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$p$-adic interpolation of logarithmic derivatives associated to certain Lubin-Tate formal groups


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**p-ADIC INTERPOLATION OF LOGARITHMIC DERIVATIVES ASSOCIATED TO CERTAIN LUBIN-TATE FORMAL GROUPS**

by John L. BOXALL

**Introduction.**

The purpose of this paper is to study the $p$-adic interpolation properties of the values of logarithmic derivatives of power series at 0 attached to certain one-dimensional formal groups over $p$-adic integer rings. The earliest results at this kind were given in Iwasawa [3], following the then unpublished work of Kubota and Leopoldt [9], who applied them to the construction of $p$-adic L-functions attached to Dirichlet characters. They were subsequently used to construct $p$-adic L-functions in other contexts, notably those attached to abelian extensions of totally real fields [1] and to elliptic curves with complex multiplication, at least when $p$ splits in the field of complex multiplication. We first recall the interpolation results of Iwasawa, Kubota and Leopoldt in a form similar to that in Lichtenbaum [10, §1]. Fix an odd prime $p$ and let $C_p$ be the completion of the algebraic closure of $Q_p$. We denote by $v : C_p^* \rightarrow Q$ the valuation normalised so that $v(p) = 1$. Let $Q_0$ be the ring of power series

$$f(T) = \sum_{n=0}^{\infty} \frac{c_n T^n}{n!} \in C_p [[T]] \mid v(c_n) \rightarrow \infty \text{ as } n \rightarrow \infty.$$ (1)

For $\beta \in Z/(p - 1) Z$ and $f \in Q_0$ define $f_{\beta}$ to the power series

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Theorem A. (i) Let \( f \in \mathbb{Q}_0 \) and \( \alpha \in \mathbb{Z} / (p-1) \mathbb{Z} \). Then there exists a unique continuous function \( C_f^{(\alpha)} : \mathbb{Z}_p \longrightarrow \mathbb{C} \) such that for each \( \beta \in \mathbb{Z} / (p-1) \mathbb{Z} 
abla 0 \),

\[
C_f^{(\alpha)} (k) = (-1)^{\alpha - \beta} \left( (1 + T) \frac{d}{dT} \right)^k f_{\alpha - \beta} (T) |_{T=0}
\]

whenever \( k \geq 0 \) and \( k \in \beta \).

(ii) If \( f \in \mathbb{O}[[T]] \), then \( f_{\beta} \in \mathbb{O}[[T]] \) for each \( \beta \) and there is a unique power series \( G_f^{(\alpha)} (X) \in \mathbb{O}[[X]] \) such that

\[
G_f^{(\alpha)} (u^s - 1) = C_f^{(\alpha)} (s)
\]

for all \( s \in \mathbb{Z}_p \).

The existence of the power series \( G_f^{(\alpha)} \) in (ii) is equivalent to the assertion that \( C_f^{(\alpha)} \) is an Iwasawa function (see Serre [14]), or that it is \( p \)-adic Mellin transform of a Mazur measure (see Lang [8, Chapter 4] or Mazur and Swinnerton-Dyer [12]).

We now explain the generalisation of Theorem A to which this article is devoted. Let \( \mathfrak{F} \) be a (commutative) one dimensional formal group over the ring of integers \( \mathbb{O}_{F_p} \) of a finite extension \( F_p \) of \( \mathbb{Q}_p \).
Let $K_p$ be another finite extension of $Q_p$, of degree $h$ and ramification index $e$, let $\mathcal{O}_{K_p}$ be the ring of integers of $K_p$, $\pi$ a uniformising parameter for $K_p$, and $q$ (a power of $p$) the cardinality of the residue class field $k_{K_p}$. We suppose that $\mathcal{S}$ is isomorphic over $\mathcal{O}$ to the basic Lubin-Tate group $\mathcal{S}_0$ associated to the polynomial $\pi T + T^q$, so that in particular the height of $\mathcal{S}$ is $h$. Let $\eta_0$ be a fixed non-trivial element of ker $[\pi]$, the $\pi$-division points of $\mathcal{S}$, and write $\lambda$ for the logarithm of $\mathcal{S}$. The Teichmuller character $\omega: \mathcal{O}_{K_p}^\times \rightarrow \mathcal{O}_{K_p}^\times$ is defined by taking $\omega(a)$ to be the unique $q - 1 - st$ root of unity in $K_p$ which satisfies $\omega(a) \equiv a (\text{mod } \pi)$. Also, for each residue class $\beta \in Z/(q - 1) Z$ we denote by $\tau(\beta)$ the Gauss sum to be defined in § 1. If $f \in \mathcal{O}[[T]]$, we define $\Delta^\beta f$ by

$$ (\Delta^\beta f)(T) = f(T) - \sum_q \frac{1}{q} f(T + s[c](\eta_0)) \quad \text{if } \beta = 0, $$

$$ = \frac{\tau(\beta)}{q} \sum_{c \neq 0} f(T + s[c](\eta_0)) \omega^{-\beta}(c) \quad \text{if } \beta \neq 0. $$

Here the sum is taken over a complete set of representatives $\{c\}$ of $k_{K_p}^\times$ in $\mathcal{O}_{K_p}$ if $\beta = 0$ and of $k_{K_p}^\times$ if $\beta = 0$. Let $e$ be the ramification degree of $K_p$ over $Q_p$. We shall prove

**Theorem B.** — Suppose that $e \leq p - 1$. Let $f \in \mathcal{O}[[T]]$ and $\alpha \in Z/(q - 1) Z$. Then there exists a constant $\Omega_p \in C_p$ with $v(\Omega_p) = \frac{1}{p - 1} - \frac{1}{e(q - 1)}$ and a unique locally analytic function $C_f^{(\alpha)}: Z_p \rightarrow C_p$ such that for each $\beta \in Z/(q - 1) Z$

$$ C_f^{(\alpha)}(k) = \frac{(-1)^{p-\beta}}{\Omega_p^k} \left( \frac{1}{\lambda'(T)} \frac{d}{dT} \right)^k (\Delta^\beta f)(T) \big|_{T=0} $$

whenever $k \geq 0$ and $k \in \beta$.

(Loosly analytic means that $C_f^{(\alpha)}$ can be expanded in a Taylor series about every $s_0 \in Z_p$).

If $\mathcal{S}$ is the multiplicative group $G_m$ (so that $h = 1$), it is easy to see that Theorem B reduces to a weaker form of Theorem A (iii);
more generally, if \( \mathcal{F} \) is an arbitrary formal group of height 1 the results of Lubin [11] imply that \( \mathcal{F} \) is isomorphic to \( \mathbb{G}_m \) and so it is possible to deduce a much stronger form of Theorem B from Theorem A; thus we shall only regard Theorem B as being of interest when the height of \( \mathcal{F} \) is \( \geq 2 \). After some preliminaries in \( \S \) 1, we define in \( \S \) 2 a subring \( B_0 \) of \( \mathbb{C}_p[[T]] \) which is the analogue of \( \mathbb{Q}_0 \) and prove the existence of a continuous function interpolating the right hand side of (4). In \( \S \) 3 we describe a condition on the \( f \in B_0 \) which ensures that \( \mathbb{C}_f^{(\alpha)} \) is locally analytic while in \( \S \) 4 we show that this condition is satisfied if \( f \in \mathcal{O} [[T]] \).

A weaker form of Theorem B in the case when the height of \( \mathcal{F} \) is 2 has been proved by Katz [6], [7] and also by Rubin [13], but our argument is more in the line of Lichtenbaum’s proof of Theorem A, and in fact there is more than a germ of these ideas in Kummer’s note [4].

