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Vanishing theorems for compact hessian manifolds


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Let $M$ be a flat affine manifold with a locally flat affine connection $D$. Among the Riemannian metrics on $M$ there is an important class of Riemannian metrics which are compatible with the flat affine structure on $M$. A Riemannian metric $g$ on $M$ is said to be \textit{Hessian} if $g$ has an expression $g = D^2 u$ where $u$ is a local $C^\infty$-function. A flat affine manifold provided with a Hessian metric is called a \textit{Hessian manifold}. A certain geometry of Hessian manifolds has been studied in Shima [10]-[14]. See also Cheng and Yau [2] and Yagi [15].

Hessian manifolds have in a certain sense some analogy with Kählerian manifolds. In this paper, being motivated by the theory of cohomology for Kählerian manifolds we study cohomology groups for Hessian manifolds.

Let $F$ be a locally constant vector bundle over $M$. We denote by $\Omega^{p,q}(F)$ the space of all sections of $(\wedge^p T^*) \otimes (\wedge^q T^*) \otimes F$, where $T^*$ is the cotangent bundle over $M$. Since the vector bundle $(\wedge^q T^*) \otimes F$ is locally constant, we can naturally define a complex

$$
\ldots \rightarrow \Omega^{p-1,q}(F) \rightarrow \Omega^{p,q}(F) \rightarrow \Omega^{p+1,q}(F) \rightarrow \ldots
$$

We denote by $H^{p,q}(F)$ the $p$-th cohomology group of the complex. Then we have the following duality theorem analogous to that of Serre [9].

\textbf{Theorem.} — \textit{Let $M$ be a compact oriented flat affine manifold of dimension $n$. Then we have}

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$H^{p,q}(F) \cong H^{n-p,n-q}((K \otimes F)^*)$, 

where $K$ is the canonical line bundle over $M$ and $(K \otimes F)^*$ is the dual bundle of $K \otimes F$.

Let $F$ be a locally constant line bundle over $M$. Choose an open covering $\{U_\lambda\}$ of $M$ such that the local triviality holds on each $U_\lambda$. Denote by $\{f_{\lambda \mu}\}$ the constant transition functions with respect to $\{U_\lambda\}$. A fiber metric $a = \{a_\lambda\}$ on $F$ is a collection of positive $C^\infty$-functions $a_\lambda$ on $U_\lambda$ such that 

$$a_\mu = f_{\lambda \mu}^2 a_\lambda.$$ 

Using this we can define a globally defined closed 1-form $A$ and a symmetric bilinear form $B$ by 

$$A = -D \log a_\lambda,$$

$$B = -D^2 \log a_\lambda,$$

and we call them the first Koszul form and the second Koszul form of $F$ with respect to the fiber metric $a = \{a_\lambda\}$ respectively.

A locally constant line bundle $F$ is said to be positive (resp. negative) if the second Koszul form is positive (resp. negative) definite with respect to a certain fiber metric. It should be remarked that if a compact connected flat affine manifold $M$ admits a locally constant positive (resp. negative) line bundle, then by a theorem of Koszul [6] $M$ is a hyperbolic affine manifold, that is, the universal covering of $M$ is an open convex cone not containing any full straight line.

Kodaira-Nakano's vanishing theorem for compact Kählerian manifolds plays an essential role in the theory of compact Kählerian manifolds. In this paper we prove the following vanishing theorem for a compact Hessian manifold analogous to that of Kodaira-Nakano.

**Theorem.** — Let $M$ be a compact connected oriented Hessian manifold. Denote by $K$ the canonical line bundle over $M$. Let $F$ be a locally constant line bundle over $M$.

(i) If $2F + K$ is positive, then

$$H^{p,q}(F) = 0 \text{ for } p + q > n.$$
(ii) If $2F + K$ is negative, then

$$H^{p,q}(F) = 0 \quad \text{for} \quad p + q < n.$$ 

As to vanishing theorem for compact hyperbolic affine manifolds we should mention the following theorem due to Koszul [7].

**Theorem.** — Let $M$ be a compact oriented hyperbolic affine manifold. Then we have

$$H^{p,q}(1) = 0 \quad \text{for} \quad p, q > 0,$$

where 1 is the trivial line bundle over $M$.

In § 1 and § 2 a Riemannian metric $g$ is not assumed to be Hessian. We define in § 1 fundamental operators $e(g), i(g), \Pi, *, \partial, \delta$ and $\Box$. In § 2 we define the Laplacian $\Box_\xi$ on $\Omega^{p,q}(F)$, and prove the duality theorem $H^{p,q}(F) \cong H^{n-p,n-q}((K \otimes F)*)$ and the cohomology isomorphisms $\mathcal{K}^{p,q}(F) \cong H^{p,q}(F) \cong H^p(P(F))$.

In § 3 we give the local expressions for geometric concepts on Hessian manifolds. In § 4 and § 5 the formulae of Weitzenböck type for $\Box$ and $\Box_\xi$ are obtained. In § 6 we prove a vanishing theorem analogous to that of Kodaira-Nakano. In § 7 we mention a vanishing theorem of Koszul type.

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### 1. The Laplacian $\Box$ on $\Omega^{p,q}$

Let $M$ be a flat affine manifold with a locally flat affine connection $D$. Then there exist local coordinate systems $\{x^1, \ldots, x^n\}$ such that $Dx^f = 0$, which will be called affine local coordinate systems. Throughout this paper the local expressions for geometric concepts on $M$ will be given in terms of affine local coordinate system. From now on we assume further that $M$ is compact, connected and oriented.

