ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 37, nº 2 (1987), p. 33-48
<http://www.numdam.org/item?id=AIF_1987__37_2_33_0>

Ann. Inst. Fourier, Grenoble
37, 2 (1987), 33-48.

DEFORMATIONS OF COHERENT FOLIATIONS ON A COMPACT NORMAL SPACE

by Geneviève POURCIN

Introduction.

Let X be a normal reduced compact analytic space with countable topology. Let Ω^1_X be the coherent sheaf of holomorphic 1-forms on X and $\Theta_X = \mathrm{Hom}_{O_X}(\Omega^1_X, O_X)$ its dual sheaf. The bracket of holomorphic vector fields on the smooth part of X induces a C-bilinear morphism $m : \Theta_X \times \Theta_X \to \Theta_X$ (section 1); therefore, for any open subset U of X, m defines a map $m_U : \Theta_X(U) \times \Theta_X(U) \to \Theta_X(U)$ which is continuous for the usual topology on $\Theta_X(U)$.

We shall study coherent foliations on X (section 1 definition 2), using the definition given in [2], this notion generalizes the notion of analytic foliations on manifolds introduced by P. Baum ([1]) (see also [8]). A coherent foliation on X defines a quotient O_X-module of Θ_X by a m-stable submodule (condition (i) of definition 2), this quotient being a non zero locally free O_X-module outside a rare analytic subset of X (condition (ii) of definition (ii)).

Then the set of the coherent foliations on X is a subset of the universal space H of all the quotient O_X-modules of Θ_X; the analytic structure of H has been constructed by A. Douady in [4].

The aim of this paper is to prove that the set of the quotient O_X-modules of Θ_X which satisfy conditions (i) and (ii) of definition 2 is an analytic subspace $\mathscr{H}$ of an open set of H and that $\mathscr{H}$ satisfies a universal property (Theorem 2). Any coherent foliation gives a point of $\mathscr{H}$, any point of $\mathscr{H}$ defines a coherent foliation but two different points of $\mathscr{H}$ can define the same foliation (cf. section 1, remark 3).

Key-words: Singular holomorphic foliations - Deformations.

In section 2 one proves that, in the local situation, m-stability is an analytic condition on a suitable Banach analytic space (of infinite dimension).

In section 3 we follow the construction of the universal space of A. Douady and we get the analytic structure of $\mathscr{H}$.

Notations :

— For any analytic space Y and any analytic space not necessarly of finite dimension Z let us denote $p_Z : Z \times Y \to Y$ the projection.

— For any $O_{Z\times Y}$-module $\mathscr{F}$ and any $z \in Z$ let us denote $\mathscr{F}(z)$ the O_Y-module which is the restriction to $\{z\} \times Y$ of $\mathscr{F}$, by definition we have for any $y \in Y$

$$\mathscr{F}(z)_y = \mathscr{F}_{(z,y)} \otimes_{O_{Z,z}} O_{Z,z}/m_z .$$

1. Coherent foliations.

Let X be a reduced connected normal analytic space with countable topology; let Ω^1_X be the coherent sheaf of holomorphic differential 1-forms on X and

$$(*) \qquad \Theta_X = \mathrm{Hom}_{O_X}(\Omega^1_X, O_X)$$

Θ_X is called the tangent sheaf on X. Let S be the singular locus of X, then S is at least of codimension two and the restriction of Θ_X to $X - S$ is the sheaf of holomorphic vector fields on the manifold $X - S$.

Bracket of two sections of Θ_X.

The bracket of two holomorphic vector fields on the manifold $X - S$ is well-defined; recall that, if $z = (z_1, \ldots, z_p)$ denotes the coordinates on $\mathbf{C}^p$, if U is an open set in $\mathbf{C}^p$ and if a and b are two holomorphic vector fields on U, with

$$a = \sum_{i=1}^{p} a_i(z) \frac{\partial}{\partial z_i}, \qquad b = \sum_{i=1}^{p} b_i(z) \frac{\partial}{\partial z_i}$$

then we have $[a,b] = c$ with

$$c = \sum_{i=1}^{p} c_i \frac{\partial}{\partial z_i} \text{ where } c_i = \sum_{j=1}^{p} \left(a_j \frac{\partial b_i}{\partial z_j} - b_j \frac{\partial a_i}{\partial z_j} \right).$$

Let $m_U : O(U)^p \times O(U)^p \to O(U)^p$ be the **C**-bilinear map which sends $((a_1,\ldots,a_p), (b_1,\ldots,b_p))$ onto $(c_1,\ldots,c_p)$; the Cauchy majorations imply the continuity of m_u for the Frechet topology of uniform convergence on compacts of U.

PROPOSITION 1. — *For every open subset* U *of* X *the restriction homomorphism*

$$\rho : H^0(U,\Theta_X) \to H^0(U-U_\cap S,\Theta_X)$$

is an isomorphism of Frechet spaces.