In a subsequent paper we shall show how these results can be used to construct \( p \)-adic L-functions attached to elliptic curves with complex multiplication, even if \( p \) is inert or ramified in the field of complex multiplication. It would be interesting to find applications of our results to other situations. For example the power series studied by Coleman [2] and to generalise Theorem B to other kinds of formal groups.

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1. Preliminaries.

Let \( p \) be an odd prime. The symbols \( \mathbb{Z}, \mathbb{Z}_p, \mathbb{Q}, \mathbb{Q}_p \) have their usual meaning and we write \( \mathbb{Z}_+ \) for the non-negative integers. Let \( \mathbb{C}_p \) denote the completion of the algebraic closure of \( \mathbb{Q}_p \), \( \mathcal{O} \) its ring of integers and \( \mathfrak{m} \) its maximal ideal. We denote by \( v: \mathbb{C}_p^* \to \mathbb{Q} \) the \( p \)-adic valuation normalised so that \( v(p) = 1 \).
If $L_p$ is a subfield of $\mathbb{C}_p$ we write $\mathbb{O}_{L_p}$ for its ring of integers, $m_{L_p}$ for its maximal ideal, and $k_{L_p}$ for the residue class field; $\mu_n(L_p)$ is the group of $n$-th roots of unity in $L_p$ and $\mu(L_p)$ the group of all roots of unity in $L_p$. Let $\omega: \mathbb{O}^* \to \mu(\mathbb{C}_p)$ be the Teichmüller character; if $a \in \mathbb{O}^*$ then $\omega(a)$ is the unique prime-to-$p$-th root of unity congruent to $a \pmod{m}$; we also use $\omega$ for its restriction to $\mathbb{O}^*_{L_p}$ for any subfield $L_p$. Let $K_p$ denote an extension of $\mathbb{Q}_p$ of degree $h$ and residue class degree $e$, and $q$ (a power of $p$) the cardinality of $k_{K_p}$. We consider a one-dimensional (commutative) formal group $\mathcal{H}$ defined over $\mathbb{O}_F$, where $F_p$ is another finite extension of $\mathbb{Q}_p$, and as in the Introduction we assume that $\mathcal{H}$ is $\mathbb{O}$-isomorphic to the Lubin-Tate group associated to the polynomial $T^h + T^e$, where $\pi$ is a uniformising parameter for $K_p$ (for the theory and basic properties of such groups see Lang [8, Chapter 8]). This implies that the absolute endomorphism ring of $\mathcal{H}$ is isomorphic to $\mathbb{O}_{K_p}$, and we may suppose that all the elements of $\text{End}(\mathcal{H})$ are defined over $F_p$, and that $K_p \subseteq F_p$. Let $\lambda(T)$ be the logarithm of $\mathcal{H}$, i.e. the unique element of $F_p[[T]]$ satisfying

$$\lambda(X + Y) = \lambda(X) + \lambda(Y) \quad \text{and} \quad \lambda(T) = T + O(T^2).$$

It is well-known that $\lambda'(T) \in 1 + T \mathbb{O}[[T]]$ and in fact that

$$\frac{1}{\lambda'(T)} = \frac{\delta}{\delta Y} \mathcal{H}(X, Y) \bigg|_{X=T, Y=0}$$

(see Lang [8, Chapter 7]). We denote by $\ker[\pi]$ the group of order $q$ which is the kernel of multiplication by $\pi$ in the group law of $\mathcal{H}$, and fix a non-trivial element $\eta_0$ of $\ker[\pi]$, so that $\ker[\pi] = \{[c](\eta_0)\}$ as $c$ runs over a set of representatives for $k_{K_p}$ in $\mathbb{O}_{K_p}$ (here $[c] \in \text{End}(\mathcal{H})$ denotes the element corresponding to $c \in \mathbb{O}_{K_p}$).

Let

$$\mathcal{H} = \{f \in \mathbb{O}[[T]] \mid f(X + Y) = f(X)f(Y) \quad \text{and} \quad f(0) = 1\}.\ (6)$$

Then $\mathcal{H}$ can be identified with $\text{Hom}_e(\mathcal{H}, G_m)$. It is evident that every $f \in \mathcal{H}$ induces an element of $\text{Hom}(T_p \mathcal{H}, T_p G_m)$, $T_p \mathcal{H}$ and
$T_p G_m$ being the Tate modules of $G$ and $G$ respectively. According to an important result of Tate [15] the induced map

$$\text{Hom}_e (G, G_m) \rightarrow \text{Hom}(T_p G, T_p G_m)$$

is an isomorphism of $\mathbf{Z}_p$-modules. From this we deduce the following facts which are vital to the following discussion.

**FACT 1:** $\mathcal{H}$ is a free $\mathbf{Z}_p$-module of rank $h$.

**FACT 2:** For each non-zero element $\eta$ of ker $[\pi]$ there exists $t \in \mathcal{H}$ such that $t(\eta)$ is a *primitive* $p$-th root of unity.

In our case, if $t \in \mathcal{H}$ then also $t \circ [a] \in \mathcal{H}$ whenever $a \in \mathcal{O}_{K_p}$, and so $\mathcal{H}$ acquires the structure of an $\mathcal{O}_{K_p}$-module, which must necessarily be free of rank one: we fix a generator $t_1$ and write $t_a$ for $t_1 \circ [a]$. Define a constant $\Omega_p$ by

$$t_a(T) = 1 + \Omega_p aT + O(T^2).$$

We denote by Diff $(\mathcal{O})$ the $\mathcal{O}$-algebra of all $\mathcal{O}$-invariant differential operators taking $\mathcal{O}[[T]]$ into itself (recall that $\mathcal{O}$-invariant means that $(Df)(T + sw) = D(f(T + sw))$ for all $D \in \text{Diff}(\mathcal{O})$, $f \in \mathcal{O}[[T]]$ and $w \in m$). It is known that Diff $(\mathcal{O})$ is the free $\mathcal{O}$-module on the operators $D_n$, $n \in \mathbf{Z}_+$ defined by the “Taylor expansion”

$$f(X + sY) = \sum_{n=0}^{\infty} (D_n f)(X) Y^n. \quad (8)$$

We now recall some properties of $\mathcal{H}$ and Diff $(\mathcal{O})$ (cf. [6], [7]).

**Lemma 1.** – (i) Each $t \in \mathcal{H}$ is a simultaneous eigenfunction for all the $D \in \text{Diff}(\mathcal{O})$; in fact $(D t)(T) = D(t(0)) t(T)$.

(ii) We have the expansion

$$t(T) = \sum_{n=0}^{\infty} (D_n t)(0) T^n.$$  

(iii) $(D_0 f)(T) = f(T)$ and $(D_1 f)(T) = \frac{1}{\lambda'(T)} f'(T)$ for all $f \in \mathcal{C}_p[[T]]$, i.e. $D_1$ is the logarithmic derivative of $\mathcal{O}$. 


(iv) If $a, b \in \mathcal{O}_{K_p}$ then $t_{a+b}(T) = t_a(T) t_b(T)$, $t_0(T) = 1$, $t_{ab}(T) = t_a([b](T)) = t_b([a](T))$, and if $b \in \mathbb{Z}_p$ then

$$t_{ab}(T) = t_a(T)^b = \sum_{n=0}^{\infty} (t_a(T) - 1)^n \binom{b}{n}.$$

**Proof.** (i) We have

$$D_t(t + s w) = D(t(t + s w)) = D(t(T)) t(w) = (D_t)(T) t(w)$$

for all $w \in m$. Putting $T = 0$ we obtain $(D_t)(w) = (D_t)(0) t(w)$ and since $w$ is arbitrary the assertion follows.

(ii) This is the special case $X = 0$, $Y = T$ of (8).