Choose an arbitrary Riemannian metric $g$ on $M$. Let $\Omega^{p,q}$ be the space of all sections of $(\wedge T^*) \otimes (\wedge T^*)$. We denote the local
expression of $\phi \in \Omega^{p,q}$ by

$$\phi = \frac{1}{p! q!} \sum \phi_{i_1 \ldots i_p \bar{i}_1 \ldots \bar{i}_q} (dx^{i_1} \wedge \ldots \wedge dx^{i_p}) \otimes (dx^{\bar{i}_1} \wedge \ldots \wedge dx^{\bar{i}_q}).$$

For simplicity let us fix some notation. We denote as follows:

$$I_p = (i_1, \ldots, i_p), \quad i_1 < i_2 < \ldots < i_p, \quad 1 \leq i_1 \leq n,$n

$$I_{n-p} = (i_{p+1}, \ldots, i_n), \quad i_{p+1} < \ldots < i_n, \quad 1 \leq i_{p+1} \leq n,$$n

and $(i_1, \ldots, i_p, i_{p+1}, \ldots, i_n)$ is a permutation of $(1, \ldots, n)$. Then with this notation we write

$$\phi = \sum_{I_p, \bar{I}_q} \phi_{I_p \bar{I}_q} dx^{I_p} \otimes dx^{\bar{I}_q},$$

where $dx^{I_p} = dx^{i_1} \wedge \ldots \wedge dx^{i_p}$.

For $\phi, \psi \in \Omega^{p,q}$ we set

$$h(\phi, \psi) = \frac{1}{p! q!} \phi_{i_1 \ldots i_p \bar{i}_1 \ldots \bar{i}_q} \psi^{i_1 \ldots i_p \bar{i}_1 \ldots \bar{i}_q} (\ast)$$

$$= \phi_{I_p \bar{I}_q} \psi^{I_p \bar{I}_q}. (\ast\ast)$$

**DEFINITION 1.1.** — The inner product of $\phi, \psi \in \Omega^{p,q}$ is

$$(\phi, \psi) = \int_M h(\phi, \psi) v,$$

where $v$ is the volume element determined by $g$.

**DEFINITION 1.2.** — We define $\ast$-operation

$$\ast : \Omega^{p,q} \rightarrow \Omega^{n-p,n-q}$$

by $$(\ast \phi)_{I_{n-p} \bar{I}_{n-q}} = (-1)^{pq} \text{sgn}(I_p I_{n-p}) \text{sgn}(\bar{I}_q \bar{I}_{n-q}) G \phi^{I_p \bar{I}_q},$$

where $\text{sgn}(I_p I_{n-p})$ is the signature of the permutation $(I_p I_{n-p})$ of $(1, \ldots, n)$ and $G = \det(g_q)$.

(\ast) Throughout this paper we use Einstein's convention on indices.

(\ast\ast) $\phi_{I_p \bar{I}_q} \psi^{I_p \bar{I}_q}$ means $\sum_{I_p, \bar{I}_q} \phi_{I_p \bar{I}_q} \psi^{I_p \bar{I}_q}$. 
DEFINITION 1.3. - Let \( \phi = \sum \phi_{lp} \bar{\psi}_q dx^p \otimes dx^q \) and \( \psi = \sum \psi_{kr} \bar{\psi}_r dx^k \otimes dx^l \).

We set \( \phi \wedge \psi = \sum \phi_{lp} \bar{\psi}_q \psi_{kr} \bar{\psi}_r (dx^p \wedge dx^k) \otimes (dx^q \wedge dx^l) \).

A straightforward calculation shows

PROPOSITION 1.1. - Let \( \phi, \psi \in \Omega^p,q \). Then

(i) \( ** \phi = (-1)^{n+p+q} \phi \),

(ii) \( \phi \wedge * \psi = (-1)^{p+q} h(\phi, \psi) v \otimes \nu \).

DEFINITION 1.4. - Considering the Riemannian metric \( g \) as an element in \( \Omega^1,1 \) we define

\[
e(g) : \Omega^{p,q} \rightarrow \Omega^{p+1,q+1},
\]

\[
i(g) : \Omega^{p,q} \rightarrow \Omega^{p-1,q-1},
\]

by \( e(g) \phi = g \wedge \phi \) for \( \phi \in \Omega^{p,q} \) and \( i(g) = (-1)^{n+p+q+1} * e(g) * \).

Then \( i(g) \) is the adjoint operator of \( e(g) \) with respect to the inner product given in Definition 1.1:

\[
(i(g) \phi, \psi) = (\phi, e(g) \psi) \quad \text{for} \quad \phi \in \Omega^{p,q}, \psi \in \Omega^{p-1,q-1}.
\]

DEFINITION 1.5. - We set

\[
\Pi = \sum_{p,q} (n-p-q) \pi_{p,q},
\]

where \( \pi_{p,q} \) is the projection from \( \sum_{r,s} \Omega^{r,s} \) onto \( \Omega^{p,q} \).

PROPOSITION 1.2. - We have

\[
[\Pi, e(g)] = -2e(g), \quad [\Pi, i(g)] = 2i(g), \quad [i(g), e(g)] = \Pi.
\]

The proof is carried out by a direct calculation and so it is omitted.
**DEFINITION 1.6.** — Define
\[ \partial : \Omega^{p,q} \to \Omega^{p+1,q} \]
by \( \partial = \sum_k (e(dx^k) \otimes \text{id}) D_k \), where \( e(dx^k) \) is a linear map from \( \Lambda^k T^* \) to \( \Lambda^{k+1} T^* \) given by \( e(dx^k) \omega = dx^k \wedge \omega \), \( \text{id} \) is the identity map on \( \Lambda^k T^* \) and \( D_k \) is the covariant derivation with respect to \( \partial/\partial x^k \) for the locally flat affine connection \( D \).

Then we have
\[ \partial \partial = 0. \quad (1.2) \]

**DEFINITION 1.7.** — Define
\[ \delta : \Omega^{p,q} \to \Omega^{p-1,q} \]
by \( \delta = (-1)^{n+1} \sqrt{G} \ast \partial \left( \frac{1}{\sqrt{G}} \ast \right) \).