Proof. — One knows that p is continuous; by the open mapping theorem it is sufficient to prove that p is bijective.

Now we may suppose that X is an analytic subspace of an open set V in $\mathbf{C}^n$; let I be the coherent ideal sheaf defining X in V; one has an exact sequence

$$(1) \qquad O \to \Theta_X \to O_X^n \xrightarrow{\alpha} \mathrm{Hom}_{O_U}(I/I^2,O_X)$$

where the map α is defined by

$$\alpha(a_1,\ldots,a_n)(f) = \sum_{i=1}^{n} a_i \left.\frac{\partial f}{\partial z_i}\right|_X$$

$z_1, \ldots, z_n$ being the coordinates in $\mathbf{C}^n$.

Because the complex space X is reduced and normal it follows from the second removable singularities theorem two isomorphisms

$$(2) \qquad \begin{array}{l} O_X(V) \approx O_X(V-S) \\ I(V) \approx I(V-S). \end{array}$$

Then the proposition 1 follows from (1) and (2). As an immediate consequence of proposition 1 we obtain the following corollary:

COROLLARY AND DEFINITION. — *It exists a unique homomorphism of sheaves of* **C**-*vector spaces*

$$m : \Theta_X \times \Theta_X \to \Theta_X$$

extending the bracket defined on X — S. *Therefore, for every open subset* U *of* X, *the induced map*

$$m_U : H^0(U,\Theta_X) \times H^0(U,\Theta_X) \to H^0(U,\Theta_X)$$

is **C**-*bilinear and continuous for the Frechet topology on* $H^0(U,\Theta_X)$. *We call bracket-map the sheaf morphism* $m : \Theta_X \times \Theta_X \to \Theta_X$.

Coherent foliations.

DEFINITION 1. — *A coherent* O_X-*submodule* T *of* Θ_X *is said to be maximal if for any open* $U \subset X$, *any section* $s \in \Theta_X(U)$ *and any nowhere dense analytic set* A *in* U

$$s \in T(U-A) \Rightarrow s \in T(U)$$

holds.

Because X is reduced and normal, then locally irreducible, T is maximal if and only if Θ_X/T has no O_X-torsion.

DEFINITION 2 [2]. — *A coherent foliation on* X *is a coherent* O_X-*submodule* T *of* Θ_X *such that:*

(i) Θ_X/T *is non zero locally free outside a nowhere dense analytic subset of* X;

(ii) T *is a subsheaf of* Θ_X *stable by the bracket-map;*

(iii) T *is maximal.*

Remarks. — 1) A coherent foliation induces a classical smooth holomorphic foliation outside a nowhere dense analytic subset of X − S.

2) If T is maximal the stability of T by the bracket-map on X is equivalent to the stability of T on X − A, for any rare analytic subset A.

3) A coherent foliation on a connected reduced normal complex space X is characterized by a quotient module F of Θ_X, without O_X-torsion, such that ker $[\Theta_X \to F]$ is stable by the bracket-map and which is a non zero locally free O_X-module outside a rare analytic subset of X.

4) Let T be a coherent O_X-submodule of Θ_X satisfying conditions (i) and (ii) of definition 2; then T is included in a maximal coherent sheaf $\hat{T}$ which is equal to T outside a rare analytic subset of X ([7] 2.7); the conditions (i) and (ii) are also fullfilled for $\hat{T}$, hence one can associate to T a maximal foliation on X. But two different T for which (i) and (ii) hold may give the same maximal sheaf $\hat{T}$.

We suppose X compact.

The purpose of this paper is to put an analytic structure on the set of all subsheaves of Θ_X satisfying conditions (i) and (ii) of Definition 2 (Theorem 2 below), that gives a versal family of holomorphic singular foliations for which a coherent extension exists.

First we have the following proposition :

PROPOSITION 2. — *Let* X *be an irreducible complex space; let* Z *be a complex space and* F *a coherent* $O_{Z\times X}$*-module. Let* F *be* Z*-flat.*

Let Z_1 *be the set of points* $z \in Z$ *such that* $F(z)$ *is a non-zero locally free* O_X*-module outside a rare analytic subset of* X.

Then Z_1 *is an open subset of* Z.

Proof. — For every $z \in Z$ let σ_z be the analytic subset of points $x \in X$ where $F(z)$ is not locally free ([3]). Put $z_0 \in Z_1$. The irreducibility of X implies that G_{z_0} is nowhere dense; fix $x_0 \in X - S \cap \sigma_{z_0}$ and denote $r > 0$ the rank of the O_{X,x_0}-module $F(z_0)$. The Z-flatness of F implies that F is $O_{Z\times X}$-free of rank r in an open neighborhood V of (z_0, x_0). Let U be the projection of V on Z. For any point z of the open set U the Z-flatness of F implies that $F(z)_{x_0}$ is O_{X,x_0}-free of rank r; then the support of the sheaf $F(z)$ contains a neighborhood of x_0; hence the irreducibility of X implies

$$\text{support } F(z) = X$$

and the proposition.