(iii) Since

$$f(X + s Y) = f(\mathcal{F}(X, Y)) = \sum_{n=0}^{\infty} \frac{Y^n}{n!} \frac{\delta^n f(\mathcal{F}(X, Y))}{\delta Y^n} \bigg|_{Y=0}$$

by the "usual" Taylor expansion, we find that $D_0 f = f$ and

$$(D_1 f)(T) = \frac{\delta f(\mathcal{F}(X, Y))}{\delta Y} \bigg|_{X=T, Y=0} = \frac{1}{\lambda'(T)} f'(T)$$

using the chain rule together with (5).

(iv) This is obvious from the definition of $\mathcal{F}$; note that the last expression is well-defined since $t_a(T) - 1$ has no constant term.

Our next task is to define the Gauss sum $\tau(\beta)$ appearing in Theorem B. Fact 2 above together with part (iv) of Lemma 1 tell us that $t_a$ induces a homomorphism from $\ker [\pi]$ onto $\mu_p(\mathbb{C}_p)$ if and only if $a \not\equiv 0 \pmod{\pi}$. In particular $t_a = t_b$ (restricted to $\ker [\pi]$) if and only if $a \equiv b \pmod{\pi}$.

**Lemma 2.** (ii) Let $\eta \in \ker [\pi]$. Then

$$\sum_{a \equiv \eta \pmod{\pi}} t_a(\eta) = 0 \quad \text{if} \quad \eta \not\equiv 0$$

$$= q \quad \text{if} \quad \eta \equiv 0.$$

(ii) Let $a \in \mathcal{O}_{K_p}$. Then

$$\sum_{\eta \in \ker [\pi]} t_a(\eta) = 0 \quad \text{if} \quad a \not\equiv 0 \pmod{\pi}$$

$$= q \quad \text{if} \quad a \equiv 0 \pmod{\pi}.$$
Proof. – (i) If \( \eta \neq 0 \) then \( \sum_a t_a(\eta) = q p^{-1} \sum_{\xi \in \eta_p} \xi = 0 \) while
\[\text{if } \eta = 0 \text{ then } \sum_a t_a(\eta) = \sum_a 1 = q \text{ by Fact 2. The proof of (ii) is similar.}\]

Now let \( \beta \in \mathbb{Z}/(q - 1) \mathbb{Z} \) with \( \beta \neq 0 \) and recall that \( \eta_0 \) is a fixed non-trivial element of \( \ker[\pi] \). Define
\[
\tau_a(\beta) = \sum' \omega^\beta(u) t_a([u](\eta_0)),
\]
where \( \sum' \) indicates that the sum is taken over a complete set of representatives of \( k_\mathbb{K}_p^* \) in \( \Theta_{\mathbb{K}_p} \) (i.e. omitting the term \( u \equiv 0 \pmod{\pi} \)); and write \( \tau(\beta) \) for \( \tau_1(\beta) \). The \( \tau_a(\beta) \)'s may be thought of as Gauss sums and we have

**Lemma 3.** – (i) \( \tau_a(\beta) = \omega^{-\beta}(a) \tau(\beta) \) if \( a \in \Theta_{\mathbb{K}_p}^* \).

(ii) \( \tau(\beta) \tau(-\beta) = (-1)^\beta q \).

Proof. – (i) We have
\[
\tau_a(\beta) = \sum'_u \omega^\beta(u) t_a([u](\eta_0))
= \sum'_u \omega^\beta(u) t_1([au](\eta_0))
= \omega^{-\beta}(a) \sum'_u \omega^\beta(au) t_1([au](\eta_0))
\]
and (i) follows.

(ii) we have
\[
\tau(\beta) \tau(-\beta) = \sum'_{u,v} \omega^\beta(u) \omega^{-\beta}(v) t_1([u](\eta_0)) t_1([v](\eta_0))
= \sum'_{u,x} \omega^\beta(u) \omega^{-\beta}(-xu) t_1([u-xu](\eta_0))
\]
(on writing \( v = -xu \))
\[
= (-1)^\beta \sum_x \omega^{-\beta}(x) \sum'_u t_1([u-xu](\eta_0)).
\]
But by Lemma 2 (ii) \( \sum_u t_1 ([u - xu] (\eta_0)) = q - 1 \) if \( x \equiv 1 \pmod{\pi} \) and \(-1\) otherwise. Therefore
\[
\tau(\beta) \tau (-\beta) = (-1)^\beta [(q - 1) + \sum_{x \not \equiv 1 (\pi)} \omega^{-\beta}(x) (-1)]
\]
\[
= (-1)^\beta [(q - 1) + (-1)(-1)]
\]
\[
= (-1)^\beta q,
\]
as claimed.

2. An interpolation theorem.

In this section we define a ring of power series \( B_0 \subseteq \mathbb{C}_p[[T]] \) and a "twisting operator" \( \Delta_{\mathbb{Z}}^{(\beta)} \) for each residue class \( \beta \mod (q - 1) \) of \( \mathbb{Z} \), and prove an interpolation theorem for the quantities
\[
(D_k^\beta \Delta_{\mathbb{Z}}^{(\beta)} f)(0), \quad k = 0, 1, 2, \ldots,
\]
where \( D_1 = \frac{1}{\lambda'(T)} \frac{d}{dT} \) and \( f \in B_0 \).

We first introduce a notational convention which will be in constant use throughout this and the next section: if \( x \in \mathbb{C}_p^h \) (resp. \( \mathbb{K}_p^h \), resp. \( \mathbb{Z}_+^h \) etc.) then we denote the \( i \)-th component of \( x \) by \( x_i \). Conversely, if a system of \( h \) elements of \( \mathbb{C}_p \) (resp. \( \mathbb{K}_p \), resp. \( \mathbb{Z}_+ \) etc.) has been denoted by a letter with suffices \( i = 1, 2, \ldots, h \) then the same letter (without a suffix) is used for the corresponding element of \( \mathbb{C}_p^h \) (resp. \( \mathbb{K}_p^h \), resp. \( \mathbb{Z}_+^h \) etc.). If \( n \in \mathbb{Z}_+^h \) we write \( n! \) for \( n! \). If \( X_1, X_2, \ldots, X_h \) are indeterminates, the monomial \( X_1^{n_1} X_2^{n_2} \ldots X_h^{n_h} \) is abbreviated to \( X^n \); however the letter \( T \) will always stand for a single indeterminate.

Let \( x \in \mathbb{C}_p^h \) be a basis for \( \mathbb{C}_{\mathbb{K}_p} \) over \( \mathbb{Z}_p \), and \( Q_0 \) the ring
\[
\left\{ F(X) \left| \sum_{n \in \mathbb{Z}_+^h} \frac{c_n X^n}{n!} \in \mathbb{C}_p \left[ [X_1, X_2, \ldots, X_h] \right] \right. \right\}
\]
Define a homomorphism \( e : \mathbb{Q}_0 \rightarrow \mathbb{C}_p[[T]] \) by setting
\[ e(X_i) = t_{X_i}(T) - 1 \]
for each \( i = 1, 2, \ldots, h \). This is well-defined since \( t_a(T) - 1 \) has no constant term.

**Definition.** - \( B_0 \) is the image of \( e \) in \( \mathbb{C}_p[[T]] \). If \( a \in \mathbb{C}_p \)
we can write \( a = \sum_{i=1}^{h} v_i x_i \) where \( v_i \in \mathbb{Z}_p \) and so (using Lemma 1
(iv)) \( t_a(T) = \prod_{i=1}^{h} t_{X_i}(T)^{v_i} = e \left( \prod_{i=1}^{h} (1 + X_i)^{v_i} \right) \). This implies at once
that \( B_0 \) does not depend on the choice of basis \( x \).

