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A value-distribution criterion for the class $L \log L$
and some related questions


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A VALUE-DISTRIBUTION CRITERION
FOR THE CLASS L LOG L,
AND SOME RELATED QUESTIONS

by M. ESSEN (¹), D. F. SHEA (²) and C. S. STANTON

1. Introduction.

Let F belong to the Nevanlinna class of functions analytic in the unit
disk U, so that

\[ T(1,F) = \lim_{r \to 1} \frac{1}{2\pi} \int_{0}^{2\pi} \log^+ |F(re^{i\theta})| \, d\theta < \infty. \]

In particular, \( \lim_{r \to 1} F(re^{i\theta}) = F(e^{i\theta}) \) exists a.e.. We shall say that
\( \text{Re} \, F \in L \log L \) if

\[ \sup_{0 < r < 1} \int_{0}^{2\pi} \log^+ |\text{Re} \, F(re^{i\theta})| \, d\theta < \infty. \]

The class \( L \log L \) is closely related to the Hardy space \( H^1(U) \), as is shown by the following classical results of Zygmund [22]:

**Theorem A.** - If \( \text{Re} \, F \in L \log L \), then \( F \in H^1 \).

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Theorem B. If $F \in H^1$ and $\text{Re } F > 0$, then $\text{Re } F \in L \log L$.

In this paper we prove some refinements of Theorem B. We state our basic results in terms of the usual Nevanlinna counting function

$$N(r,w) = N(r,w;F) = \int_0^r n(t,w) \, dt/t$$

where $n(t,w) = \sum_{|z| < t} 1$ and $\{z_v\} = f^{-1}(w)$. Our main result is

Theorem 1. Let $F \in H^1(U)$. The following are equivalent:

1. $\int_{-\infty}^{\infty} N(1,iv) \log^+ |v| \, dv < \infty$.
2. $\text{Re } F \in L \log L$.
3. $\int_0^{2\pi} |\text{Re } F(e^{i\theta})| \log^+ |F(e^{i\theta})| \, d\theta < \infty$.

Remark 1. We note that if $\text{Re } F > 0$, then $N(1,iv) = 0$ for all $v \in \mathbb{R}$. Hence Theorem B follows from the equivalence of (1.1a) and (1.1b).

Remark 2. We could replace the integration over the imaginary axis in (1.1a) by integration over any vertical line, i.e., for any real $u$ (1.1a) is equivalent to

(1.1a') $\int_{-\infty}^{\infty} N(1,u+iv) \log^+ |v| \, dv < \infty$.

Once the Theorem has been proved, this follows immediately since $N(1,u+iv;F) = N(1,iv;F-u)$ and $\text{Re } (F-u) \in L \log L$ if and only if $\text{Re } F \in L \log L$.

In Sections 3 through 6, we give some further refinements of Theorem B: from a geometrical condition on the range of $F$, we can deduce that $\text{Re } F \in L \log L$. To apply Theorem 1, we need a criterion to decide whether $F \in H^1(U)$. In this context, our main tool is a more general result which may have independent interest: in terms of harmonic measure, it gives a necessary and sufficient condition for $F \in H^p(U), 0 < p < \infty$ (cf. Theorem 7 in Section 5). We also consider cases when the hypothesis $F \in H^1(U)$ in Theorem 1 is omitted. The material in Sections 5 and 6 overlaps with certain work of Burkholder in [4], [5].
The starting-point of our work was a study of the relation between the classical criterion of Zygmund and the following result of A. Baernstein. Let $f \in L^1(T)$ be a given real-valued function and consider

$$(1.2) \quad F(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\phi}) \frac{e^{i\phi} + z}{e^{i\phi} - z} d\phi.$$ 

We have $F(e^{i\theta}) = f(e^{i\theta}) + if'(e^{i\theta})$ a.e. on $T$. Let $g$ be the symmetric decreasing rearrangement of $f$ and $\tilde{g}$ the conjugate function of $g$ with mean value zero. In particular, we have

$$\|g\|_1 = \|f\|_1, \quad \int_T |g| \log^+ |g| = \int_T |f| \log^+ |f|.$$ 

In [2], Baernstein proved that

$$(1.3) \quad \|f\|_1 \leq \|\tilde{g}\|_1.$$ 

Thus, when $f \in L^1(T)$ is given, (1.3) implies a sufficient condition for $F$ to be in $H^1$, namely: $F \in H^1$ if $\|\tilde{g}\|_1 < \infty$. However, this consequence of (1.3) does not actually yield a new criterion for $F \in L^1$, in view of the following consequence of Theorem 1:

