

ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 37, n° 1 (1987), p. 175-185

http://www.numdam.org/item?id=AIF_1987__37_1_175_0

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ON THE ANGLES BETWEEN CERTAIN ARITHMETICALLY DEFINED SUBSPACES OF \mathbf{C}^n

by Robert BROOKS(*)

In this note, we consider the following problem: Let $\{v_i\}$ and $\{w_j\}$ be two sets of unitary bases for \mathbf{C}^n . The bases $\{v_i\}$ and $\{w_j\}$ are about as "independent as possible" if, for all i and j , $|\langle v_i, w_j \rangle|$ is on the order of $\frac{1}{\sqrt{n}}$. For θ some fixed number, for instance $\frac{1}{5}$, we consider linear spaces V^θ (resp. W^θ) spanned by $[\theta \cdot n]$ of the vectors in the set $\{v_i\}$ (resp. $\{w_j\}$), where $[\]$ denotes the greatest integer function. What can one say about the angle between V^θ and W^θ , as n tends to infinity?

In view of the paper [5], we may view such a question as relating to the prediction theory of such subspaces, although we do not see a direct connection between the methods of [5] and the present paper.

Let us consider the following special cases: In the first case, let $\{v_i\}$ be the standard basis for \mathbf{C}^n , and let $\{w_j\}$ be the "Fourier transform" of this basis

$$w_j = \frac{1}{\sqrt{n}} (\zeta^j, \zeta^{2j}, \dots, \zeta^{nj})$$

(*) Partially supported by NSF grant DMS-83-15522 ; Alfred P. Sloan fellow.

where $\zeta = e^{2\pi i/n}$ is a primitive n -th root of 1. Then clearly $|\langle v_i, w_j \rangle| = \frac{1}{\sqrt{n}}$ for all i, j .

For a number α , let us denote by $[[\alpha]]$ the distance from α to the nearest integer

$$[[\alpha]] = \inf_{n \in \mathbb{Z}} |\alpha - n|.$$

Let V^θ and W^θ denote the spaces spanned by

$$\left\{ v_i : \left[\left[\frac{i}{n} \right] \right] < \theta \right\} \quad \text{and} \quad \left\{ w_j : \left[\left[\frac{j}{n} \right] \right] < \theta \right\}$$

respectively. For σ_n a permutation of the integers (mod n), let $W_{\sigma_n}^\theta$ be the space spanned by $\left\{ w_j : \left[\left[\frac{\sigma_n(j)}{n} \right] \right] < \theta \right\}$. Then we will show :

THEOREM 1. – (a) *For any θ , the angle between V^θ and W^θ tends to 0 as n tends to ∞ .*

(b) *If the permutations σ_n are “sufficiently mixing”, then the angle between V^θ and $W_{\sigma_n}^\theta$ stays bounded away from 0 as n tends to ∞ .*

By “sufficiently mixing”, we mean that, for all i , we do not have both $\left[\left[\frac{\sigma_n(i)}{n} \right] \right] < \theta$ and $\left[\left[\frac{\sigma_n(i+1)}{n} \right] \right] < \theta$. Clearly, weaker hypotheses on the σ_n would also allow us to conclude (b), but we will not explore this question here.

Now let us consider the following different example: for a prime p , let χ denote an even multiplicative character (mod p). Then set $\{v_i\}, \{w_j\}$ to be the following bases for \mathbb{C}^{p+1} :

$$v_j = \frac{1}{\sqrt{p}} (1, \zeta^j, \dots, \zeta^{(n-1)j}, 0) \quad j = 0, \dots, p-1$$

$$v_p = (0, \dots, 0, 1)$$

$$w_k = \frac{1}{\sqrt{p}} (0, \chi(1) \zeta^{-k}, \chi(2) \zeta^{-2k}, \dots, \chi(n-1) \zeta^{-(n-1)k}, 1)$$

$$k = 0, \dots, p-1$$

$$w_p = (1, 0, \dots, 0)$$

where \bar{m} denotes the reciprocal of $m \pmod{p}$. Note that

$$\langle v_j, w_k \rangle = \frac{1}{p} \sum_{x=1}^{p-1} \overline{\chi(k)} \zeta^{(jx+k\bar{x})} = \frac{1}{p} S_\chi(j, k, p)$$

where $S_\chi(j, k, p)$ is a Kloosterman sum. The fact that the bases $\{v_k\}$, $\{w_k\}$ are about as “independent as possible” is a deep result of A. Weil [7] that $|S_\chi(j, k, p)| < 2\sqrt{p}$.