**Proposition 1.3.** — \( \delta \) is the adjoint operator of \( \partial \) with respect to the inner product given in Definition 1.1;
\[ (\partial \phi, \psi) = (\phi, \delta \psi) \quad \text{for} \quad \phi \in \Omega^{p,q}, \psi \in \Omega^{p+1,q}. \]

In Proposition 2.1 we prove the above fact in more general situation and so we omit the proof.

**Definition 1.8.** — We define
\[ \Box : \Omega^{p,q} \to \Omega^{p,q} \]
by \( \Box = \partial \delta + \delta \partial \), and call it the Laplacian. \( \phi \in \Omega^{p,q} \) is said to be \( \Box \)-harmonic if \( \Box \phi = 0 \).

**2. The Laplacian \( \Box \) on \( \Omega^{p,q}(F) \).**

Let \( F \) be a locally constant vector bundle over \( M \). Choose an open covering \( \{ U_\lambda \} \) of \( M \) such that the local triviality holds
on each $U_\lambda$. Let $\{\xi^1_\lambda, \ldots, \xi^m_\lambda\}$ be fiber coordinate systems such that the transition functions $\{f_{\lambda i}\}$ defined by

$$\xi^i_\lambda = \sum_j f_{\lambda i j} \xi^j$$

are constants. A fiber metric $a = \{a_\lambda\}$ on $F$ is a collection of $m \times m$ positive definite symmetric matrices $a = (a_{\lambda ij})$ such that each $a_{\lambda ij}$ is a $C^\infty$-function on $U_\lambda$ and

$$a_\lambda = f_{\mu \lambda} a_\mu f_{\mu \lambda}$$

holds. Let $\Omega^{p \cdot q}(F)$ denote the space of all sections of $(\wedge^p T^*) \otimes (\wedge^q T^*) \otimes F$. Using fiber coordinate systems $\{\xi^i_\lambda\}$ we express an element $\phi \in \Omega^{p \cdot q}(F)$ as $\phi = \{\phi^i_\lambda\}$.

**Definition 2.1.** Define

$$\partial : \Omega^{p \cdot q}(F) \to \Omega^{p+1 \cdot q}(F)$$

by $\partial \{\phi^i\} = \{\partial \phi\}$.$(*)$

We have then

$$\partial \partial = 0.$$  \hspace{1cm} (2.1)

**Definition 2.2.** The inner product of $\phi, \psi \in \Omega^{p \cdot q}(F)$ is

$$(\phi, \psi) = \int_M \sum a_{ij} h(\phi^i, \psi^j) v.$$

**Definition 2.3** — Define

$$\delta_a : \Omega^{p \cdot q}(F) \to \Omega^{p-1 \cdot q}(F)$$

by $\delta_a \{\phi^i\} = \left\{(-1)^{n+1} \sum_{i,k} \sqrt{G} a^{ij} \ast \partial \left( \frac{a_{jk}}{\sqrt{G}} \phi^k \right) \right\}$, where $a^{ij}$ is the $(i,j)$-component of $(a_{ij})^{-1}$.

$(*$) For brevity the subscripts $\lambda, \mu, \ldots$ are dropped where no confusion will arise.
PROPOSITION 2.1. — \( \delta_a \) is the adjoint operator of \( \partial \) with respect to the inner product given in Definition 2.2:

\[
(\partial \phi, \psi) = (\phi, \delta_a \psi) \quad \text{for} \quad \phi \in \Omega^{p-1,q}(F), \quad \psi \in \Omega^{p,q}(F).
\]

Proof. — Since \( \sum_{i,j} a_{ij} \phi^i \wedge * \psi^j \) is globally defined on \( M \), there exists \( (n-1) \)-form \( \omega \) on \( M \) such that \( \omega \otimes v = \sum a_{ij} \phi^i \wedge * \psi^j \). Then

\[
\partial(\omega \otimes v) = (\alpha \wedge \omega + d\omega) \otimes v,
\]
where \( \alpha = d \log \sqrt{G} \), and

\[
\partial (\sum a_{ij} \phi^i \wedge * \psi^j)
= (-1)^{pq} \sum a_{ij} h(\partial \phi^i, \psi^j) \otimes v + (-1)^{n-q} \sum \phi^i \wedge * \partial(a_{ij} \psi^j).
\]

Since

\[
\delta_a \psi^j = -(-1)^{n+1} (\alpha \wedge * \psi^j) + (-1)^{n+1} \sum a^k \wedge \partial(a_{kj} \psi^j),
\]
we have

\[
(\alpha \wedge \omega + d\omega) \otimes v
= (-1)^{pq} \sum a_{ij} h(\partial \phi^i, \psi^j) \otimes v + (-1)^{n-q} \sum a_{ij} \phi^i \wedge * (\alpha \wedge * \psi^j)
+ (-1)^{q+1} \sum a_{ij} \phi^i \wedge * \delta_a \psi^j
= (-1)^{pq} \sum a_{ij} h(\partial \phi^i, \psi^j) \otimes v + (\alpha \wedge \omega) \otimes v
+ (-1)^{pq-1} \sum a_{ij} h(\phi^i, \delta_a \psi^j) \otimes v,
\]
and so

\[
d\omega = (-1)^{pq} (\sum a_{ij} h(\partial \phi^i, \psi^j) - \sum a_{ij} h(\phi^i, \delta_a \psi^j)) v.
\]

Therefore

\[
0 = \int_M d\omega = (-1)^{pq} ((\partial \phi, \psi) - (\phi, \delta_a \psi)).
\]

Q.E.D.