**Corollary 1.** $f \in L^1(T) \Leftrightarrow g \in L^1 \log L$.

**Proof.** Assume that $\tilde{g} \in L^1(T)$. From the discussion in Section 6 in Baernstein [2], we see that the analytic function $G$ associated to $g$ by (1.2) maps $U$ univalently onto a Steiner symmetric domain, i.e., a domain with the property that for all $u \in \mathbb{R}$,

$$G(U) \cap \{\text{Re } w = u\} = \{w = u + iv : |v| < b(u)\}.$$ 

If there exists $u_0$ such that $b(u_0) < \infty$, condition (1.1a') will hold for $u = u_0$, and Theorem 1 implies that $g \in L \log L$ which is equivalent to $f \in L \log L$. But such a $u_0$ exists because if $b(u) = \infty$ for all $u \in \mathbb{R}$, $G$ would map $U$ onto the whole complex plane.

The converse assertion is simply Zygmund’s Theorem A; Corollary 1 is proved.

**Remark.** In a private discussion, Lennart Carleson has shown us a simple real-variable proof of Corollary 1.
In Section 2, we deduce Theorem 1 from a general identity:

**THEOREM 2.** — Let $F$ be analytic in $U$ and let $\Phi$ be subharmonic in $C$ with $\Phi(F(0))$ finite. Then

$$
(1.4) \quad (2\pi)^{-1} \int_{0}^{2\pi} \Phi(F(re^{i\theta})) \, d\theta = \int_{C} N(r, w) \, d\mu(w) + \Phi(F(0)),
$$

$0 < r < 1$,

where $\mu$ is the Riesz measure of $\Phi$.

For each choice of $\Phi$ in Theorem 2, we get a formula connecting a mean of $F$ over a circle in $U$ with an integral of $N(r, \cdot)$ over the range of $F$. Examples of such formulas will be given in Section 2. The present proof of Theorem 1 avoids some lengthy estimates in the original proof of Essén and Shea, as announced in [8]; this simplification is made possible because of Stanton’s proof of (1.4) in [19]. We apply (1.4) here with $\Phi(w) = |\text{Re} \, w| \log (1 + |w|^2)$, cf. (2.5) below.

If $\mu$ is a positive finite measure on $C$ and $\Phi$ is the logarithmic potential of $\mu$, then (1.4) is a classical formula of Frostman, cf. [16, p. 177].

We can extend Theorem 2 in a number of ways. For example, it holds when $\Phi$ is $\delta$-subharmonic, i.e. the difference of two subharmonic functions, with $\mu$ a signed measure. An identity like (1.4) is true for $\Phi$ meromorphic (in the disk or in the plane) provided $\Phi$ has sufficiently small growth at infinity. The theorem also can be extended to analytic functions mapping the polydisk or ball of $C^n$ into $C$. Details of these extensions are given in Section 7.

2. Proofs of Theorems 1 and 2.

We need the following well-known facts on the Nevanlinna counting function (for further information and references, cf. Section 4 in Essén and Shea [7]):

$N(1, w, F) = \lim_{r \to 1} N(r, w, F)$ exists and is uniformly bounded except near $F(0)$. The upper regularization $N(w) = N(w, F)$ of $N(1, w, F)$, defined by

$$
N(w) = \limsup_{\zeta \to w} N(1, \zeta, F),
$$
is subharmonic in $C \setminus \{F(0)\}$ and coincides with $N(1,w,F)$ off a set of $w$-values of logarithmic capacity zero. The function $N(w) + \log |w - F(0)|$ can be defined at $F(0)$ to be subharmonic in $C$.

Throughout the paper, the $H^p$-norms are defined as in Duren [6], p. 35. The class $h^p$ of harmonic functions in $U$ is defined in [6], p. 2. The set of interior points of a set $K$ is denoted by $K^0$.