Denoting by V^θ and W_χ^θ the vectors spanned by

$$\{v_i : \llbracket i/p \rrbracket < \theta\} \quad \text{and} \quad \{w_j : \llbracket j/p \rrbracket < \theta\}$$

respectively, our second result is:

THEOREM 2. — *For θ sufficiently small, the angle between V_χ^θ and W_χ^θ stays bounded away from 0 as p tends to ∞ , uniformly with respect to χ .*

Our proof of Theorem 2 relies on the deep theorem of Selberg [6] that, when Γ_n is a congruence subgroup of $\text{PSL}(2, \mathbf{Z})$, then the first eigenvalue $\lambda_1(\mathbf{H}^2/\Gamma_n)$ of the spectrum of the Laplacian satisfies

$$\lambda_1(\mathbf{H}^2/\Gamma_n) \geq \frac{3}{16}.$$

Another important ingredient in Theorem 2 is our recent work [3] on the behavior of λ_1 in a tower of coverings. Indeed it is not difficult to find an extension of Theorem 2 which is actually equivalent, given [3], to Selberg’s theorem, at least after replacing “ $\frac{3}{16}$ ” by “some positive constant”.

The main number-theoretic input into Selberg’s theorem is the Weil estimate. Theorem 1 shows that, by contrast, the conclusion of Theorem 2 cannot be achieved directly by appealing to the Weil estimate, and suggests an interpretation of Selberg’s theorem in terms of the random distribution of Kloosterman sums.

The proof of Theorem 1 is completely elementary.

We would like to thank Peter Sarnak for useful discussions, and Alice Chang for showing us the paper [5] and for her suggestions.

1. A Lemma.

In this section, we give a simple lemma in linear algebra which is the key to proving Theorems 1 and 2.

Suppose U and T are unitary matrices acting on \mathbf{C}^n . For a given value δ , let U^δ (resp. T^δ) be the subspace spanned by the eigenvectors of U (resp. T) whose eigenvalues λ satisfy $|\lambda - 1| < \delta$. Let U_\perp^δ and V_\perp^δ denote the perpendicular subspaces.

Denote by $k(U, T)$ the expression

$$k(U, T) = \inf_{\|X\|=1} \max(\|U(X) - X\|, \|T(X) - X\|).$$

Let $\alpha(\delta)$ denote the cosine of the angle between U^δ and T^δ :

$$\alpha(\delta) = \sup_{X \in U^\delta, Y \in V^\delta} \frac{|\langle X, Y \rangle|}{\|X\| \|Y\|}.$$

The main result of this section is:

$$\text{LEMMA. } -\delta \sqrt{\frac{1 - \alpha^2}{2}} \leq k(U, T) \leq \sqrt{\delta^2 \alpha^2 + 4(1 - \alpha^2)}.$$

Proof. — To show the right-hand inequality, let X be a unit-length vector in U^δ such that its orthogonal projection Y onto T^δ is of maximum length $\alpha(\delta)$.

Since $X \in U^\delta$, we have $\|U(X) - X\| \leq \delta$. Writing

$$X = Y + Y^\perp, Y^\perp \in T_\perp^\delta,$$

we see that

$$\begin{aligned} \|T(X) - X\|^2 &= \|T(Y) - Y\|^2 + \|T(Y^\perp) \\ &\quad - Y^\perp\|^2 \leq \delta^2 \cdot \alpha^2 + 4(1 - \alpha^2). \end{aligned}$$

So $k(U, T) \leq \max(\delta, \sqrt{\delta^2 \alpha^2 + 4(1 - \alpha^2)})$. When $\delta < 2$, the second term on the right is $\geq \delta$. When $\delta \geq 2$, then $\alpha = 1$ and again the second term is $\geq \delta$.

