ldots, \sigma\}\) such that
\( |\beta| \leq k \) (\( e(k) \) is the number of columns of \( C_{\ell,k,a} \)). By Corollary 14.10
and Lemma 8.1.1 (2), \( \text{rank} C_{\ell,k,a} = e(k) \). Then, by Cramer's rule,
for all \((\beta, j) \in \mathbb{N}^m \times \{1, \ldots, \sigma\}\), \( |\beta| \leq k \), we obtain \( \zeta_{\beta,j} \), \( \omega_{\beta,j} \in \mathcal{C}(U) \) (\( U \) is a
product coordinate neighborhood of \( a_0 \) in \( M^0 \)) such that, if \( a \in X - Y \),
then \( \omega_{\beta,j}(a) \neq 0 \) and \( H_{\beta,j,a} = \frac{\zeta_{\beta,j,a}}{\partial_{\beta,j,a}} \), as required. \( \square \)

We can now complete the proof of Theorem 14.2. Since the projection
of \( Z \) onto \( N' \) is finite, then, by the finite coherence theorem of Grauert
and Remmert [32, Ch. IV, Thm. 7], we can assume there exist
\( \xi_1, \ldots, \xi_\rho \in \mathcal{O}(N) \) satisfying the following condition: For all
\( b \in Z \) and \( G \in \mathcal{O}_b \), there exist \( G_1, \ldots, G_\rho \in \mathcal{O}_{\pi(b)} \) such that
\( G - \sum_{h=1}^\rho \hat{\pi}_h^* (G_h) \cdot \xi_{h,b} \in \mathcal{I}_{Z,b} \cdot \mathcal{O}_b \), where \( \mathcal{I}_Z \) denotes the sheaf of germs
of analytic functions which vanish on \( Z \).

Let \( a \in X - Y \), \( a = (a^1, \ldots, a^\sigma) \). By Lemma 14.11 (1), there exist
unique \( p \times q \) matrices \( C_{h,a} \), \( h = 1, \ldots, \rho \), \( j = 1, \ldots, \sigma \), and unique
\( p \times r \) matrices \( D_{\ell,j,a} \), \( \ell = 1, \ldots, \sigma \), all with entries in \( \mathcal{O}_{\theta(a)} \), such that,
for all \( i = 1, \ldots, s \),
\[
(\xi_{h,\theta(a)} \circ \partial_{\theta(a)}) \cdot A_{d} = \sum_{j=1}^\sigma \hat{\eta}_{j,d}^* \cdot (C_{h,a} \circ \partial_{\theta(a)}),
\]
\[
\hat{\eta}_{\ell,d}^* \cdot B_{d} = \sum_{j=1}^\sigma \hat{\eta}_{j,d}^* \cdot (D_{\ell,j,a} \circ \partial_{\theta(a)}).
\]
By Lemmas 14.11 (2) and 7.2 (3) and Remark 7.6, there exists \( \lambda \in \mathbb{N} \)
satisfying the following condition: Let \( a \in X - Y \). Suppose that
\( G_h \in \mathcal{O}_{\theta(a)}^\mathcal{P} \), \( h = 1, \ldots, \rho \), \( H_{\ell} \in \mathcal{O}_{\theta(a)}^\mathcal{P} \), \( \ell = 1, \ldots, \sigma \), and \( \sum_{h=1}^\rho C_{h,a} \cdot G_h + \sum_{\ell=1}^\sigma D_{\ell,j,a} \cdot H_{\ell} \in m_{\theta(a)}^{k+\gamma} \cdot \partial_{\theta(a)}^\mathcal{P} \), \( j = 1, \ldots, \sigma \). Then there exist \( G_h' \in \mathcal{O}_{\theta(a)}^\mathcal{P} \) and
such that $\sum_{h}^{\sigma} C_{h_j, a} \cdot G_{h} + \sum_{r}^{\eta} D_{r_j, a} \cdot H_{r} = 0$, $j = 1, \ldots, \sigma$, and

$$G_{h} - G_{h} \in m_{\theta(a)}^{k} \cdot \hat{\theta}_{\theta(a)}^{q}, \quad H_{r} - H_{r} \in m_{\theta(a)}^{k} \cdot \hat{\theta}_{\theta(a)}^{q}.$$ 