DEFINITION 2.4. — We define

\[
\square_a : \Omega^{p,q}(F) \rightarrow \Omega^{p,q}(F)
\]
by $\Box_a = \partial \delta_a + \delta_a \partial$, and call it the Laplacian. $\phi \in \Omega^{p,q}(F)$ is said to be $\Box_a$-harmonic if $\Box_a \phi = 0$.

**Definition 2.5.** We set

$$\mathcal{H}^{p,q}(F) = \{ \phi \in \Omega^{p,q}(F) | \Box_a \phi = 0 \}.$$ 

**Theorem 2.2.** We have the following duality:

$$\mathcal{H}^{p,q}(F) \cong \mathcal{H}^{n-p,n-q}((K \otimes F)^*),$$

where $K$ is the canonical line bundle over $M$ and $(K \otimes F)^*$ is the dual bundle of $K \otimes F$.

**Proof.** For $\psi = \{ \psi^i \} \in \Omega^{p,q}(F)$ we set

$$\psi_i^* = \sum_i a_{ij} \sqrt{G} t^j \psi^j. \quad (2.2)$$

Then we have $\psi^* = \{ \psi_i^* \} \in \Omega^{n-p,n-q}((K \otimes F)^*)$. It follows from Proposition 1.1 (i)

$$\psi^j = (-1)^{p+q} \sum_i \sqrt{G} d^i \star \psi_i^*. \quad (2.3)$$

Thus the map $\psi \mapsto \psi^*$ is a linear isomorphism from $\Omega^{p,q}(F)$ onto $\Omega^{n-p,n-q}((K \otimes F)^*)$.

Let $\phi \in \Omega^{p,q}(F)$ and $\psi^* \in \Omega^{n-p,n-q}((K \otimes F)^*)$. Then

$$\sum_i \sqrt{G} \phi^i \wedge \psi_i^*$$

is globally defined on $M$. Hence there exists a $C^\infty$-function $k(\phi, \psi^*)$ on $M$ such that

$$\sum_i \sqrt{G} \phi^i \wedge \psi_i^* = k(\phi, \psi^*) v \otimes v.$$

We set

$$\langle \phi, \psi^* \rangle = (-1)^{pq} \int_M k(\phi, \psi^*) v.$$

Since

$$k(\phi, \psi^*) v \otimes v = \sum_{i,j} a_{ij} \phi^i \wedge \psi^j = (-1)^{pq} \sum_{i,j} a_{ij} h(\phi^i, \psi^j) v \otimes v,$$

we have

$$\langle \phi, \psi^* \rangle = \langle \phi, \psi \rangle$$

for $\phi, \psi \in \Omega^{p,q}(F)$. 

Define the inner product of \( \psi^*, \phi^* \in \Omega^{n-p,n-q} ((K \otimes F)^*) \) by
\[
(\psi^*, \phi^*) = \int_M \sum G d^H h(\psi_i^*, \phi_j^*) v.
\]

Since
\[
\sum G d^H h(\psi_i^*, \phi_j^*) v \otimes v = \sum a_{ij} h(\psi^i, \phi^j) v \otimes v
\]
\[
= (-1)^{pq} \sum a_{ij} \phi^j \wedge \psi^i = \sum a_{ij} h(\phi^j, \psi^i) v \otimes v,
\]
we obtain
\[
(\psi^*, \phi^*) = (\phi, \psi) \quad \text{for} \quad \phi, \psi \in \Omega^{p,q}(F).
\]

Let \( \phi \in \Omega^{p-1,q}(F) \) and \( \psi^* \in \Omega^{n-p,n-q} ((K \otimes F)^*) \). Then
\[
\sum \sqrt{G} \phi^i \wedge \psi_i^* \quad \text{is globally defined on} \ M \quad \text{and hence there exists} \quad (n-1)\text{-form} \ \omega \ \text{on} \ M \quad \text{such that}
\]
\[
\sum_i \sqrt{G} \phi^i \wedge \psi_i^* = \omega \otimes v.
\]
Since
\[
\partial \left( \sum \sqrt{G} \phi^i \wedge \psi_i^* \right)
\]
\[
= \sum_i \left\{ \alpha \wedge \sqrt{G} \phi^i \wedge \psi_i^* + \sqrt{G} \partial \phi^i \wedge \psi_i^* + (-1)^{p-1} \sqrt{G} \phi^i \wedge \partial \psi_i^* \right\}
\]
\[
= (\alpha \wedge \omega) \otimes v + \sum_i \left\{ k(\partial \phi^i, \psi_i^*) + (-1)^{p-1} k(\phi^i, \partial \psi_i^*) \right\} v \otimes v,
\]
and
\[
\partial (\omega \otimes v) = (\alpha \wedge \omega + d\omega) \otimes v,
\]
we obtain
\[
d\omega = \sum_i \left\{ k(\partial \phi^i, \psi_i^*) + (-1)^{p-1} k(\phi^i, \partial \psi_i^*) \right\} v.
\]
Therefore
\[
0 = \int_M d\omega
\]
\[
= (-1)^{pq} \langle \partial \phi, \psi^* \rangle + (-1)^{p-1+q(p-1)} \langle \phi, \partial \psi^* \rangle.
\]
This implies
\[ \langle \partial \phi, \psi^* \rangle = (-1)^{p+q} \langle \phi, \partial \psi^* \rangle. \]

Using these facts we obtain
\[ (\phi^*, \partial \psi^*) = \langle \phi, \partial \psi^* \rangle = (-1)^{p+q} \langle \partial \phi, \psi^* \rangle = (-1)^{p+q} (\partial \phi, \psi) \]
\[ = (-1)^{p+q} (\phi, \delta_a \psi) = (-1)^{p+q} (\phi^*, (\delta_a \psi)^*), \]

hence
\[ \partial \psi^* = (-1)^{p+q} (\delta_a \psi)^* \quad \text{for} \quad \psi \in \Omega^p,q(F). \quad (2.4) \]