To get the left-hand inequality, let X be a vector of length 1 minimizing $\sup(\|U(X) - X\|, \|T(X) - X\|)$. Write

$$X = X_U + X_T + X_1$$

where $X_U \in U^\delta$, $X_T \in T^\delta$, and $X_1 \in U_1^\delta \cap T_1^\delta$. Then

$$\|U(X) - X\|^2 \geq \delta^2 [(1 - \alpha^2) \|X_T\|^2 + \|X_1\|^2]$$

$$\|T(X) - X\|^2 \geq \delta^2 [(1 - \alpha^2) \|X_U\|^2 + \|X_1\|^2]$$

and so

$$\delta^2 (1 - \alpha^2) \|X\|^2 \leq \|U(X) - X\|^2 + \|T(X) - X\|^2 \leq 2k^2(U, T)$$

and so $k(U, T) \geq \delta \sqrt{\frac{1 - \alpha^2}{2}}$.

From the left-hand estimate, we see that for δ fixed, and hence for δ arbitrarily small, a lower bound for $1 - \alpha^2$ gives a lower bound for $k(U, T)$. From the right-hand side, we see that a lower bound for $k(U, T)$ gives, for $\delta \ll k(U, T)$, a lower bound for $1 - \alpha^2$.

2. Proof of Theorem 1.

Let $v_i = (0, 0, \dots, 1, 0, \dots, 0)$ be the standard basis for \mathbf{C}^n and let

$$w_j = \frac{1}{\sqrt{n}} (\zeta^j, \zeta^{2j}, \dots, \zeta^{nj}).$$

Let V be the unitary transformation whose eigenvectors are the v_i 's, with $V(v_i) = \zeta^i v_i$. Of course, the matrix for V is simply the diagonal matrix

$$V = \begin{pmatrix} \zeta^1 & & 0 \\ & \zeta^2 & \\ 0 & & \zeta^n \end{pmatrix}.$$

Similarly, let W be the unitary transformation whose eigenvectors are the w_j 's, with $W(w_j) = \zeta^j \cdot w_j$. We compute:

LEMMA. —

$$W = \begin{pmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & 0 \\ 1 & 0 & 0 & . & . & . & 0 \end{pmatrix}.$$

Proof. — $W = EVE^{-1}$, where $E = (e_{ij})$ is given by

$$e_{ij} = \frac{1}{\sqrt{n}} \xi^{ij}.$$

The lemma now follows by routine calculation.

To prove Theorem 1 (a) it suffices, from the lemma of § 1, to show that $k(V, W)$ tends to 0 as n tends to infinity.

But $V - I$ has the matrix expression

$$\begin{pmatrix} \xi - 1 & & & 0 \\ & \xi^2 - 1 & & \\ & & \cdot & \\ 0 & & & \cdot \xi^n - 1 \end{pmatrix}$$

so that any element in V^θ satisfies

$$\|(V - I)(v)\| \leq 2 \left| \sin \left(\frac{\theta}{2} \right) \right| \|v\|. \quad (*)$$

Now consider the vector v_n whose j th coordinate is 1 for $[j/n] < \theta$, and is 0 otherwise. Then we have that $v_n \in V^\theta$, so that, by (*) we have

$$\|(V - I)(v_n)\| \leq 2 \left| \sin \left(\frac{\theta}{2} \right) \right| \|v_n\|.$$

On the other hand, from the lemma, we compute easily that

$$\|(W - I)(v_n)\| = \sqrt{2}.$$

Since $\|v_n\| = \sqrt{2[n \cdot \theta] + 1}$, where $[]$ denotes the greatest integer function, we have that

$$k(V, W) \leq \sup \left(2 \left| \sin \left(\frac{\theta}{2} \right) \right|, \frac{1}{\sqrt{[n \cdot \theta] + \frac{1}{2}}} \right).$$

It is then evident that as $n \rightarrow \infty$, we may choose $\theta \rightarrow 0$ such that the right-hand side $\rightarrow 0$, establishing Theorem 1 (a).