For any analytic space S $m_S : p_S^*\Theta_X \times p_S^*\Theta_X \to p_S^*\Theta_X$ denotes the pull back of m by the projection $p_S : S \times X \to X$ (i.e. the bracket map in the direction of the fibers of the projection $S \times X \to S$). Our aim is the proof of the following theorem :

THEOREM 1. — *Let* X *be a compact connected normal space. There exist an analytic space* $\tilde{H}$ *and a coherent* $O_{\tilde{H}\times X}$*-submodule* $\tilde{T}$ *of* $p_{\tilde{H}}^*\Theta_X$ *such that :*

(i) $p_{\tilde{H}}^*\Theta_X/\tilde{T}$ *is* $\tilde{H}$*-flat;*

(iii) $\tilde{T}$ *is a* $m_{\tilde{H}}$*-stable submodule of* $p_{\tilde{H}}^*\Theta_X$;

(iii) $(\tilde{H},\tilde{T})$ *is universal for properties* (i) *and* (ii).

As a corollary of proposition 2 and theorem 1 we obtain :

THEOREM 2. — *Let* X *be a compact connected normal space and* r *a positive integer. There exist an analytic space* $\mathscr{H}$ *and a coherent* $O_{\mathscr{H}\times X}$-*submodule* $\mathscr{T}$ *of* $p_{\mathscr{H}}^{*}\Theta_X$ *such that:*

(i) $p_{\mathscr{H}}^{*}\Theta_X/\mathscr{T}$ *is* $\mathscr{H}$-*flat;*

(ii) $\mathscr{T}$ *is* $m_{\mathscr{H}}$-*stable and for any* $h \in \mathscr{H}\Theta_X/\mathscr{T}(h)$ *is a locally free* O_X-*module of rank* r *outside a rare analytic subset of* X;

(iii) $(\mathscr{H},\mathscr{T})$ *is universal, i.e. for any analytic space* S *and any coherent* $O_{S\times X}$-*submodule* $\mathscr{F}$ *of* $p_S^{*}\Theta_X$ *such that*

— $p_S^{*}\Theta_X/\mathscr{F}$ *is* S-*flat;*

— $\mathscr{F}$ *is* m_S-*stable and for any* $s \in S$ $\Theta_X/\mathscr{F}(s)$ *is a locally free* O_X-*module of rank* r *outside a rare analytic subset of* X *then it exists a unique morphism* $f: S \to \mathscr{H}$ *satisfying*

$$(f\times I_X)^{*}(p_{\mathscr{H}}^{*}\Theta_X/\mathscr{T}) = p_S^{*}\Theta_X/\mathscr{F}.$$

We shall use the following theorem and Douady ([4]):

THEOREM. — *Let* X *be a compact analytic space and* $\mathscr{E}$ *a coherent* O_X-*module; there exist an analytic space* H *and a quotient* $O_{H\times X}$-*module* $\mathscr{R}$ *of* $p_H^{*}\mathscr{E}$ *such that:*

(i) $\mathscr{R}$ *is* H-*flat;*

(ii) *for any analytic space* S *and any quotient* $O_{S\times H}$-*module* $\mathscr{F}$ *of* $p_S^{*}\mathscr{E}$ *which is* S-*flat, it exists a unique morphism* $f: S \to H$ *satisfying*

$$(f\times I_X)^{*}\mathscr{R} = \mathscr{F}.$$

2. Local deformations.

One uses notations and results of [4]; the notions of infinite dimensional analytic spaces, called Banach analytic spaces, and of anaflatness are defined respectively in ([4] § 3) and in ([4] § 8).

In this section we fix an open subset U of $\mathbf{C}^n$, two compact polycylinders of non-empty interior K and K′ satisfying

$$K' \subset \mathring{K} \subset K \subset U$$

and a reduced normal analytic subspace X of U. Let B(K) be the Banach algebra of those continuous functions on K which are analytic on the interior $\mathring{K}$ of K; one defines B(K′) in an analogous way.