By the same way we have
\[ (\psi^*, \delta_a \phi^*) = (\partial \psi^*, \phi^*) = \langle \phi, \partial \psi^* \rangle = (-1)^{p+q} \langle \partial \phi, \psi^* \rangle \]
\[ = (-1)^{p+q} (\partial \phi, \psi) = (-1)^{p+q} ((\partial \phi)^*, \psi^*), \]

hence
\[ \delta_a \phi^* = (-1)^{p+q} (\partial \phi)^*. \]

Thus
\[ \delta_a \psi^* = (-1)^{p+q+1} (\partial \psi)^* \quad \text{for} \quad \psi \in \Omega^{p,q}(F). \quad (2.5) \]

(2.4) and (2.5) imply that \( \psi^* \) is harmonic if and only if \( \psi \) is harmonic.

Q.E.D.

**Definition 2.6.** — We set
\[ H^{p,q}(F) = \{ \phi \in \Omega^{p,q}(F) \mid \partial \phi = 0 \} / \{ \partial \psi \mid \psi \in \Omega^{p-1,q}(F) \}. \]

A \( q \)-form \( \omega \) on \( M \) is said to be \textit{D-parallel} if \( D \omega = 0 \). Let us denote by \( P^q(F) \) the sheaf over \( M \) of germs of \( F \)-valued D-parallel \( q \)-forms.

**Definition 2.7.** — We denote by \( H^p(P^q(F)) \) the \( p \)-th cohomology group of \( M \) with coefficients on \( P^q(F) \).

**Theorem 2.3.** — We have the following isomorphisms:
\[ \gamma F^{p,q}(F) \cong H^{p,q}(F) \cong H^p(P^q(F)). \]
Proof. — By the theory of harmonic integral we have
\[ H^p,q(F) \cong H^p,q(F) . \]
Let \( A^{p,q}(F) \) denote the sheaf over \( M \) of germs of sections of
\[ (\wedge^p T^*) \otimes (\wedge^q T^*) \otimes F. \]
Then
\[ 0 \rightarrow P^q(F) \rightarrow A^{0,q}(F) \rightarrow A^{1,q}(F) \rightarrow A^{2,q}(F) \rightarrow \ldots \]
is a fine resolution of \( P^q(F) \). Thus we have \( H^p,q(F) \cong H^p(P^q(F)) \).
Q.E.D.

3. Hessian metrics on affine local coordinate systems.

Let \( M \) be a Hessian manifold with a locally flat affine connection \( D \) and a Hessian metric \( g \). We denote by \( \nabla \) the Riemannian connection for \( g \). In this section we shall express various geometric concepts on the Hessian manifold \( M \) in terms of affine local coordinate systems. Let us denote by \( D_k \) and \( \nabla_k \) the covariant derivations with respect to \( \partial/\partial x^k \) for \( D \) and \( \nabla \) respectively. Since the Christoffel symbol \( \Gamma^i_{jk} \) for \( g \) is the difference between the components of affine connections \( \nabla \) and \( D \), we may consider that \( \Gamma^i_{jk} \) is a tensor field. We have then
\[ \Gamma^i_{jk} = \frac{1}{2} g^{ik} D_k g_{jq} , \quad (3.1) \]
\[ D_k g_{jq} = 2 \Gamma^i_{jk} , \quad D_k g^{ij} = -2 \Gamma^{ij}_{k} , \]
\[ \Gamma^{jk}_{i} = \Gamma^{jk}_{i} = \Gamma^{jk}_{i} \cdot \]

Definition 3.1. — We define a 1-form \( \alpha \) and a symmetric bilinear form \( \beta \) by
\[ \alpha = D \log \sqrt{G} , \]
\[ \beta = D^2 \log \sqrt{G} , \]
where \( G = \det(g_{ij}) \), and call them the first Koszul form and the second Koszul form of \( M \) respectively.
Then we have
\[ \alpha_i = \Gamma^r_{ir}, \quad (3.2) \]
\[ \beta_{ij} = D_j \Gamma^r_{ir}. \]

**Definition 3.2.** - Let \( \gamma_k \) be the derivation of the algebra of tensor fields defined by
\[ \gamma_k = \nabla_k - D_k. \]

Let \( T^p_q \) be the space of tensor fields of type \((p, q)\) defined on \( M \).

**Definition 3.3.** - We define certain covariant derivations \( \nabla'_k, \nabla''_k \) on \( T^p_q \otimes T^q_s \) by
\[ \nabla'_{k} = (2\gamma_k) \otimes \text{id} + D_k, \]
\[ \nabla''_{k} = \text{id} \otimes (2\gamma_k) + D_k, \]
where \( \text{id} \) are the identity transformations.

Notice that
\[ \nabla_k = \frac{1}{2} (\nabla'_k + \nabla''_k), \quad \text{where} \quad k = \bar{k}. \]

**Lemma 3.1.** - For the Hessian metric \( g \) we have
\[ \nabla'_k g_{ij} = 0, \quad \nabla''_k g_{ij} = 0, \]
\[ \nabla'_k g_{i\bar{j}} = 0, \quad \nabla''_k g_{i\bar{j}} = 0. \]

**Proof.** - By (3.1) we obtain
\[ \nabla'_k g_{ij} = D_k g_{ij} - 2\Gamma^m_{ki} g_{mj} = 2\Gamma_{i\bar{k}j} - 2\Gamma_{\bar{j}ki} = 0. \]
Similarly we can prove the other equalities.

Q.E.D.