Let \( \beta \in \mathbb{Z}/(q - 1)\mathbb{Z} \). One would like to define the \( \Theta \)-linear
operator \( \Delta_{\beta} : B_0 \rightarrow \mathbb{C}_p[[T]] \) by
\[
(\Delta_{\beta} f)(T) = f(T) - \frac{1}{q} \sum_{u} f(T) + \mu_{\mathbb{Z}_p}(\eta) \quad \text{if} \quad \beta = 0,
\]
\[
= \frac{\tau(\beta)}{q} \sum_{u} f(T) + \mu_{\mathbb{Z}_p}(\eta) \omega^{-\beta}(u) \quad \text{if} \quad \beta \neq 0.
\]
However at first sight it is not clear whether this is well-defined owing
to the possible presence of denominators in the coefficients of \( f \).
The fact that it is follows from

**Lemma 4.** - Let \( \eta \in \ker [\pi] \) and \( f \in B_0 \). Then \( f(T + \mu_\eta) \)
is a well-defined element of \( B_0 \), whence \( \Delta_{\beta} \) is a well-defined
operator taking values in \( B_0 \).

**Proof.** - Let \( f = e(F) \) with \( F \in \mathbb{Q}_0 \). Define \( \xi \in (\mu_{\mathbb{Z}_p}(\mathbb{C}_p))^h \)
by \( \xi_i = t_{X_i}(\eta) \) for each \( i = 1, 2, \ldots, h \). It is well-known that
\[
v(\xi_i - 1) \geq \frac{1}{p - 1} \quad \text{and} \quad v(n!) \leq \frac{1}{p - 1} \quad \text{if} \quad n \in \mathbb{Z}_p^h .
\]
These estimates imply that
\[
F_{\xi}(X_1, \ldots, X_h) := F((\xi_1 - 1) + \xi_1 X_1, \ldots, (\xi_h - 1) + \xi_h X_h)
\]
is an element of \( \mathbb{Q}_0 \). Now since
we have
\[ F(t_{x_1}(T + s\eta) - 1, \ldots, t_{x_h}(T + s\eta) - 1) = F(t_{x_1}(T) - 1, \ldots, t_{x_h}(T) - 1) \quad (9) \]
and so we would like to define \( f(T + s\eta) \) by the right hand side of (9). To do this we need to check that this is independent of the choice of \( F \). One way to do this is as follows; it suffices to consider the case \( f = 0 \): now \( F \in \mathbb{Q}_0 \) implies that \( F \) converges at all \( z \in \mathbb{C}_p \) with \( v(z_i) \geq \frac{1}{p-1} \) and \( y \rightarrow t_{x_i}(y) - 1 \) defines a homeomorphism from an open subset \( \mathcal{A} \) of \( \mathbb{C}_p \) containing the origin into \( \mathbb{C}_p \). Hence
\[ F(t_{x_1}(T) - 1, \ldots, t_{x_h}(T) - 1) = 0 \quad \text{in } \mathbb{C}_p[[T]] \]
implies that
\[ F(t_{x_1}(y) - 1, \ldots, t_{x_h}(y) - 1) = 0 \]
for all \( y \in \mathcal{A} \), i.e.
\[ F((\xi_1 - 1) + \xi_1(t_{x_1}(y') - 1), \ldots, (\xi_h - 1) + \xi_h(t_{x_h}(y') - 1)) = 0 \]
and therefore
\[ F(t_{x_1}(y') - 1, \ldots, t_{x_h}(y') - 1) = 0 \]
for all \( y' \) such that \( y' + s\eta \in \mathcal{A} \). Since a power series that vanishes on an open set on which it converges vanishes identically, we conclude that
\[ F(t_{x_1}(T) - 1, \ldots, t_{x_1}(T) - 1) = 0 \]
which is what is required.

The following lemma shows that any \( f \in B_0 \) can be decomposed as a sum of functions on which the \( \Delta^{(\beta)}_\sigma \)'s act especially simply.

**Lemma 5.** - Let \( f \in B_0 \) and for \( a \in \Theta_{\kappa_p} \) define
\[ F_a(T) = \sum_{\eta \in \ker[\pi]} f(T + s\eta) t_a(\eta); \quad (10) \]
then

(i) \((\Delta^{(\beta)} F_a) (T) = \omega^\beta (a) F_a(T)\) if \(a \not\equiv 0 \pmod{\pi}\)

\(= 0\) if \(a \equiv 0 \pmod{\pi}\).

(ii) We have \(f(T) = \frac{1}{q} \sum_{a \mod \pi} F_a(T)\).

(iii) If \(\beta \neq 0\) then

\((\Delta^{(\beta)} f) (T) = \frac{1}{q} \sum_a' \omega^\beta (a) F_a(T) = \frac{1}{q} \sum_a' (\Delta^{(\beta)} F_a) (T)\).

**Proof.** – (i) Suppose that \(\beta \neq 0\). Then

\[
(\Delta^{(\beta)} F_a) (T)
= \sum_{\eta \in \ker[\pi]} (\Delta^{(\beta)} f (T + \eta)) t_a (\eta)
= \frac{\tau(\beta)}{q} \sum_{\eta} \sum_{u} f (T +\eta, [u] (\eta_0)) t_a (\eta_0) \omega^{\beta} (u)
= \frac{\tau(\beta)}{q} \sum_{u} \sum_{v} f (T +\eta, [u + v] (\eta_0)) t_a ([v] (\eta_0)) \omega^{\beta} (u)
= \frac{\tau(\beta)}{q} \sum_{x} \sum_{u} f (T +\eta, [x] (\eta_0)) t_a ([x - u] (\eta_0)) \omega^{\beta} (u)
\]

(writing \(u + v = x\))

\[
= \frac{\tau(\beta)}{q} \left( \sum_{x} f (T +\eta, [x] (\eta_0)) t_a ([x] (\eta_0)) \right) \left( \sum_{u} t_a ([ - u] (\eta_0)) \omega^{-\beta} (u) \right)
= \frac{\tau(\beta)}{q} F_a (T) (-1)^\beta \tau_a (-\beta)
= \omega^\beta (a) F_a (T)
\]

by Lemma 3. The case \(\beta = 0\) of (i) as well as parts (ii) and (iii) require similar calculations and will be omitted.

We shall now state and prove the main result of this section.
THEOREM 6. - Let $f \in B_0$ and $\alpha \in \mathbb{Z}/(q - 1)\mathbb{Z}$. Then there exists a unique continuous function $C_{f}^{(\alpha)} : \mathbb{C}_p \rightarrow \mathbb{C}_p$ such that for each $\beta \in \mathbb{Z}/(q - 1)\mathbb{Z}$

$$C_{f}^{(\alpha)}(k) = (-1)^{\alpha - \beta} \frac{D_{1}^{k}(\Delta_{\alpha}^{(\alpha - \beta)} f)(0)}{\Omega_{p}^{k}}$$

whenever $k \geq 0$ and $k \in \beta$.