For every coherent sheaf $\mathscr{F}$ on U, one knows that it exists finite free resolutions of $\mathscr{F}$ in a neighborhood of K; for such a resolution

$$(L.) \quad O \to L_n \to L_{n-1} \to \cdots \to L_0$$

let us consider the complex of Banach spaces

$$B(K,L.) = B(K) \otimes_{O(K)} H^0(K,L.)$$

and the vector space

$$B(K,\mathscr{F}) = \operatorname{coker}\,[B(K;L_1) \to B(K,L_0)]\,.$$

DEFINITION 1 ([4] §7, [5]). — *K is $\mathscr{F}$-privileged if and only if it exists a finite free resolution* L. *of $\mathscr{F}$ on a neighborhood of* K *such that the complex* B(K,L.) *is direct exact.*

Then this is true for every finite free resolution; therefore $B(K,\mathscr{F})$ is a Banach space which does not depend of the resolution; $\mathscr{F}$-privileged polycylinders give fundamental systems of neighborhoods at every point of U. For a more geometric definition of privilege, the reader can refer to ([6]).

In the following, we always suppose that the two polycylinders K and K′ are Θ_X-privileged, Θ_X being the tangent sheaf defined by 1 − (*).

Let G_K be the Banach analytic space of those B(K)-submodules Y of $B(K,\Theta_X)$ (or equivalently of quotient modules) for which it exists an exact sequence of B(K)-modules

$$O \to B(K)^{r_n} \to \cdots \to B(K)^{r_0} \to B(K,\Theta_X) \to B(K,\Theta_X)/Y \to O$$

which is a direct sequence of Banach vector spaces.

A universal sheaf R_X on $G_X \times \mathring{K}$ is constructed in [4]; R_K satisfies the following proposition:

PROPOSITION 1 ([4] § 8 n° 5). — (i) R_K *is* G_K*-anaflat.*

(ii) *For every Banach analytic space Z and for every Z-anaflat quotient $\mathscr{F}$ of $p_Z^*\Theta_X$ it exists a natural morphism $\varphi : Z \to G_K$ such that*

$$(\varphi \times I_{\mathring{K}})^* R_K = \mathscr{F}_{S\times\mathring{K}}\,.$$

Recall that the Z-anaflatness generalizes to the infinite dimensional space Z the notion of flatness; pull back preserves anaflatness.

Let $G_{K,K'}$ be the set of the B(K)-submodules E of $B(K,\Theta_X)$, element of G_K, such that $E \otimes_{B(K)} B(K')$ gives an element of $G_{K'}$.

PROPOSITION 2. — (i) $G_{K,K'}$ *is an open subset of* G_K.

(ii) *Let* $\mathscr{R}$ *be the pull back of* R_K *by the inclusion* $G_{K,K'} \hookrightarrow G_K$. *Then the map from* $G_{K,K'}$ *to* $G_{K'}$ *which maps every* B(K)*-module* E *element of* $G_{K,K'}$ *onto the* B(K')*-module* $E \otimes_{B(K)} B(K')$ *is given by a unique morphism*

$$\rho_{K,K'} : G_{K,K'} \to G_{K'}$$

satisfying

$$\rho^*_{K,K'} R_{K'} = \mathscr{R}.$$

Proof. — Proposition 2 follows from ([4] 14 prop. 4).

Let $\rho_1 : B(K,\Theta_X) \times B(K,\Theta_X) \to \Theta_X(\mathring{K}) \times \Theta_X(\mathring{K})$ and

$$\rho_2 : \Theta_X(\mathring{K}) \to B(K',\Theta_X)$$

be the restriction homomorphisms and

$$m : \Theta_X(\mathring{K}) \times \Theta_X(\mathring{K}) \to \Theta_X(\mathring{K})$$

the bracket map.

Let

$$m_{K,K'} : B(K,\Theta_X) \times B(K,\Theta_X) \to B(K',\Theta_X)$$

be the continuous C-bilinear map defined by

$$m_{K,K'} = \rho_2 \circ m \circ \rho_1 .$$

DEFINITION 2. — *A* B(K)*-submodule* Y *of* $B(K,\Theta_X)$ *is said to be* $m_{K,K'}$*-stable if it verifies :*

(i) Y *is an element of* $G_{K,K'}$,

(ii) *for every f and g in* Y *one has*

$$m_{K,K'}(f,g) \in \rho_{K,K'}(Y).$$

Then, if $\mathscr{G}$ is a *m*-stable O_X-submodule of Θ_X such that K and K' are $\mathscr{G}$-privileged, $B(K,\mathscr{G})$ is $m_{K,K'}$-stable ; the converse is not necessarly true ; however we have the following proposition :

PROPOSITION 3. — *Let* Y *be a* $m_{K,K'}$*-stable* B(K)*-submodule of* $B(K,\Theta_X)$; *then* Y *defines in a natural way a coherent* O_X*-submodule of* Θ_X *on* $\mathring{K}$, *the restriction to* $\mathring{K}'$ *of which is m-stable (i.e. stable by the bracket-map).*

Proof. — Let B_Y be the privileged B_K-module given by Y ([6]); the restriction to $\mathring{K}$ of B_Y is a coherent sheaf; therefore one has ([6] th. 2.3 (ii) and prop. 2.11)

$$Y = \mathring{H}(K,B_Y)$$

and the restriction homomorphism

$$i : Y = H^0(K,B_Y) \to H^0(\mathring{K},B_Y)$$

is injective and has dense image; therefore the restriction $B_{Y|\mathring{K}}$ is a submodule of Θ_X ([4] § 8 lemme 1(*b*)), hence $H^0(\mathring{K}',B_Y)$ is a closed subspace of the Frechet space $H^0(\mathring{K}',\Theta_X)$.