To establish 1 (b), we first notice from the computation of the lemma that whenever σ_n is sufficiently mixing,

$$\| (W_{\sigma_n} - I) v \| = (\sqrt{2}) \| v \|$$

for $v \in V^\theta$. Fixing θ , for $v \in V^\theta$, let us write

$$v = w + w^\perp, w \in W_{\sigma_n}^\theta, w^\perp \in (W_{\sigma_n}^\theta)^\perp.$$

$$\begin{aligned} 2 \| v \|^2 &= \| W_{\sigma_n}(v) - v \|^2 = \| W_{\sigma_n}(w) - w \|^2 + \| W_{\sigma_n}(w^\perp) - w^\perp \|^2 \\ &\leq 4 \sin^2(\pi\theta) \cdot \| w \|^2 + 4 \| w^\perp \|^2 = 4 \sin^2(\pi\theta) \cdot \| w \|^2 \\ &\qquad\qquad\qquad + 4 (\| v \|^2 - \| w \|^2) \end{aligned}$$

from which we see that

$$\begin{aligned} 4(1 - \sin^2(\pi\theta)) \| w \|^2 &\leq 2 \| v \|^2 \quad \text{so that} \quad \frac{\| w \|}{\| v \|} \leq \frac{1}{(\sqrt{2})} \cos(\pi\theta), \\ &\qquad\qquad\qquad \alpha \leq \left(\frac{1}{\sqrt{2}} \right) \cos(\pi\theta). \end{aligned}$$

Choosing θ smaller than $\frac{1}{4}$ then establishes Theorem 1 (b).

3. Proof of Theorem 2.

We begin this section with a quick review of the result of [3]. For M a compact manifold, and $M^{(i)}$ a family of finite covering spaces of M , we seek conditions of a combinatorial nature on $\pi_1(M), \pi_1(M^{(i)})$ which govern the asymptotic behavior of $\lambda_1(M^{(i)})$ as i tends to infinity.

To state the main result of [3], let us assume that the $M^{(i)}$'s are normal coverings of M , so that the group $\pi^i = \pi_1(M)/\pi_1(M^{(i)})$ are defined. Let us also fix generators g_1, \dots, g_k for $\pi(M)$ - note that g_1, \dots, g_k also generate all the π^i 's.

Let H_i denote orthogonal complement to the constant function in $L^2(\pi^i)$, which carries an obvious unitary structure preserved by the action of π^i .

If H is any space on which π acts unitarily, denote by $k(H)$

the “Kazhdan distance” from H to the trivial representation defined by

$$k(H) = \inf_{\|X\|=1} \sup_{i=1, \dots, k} \|g_i(X) - X\|.$$

Then we have :

THEOREM ([3]). — *The following two conditions are equivalent :*

- a) *There exists $c > 0$ such that $\lambda_1(M^{(i)}) > c$ for all i*
- b) *There exists $k > 0$ such that $k(H_i) > k$ for all i .*

We may now extend this result in the following way: we observe that each non-trivial representation of π^i occurs as an orthogonal direct summand in H_i , and furthermore that

$$k\left(\bigoplus_{i=1}^n H_i\right) = \inf k(H_i).$$

Hence we may rephrase the Theorem as follows:

COROLLARY. — *The following two conditions are equivalent :*

- a) *There exist $c > 0$ such that $\lambda_1(M^{(i)}) > c$ for all i .*
- b) *There exist $k > 0$ such that for all i and for every non-trivial irreducible unitary representation H of π^i , $k(H) > k$.*

We now observe that, using the technique of [1] and [2], we may weaken the hypothesis that M be compact. To explain this briefly, let us assume that M has finite volume, and let F be a fundamental domain for M in \tilde{M} .

Recall from [1] that M satisfies an “isoperimetric condition at infinity” if there is a compact subset K of F such that $h(F - K) > 0$ where h denote the Cheeger isoperimetric constant, with Dirichlet conditions on ∂K and Neumann conditions on $\partial F - \partial K$.