Proof. - The uniqueness is clear, since $\mathbb{Z}_+$ is dense in $\mathbb{Z}_p$. Evidently $t_{\alpha} \in B_0$ for all $\alpha \in \mathcal{E}_{\mathbb{K}_{p}}$, and we claim that

$$(\Delta_{\alpha}^{(\beta)} t_{\alpha})(T) = (-1)^{\beta} \omega^{\beta}(a) t_{\alpha}(T) \quad \text{if} \quad a \not\equiv 0 \pmod{\pi}$$

$$= 0 \quad \text{if} \quad a \equiv 0 \pmod{\pi}$$

for each $\beta \in \mathbb{Z}/(q - 1)\mathbb{Z}$. Indeed suppose that $a \not\equiv 0 \pmod{\pi}$ and $\beta \neq 0$. Then we compute

$$(\Delta_{\alpha}^{(\beta)} t_{\alpha})(T) = \frac{\tau(\beta)}{q} \sum_{u} t_{\alpha}(T + [u] (\eta_{0})) \omega^{-\beta}(u)$$

$$= \frac{\tau(\beta)}{q} t_{\alpha}(T) \sum_{u} t_{\alpha}([u] (\eta_{0})) \omega^{-\beta}(u)$$

$$= \frac{\tau(\beta)}{q} t_{\alpha}(T) \tau_{\alpha}(-\beta)$$

$$= \omega^{\beta}(a) t_{\alpha}(T) (-1)^{\beta}$$

by Lemma 3. The other cases are similar.

Now $(D_{1} t_{\alpha})(0) = \Omega_{p} a$ by (7) and Lemma 1 (ii). Hence

$$(-1)^{\alpha - \beta} \frac{D_{1}^{k}(\Delta_{\alpha}^{(\alpha - \beta)} t_{\alpha})(0)}{\Omega_{p}^{k}} = \omega^{\alpha - \beta}(a) a^{k} \quad \text{if} \quad a \not\equiv 0 \pmod{\pi}$$

$$= 0 \quad \text{if} \quad a \equiv 0 \pmod{\pi}.$$  

Define $\langle a \rangle$, for $a \in \mathcal{E}_{\mathbb{K}_{p}}$, by $\langle a \rangle \omega(a) = a$ if $a \not\equiv 0 \pmod{\pi}$ and $\langle a \rangle = 0$ if $a \equiv 0 \pmod{\pi}$. Then $\langle a \rangle \equiv 1 \pmod{\pi}$ if $a \not\equiv 0 \pmod{\pi}$ and so $\langle a \rangle^{s}$ is well-defined for all $s \in \mathbb{Z}_p$; we interpret $\omega^{a}(a)$ and $\langle a \rangle^{s}$ as 0 if $a \equiv 0 \pmod{\pi}$. With these conventions, we have

$$C_{t_{\alpha}}^{(\alpha)}(s) = \omega^{a}(a) \langle a \rangle^{s}.$$
Now let $f \in B_0$. Then we can write

$$f = \sum_{n \in \mathbb{Z}_+^h} \frac{c_n(t_x - 1)^n}{n!}$$

(11)

where $(t_x - 1)^n$ is an abbreviation for

$$(t_{x_1}(T) - 1)^{n_1} \cdots (t_{x_h}(T) - 1)^{n_h},$$

and $v(c_n) \to \infty$ as $n \to \infty$. If $$\binom{n}{r} = \prod_{i=1}^h \binom{n_i}{r_i}$$ for each $n, r \in \mathbb{Z}_+^h$ then

$$(t_x - 1)^n = \sum_{r=0}^n (-1)^i \binom{n}{r} t_{x \cdot r}$$

where $x \cdot r = \sum_{i=1}^h x_i r_i$, so that

$$C_{(t_x - 1)^n}(s) = \sum_{r=0}^n (-1)^i \binom{n}{r} \omega^\alpha(x \cdot r) (x \cdot r)^k.$$

(12)

In view of this, and the fact that $v(c_n) \to \infty$, the theorem will be proved if we can show that

$$C_{(t_x - 1)^n}(s) \equiv 0 \pmod{n!}$$

(13)

whenever $n \in \mathbb{Z}_+^h$. Let $k \in \mathbb{Z}_+$ satisfy $k \in \alpha$, and $ev(n!) \leq k$, where $e$ is the ramification degree of $K_p$ over $Q_p$; then

$$C_{(t_x - 1)^n}(k) = \sum_{r=0}^n (-1)^i \binom{n}{r} \omega^\alpha(x \cdot r) (x \cdot r)^k$$

\equiv \sum_{r=0}^n (-1)^i \binom{n}{r} (x \cdot r)^k \pmod{n!}$$

(so that this congruence also holds modulo $n!$) and

$$\sum_{r=0}^n (-1)^i \binom{n}{r} (x \cdot r)^k = \left[ \left( \frac{d}{dz} \right)^k \Pi_{i=1}^h (\exp(x_i z) - 1)^{n_i} \right]_{z=0}. $$
Since the set of integers \( k \) described above is dense in \( \mathbb{Z}_p \), the theorem follows from the following lemma (cf. [3 § 3.5]):

**Lemma 7.** For all \( k \in \mathbb{Z}_+ \) we have the congruence

\[
\delta_k(n) := \left[ \left( \frac{d}{dz} \right)^k \prod_{i=1}^{\eta} (\exp(x_iz) - 1)^{n_i} \right]_{z=0} \equiv 0 \pmod{n!}.
\]

**Proof.** If \( k < \sum n_i \) then \( \delta_k(n) = 0 \) and the assertion is trivial. On the other hand

\[
\delta_{k+1}(n) = \left[ \left( \frac{d}{dz} \right)^{k+1} \prod_{i=1}^{\eta} (\exp(x_iz) - 1)^{n_i} \right]_{z=0}
\]

\[
= \sum_{i=1}^{\eta} x_i n_i (\delta_k(n^{(i)}) + \delta_k(n))
\]

where \( n^{(i)} \) is obtained from \( n \) by replacing \( n_i \) by \( n_i - 1 \). Hence if \( \delta_k(n) \equiv 0 \pmod{n!} \) and \( \delta_k(n^{(i)}) \equiv 0 \pmod{n^{(i)}!} \) then

\[
\delta_{k+1}(n) \equiv 0 \pmod{n!}
\]

as required.

### 3. An analyticity theorem

The function \( C_f^{(\alpha)} \) introduced in the previous section does not appear to have any analyticity properties for an arbitrary \( f \in B_0 \); however we can prove that \( C_f^{(\alpha)} \) is locally analytic if the coefficients \( c_n \) in (11) can be chosen to tend to zero sufficiently fast. More precisely we have

**Theorem 8.** Let notation be as in Theorem 6. Suppose that there exist real numbers \( A, B \) with \( B > 0 \) such that

\[
v(c_n) \geq A + Bv(n!)
\]

for all \( n \in \mathbb{Z}_+^h \). Then \( C_f^{(\alpha)} \) is locally analytic on \( \mathbb{Z}_p \) in the following sense: at every \( s_0 \in \mathbb{Z}_p \), it has a power series expansion
\[ C_f^{(a)}(s) = \sum_{k=0}^{\infty} \sigma_k(s_0)(s-s_0)^k \]

with non-zero radius of convergence.

**Proof.** – Suppose first that \( f(T) = t_a(T) \) for some \( a \in \Theta_{k_0}. \)

We saw in the proof of Theorem 6 that \( C_f^{(a)}(s) = \omega^x(a) \langle a \rangle^x \) if \( a \not\equiv 0 \pmod{\pi} \) while \( C_f^{(a)}(s) = 0 \) if \( a \equiv 0 \pmod{\pi} \). Hence

\[
C_f^{(a)}(s) = \sum_{k=0}^{\infty} \frac{\omega^x(a)(\log p \langle a \rangle^x \langle a \rangle^x_0)}{k!} (s-s_0)^k \quad \text{if} \quad a \not\equiv 0 \pmod{\pi} \quad (14)
\]

so that if we now take \( f(T) = (t_x(T) - 1)^n = \prod_{i=1}^{h} (t_x(T) - 1)^{n_i} \) then by (12) and (14)

\[
C_f^{(a)}(s) = \sum_{k=0}^{\infty} \delta(n, k, s_0)(s-s_0)^k
\]

where

\[
\delta(n, k, s_0) = \sum_{r=0}^{n} (-1)^{\eta - n} \binom{n}{r} \omega^x(x \cdot r)(x \cdot r)^x_0 \\
\frac{(\log p \langle x \cdot r \rangle)^k}{k!} \quad \text{if} \quad x \cdot r \not\equiv 0 \pmod{\pi} \quad (15)
\]

\[
= 0 \quad \text{if} \quad x \cdot r \equiv 0 \pmod{\pi}.
\]

Now suppose that \( f(T) = \sum_{n \in \mathbb{Z}_{k_0}^+} c_n \frac{(t_x(T) - 1)^n}{n!} \) as in (11).