Let us show that $m_{K,K'}$ induces a C-bilinear continuous map

$$\mathring{m} : H^0(\mathring{K},B_Y) \times H^0(\mathring{K},B_Y) \to H^0(\mathring{K}',B_Y).$$

Take t_1, t_2 two elements of $H^0(\mathring{K},B_Y)$ and (t_1^n) and (t_2^n) two sequences of elements of Y with

$$\lim_{n\to\infty} t_i^n = t_i, \qquad i = 1,2.$$

Because the bracket-map $m : H^0(\mathring{K},\Theta_X) \times H^0(\mathring{K},\Theta_X) \to H^0(\mathring{K},\Theta_X)$ is continuous one has

$$\lim_{n\to\infty} m(t^n_{1|\mathring{K}}, t^n_{2|\mathring{K}}) = m(t_1,t_2) \in H^0(\mathring{K},\Theta_X).$$

Therefore the $m_{K,K'}$-stability of Y implies for every m

$$m_{K,K'}(t_1^n,t_2^n) \in B(K',B_Y) \subset H^0(\mathring{K}',B_Y)$$

then $m(t_1,t_2)|_{\mathring{K}'} \in H^0(\mathring{K}',B_Y)$ follows.

In order to prove the proposition it is sufficient to remark that, for every polycylinder $K'' \subset \mathring{K}'$, the restriction homomorphism

$$H^0(\mathring{K}',B_Y) \to H^0(\mathring{K}'',B_Y)$$

has a dense image. Q.E.D.

Recall some properties of infinite dimensional spaces : let V be an open subset of a Banach **C**-vector space; let F be a Banach vector space and $f: V \to F$ an analytic map. Let $\mathcal{X}$ the Banach analytic space defined by the equation $f = 0$; $\mathcal{X}$ is a local model of general Banach analytic space; the morphisms from $\mathcal{X}$ into a Banach vector space G extend locally in analytic maps on open subsets of V; for such a morphism $\varphi : \mathcal{X} \to G$ the equation $\varphi = 0$ defines in a natural way a Banach analytic subspace of $\mathcal{X}$; the morphisms from a Banach analytic space $\mathcal{Y}$ into $\mathcal{X}$ are exactly the morphisms $\psi : \mathcal{Y} \to V$ such that $f \circ \psi = 0$.

PROPOSITION 4. — *Let* $S_{K,K'}$ *be the subset of elements of* $G_{K,K'}$ *which are* $m_{K,K'}$*-stable. Then* $S_{K,K'}$ *is a Banach analytic subspace of* $G_{K,K'}$.

Proof. — Let $Y_0 \in S_{K,K'}$ and $Y'_0 = \rho_{K,K'}(Y_0)$; let G_0 (resp. G'_0) a closed **C**-vector subspace of $B(K,\Theta_X)$ (resp. $B(K',\Theta_X)$) which is a topological supplementary of Y_0 (resp. Y'_0). Let U_0 (resp. U'_0) the set of closed **C**-vector subspaces of $B(K,\Theta_X)$ (resp. $B(K',\Theta_X)$) which are topological supplementaries of G_0 (resp. G'_0); we identify U_0 and $L(Y_0,G_0)$, hence $U_0 \cap G_K$ is a Banach analytic subspace of U_0([4] § 4).

For every Y in U_0 one denotes $p_Y : B(K,\Theta_Y) = Y \oplus G_0 \to G_0$ the projection and $j_Y : Y_0 \to Y \subset B(K,\Theta_X)$ the reciprocal map of the restriction to Y of the projection $B(K,\Theta_X) = Y_0 \oplus G_0 \to Y_0$.

Then the two maps

$$\begin{aligned} p^K &: G_K \to L(B(K,\Theta_X),G_0) \\ j^K &: G_K \to L(Y_0,B(K,\Theta_X)) \end{aligned}$$

defined by $p^K(Y) = p_Y$ and $j^K(Y) = j_Y$ are induced by morphisms ([4] § 4, n° 1); associated to the polycylinder K′ we have in the same way morphisms $p^{K'}$ and $j^{K'}$. Put $W_0 = G_{K,K'} \cap U_0 \cap \rho_{K,K'}^{-1}(U'_0)$; W_0 is an open subset of $G_{K,K'}$. Let be