**Proof of Theorem 1.** — Let $r$ be fixed, $0 < r < 1$. Then there exists a compact set $K$ such that $\text{supp } N(r, \cdot) \subset K^0$ and we have

$$
\Phi(\zeta) = \int_K \log |\zeta - w| \, d\mu(w) + h(\zeta),
$$

where $h$ is harmonic in $K^0$. From Jensen's formula

$$(2\pi)^{-1} \int_0^{2\pi} \log |F(re^{i\theta}) - w| \, d\theta = N(r,w) + \log |F(0) - w|,$$

we deduce that

$$(2\pi)^{-1} \int_0^{2\pi} \Phi(F(re^{i\theta})) \, d\theta
= \int_K \log |F(re^{i\theta}) - w| \, d\theta/(2\pi) + h(F(0))
= \int_K (N(r,w) + \log |F(0) - w|) \, d\mu(w) + h(F(0))
= \int_K N(r,w) \, d\mu(w) + \Phi(F(0)).$$

The theorem is proved.

Next, we give a list of some special subharmonic functions $\Phi$, their associated Riesz measures $\mu$ and the formulas which follow from (1.4). Most of these formulas are known; (2.5) is new and is basic to our proof of Theorem 1.

We write $w = u + iv$. $\delta_u$ and $\delta_v$ are Dirac measures supported by the $v$- and $u$-axis, respectively. (Formally, we should write $\delta_{u=0} \otimes 1$ and $1 \otimes \delta_{v=0}$). We put $k(t) = (2 + t)(1 + t)^{-2}$ and $C(t) = 2\pi t \log (1 + t)$.

| $\Phi$ | $|u|$ | $|v|$ | $|w|$ | $|u| \log (1 + |u|)$ | $|u| \log (1 + |w|^2)$ |
|--------|------|------|------|----------------|-----------------|
| $\Delta \Phi = 2\pi \, d\mu$ | $2\delta_u$ | $2\delta_v$ | $|w|^{-1}$ | $k(|u|)$ | $2 \delta_u \log (1 + v^2) + 4|u|k(|w|^2)$ |
Applying (1.4), we obtain

\[ (2.1) \int_{0}^{2\pi} |\text{Re } F(e^{i\theta})| d\theta = 2 \int_{-\infty}^{\infty} N(r,iv) \, dv + 2\pi |\text{Re } F(0)|, \]

\[ (2.2) \int_{0}^{2\pi} |\text{Im } F(e^{i\theta})| d\theta = 2 \int_{-\infty}^{\infty} N(r,u) \, du + 2\pi |\text{Im } F(0)|, \]

\[ (2.3) \int_{0}^{2\pi} |F(e^{i\theta})| d\theta = \iint_{C} N(r,w) \, dw \, dv / |w| + 2\pi |F(0)|, \]

\[ (2.4) \int_{0}^{2\pi} (|\text{Re } F| \log (1 + |\text{Re } F|))(e^{i\theta}) \, d\theta \]
\[ = \iint_{C} N(r,w)k(|u|) \, du \, dv + C(|\text{Re } F(0)|), \]

\[ (2.5) \int_{0}^{2\pi} (|\text{Re } F| \log (1 + |F|^2))(e^{i\theta}) \, d\theta \]
\[ = 4 \iint_{C} N(r,w)|u|k(|w|^2) \, du \, dv \]
\[ + 2 \int_{-\infty}^{\infty} N(r,iv) \log (1 + v^2) \, dv + 2\pi |\text{Re } F(0)| \log (1 + |F(0)|^2). \]

The equation \( \Delta \Phi = 2\pi d\mu \) is interpreted in the distributional sense. This means that for all \( \psi \in C_0^\infty (\mathbb{R}^2) \), we have

\[ (2.6) \iint_{C} \Phi \Delta \psi = 2\pi \iint_{C} \psi \, d\mu, \]

(cf. Lemmas 3.6 and 3.8 in Hayman and Kennedy [12]).