Then Theorem 8 will be proved if we can show that

\[
\sum_{n \in \mathbb{Z}_{k_0}^+} c_n \delta(n, k, s_0) \cdot \frac{1}{n!} \quad (16)
\]

converges for all \( k \in \mathbb{Z}_+ \) and, if its sum is denoted by \( \sigma_k(s_0) \), then

\[
v(\sigma_k(s_0)) \geq A + B'k
\]

for some constant \( B' \) depending only on \( B \).
If \( F : \mathbb{Z}_p^h \rightarrow \mathbb{C}_p \) is any function and \( n \in \mathbb{Z}_p^h \) we define \( M(n, F) \) to be the expression

\[
M(n, F) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} F(r)
\]

where the sum is taken over all \( r \in \mathbb{Z}_p^h \) with \( 0 \leq r_i \leq n_i \) for each \( i \). In particular

\[
\delta(n, k, s_0) = M(n, G)
\]

where \( G \) is defined by

\[
G(p) = \omega^a(x \cdot p) (x \cdot p)^s_0 \frac{(\log_p < x \cdot p >)^k}{k!}
\]

if \( x \cdot p \equiv 0 \pmod{\pi} \)

\[
= 0 \quad \text{if} \quad x \cdot p \equiv 0 \pmod{\pi}.
\]

The proof of assertions (16) and (17) will require a couple of lemmas.

**Lemma 9.** (i) The map \( F \mapsto M(n, F) \) is \( \mathbb{C}_p \)-linear and if \( F \) takes values in \( \mathbb{O} \) then \( M(n, F) \in \mathbb{O} \).

(ii) If \( F \) and \( F' \) take values in \( \mathbb{O} \) and \( F(p) \equiv F'(p) \pmod{\pi^m} \) for some \( m \geq 0 \) and all \( p \in \mathbb{Z}_p^h \) then

\[
M(n, F) \equiv M(n, F') \pmod{\pi^m}
\]

for all \( n \in \mathbb{Z}_p^h \).

(iii) Let \( F \) be defined by

\[
F(p) = \omega^a(x \cdot p) (x \cdot p)^s
\]

if \( x \cdot p \equiv 0 \pmod{\pi} \) and \( F(p) = 0 \) if \( x \cdot p \equiv 0 \pmod{\pi} \). Then

\[
v(M(n, F)) \geq \max \left( \frac{q}{e}, v(n !) \right), \quad \text{where} \quad e \quad \text{is the ramification degree of} \quad K_p \quad \text{over} \quad Q_p.
\]

**Proof.** Assertion (i) is trivial. (ii) follows from (i) by considering the function \( \frac{1}{\pi^m} (F - F') \). To prove (iii) observe first that if

\[
F_0(p) = \omega^a(x \cdot p) (x \cdot p)^s \quad \text{for} \quad x \cdot p \equiv 0 \pmod{\pi} \quad \text{and} \quad F_0(p) = 0 \quad \text{otherwise}, \quad \text{then} \quad v(M(n, F_0)) \geq v(n !);
\]

indeed this is just the content of equation (13) of the previous section. But \( F \) is an \( \mathbb{O} \)-linear
combination of functions of this form so that \( v(M(n, F)) \geq v(n !) \) also. On the other hand \( (x \cdot p \equiv 1 \mod \pi) \) if \( x \cdot p \neq 0 \mod \pi \) and so \( F(\rho) \equiv 0 \mod \pi^\rho \) for all \( \rho \) so that \( v(M(n, F)) \geq \lambda e^{-1} \) by (ii).

**Lemma 10.** — For \( k, \lambda \in \mathbb{Z}_+ \) let \( \epsilon_{k, \lambda} \) be defined by the expansion

\[
(\log(1 + T))^k = \sum_{\lambda=0}^{\infty} \epsilon_{k, \lambda} T^\lambda,
\]

where \( \log(1 + T) = \sum_{\lambda=1}^{\infty} (-1)^{\lambda+1} \frac{T^\lambda}{\lambda} \). Then we have \( \epsilon_{k, \lambda} = 0 \) if \( \lambda < k \) and \( v(\epsilon_{k, \lambda}) \geq -k \frac{\log \lambda - \log k}{\log p} \) if \( \lambda \geq k \).

**Proof.** — It is clear that \( \epsilon_{k, \lambda} = 0 \) if \( \lambda < k \). On the other hand if \( \lambda \geq k \) then

\[
\epsilon_{k, \lambda} = \sum_{m_1 \geq 1} \frac{(-1)^{m_1 + m_2 + \ldots + m_k - k}}{m_1 m_2 \ldots m_k}
\]

and so

\[
v(\epsilon_{k, \lambda}) \geq \min_{m_1 \geq 1} v\left(\frac{1}{m_1 m_2 \ldots m_k}\right).
\]

Therefore we need to estimate \( v(m_1 m_2 \ldots m_k) \). Now by an elementary inequality \( \left( \prod_{i=1}^{k} m_i \right)^{\frac{1}{k}} \leq \frac{1}{k} \sum_{i=1}^{k} m_i = \frac{\lambda}{k} \) and so

\[
\sum_{i=1}^{k} \log m_i \leq k \left( \log \frac{\lambda}{k} \right).
\]

But \( v(m_i) \leq \frac{\log m_i}{\log p} \) for all \( i \) and so

\[
v(m_1 m_2 \ldots m_k) = \sum_{i=1}^{k} v(m_i) \leq \frac{\sum_{i=1}^{k} \log m_i}{\log p} \leq k \left( \frac{\log (\lambda/k)}{\log p} \right)
\]

and the result follows.
We can now estimate \( \delta(n, k, s_0) = M(n, G) \) where \( G \) is the function in (18). In fact

\[
G(p) = \frac{1}{k!} \omega^a(x \cdot \rho) \langle x \cdot \rho \rangle^{s_0} \sum_{\xi = k}^\infty \epsilon_{k, \xi} (\langle x \cdot \rho \rangle - 1)^\xi \text{ if } x \cdot \rho \neq 0
\]

\[
= 0 \text{ if } x \cdot \rho \equiv 0 \pmod{\pi},
\]

and so combining Lemmas 9 (iii) and 10 we find that

\[
v(\delta(n, k, s_0)) \geq \min_{\xi \geq k} \left\{ \max \left( \frac{\xi}{e} + v(\epsilon_{k, \xi}), v(n!) + v(\epsilon_{k, \xi}) \right) \right\} - v(k!)
\]

whence

\[
v\left( \frac{\delta(n, k, s_0)}{n!} \right) \geq \min_{\xi \geq k} \left\{ \max \left( \frac{\xi}{e} - v(n!) - k \left( \frac{\log \xi - \log k}{\log p} \right), - k \left( \frac{\log \xi - \log k}{\log p} \right) \right) \right\} - v(k!).
\]

It is easy to see that the minimum is attained at

\[
\xi = \frac{e}{\log p} k \quad \text{if } e \geq \log p \text{ and } (\log p) v(n!) \leq k,
\]

\[
\xi = k \quad \text{if } e \leq \log p \text{ and } ev(n!) \leq k,
\]

and \( \xi = ev(n!) \) if \( \inf(e, \log p) v(n!) \geq k \).