**Definition 3.4.** - Considering \( \gamma_i \) as tensor fields of type \((1, 1)\) we define tensor fields \( \gamma \) and \( S \) by
\[ \gamma = \sum_i \gamma_i \otimes dx^i, \]
\[ S = D\gamma. \]
The component of $S$ is given by

$$S_{ijkl}^i = D_k \Gamma_{ij}^i.$$

**Lemma 3.2.** — $S_{ijkl}^i = S^i_k = S^i_l = S^i_{ljk}.$

**Proof.** — Let $g^i_{ij} = D_i D_j u$. By (3.1) we have

$$S^i_{ijkl} = g^i_{ip} D_k \Gamma_{jl}^{pq} g^p_{qj} = g^i_{ip} (D_k g^p_{qj}) \Gamma_{qjl} + g^p_{qj} D_k \Gamma_{qjl}$$

$$= -2 \Gamma_{ik}^{qr} \Gamma_{qjl} + D_k \Gamma_{qlj} = -2 g^q_{qr} \Gamma_{ikr} \Gamma_{qjl} + D_k \Gamma_{qjl}$$

$$= \frac{1}{2} D_i D_j D_k D_l u - \frac{1}{2} g^q_{qr} (D_r D_i D_k u) (D_q D_l D_\mu).$$

This proves the Lemma. Q.E.D.

**Lemma 3.3.** — $\beta_{ij} = S_{rij}^r = S_{ij}^{rr}.$

**Proof.** — $\beta_{ij} = D_i \alpha_i = D_i \alpha_j = D_j \Gamma_{ij}^{rr} = S_{rij}^r.$ By Lemma 3.2 we have $S_{rij}^r = g^{rp} S_{prlj}^r = g^{rp} S_{lipr}^r = S_{ij}^{rr}.$ Q.E.D.

4. The local expression for $\Box$.

From now on we always assume that $M$ is a compact connected oriented Hessian manifold.

**Proposition 4.1.** — Let $\phi \in \Omega^{p,q}$. Then we have

$$(\partial \phi)^{i_k \ldots i_p+1} \overline{1}_q = \sum_{a} (-1)^{a-1} \nabla_{i_o} \phi_{i_1 \ldots i_o \ldots i_{p+1}} \overline{1}_q,$$

where $\hat{i_o}$ means "omit $i_o".

**Proof.** — By Definition 1.6 we have

$$(\partial \phi)^{i_p+1} \overline{1}_q = \sum_{a=1}^{p+1} (-1)^{a-1} D_{i_a} \phi_{i_1 \ldots i_o \ldots i_{p+1}} \overline{1}_q. \quad (4.1)$$

Using this and (3.1) we obtain the proposition. Q.E.D.
**Proposition 4.2.** Let $\phi \in \Omega^{p,q}$. Then we have

$$(\delta \phi)_{p-1} \bar{I}_q = -g^{\bar{r} \bar{s}} \bar{\nabla}^r \phi_{\bar{s} \bar{p}} \bar{I}_q + \alpha^r \phi_{n \bar{p} \bar{I}_q}.$$ 

**Proof.** Let $\psi \in \Omega^{p-1,q}$. By (4.1) and Green's theorem we have

$$(\phi, \partial \psi) = -\int_M D_r (\phi^{r} \bar{I}_q \sqrt{G}) \frac{1}{\sqrt{G}} \psi_{1 \bar{p} \bar{I}_q} v.$$ 

Thus we obtain

$$(\delta \phi)^{p-1} \bar{I}_q = -D_r \phi^{r} \bar{I}_q - \alpha^r \phi^{r} \bar{I}_q$$

$$= -\nabla_r \phi^{r} \bar{I}_q + \alpha^r \phi^{r} \bar{I}_q.$$ 

This completes the proof.

Q.E.D.

**Theorem 4.1.** Let $\phi \in \Omega^{p,q}$. Then we have

$$(\Box \phi)_{p \bar{I}_q} = -g^{\bar{r}} \bar{\nabla}^r \phi_{\bar{s} \bar{p} \bar{I}_q} + \alpha^r \bar{\nabla}^r \phi_{\bar{s} \bar{p} \bar{I}_q} - \sum_s \beta_s^r \phi_{l_1 \ldots (s) \ldots l_p \bar{I}_q}$$

$$+ 2 \sum_{s, r} S^{l_r}_{l_1 \ldots (s) \ldots l_p \bar{I}_q} \phi_{l_1 \ldots (s) \ldots l_r \ldots \bar{I}_q},$$

where $(s)_o$ means "substitute $s$ for $o$-th place".

**Proof.** Using Proposition 4.1, Proposition 4.2 and $\nabla^r \alpha^l = \beta^l_1$ we obtain

$$(\partial \delta \phi)_{p \bar{I}_q} = -g^{\bar{r}} \sum_s \nabla^r \phi_{l_1 \ldots (s) \ldots l_p \bar{I}_q} + \sum_s \beta_s^r \phi_{l_1 \ldots (s) \ldots l_p \bar{I}_q}$$

$$+ \sum_s \alpha^r \nabla^r \phi_{l_1 \ldots (s) \ldots l_p \bar{I}_q},$$

$$(\delta \partial \phi)_{p \bar{I}_q} = -g^{\bar{r}} (\bar{\nabla}^r \phi_{l_1 \ldots l_p \bar{I}_q} - \sum_s \nabla^r \phi_{l_1 \ldots (s) \ldots l_p \bar{I}_q})$$

$$+ \alpha^r (\bar{\nabla}^r \phi_{l_1 \ldots l_p \bar{I}_q} - \sum_s \nabla^r \phi_{l_1 \ldots (s) \ldots l_p \bar{I}_q}),$$
Let us calculate the third term on the right-hand of the above formula. Since \([\nabla^i, \nabla^j]\) is a derivation of the algebra of tensor fields which maps every function to 0 and since

\[
[\nabla^i, \nabla^j] \phi_k = 2S_{ijk} \xi_p,
\]

\[
[\nabla^i, \nabla^j] \xi_k = -2S_{ijk} \xi_p,
\]

we have

\[
[\nabla^i, \nabla^j] \phi_{i_1\ldots(i)\ldots p\ldots q} = \sum \chi \frac{2S^{m}}{l_{a} l_{r}} \phi_{i_1\ldots(m)\ldots p\ldots q} + 2S^{m}_{l_{a} l_{r}} \phi_{i_1\ldots(m)\ldots p\ldots q} - \sum \chi \frac{2S^{m}}{r_{a} l_{r}} \phi_{i_1\ldots(m)\ldots p\ldots q}.
\]