We illustrate the computation of these formulas by deriving the one needed for (2.5). We choose \( \Phi(w) = |u| \log (1 + |w|^2) \). Then if \( u \neq 0 \), \( \Phi \in C^\infty \) near \( u + iv \) and \( \Delta \Phi = 4|u|k(|w|^2) \).

Let \( \psi \in C_0^\infty (\mathbb{R}^2) \). From Green's theorem, we deduce that

\[ \iint_{\{u > 0\}} \Phi \Delta \psi = \iint_{\{u > 0\}} \psi(w)4uk(|w|^2) \, du \, dv + \int_{-\infty}^{\infty} \psi(iv) \log (1 + v^2) \, dv, \]
\[ \iint_{\{u < 0\}} \Phi \Delta \psi = \iint_{\{u < 0\}} \psi(w)4|u|k(|w|^2) \, du \, dv + \int_{-\infty}^{\infty} \psi(iv) \log (1 + v^2) \, dv. \]
Adding these two formulas, and using (2.6), we obtain

\[ 2\pi \int \int_{C} \psi \, d\mu = \int \int_{C} \psi(w)4|u|k(|w|^2)\, du \, dv + 2 \int_{-\infty}^{\infty} \psi(iv) \log (1+v^2) \, dv, \]

which is the fifth formula in the table above.

We rewrite (2.5) in the following way:

\[ I_1(r) = 4I_2(r) + 2I_3(r) + 2\pi |\text{Re} F(0)| \log (1+|F(0)|^2). \]

Let \( I_j = \sup I_j(r), 0 < r < 1, j = 1, 2, 3. \)

**Lemma 1.** Let \( F \) be analytic in \( U \). Then \( F \in H^1(U) \) if and only if \( \text{Re} F \in h^1 \) and \( I_2 \) is finite.

**Proof.** Let \( F \in H^1(U) \). From the inequality \( k(t) < t^{-1}, t > 0 \), it follows that \( |u|k(|w|^2) \leq |u||w|^{-2} \leq |w|^{-1} \) and \( I_2 \) must be finite since we have (2.3). Trivially, we have \( \text{Re} F \in h^1 \).

Conversely, if \( I_2 \) is finite, we use the subharmonicity of \( N(r, w) \) in \( C\{F(0)\} \) to deduce that

\[
\int_{|w| > 2|F(0)|} N(r,w) \, dw \leq 4 \int_{-\infty}^{\infty} (\pi u^2)^{-1} \int_{\zeta - u < |w|/2} N(r,\zeta) \, d\xi \, d\eta
\]

\[
\leq (4/\pi) \int_{D_+} N(r,\zeta)(\xi^2 - 3\eta^2)^{1/2} \, d\xi \, d\eta/|\zeta|^2
\]

\[
\leq (4/\pi) \int_{C} N(r,\zeta)|\xi| \, d\xi \, d\eta/|\zeta|^2 = I_4(r),
\]

where \( D_+ = \{\zeta = \xi + i\eta : \xi^2 \geq 3\eta^2\} \).

If \( I_2 \) is finite, \( \sup_{0 < r < 1} I_4(r) \) will also be finite. It is now clear from (2.2) that \( \text{Im} F \in h^1 \). Since \( \text{Re} F \in h^1 \), we must have \( F \in H^1(U) \) and the lemma is proved.

**Proof of Theorem 1.** The proof will show that

\[(1.1a) \rightarrow (1.1b) \rightarrow (1.1c) \rightarrow (1.1a).\]

To prove \( (a) \rightarrow (b) \), we first note that it follows from \( (a) \) that \( I_3 \) is finite and from Lemma 1 that \( I_2 \) is finite. Thus, by (2.5), \( I_1 \) is finite and we have proved \( (b) \).
To prove \((b) \rightarrow (c)\), we consider the following simple chain of inequalities:

\[
|u| \log (1 + u^2 + v^2) \leq 2|u| \log (1 + |u| + |v|) \leq 2(|u| \log (1 + |u| + |v|)).
\]

If we now choose \(u = \text{Re } F\), \(v = \text{Im } F\) and integrate with respect to \(\theta\), we obtain

\[
I_1 \leq 2 \int_0^{2\pi} (|\text{Re } F| \log (1 + |\text{Re } F|) + |\text{Im } F|)(e^{i\theta}) d\theta < \infty,
\]

and have proved \((c)\).