Taking each of these cases in turn, we have

\[
v\left( \frac{\delta(n, k, s_0)}{n!} \right) \geq k \frac{\log p}{e} - v(n!) - k \left( \frac{\log (e/\log p)}{\log p} \right) - v(k!),
\]

or

\[
\geq \frac{k}{e} v(n!) - v(k!),
\]

or

\[
\geq - k \left( \frac{\log (ev(n!)) - \log k}{\log p} \right) - v(k!).
\]
Hence if \( v(c_n) \geq A + Bv(n !) \) for some \( B > 0 \), then

\[
v \left( c_n \frac{\delta(n, k, s_0)}{n !} \right) \to \infty \quad \text{as} \quad n \to \infty,
\]

(\( k \) fixed) and so (16) is proved.

We now turn to the proof of (17). For \( \xi > 0 \) and \( k \in \mathbb{Z}_+ \) define

\[
v_k(\xi) = A + B\xi + \frac{k}{\log p} - \xi - k \frac{\log(e/\log p)}{\log p} - v(k !)
\]

if \( e \geq \log p \) and \( (\log p) \xi \leq k \),

\[
= A + B\xi + \frac{k}{\log p} - \xi - v(k !)
\]

if \( e \leq \log p \) and \( e \xi \leq k \),

\[
= A + B\xi - k \left( \frac{\log e\xi - \log k}{\log p} \right) - v(k !)
\]

if \( \inf(e, \log p) \xi \geq k \).

Then if \( a_k(s_0) = \sum_{n \in \mathbb{Z}_+^h} c_n \frac{\delta(n, k, s_0)}{n !} \) we have

\[
v(a_k(s_0)) \geq \min_{n \in \mathbb{Z}_+^h} (v_k(v(n !))) \geq \inf_{\xi > 0} y_k(\xi).
\]

We may suppose that \( B \leq \inf(\log p, e) \). Then \( y_k(\xi) \) is decreasing in \( 0 \leq \xi \leq \frac{k}{\inf(\log p, e)} \) and a routine computation shows that \( y_k(\xi) \) has a unique minimum at \( \xi = \frac{k}{B \log p} \) which is greater than

\[
\frac{k}{\inf(\log p, e)}.
\]

Hence

\[
\inf_{\xi > 0} y_k(\xi) = y_k \left( \frac{k}{B \log p} \right)
\]

\[
= A + k \left( \frac{1 + \log B + \log \log p - \log e}{\log p} \right) - v(k !).
\]
Recalling that \( v(k!) \leq \frac{k}{p - 1} \) we see that

\[
v(\sigma_k(s_0)) \geq A + k \left( \frac{1 + \log B + \log \log p - \log e}{\log p} \right) - \frac{k}{p - 1}
\]

for all \( k \), and so we may take

\[
B' = \frac{1 + \log B + \log \log p - \log e}{\log p} - \frac{1}{p - 1}
\]

in (17) and so complete the proof of Theorem 8.


We preserve the notation of the previous sections. The purpose of this section is to prove

**Theorem 11.** Suppose that the ramification index \( e \) of \( K_p \) over \( \mathbb{Q}_p \) is less than or equal to \( p - 1 \). Then there exists a constant \( \delta > 0 \) with the following property: if \( f(T) = \sum_{n=0}^{\infty} a_n T^n \in \mathbb{C}_p[[T]] \) satisfies \( v(a_n) \geq A - n\delta \) for some \( A \in \mathbb{R} \), then \( f(T) \in B_0 \) and the function \( s \mapsto C_f^{(e)}(s) \) is locally analytic for every \( \alpha \in \mathbb{Z}/(q - 1)\mathbb{Z} \).

In particular since \( \chi'(T) \in \Theta[[T]] \), it is easy to see that if \( f \in \mathbb{C}_p[[T]] \) satisfies \( D_k^e f(T) \in \Theta[[T]] \) for some \( k \geq 0 \), then Theorem 11 can be applied to it, and so we have

**Corollary 12.** Let \( f \in \mathbb{C}_p[[T]] \) be such that

\[(D^k f)(T) \in \Theta[[T]]\]

for some \( k \geq 0 \). Then \( f \in B_0 \) and \( s \mapsto C_f^{(e)}(s) \) is a locally analytic function for every \( \alpha \in \mathbb{Z}/(q - 1)\mathbb{Z} \).

Theorem B of the introduction is evidently the special case \( k = 0 \) of Corollary 12 (except for the statement

\[
v(\Omega_p) = \frac{1}{p - 1} - \frac{1}{e(q - 1)}
\]

which is Lemma 13 below).
We now begin the proof of Theorem 11. Recall our hypothesis that \( \mathcal{S} \) is isomorphic to the Lubin-Tate group \( \mathcal{S}_0 \). If \( h = 1 \), then Lubin [11, Theorem 4.3.2] tells us that \( \mathcal{S} \) is isomorphic to \( \mathbb{G}_m \) and so a much stronger result can be deduced from Theorem A (ii); and we shall therefore suppose that \( h > 1 \).

Let \( \tau: \mathcal{S} \rightarrow \mathcal{S}_0 \) be an isomorphism. Then

\[
\tau(T) = \tau_0 T + O(T^2) \in \mathcal{O}[[T]]
\]

with \( \tau_0 \) a unit of \( \mathcal{O} \). Thus \( \mathcal{O}[[T]] = \mathcal{O}[[\tau(T)]] \) and so \( \tau(B_0) = B_0' (= B_0 \text{ defined with } \mathcal{S}_0 \text{ in place of } \mathcal{S}) \). Also if \( \lambda_0 \) denotes the logarithm of \( \mathcal{S}_0 \), and \( W = \tau(T) \) then \( \lambda(T) = \tau_0^{-1} \lambda_0(W) \) and so by the chain rule

\[
\frac{1}{\lambda'(T)} \frac{df(T)}{dT} = \frac{1}{\tau_0^{-1} \lambda_0'(W)} \frac{df \tau^{-1}(W)}{dW}.
\]

Since \( f(T + s \eta) = f(\tau^{-1}(W) + s_0 \tau^{-1}(\eta)) \) it suffices to prove the theorem for the group \( \mathcal{S}_0 \) and we assume that \( \mathcal{S} = \mathcal{S}_0 \) for the rest of this section; thus in what follows \( \lambda, t_a \) etc are associated to \( \mathcal{S}_0 \).

We write \( \lambda(T) = \sum_{n=1}^\infty \lambda_n T^n \) with \( \lambda_1 = 1 \).

We need

**Lemma 13.** — (i) \( \lambda_n = 0 \) unless \( n \equiv 1 \) (mod \( q - 1 \)), \( \lambda_n \in \mathcal{O}_{K_p} \) except perhaps when \( n \equiv 0 \) (mod \( q \)), and \( v(\lambda_q) = -e^{-1} \).

(ii) The map \( (\Omega_p a) \mapsto (D_n t_a) (0) (a \in \mathcal{O}_{K_p} \) is a polynomial in \( \Omega_p a \) of degree \( n \) with coefficient in \( K_p \); the coefficient of \( (\Omega_p a)^k \) being 0 unless \( k \equiv n \) (mod \( q - 1 \)).

(iii) We have

\[
t_a(T) = \sum_{k=0}^{q-1} \frac{(\Omega_p a)^k}{k!} T^k + \left[ \frac{(\Omega_p a)^q}{q!} + \lambda_q \Omega_p a \right] T^q + O(T^{q+1}).
\]

(iv) \( v(\Omega_p) = \frac{1}{p - 1} - \frac{1}{e(q - 1)} \).