$$\varphi_1 = p^{K'} \circ \rho_{K,K'} : W_0 \to L(B(K',\Theta_X), G'_0)$$

and $\Delta : G_K \to L(Y_0 \underset{\pi}{\otimes} Y_0, B(K',\Theta_X))$ the morphism defined by

$$\Delta(Y) = m_{K,K'} \circ (j_Y \times j_Y).$$

Let be $\varphi_2 = \Delta \circ j^K : W_0 \to L(Y_0 \underset{\pi}{\otimes} Y_0, B(K',\Theta_X))$; φ_1 and φ_2 are morphisms; let

$$\varphi : W_0 \to L(Y_0 \underset{\pi}{\otimes} Y_0, G'_0)$$

be the morphism defined by

$$\varphi(Y) = \varphi_2(Y) \circ \varphi_1(Y).$$

We have $W_0 \cap S_{K,K'} = \varphi^{-1}(0)$, hence $S_{K,K'} \cap W_0$ is a Banach analytic subspace of W_0; following ([4] § 4, n° 1 (i) and (ii)) one easily proves that the analytic structures obtained in the different charts of G_K and $G_{K'}$ patch together in an analytic structure on $S_{K,K'}$; that proves proposition 4.

Remark 1. — With the previous notations the morphisms of Banach analytic spaces $g : Z \to S_{K,K'} \cap W_0$ are the morphisms $g : Z \to W_0$ satisfying $\varphi \circ g = 0$.

Let $\iota : S_{K,K'} \to G_K$ be the inclusion and $R_{K,K'}$ the pullback of R_K by ι; $R_{K,K'}$ is $S_{K,K'}$-anaflat; by construction $R_{K,K'}$ is a quotient of $p^*_{S_{K,K'}}\Theta_X$, then put

$$R_{K,K'} = p^*_{S_{K,K'}}\Theta_X/T_{K,K'}.$$

By anaflatness one obtains for every $s \in S_{K,K'}$ an exact sequence of coherent sheaves on $\mathring{K}$:

$$O \to T_{K,K'}(s) \to \Theta_X \to R_{K,K'}(s) \to 0.$$

From the definition of the analytic structure of $S_{K,K'}$ and from proposition 3 one deduces the following theorem:

THEOREM 3. — (i) *For every $s \in S_{K,K'}$ the restriction to $\mathring{K}'$ of the coherent subsheaf* $T_{K,K'}(s)$ of Θ_X *is stable by the bracket-map.*

(ii) *For every Banach analytic space* Z *and every quotient* $\mathscr{F} = p_Z^*\Theta_X/T$ *of* $p_Z^*\Theta_X$ *by a* $O_{Z\times X}$*-submodule* T *such that*

— $\mathscr{F}$ *is* Z*-anaflat.*

— T *is* m_Z*-stable and for any* $z \in Z$ *the polycylinders* K *et* K' *are* $\mathscr{F}(z)$*-privileged;*

then the unique morphism $g : Z \to G_K$ *satisfying*

$$(g \times I_{\mathring{K}})^* R_K = \mathscr{F}$$

factorizes through $S_{K,K'}$ *(i.e. it exists a unique morphism* $f : Z \to S_{K,K'}$ *with* $r \circ f = g$).

Remark 2. — We don't know if the restriction of $R_{K,K'}$ to $S_{K,K'} \times \mathring{K}'$ is $m_{S_{K,K'}}$-stable; but if S is a finite dimensional analytic space then the pull back of $R_{K,K'}$ by any morphism $S \to S_{K,K'}$ is m_S-stable.

3. Proof of theorem 1.

In this section X denotes a compact reduced normal space and Θ_X its tangent sheaf. Let H be the universal space of quotient O_X-modules of Θ_X and $\mathscr{R}$ the H-flat universal sheaf on H × X ([4]). Put $\mathscr{R} = p_H^*\Theta_X/\mathscr{T}$, $\mathscr{T}$ being a coherent submodule of $p_H^*\Theta_X$; for any $h \in H$ $\mathscr{T}(h)$ is a coherent submodule of Θ_X. We shall construct the space $\tilde{H}$ as an analytic subspace of an open subset of H.

1. *Refining of a privileged « cuirasse ».*

Let M be a Θ_X-privileged « cuirasse »» ([4] § 9, n° 2); M is given by,

(i) a finite family $(\varphi_i)_{i \in I}$ of charts of X, i.e. for every $i \in I$ φ_i is an isomorphism from an open set $X_i \subset X$ onto a closed analytic subspace of an open set U_i in $\mathbf{C}^{n_i}$,