Let $\ell(k) = \ell_{1}(k + \lambda)$, $k \in \mathbb{N}$. We claim that $\ell(k, a) \leq \ell(k)$ for all $a \in X - Y$ and $k \in \mathbb{N}$: Let $a \in X - Y$ and let $G \in \hat{\theta}_{\theta(a)}^{q}$. Suppose that $A_{d_{r}} \cdot (G \circ \hat{\theta}_{d_{r}}) + B_{d_{r}} \cdot H_{r} \in m_{\theta(a)}^{k} \cdot \hat{\theta}_{\theta(a)}^{q}$, where $H_{r} \in \hat{\theta}_{\theta(a)}^{q}$, $i = 1, \ldots, s$. There exist $G_{1}, \ldots, G_{p} \in \hat{\theta}_{\theta(a)}^{q}$ such that $G - \sum_{h}^{\sigma} \hat{\theta}_{\theta(a)}^{q}(G_{h}) \in \mathcal{I}_{\phi(a)} \cdot \hat{\theta}_{\theta(a)}^{q}$.

Also, there exist unique $H_{1}, \ldots, H_{\sigma} \in \hat{\theta}_{\theta(a)}^{q}$ such that $H_{1} = \sum_{h}^{\sigma} \hat{\theta}_{\theta(a)}^{q}(H_{r} \circ \hat{\theta}_{d_{r}})$, $i = 1, \ldots, s$. Thus, for each $i = 1, \ldots, s$,

$$A_{d_{r}} \cdot (G \circ \hat{\theta}_{d_{r}}) + B_{d_{r}} \cdot H_{r} = \sum_{j=1}^{\sigma} \hat{\theta}_{\theta(a)}^{q} \left( \left( \sum_{h=1}^{\sigma} C_{h_j, a} \cdot G_{h} + \sum_{r=1}^{\eta} D_{r_j, a} \cdot H_{r} \right) \circ \hat{\theta}_{d_{r}} \right).$$

By Corollary 14.10, $\sum_{h=1}^{\sigma} C_{h_j, a} \cdot G_{h} + \sum_{r=1}^{\eta} D_{r_j, a} \cdot H_{r} \in m_{\phi(a)}^{k} \cdot \hat{\theta}_{\theta(a)}^{q}$, $j = 1, \ldots, \sigma$. Thus there exist $G_{1}, \ldots, G_{p} \in \hat{\theta}_{\theta(a)}^{q}$ and $H_{1}, \ldots, H_{\sigma} \in \hat{\theta}_{\theta(a)}^{q}$ such that $\sum_{h=1}^{\sigma} C_{h_j, a} \cdot G_{h} + \sum_{r=1}^{\eta} D_{r_j, a} \cdot H_{r} = 0$, $j = 1, \ldots, \sigma$, and each $G_{h} - G_{h} \in m_{\phi(a)}^{k+1} \cdot \hat{\theta}_{\theta(a)}^{q}$. Put $G' = \sum_{h=1}^{\sigma} \hat{\theta}_{\theta(a)}^{q}(G_{h}) \cdot \hat{\theta}_{\theta(a)}^{q}$. Then $A_{d_{r}} \cdot (G \circ \hat{\theta}_{d_{r}})$

$\in \text{Im} \hat{\theta}_{d_{r}}$, $i = 1, \ldots, s$, and $G - G' \in m_{\phi(a)}^{k+1} \cdot \hat{\theta}_{\theta(a)}^{q}$, as claimed. This completes the proof of Theorem 14.2. □