The implication \((c) \rightarrow (a)\) is an immediate consequence of (2.5).

Remark. — It is easy to prove Zygmund’s Theorem A that (1.1b) implies \(F \in H^1(U)\), using these methods. This is immediate from (2.3), (2.4) and the inequality \(k(|u|) \geq (|u|+1)^{-1} \geq (2|w|)^{-1}\) (valid for \(|w| \geq 1\)), which yield

\[
\frac{1}{2\pi} \int_0^{2\pi} |F(re^{i\theta})| d\theta \leq \frac{1}{\pi} \int_0^{2\pi} |\text{Re } F| \log (1 + |\text{Re } F|)(re^{i\theta}) d\theta + C_1
\]

for \(C_1 = T(1,F) + |F(0)| + 1\).

In the opposite direction, (2.3) and (2.5) together with \(|u|k(|w|^2) \leq |w|^{-1}\) imply

\[
\frac{1}{2\pi} \int_0^{2\pi} |\text{Re } F| \log (1 + |F|^2)^{1/2}(re^{i\theta}) d\theta \leq \frac{1}{\pi} \int_0^{2\pi} |F(re^{i\theta})| d\theta + C_2,
\]

\[
C_2 = \frac{1}{\pi} \int_{-\infty}^{\infty} N(1,iv) \log (1 + v^2)^{1/2} dv + |\text{Re } F(0)| \log (1 + |F(0)|).
\]

With heavy restrictions on \(F(U)\), such as \(\text{Re } F > 0\), inequalities of this type are classical (cf. [22], p. 256).

Let us finally give some further examples of formulas which are immediate consequences of Theorem 2. Successively choosing \(\Phi(w)\) as \(\log^+ |w|\), \(\log (1 + |w|^2)\), \(|w|^p\) with \(p > 0\), \(|u|^p\) with \(p > 1\) and as \(|w|A(\text{arg } w)\) where

\[
A(\phi) = \begin{cases} 
(1/2)\phi \sin \phi & |\phi| \leq \pi/2, \\
(1/2)(\pi - \phi) \sin \phi - \cos \phi & \pi/2 \leq \phi \leq 3\pi/2,
\end{cases}
\]
we obtain

\[
(2.7) \int_{0}^{2\pi} \log^+ |F(re^{i\theta})| \, d\theta = \int_{0}^{2\pi} N(r,e^{i\theta}) \, d\varphi + 2\pi \log^+ |F(0)|,
\]

\[
(2.8) \int_{0}^{2\pi} \log (1 + |F(re^{i\theta})|^2) \, d\theta
\]

\[
= 4 \int_{C} N(r,w)(1 + |w|^2)^{-2} \, du \, dv + 2\pi \log (1 + |F(0)|^2)
\]

\[
(2.9) \int_{0}^{2\pi} |F(re^{i\theta})|^p \, d\theta = p^2 \int_{C} N(r,w)|w|^{p-2} \, du \, dv
\]

\[
+ 2\pi |F(0)|^p, \quad p > 0.
\]

\[
(2.10) \int_{0}^{2\pi} \text{Re} \, F(re^{i\theta})|p \, d\theta
\]

\[
= p(p-1) \int_{C} N(r,w)|u|^{p-2} \, du \, dv + 2\pi |\text{Re} \, F(0)|^p, \quad p > 1,
\]

\[
(2.11) \int_{0}^{2\pi} (|F|A(\text{arg} \, F))(re^{i\theta}) \, d\theta
\]

\[
= \int_{C} N(r,w)|u| |w|^{-2} \, du \, dv + 2\pi |F(0)|A(\text{arg} \, F)).
\]

Remarks. — Equation (2.7) is Cartan's identity (see Hayman [11], p. 8). Equation (2.8) is a version of a classical formula for the Ahlfors characteristic (see (3.1), p. 173, in Nevanlinna [16]). Equations (2.9), (2.1) and (2.2) are classical (see e.g. Lehto [15], pp. 12, 14). Baernstein derives (2.1), (2.2), (2.9) and (2.10) from Cartan's identity in [2].