(ii) for every $i \in I$ a Θ_X-privileged polycylinder $K_i \subset U_i$ (i.e. a $\varphi_{i*}\Theta_X$-privileged polycylinder). and an open set $V_i \subset X_i$ satisfying

$$\bar{V}_i \subset \varphi_i^{-1}(\mathring{K}_i) \subset X_i$$

$$X = \bigcup_{i \in I} V_i$$

(iii) for every $(i,j) \in I \times J$ a chart φ_{ij} defined on $X_i \cap X_j$ with values in an open $U_{ij} \subset \mathbf{C}^{n_{ij}}$ and a finite family (K_{ij_α}) of Θ_X-privileged polycylinders in U_{ij} such that conditions

$$\bar{V}_1 \cap \bar{V}_j \subset \bigcup_\alpha \psi_{ij}^{-1}(K_{ij\alpha})$$

$$\varphi_{ij}^{-1}(K_{ij\alpha}) \subset \varphi_i^{-1}(\mathring{K}_i) \cap \varphi_j^{-1}(\mathring{K}_j)$$

are fullfilled.

As in ([4]) let us denote H_M the open subset of the elements F of H for which M is F-privileged (i.e. all the polycylinders K_i, $K_{ij\alpha}$ are F-privileged); we shall construct $\tilde{H}$ as union of open subsets $\tilde{H} \cap H_M$.

— For any Θ_X-privileged polycylinder K let us denote G_K (§ 2) the Banach analytic space of quotients of $B(K,\Theta_X)$ with finite direct resolution.

For every $i \in \mathrm{I}$ let G_i be the open subset of $\mathrm{G}_{\mathrm{K}_i}$ on which, for any α, the restriction homomorphisms $\mathrm{B}(\mathrm{K}_i) \to \mathrm{B}(\mathrm{K}_{ij\alpha})$ induce morphisms $\mathrm{G}_i \to \mathrm{G}_{\mathrm{K}_{ij\alpha}}$. The Douady construction of H_M gives a natural injective morphism

$$i : \mathrm{H}_\mathrm{M} \to \prod_{i \in \mathrm{I}} \mathrm{G}_i .$$

Definition 5. — *A refining of the « cuirasse »* M *is given by a family* $(\mathrm{K}'_i)_{i \in \mathrm{I}}$ *of polycylinders satisfying :*

(i) *for every* i $\varphi_i(\mathrm{V}_i) \subset \mathring{\mathrm{K}}'_i \subset \mathrm{K}'_i \subset \mathring{\mathrm{K}}_i$,

(ii) *for every* i, j, α $\varphi_{ij}^{-1}(\mathrm{K}_{ij\alpha}) \subset \varphi_i^{-1}(\mathring{\mathrm{K}}'_i) \cap \varphi_j^{-1}(\mathring{\mathrm{K}}'_j)$,

(iii) *for every* i K'_i *is* Θ_X*-privileged.*

We denote by $\mathrm{M}((\mathrm{K}'_i))$ such a refining; for any coherent sheaf $\mathscr{F}$ on X we shall say that $\mathrm{M}((\mathrm{K}'_i))$ is $\mathscr{F}$-privileged if M is $\mathscr{F}$-privileged and if, for every i, K'_i is $\mathscr{F}$-privileged.

Lemma 1. — (i) *Let* $\mathscr{F}$ *be a coherent sheaf such that* M *is* $\mathscr{F}$*-privileged; then it exists a* $\mathscr{F}$*-privileged refining of* M.

(ii) *Let* $\mathrm{M}((\mathrm{K}'_i))$ *a refining of* M; *then the set of quotient* $\mathscr{F}$ *of* Θ_X *such that* $\mathrm{M}((\mathrm{K}'_i))$ *is* $\mathscr{F}$*-privileged is open in* H_M.

Proof. — (i) follows from ([4] § 7, n° 3 corollary of prop. 6) and (ii) is an immediate consequence of flatness and privilege.

2. Now we fix a Θ_X-privileged « cuirasse » $\mathrm{M} = \mathrm{M}(\mathrm{I},(\mathrm{K}_i),(\mathrm{V}_i),(\mathrm{K}_{ij\alpha}))$ and a Θ_X-privileged refining $\mathrm{M}((\mathrm{K}'_i))$ of M.

Lemma 2. — *Let* H'_M *be the subset of* H_M *the points of which are quotients* $\Theta_\mathrm{X}/\mathrm{T}$ *satisfying :*

(i) $\mathrm{M}((\mathrm{K}'_i))$ *is* $\Theta_\mathrm{K}/\mathrm{T}$*-privileged,*

(ii) T *is a subsheaf of* Θ_X *stable by the bracket-map.*

Then H'_M *is an analytic subspace of an open subset of* H_M.

Proof. — Using notations of section 2 one puts for every $i \in \mathrm{I}$

$$\mathrm{G}'_i = \mathrm{G}_{\mathrm{K}_i,\mathrm{K}'_i} \cap \mathrm{G}_i$$

G'_i is an open subset of G_i and $\mathrm{G}_{\mathrm{K}_i}$; put $\mathrm{S}_i = \mathrm{S}_{\mathrm{K}_i,\mathrm{K}'_i} \cap \mathrm{G}'_i$.