Remark 14.12. - (1) Let $a = (a_{1}, \ldots, a_{t}) \in X - Y$. Let $G \in \hat{\theta}_{\phi(a)}^{q}$ and let $H \in \bigoplus_{i=1}^{s} \hat{\theta}_{d_{i}}^{q}$, $H = (H_{1}, \ldots, H_{t})$. Let $f = \Phi_{a}(G) + \hat{\Phi}_{a}(H) \in \bigoplus_{i=1}^{s} \hat{\theta}_{d_{i}}^{q}$; i.e., $f = (f_{1}, \ldots, f_{s})$, where each $f_{i} = A_{d_{i}} \cdot (G \circ \hat{\theta}_{d_{i}}) + B_{d_{i}} \cdot H_{i}$. Suppose that $f_{i} \in \hat{\theta}_{d_{i}}^{q}$, $i = 1, \ldots, s$. Then, for all $k \in \mathbb{N}$, there exists $g \in \mathcal{O}_{\phi(a)}^{q}$ and $h \in \bigoplus_{i=1}^{s} \mathcal{O}_{d_{i}}^{q}$ such that $f = \Phi_{a}(g) + \hat{\Phi}_{a}(h)$, $g \in m_{\phi(a)}^{k+1} \cdot \hat{\theta}_{\phi(a)}^{q}$, and $h - H \in \bigoplus_{i=1}^{s} m_{d_{i}}^{k} \cdot \hat{\theta}_{d_{i}}^{q}$. We use the notation introduced above. Let $G_{1}, \ldots, G_{p} \in \hat{\theta}_{\theta(a)}^{q}$ such that $G - \sum_{h=1}^{\sigma} \hat{\theta}_{\theta(a)}^{q}(G_{h}) \in \mathcal{I}_{\phi(a)} \cdot \hat{\theta}_{\theta(a)}^{q}$, and let $H_{1}, \ldots, H_{\sigma} \in \hat{\theta}_{\theta(a)}^{q}$ such that $H' = \sum_{i=1}^{\sigma} \hat{\theta}_{\theta(a)}^{q}(H_{r} \circ \hat{\theta}_{d_{r}})$, $i = 1, \ldots, s$. By
Lemma 14.9, \( \sum_{h} c_{h_i} \cdot G_{h_i} + \sum_{\ell} d_{\ell_i} \cdot H_{\ell_i} \in \mathcal{O}_0^{r(a)} \), \( j = 1, \ldots, \sigma \). By Krull's theorem, there exist \( g_1, \ldots, g_p \in \mathcal{O}_0^{q(a)} \) and \( h_1, \ldots, h_s \in \mathcal{O}_0^{r(a)} \) such that
\[
\sum_{h} c_{h_i} \cdot G_{h_i} + \sum_{\ell} d_{\ell_i} \cdot H_{\ell_i} = \sum_{h} c_{h_i} \cdot G_{h_i} + \sum_{\ell} d_{\ell_i} \cdot H_{\ell_i},
\]
j = 1, \ldots, \sigma, and each \( g_h - G_h \in m_{0(a)}^{k}, \partial_{0(a)}, h_r - H_r \in m_{0(a)}^{k}, \partial_{0(a)} \). Put
\[
g = \sum_{h} z_{0,0} (g_h \circ \hat{\theta}_{0(a)}), \quad h_i = \sum_{\ell} \hat{h}_{\ell_i,a_i} (h_r \circ \hat{\theta}_{a_i}), \quad i = 1, \ldots, s,
\]
and \( h = (h^1, \ldots, h^s) \).

(2) Let \( a = (a^1, \ldots, a^s) \in \mathcal{X} - \mathcal{Y} \). Then \( \mathcal{R}_a = \{ G \in \partial_{\Phi(a)}^q : \Phi_\mathcal{a}(G) \in \mathcal{I}_a \} \) is generated by \( \mathcal{R}_a \cap \mathcal{O}_{\Phi(a)}^q \) (cf. Corollary 12.17).

**Remark 14.13.** Let \( \mathcal{X} \) be an analytic space over \( K \). It follows from theorems of Buchsbaum and Eisenbud [9, Thms. 1.2, 2.1] and [37, I.5.1] that \( \{ x \in \mathcal{X} : \mathcal{O}_{X,x} \text{ is Cohen-Macauley} \} \) is open in \( \mathcal{X} \). (We are grateful to David Eisenbud for the reference.) We say that \( \mathcal{X} \) is Cohen-Macauley if, for all \( x \in \mathcal{X} \), \( \mathcal{O}_{X,x} \) is a Cohen-Macauley ring. Thus, a Cohen-Macauley real analytic space admits a Cohen-Macauley complexification.

Our proof of Theorem 14.2 extends to the case that \( M \) is a Cohen-Macauley analytic space with essentially no change: We can assume that \( K = \mathbb{C} \). The equalities of Remark 14.4 remain valid. In Lemma 14.11, we can assume that \( M \) is embedded in an open subspace \( \mathcal{W} \) of \( C^r \), and that \( \mathcal{O}_M = \mathcal{O}_W/L \cdot \mathcal{O}_W' \), where \( L \) is a \( 1 \times r \) matrix with entries in \( \mathcal{O}(W) \); the same proof goes through using the formalism of 8.2, 8.3 with \( B = L \) rather than \( B = 0 \).


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E. Bierstone and P. D. Milman,
University of Toronto
Dept. of Mathematics
Toronto, Canada M5S 1A1.