When M is a Riemann surface with finite area and a complete metric of constant negative curvature, then it is easily seen that M satisfies an isoperimetric condition at infinity.

The technique of [1] and [2] then applies directly to show how to adapt the arguments of the compact case to the case when M satisfies an isoperimetric condition at infinity.

We now apply these considerations to the manifolds

$$\mathbf{M}^{(n)} = \mathbf{H}^2/\Gamma_n, \text{ where } \Gamma_n \subset \text{PSL}(2, \mathbf{Z})$$

is the congruence subgroup

$$\Gamma_n = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{n} \right\}.$$

According to the theorem of Selberg [6] mentioned above,

$$\lambda_1(\mathbf{H}^2/\Gamma_n) > \frac{3}{16}.$$

Let us fix generators

$$V = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad W = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

for $\text{PSL}(2, \mathbf{Z})$, and observe that \mathbf{H}^2/Γ_n is a finite area Riemann surface covering $\mathbf{H}^2/\text{PSL}(2, \mathbf{Z})$, with covering group

$$\pi^n = \text{PSL}(2, \mathbf{Z}/n).$$

It follows from the corollary that there is a constant $k > 0$ such that, for \mathbf{H} any non-trivial irreducible representation of $\text{PSL}(2, \mathbf{Z}/n)$, we have $k(\mathbf{H}) > k$.

We now let n be a prime p , and fix a Dirichlet character $\chi \pmod{p}$. We will assume that $\chi(-1) = 1$. We now consider the following representation \mathbf{H}_χ , which is the representation associated to χ in the continuous series of representations of $\text{PSL}(2, \mathbf{Z}/n)$: The representation of \mathbf{H}_χ is the set of all functions f on

$$\mathbf{Z}/p \times \mathbf{Z}/p - \{0\}$$

which transform according to the rule

$$f(tx, ty) = \chi(t) f(x, y), \quad t \in (\mathbf{Z}/p)^* \quad (*)$$

and where $\text{PSL}(2, \mathbf{Z}/p)$ acts on f by the rule

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} f(x, y) = f(ax + cy, bx + dy).$$

We may take as a basis for \mathbf{H}_χ the functions

$$f_a(x, 1) = 1 \quad \text{if } x = a \\ = 0 \quad \text{otherwise}$$

$$f_a(1, 0) = 0$$

for $a = 0, \dots, p-1$ and

$$f_\infty(x, 1) = 0 \quad \text{for } x = 0, \dots, p-1$$

$$f_\infty(1, 0) = 1$$

using (*) to extend the f_a 's to all values of x, y .

Then an orthonormal basis of eigenvectors of V is given by

$$v_b = \frac{1}{\sqrt{p}} \left(\sum_{x=0}^{p-1} \zeta^{bx} \cdot f_x \right) \quad V(v^b) = \zeta^b v_b$$

$$v_\infty = f_0 \quad V(v_\infty) = v_\infty$$

and an orthonormal basis of eigenvectors of W is given by

$$w_b = \frac{1}{\sqrt{p}} \left(\sum_{x=0}^{p-1} \zeta^{-bx} \chi(x) f_{\bar{x}} \right) \quad W(w_b) = \zeta^b w_b$$

$$w_\infty = f_0 \quad W(w_\infty) = w_\infty$$

where \bar{x} is the multiplicative inverse of $x \pmod{p}$, and $\bar{0} = \infty$.

When χ is the trivial character, the vector

$$\sqrt{\frac{p}{p+1}} v_0 + \frac{1}{\sqrt{p+1}} v_\infty = \sqrt{\frac{p}{p+1}} w_0 + \frac{1}{\sqrt{p+1}} w_\infty$$

splits off as a trivial representation, but for all other characters χ , H_χ is irreducible [4].

Theorem 2 is now an immediate consequence of the corollary above, the lemma of § 1, and Selberg's theorem.

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Manuscrit reçu le 9 juillet 1985

révisé le 14 mai 1986.

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