One knows that the category of Banach analytic spaces has finite products, kernel of double arrows and hence fiber products (for all this notions the reader can refer to ([4] § 3, n° 3). Then $\prod_{i\in I} S_i$ is a Banach analytic subspace of $\prod_{i\in I} G'_i$; since $\prod_{i\in I} G'_i$ is an open subset of $\prod_{i\in I} G_i$ it follows from (§ II Theorem 3)

$$H'_M = H_M \times_{\prod_{i\in I} G_i} \prod_{i\in I} S_i$$

and the lemma is proved.

– Now let R'_M (resp. T'_M) be the pull back of $\mathcal{R}$ (resp. $\mathcal{T}$) by the inclusion morphism $H'_M \times X \to H \times X$; R'_M is the quotient of $p^{*'}_{H_M}\Theta_X$ by T'_M (the sheaves T'_M and ker $[p^{*'}_{H_M}\Theta_X \to R'_M]$ are H'_M-flat and equal on the fibers $\{h\}\times X$).

Lemma 3. – *T'_M is a $m_{H'_M}$-stable submodule of $p^{*}_{H'_M}\Theta_X$.*

The proof follows immediatly of the remark 2 of paragraph 2 and of

$$X = \bigcup_{i\in I} V_i = \bigcup_{i\in I} \varphi_i^{-1}(\mathring{K}'_i).$$

– Using the universal property of H_M, Theorem 3 § 2 and the commutative diagram

$$\begin{array}{ccc} H'_M \times X & \to & H_M \times X \\ \downarrow & & \downarrow \\ \left(\prod_{i\in I} G'_i\right) \times X & \to & \left(\prod_{i\in I} G_i\right) \times X \end{array}$$

we obtain the following proposition:

Proposition 1. – *Let Z be an analytic space and T_Z a coherent subsheaf of $p_Z^*\Theta_X$ satisfying:*

(i) *$p_Z^*\Theta_X/T_Z$ is Z-flat.*

(ii) *For every $z\in Z$ the cuirasse $M((K'_i))$ is $\Theta_X/T_Z(z)$-privileged.*

(iii) *T_Z is a m_Z-stable submodule of $p_Z^*\Theta_X$.*

Then the unique morphism $g : Z \to H$ *such that*

$$(g \times I_X)^* \mathscr{R} = p_Z^* \Theta_X / T_Z$$

factorizes through H'_M *and verifies*

$$(g \times I_X)^* T'_M = T_Z .$$

3. *End of the proof of Theorem* 1.

Notations are those of the previous proposition; the unicity of g implies the unicity of its factorization through the subspace H'_M of H. Hence, when the refinings of a given M are varying, one obtains analytic spaces H'_M which patch together in an analytic subspace of an open subset of H_M.

When the « cuirasse » M varies in the family of all the Θ_X-privileged « cuirasse » the spaces H_M form an open covering of H; then the universal property of the H_M 's implies that $\tilde{H} = \bigcup_M H'_M$ is an analytic subspace of an open subset of H. Theorem 4 is proved.

Remark. — More generally if X is not compact, let Θ be a coherent sheaf on X and $m : \Theta \times \Theta \to \Theta$ a sheaf morphism inducing for each open set U a continuous C-bilinear map $m_U : \Theta(U) \times \Theta(U) \to \Theta(U)$; let H be the Douady space of the coherent quotients of Θ with compact support ([4]). We get a universal analytic structure on the subset of those quotients which are m-stable.

BIBLIOGRAPHY

[1] P. Baum, Structure of foliation singularities, *Advances in Math.*, 15 (1975), 361-374.

[2] G. Bohnhorst and H. J. Reiffen, Holomorphe blatterungen mit singularitäten, *Math. Gottingensis*, heft 5 (1985).

[3] H. Cartan, *Faisceaux analytiques cohérents*, C.I.M.E., Edizioni Cremonese, 1963.

[4] A. Douady, Le problème des modules pour les sous-espaces analytiques..., *Ann. Inst. Fourier*, XVI, Fasc. 1 (1966), 1-96.

[5] B. Malgrange, Frobenius avec singularités-codimension 1, *Pub I.H.E.S.*, n° 46 (1976).

[6] G. Pourcin, Sous-espaces privilégiés d'un polycylindre, *Ann. Inst. Fourier*, XXV, Fasc. 1 (1975), 151-193.
[7] Y. T. Siu et G. Trautmann, Gap-sheaves and extension of coherent analytic subsheaves, *Lect. Notes*, 172 (1971).
[8] T. Suwa, Singularities of complex analytic foliations, *Proceedings of Symposia in Pure mathematics*, Vol. 40 (1983), Part. 2.

Manuscrit reçu le 12 mars 1986.

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