Thus, by Lemma 3.2 and 3.3 we obtain

\[
g^{\alpha \beta} \sum \chi [\nabla^i, \nabla^j] \phi_{i_1\ldots(i)\ldots p\ldots q} = 2 \sum \chi \beta^{m}_{l_{a} l_{r}} \phi_{i_1\ldots(m)\ldots p\ldots q} - 2 \sum \chi S^{m}_{l_{a} l_{r}} \phi_{i_1\ldots(m)\ldots p\ldots q}.
\]

This completes the proof.

Q.E.D.

**Example.** — For the Hessian metric \(g\) we have

\[
(\Box g)_{ij} = -\beta_{ij}.
\]
Thus the Hessian metric $g$ is $\Box$-harmonic if and only if the second Koszul form $\beta = 0$. Therefore, by [12] the following conditions are equivalent:

(i) $g$ is $\Box$-harmonic.

(ii) The first Koszul form $\alpha = 0$.

(iii) The second Koszul form $\beta = 0$.

(iv) $g$ is locally flat.

5. The local expression for $\Box_a$.

Let $F$ be a locally constant line bundle over a compact connected oriented Hessian manifold $M$, and let $a$ be a fiber metric on $F$.

**Proposition 5.1.** — We have

$$\delta_a = \delta + i(A),$$

where $A = - D \log a$ and $(i(A) \phi)_l p - 1 \bar{q} = A^r \phi_{r lp - 1 \bar{q}}$ for $\phi \in \Omega^{p,q}(F)$.

**Proof.** — By Definition 1.2, 1.7 and 2.3 we have

$$\delta_a = (-1)^{n+1} \frac{\sqrt{G}}{a} \cdot \partial \left( \frac{a}{\sqrt{G}} \right),$$

$$= (-1)^n \cdot e(A) \ast + (-1)^{n+1} \sqrt{G} \cdot \partial \left( \frac{1}{\sqrt{G}} \right),$$

$$= i(A) + \delta,$$

where

$$(e(A) \phi)_{l_1 \ldots l_p + 1 \bar{q}} = \sum_{\sigma} (-1)^{\sigma - 1} A_{l_\sigma} \phi_{l_1 \ldots \bar{q} \ldots l_p + 1 \bar{q}}$$

for $\phi \in \Omega^{p,q}(F)$. Q.E.D.
DEFINITION 5.1. — For $\phi \in \Omega^{p,q}(F)$ we set
\[
\nabla_{\tau}^{(a)} \phi = \frac{1}{a} \nabla_{\tau} (a \phi).
\]

THEOREM 5.1. — Let $\phi \in \Omega^{p,q}(F)$. Then we have
\[
(\Box_{\sigma})_{1_p \bar{1}_q} = -g^{\sigma\tau} \nabla_{\tau}^{(a)} \nabla_{\sigma} \phi_{1_p \bar{1}_q} + \alpha^a \nabla_{\sigma} \phi_{1_p \bar{1}_q} + \sum_{\sigma} (-\beta_{\sigma} + B_{\sigma} \phi_{1_p \bar{1}_q} + S_{\sigma} \phi_{1_p \bar{1}_q} + 2 \sum_{\sigma, \tau} S_{\sigma} \phi_{1_p \bar{1}_q} + \phi_{1_p \bar{1}_q} \phi_{1_p \bar{1}_q} + \phi_{1_p \bar{1}_q} \phi_{1_p \bar{1}_q}.
\]

Proof. — By Proposition 5.1 we have
\[
\Box_{\sigma} = \Box + i(A) \partial + \partial i(A).
\]
A straightforward calculation shows
\[
(i(A) \partial + \partial i(A))_{1_p \bar{1}_q} = g^{\sigma\tau} \nabla_{\tau} \phi_{1_p \bar{1}_q} + \sum_{\sigma} (-\beta_{\sigma} + B_{\sigma} \phi_{1_p \bar{1}_q} + S_{\sigma} \phi_{1_p \bar{1}_q} + 2 \sum_{\sigma, \tau} S_{\sigma} \phi_{1_p \bar{1}_q} + \phi_{1_p \bar{1}_q} \phi_{1_p \bar{1}_q} + \phi_{1_p \bar{1}_q} \phi_{1_p \bar{1}_q}.
\]
Thus our assertion follows from the above facts and Theorem 4.1.

Q.E.D.

6. A vanishing theorem of Kodaira-Nakano type.

Let $\theta$ be a symmetric covariant tensor field of degree 2. Considering $\theta$ as an element in $\Omega^{1,1}$ we define
\[
e(\theta) : \Omega^{p,q} \longrightarrow \Omega^{p+1,q+1},
\]
\[
i(\theta) : \Omega^{p,q} \longrightarrow \Omega^{p-1,q-1},
\]
by $e(\theta) \phi = \theta \wedge \phi$ for $\phi \in \Omega^{p,q}$ and $i(\theta) = (-1)^{n+p+q+1} e(\theta)^{*}$. Then $i(\theta)$ is the adjoint operator of $e(\theta)$ with respect to the inner product in Definition 1.1 and 2.2.
In this section we always assume that $F$ is a locally constant line bundle over $M$.