**Remark.** — Part (iv) of this lemma is true for any \( \mathcal{S} \) (so long as \( e \leq p - 1 \)). Indeed, once the assertion has been proven for \( \mathcal{S}_0 \) the
isomorphism \( \tau: \mathcal{F} \to \mathcal{F}_0 \) above shows that, with suitable normalisation and obvious notation, we have \( \tau_0 \Omega_p (\mathcal{F} = \Omega_p (\mathcal{F}_0) \).

**Proof.** — (i) For the proof of this see Lang [8, Chapter 8]. The final statement follows from the proof given there.

(ii) We have the commutative triangle

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{t_a(T)} & \mathbb{G}_m \\
\downarrow \lambda(T) & & \downarrow \exp(\Omega_p a \lambda(T)) \\
\mathbb{G}_a & & 
\end{array}
\]

where \( z \) is a local parameter for the additive group \( \mathbb{G}_a \). This says precisely that \( t_a(T) = \exp(\Omega_p a \lambda(T)) \). Expanding \( \exp(\Omega_p a \lambda(T)) \) in powers of \( T \) and using (i) we obtain (ii).

(iii) This follows easily from parts (i) and (ii).

(iv) Put \( x = \frac{1}{p-1} - \frac{1}{e(q-1)} \). We first claim that \( v(\Omega_p) \leq x \).

Indeed Fact 2 of § 1 tells us that there exists \( \eta \in \ker [\pi] \) and \( \xi \in \mu_p(\mathbb{C}_p) \) with \( \xi \neq 1 \) such that \( t_1(\eta) = \xi \). Suppose that \( v(\Omega_p) > x \). Since \( v(\eta) = \frac{1}{e(q-1)} \) and \( v(n!) \leq \frac{n-1}{p-1} \) we have

\[
v \left( \left( \frac{\Omega_p \eta^n}{n!} \right) \right) > \frac{1}{p-1}
\]

for any \( n > 0 \). But one has \( v(\eta^q) = \frac{q}{e(q-1)} > \frac{1}{e} \); so, using our hypothesis \( e \leq p-1 \) and referring to the expression

\[
t_1(\eta) = \sum_{k=0}^{q-1} \frac{(\Omega_p \eta)^k}{k!} + \eta^q u, \quad \text{with } u \in \mathcal{O}
\]

obtained by substituting \( a = 1 \) and \( T = \eta \) in part (iii) we find that

\[
v(\xi - 1) = v(t_1(\eta) - 1) > \frac{1}{p-1}
\]

which is false. Hence \( v(\Omega_p) \leq x \) and so \( v(\lambda_q \Omega_p) < 0 \) by (i). But the coefficient of \( T^q \) in \( t_1(T) \) is...
which lies in \( \mathcal{O} \), so that
\[
v \left( \frac{\Omega_p^q}{q!} \right) = v(\lambda_q \Omega_p) = -\frac{1}{e} + v(\Omega_p).
\]
Since it is easy to see that
\[
v(q!) = \frac{q - 1}{p - 1}
\]
it follows that
\[
v(\Omega_p) = \frac{1}{p - 1} - \frac{1}{e(q - 1)}
\]
as asserted.

We now begin the proof of Theorem 11 itself. Consider the auxiliary function
\[
\Phi(T) = \frac{1}{q - 1} \sum_{\xi \in \mu_{q-1}} t_{\xi}(T) \xi^{-1}
\]
where the sum is taken over all \( \xi \in \mu_{q-1}(K_p) \). In view of parts (i) and (ii) of Lemma 13, and the fact that \( \sum_{\xi \in \mu_{q-1}} \xi^n = q - 1 \) or 0 according as to whether \( n \equiv 0 \pmod{q - 1} \) or not, we find that the coefficient of \( T^n \) in \( \Phi(T) \) is 0 unless \( n \equiv 1 \pmod{q - 1} \), in which case it is the same as the coefficient of \( T^n \) in \( t_1(T) \).

Now consider the case \( f(T) = T \). We can certainly write
\[
T = \sum_{n=0}^{\infty} a_n \frac{\Phi(T)^n}{n!}
\]
with \( a_n \in C_p \) and \( a_n = 0 \) if \( n \not\equiv 1 \pmod{q - 1} \).

Now
\[
\frac{\Phi(T)}{\Omega_p} = T + \frac{T^q}{\Omega_p} \Psi(T^{q-1})
\]
for some \( \Psi(T) \in \mathcal{O} \) \( \ll [T] \) and therefore if \( k \in \mathbb{Z}_+ \),
\[
\left( \frac{\Phi(T)}{\Omega_p} \right)^{k(q-1)+1} = T^{k(q-1)+1}
\]
\[
+ \frac{\beta_1^{(k)}}{\Omega_p} T^{(k+1)(q-1)+1} + \ldots + \frac{\beta_r^{(k)}}{\Omega_p} T^{(k+r)(q-1)+1} + \ldots
\]
with \( \beta_r^{(k)} \in \mathcal{O} \) and in fact \( \beta_r^{(k)} \Omega_p^{(k(q-1)+1)-r} \in \mathcal{O} \) if
Thus, proceeding inductively we find that if we write

\[ T = \sum_{k=0}^{\infty} d_k \frac{\Phi(T)^{k(q-1)+1}}{\Omega_p^{k(q-1)+1}} \]

then

\[ d_0 = 1, \quad \Omega_p d_1 \in \mathcal{O}, \quad \Omega_p^2 d_2 \in \mathcal{O}, \ldots \]

and in general \( \Omega_p^k d_k \in \mathcal{O} \). For simplicity we write \( \Lambda \) for \( \Omega_p^{q-1} \). Then

\[ \frac{T}{\Lambda} \in \mathcal{O} \left[ \left[ \frac{\Phi(T)}{\Lambda^q} \right] \right] \]

and therefore, if we take

\[ \delta = v(\Lambda) = \left( \frac{1}{p-1} - \frac{1}{e(q-1)} \right) \frac{1}{q-1}, \]

then any \( f \) satisfying the hypotheses of Theorem 11 will lie in \( C_p \cdot \mathcal{O} \left[ \left[ \frac{\Phi(T)}{\Lambda^q} \right] \right] \). Hence to complete the proof of Theorem 11 it suffices to show that every \( f \in \mathcal{O} \left[ \left[ \frac{\Phi(T)}{\Lambda^q} \right] \right] \) satisfies the hypothesis of Theorem 8. Now clearly \( \Phi(T) \in \mathcal{O} \left[ \left[ t_{x_1}(T) - 1, \ldots, t_{x_h}(T) - 1 \right] \right] \) from which it follows that any \( f \in \mathcal{O} \left[ \left[ \frac{\Phi(T)}{\Lambda^q} \right] \right] \) can be written in the form \( \sum_{n \in \mathbb{Z}_+^h} c_n \frac{(t_x - 1)^n}{n!} \) with

\[ v \left( \frac{c_n}{n!} \right) \geq v \left( \Lambda^{-\sum n_i(q-1)} \right) = -\frac{q}{q-1} \left( \sum_{i=1}^{h} n_i \right) \left( \frac{1}{p-1} - \frac{1}{e(q-1)} \right). \]

Recalling that \( v(n!) \geq \frac{\sum n_i}{p-1} + O \left( \log \prod_{i=1}^{h} n_i \right) \) we find that \( v(c_n) \geq A + Bv(n!) \) for any
and suitable $A$ depending on $B$. Now $\frac{(p - 1)q}{e(q - 1)} > 1$ since $e \leq p - 1$ and so the hypotheses of Theorem 8 are satisfied and the proof of Theorem 11 is complete.

**BIBLIOGRAPHIE**


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