**Proposition 6.1.** — We have

(i) $[\Box_a, e(g)] = e(B + \beta)$,

(ii) $[\Box_a, i(g)] = -i(B + \beta)$.

The proof follows from a straightforward calculation and so it is omitted.

**Proposition 6.2.** — Suppose $\Box_a \phi = 0$. Then we have

(i) $(e(B + \beta) i(g) \phi, \phi) \leq 0$.

(ii) $(i(g) e(B + \beta) \phi, \phi) \geq 0$.

(iii) $([i(g), e(B + \beta)] \phi, \phi) \geq 0$.

**Proof.** — By Proposition 6.1 (i) we have $\Box_a e(g) \phi = e(B + \beta) \phi$.

Thus we have

$0 \leq (\Box_a e(g) \phi, e(g) \phi) = (e(B + \beta) \phi, e(g) \phi) = (i(g) e(B + \beta) \phi, \phi)$,

which implies (ii). By the same way, since $\Box_a i(g) \phi = -i(B + \beta) \phi$ we obtain

$0 \leq (\Box_a i(g) \phi, i(g) \phi) = (-i(B + \beta) \phi, i(g) \phi)$

$= (\phi, -e(B + \beta) i(g) \phi)$,

which shows (i). (iii) follows from (i) and (ii).

Q.E.D.

**Theorem 6.1.** — Let $M$ be a compact connected oriented Hessian manifold. Denote by $K$ the canonical line bundle over $M$. Let $F$ be a locally constant line bundle over $M$.

(i) If $2F + K$ is positive, then

$H^{p,q}(F) = 0$ for $p + q > n$.

(ii) If $2F + K$ is negative, then

$H^{p,q}(F) = 0$ for $p + q < n$. 

Proof. – Suppose $2F + K$ is negative. Then $B + \beta$ is negative definite. Therefore $g' = -(B + \beta)$ gives a Hessian metric on $M$. If we denote by $\beta'$ the Koszul form on $M$ with respect to $g'$, then there exists a positive $C^\infty$-function $f$ on $M$ such that

$$\beta' = \beta + D^2 \log f.$$ 

If $B$ is a Koszul form of $F$ with respect to a fiber metric $a = \{a_\lambda\}$, then the Koszul form $B'$ of $F$ with respect to the fiber metric $a' = \{fa_\lambda\}$ satisfies

$$B' + \beta' = B + \beta = -g'.$$

Therefore if we use $-(B + \beta)$ as a Hessian metric, the formula in Proposition 6.2 (iii) is reduced to

$$([i(g), - e(g)] \phi, \phi) \geq 0 \quad \text{for} \quad \phi \in \mathfrak{g}^{p,q}(F).$$

Thus by Proposition 1.2 we have

$$(n - p - q) (\phi, \phi) \leq 0 \quad \text{for} \quad \phi \in \mathfrak{g}^{p,q}(F).$$

Therefore, if $n - p - q > 0$ then $\phi = 0$. Hence (ii) is proved. (i) follows from (ii) and Theorem 2.2.

Q.E.D.

7. A vanishing theorem of Koszul type.

In this section we mention a vanishing theorem of Koszul type. Let $M$ be a compact oriented hyperbolic affine manifold. Then there exists a canonical Hessian metric $g$ and a unique Killing vector field $H$ on $M$ such that

$$D_X H = X, \quad (7.1)$$

for all vector field $X$ on $M$ [7]. The following theorem is essentially due to Koszul.

Theorem 7.1. – Let $F$ be a locally constant vector bundle over a compact hyperbolic affine manifold. If there exist a fiber metric $a = \{(a_\eta)\}$ and a constant $c (\neq -2q)$ such that

$$Ha_\eta = ca_\eta,$$

then

...
then we have

\[ H^{p,q}(F) = 0, \quad \text{for} \quad p > 0 \quad \text{and} \quad q \geq 0. \]

The proof of this theorem is nearly the same as Koszul [7], and so we omit the proof.

**Corollary 7.1.** — Let \( M \) be a compact oriented hyperbolic affine manifold. Then we have

\[ H^{p,q}(1) = 0, \quad \text{for} \quad p, q > 0, \]

where \( 1 \) is the trivial vector bundle over \( M \).

The tensor bundle \( \bigotimes T \bigotimes T^* \) satisfies the condition of Theorem 7.1 if \( q - r + s \neq 0 \).

We give another example of locally constant vector bundle over \( M \) which satisfies the conditions of Theorem 7.1. Let \( \Omega \) be an open convex cone in \( \mathbb{R}^n \) with vertex 0 not containing any full straight line. Suppose that a discrete subgroup \( \Gamma \) of \( GL(n, \mathbb{R}) \) acts properly discontinuously and freely on \( \Omega \) such that \( M = \Gamma \backslash \Omega \) is compact. Assume further that there exist a linear mapping from \( \Omega \) to the space of all \( m \times m \) positive definite real symmetric matrices and a homomorphism from \( \Gamma \) to \( GL(m, \mathbb{R}) \), which are denoted by the same letter \( \rho \), such that

\[ \rho(\gamma x) = \rho(\gamma) \rho(x) \rho(\gamma)^t \quad \text{for} \quad \gamma \in \Gamma, \ x \in \Omega. \]

We denote by \( F_\rho \) the vector bundle over \( M \) associated with the universal covering \( \Omega \longrightarrow M \) and \( \rho \). Let \( U \) be an evenly covered open set in \( M \). Choosing a section \( \sigma \) on \( U \) we set

\[ a = (\rho \circ \sigma)^{-1}. \]

Then \( a \) is a fiber metric on \( F_\rho \) and we have

\[ Ha = -a. \]

Therefore

**Corollary 7.2.** — We have

\[ H^{p,q}(F_\rho) = 0 \quad \text{for} \quad p > 0 \quad \text{and} \quad q \geq 0. \]
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