3. The class \( L \log L \) and estimates of harmonic measure.

What more can we say about the connection between the closed set \( E \) on which \( N(z) = N(z,F) \) vanishes, and the integrability condition (1.1 b) ? From now on, we assume that \( F(0) = 0 \).

We need an idea of M. Benedicks [3], developed to study positive harmonic functions vanishing on the boundary in sets of the form \( C \setminus E_0 \), where \( E_0 \) is a subset of the imaginary axis. Our set \( E \) is not necessarily restricted in this way.
Following Benedicks, we introduce a function $\beta_E$ which measures how "thin" the set $E$ is at infinity near the imaginary axis. If $z \neq 0$, let $K_z$ be the open square in the plane with centre at $z$, sides parallel to the axis and with side length $|z|$. Let $\Omega_z = K_z \setminus E$. In $\Omega_z$, we consider the harmonic function $V^z$ which has boundary values 1 on $\partial K_z$ and 0 on $E \cap K_z$. We define $\beta_E(z) = V^z(z)$.

**Theorem 3.** — Let $F \in H^1(U)$ and assume that $F(0) = 0$. A sufficient condition for $\Re F$ to be in $L \log L$ is that

$$\int_{|b| > 1} \beta_E(iy) \log |y| \, dy/y < \infty.$$  

In Section 4, we shall give examples of conditions on the omitted set $E$ which ensure that (3.1) holds.

In the proof of Theorem 3, we need

**Lemma 2.** — Assume that $F \in H^p(U)$ for some $p > 0$ and that $F(0) = 0$. Then

$$N(z,F) \leq C_p \|F\|_p |z|^{-p}, \quad z \neq 0,$$

where $C_p = p^{-1}$, $0 < p \leq 1$, and $C_p \leq 4$, $p > 1$.

**Proof.** — For any $F$ in the Nevanlinna class with $F(0) = 0$, it follows from Jensen’s theorem that we have

$$N(w,F) \leq (2\pi)^{-1} \int_0^{2\pi} \log (1 + |F(e^{i\theta})| |w|^{-1}) \, d\theta.$$  

When $0 < p \leq 1$, (3.2) is an immediate consequence of (3.3) and the inequality

$$\log (1 + u) \leq u^p/p, \quad 0 < p \leq 1, \quad u > 0.$$  

When $p > 2$, we use the fact that $N(z)$ is subharmonic in $\mathbb{C} \setminus \{0\}$ to deduce that, if $\rho = 2/p$,

$$N(z) \leq (\pi \rho^2 |z|^2)^{-1} \iint_{|w-z| < \rho |z|} N(w) \, du \, dv$$

$$\leq |z|^{-p} \rho^{-2} (1-\rho)^{-p} \iint_C N(w) |w|^{p-2} \, du \, dv/\pi$$

$$\leq C_p \|F\|_p |z|^{-p}, \quad C_p = (1-2/p)^{2-p}/2.$$
In the last step, we used (2.9). The argument is similar when \( 1 < p \leq 2 \), with \( \rho = 1 \) and \( C_p = (2/p^2)2^{2-p} \).

**Proof of Theorem 3.** — Using the maximum principle, we deduce from Lemma 2 that

\[
N(\zeta) \leq 2\|F\|_{1}V^y(\zeta)/|y|, \quad \zeta \in \Omega_y,
\]

\[
N(iy) \leq 2\|F\|_{1}\beta_E(iy)/|y|, \quad y \neq 0.
\]

Hence, Theorem 3 is an immediate consequence of Theorem 1.

In the study of the function \( \beta_E(z) \), we need two lemmas of Hayman and Pommerenke [13].

**Lemma A.** — Let \( E_1 \) be a compact subset of \( \{z:|z|<R/2\} \) and let \( \omega_{E_1} \) be the harmonic measure of \( E_1 \) in \( \{z:|z|<R\}\setminus E_1 \). Then

\[
(3.4) \quad \omega_{E_1}(z) \geq \alpha(R,E_1), \quad |z| \leq R/2,
\]

where \( \alpha(R,E_1) = \log (5/4)/\log (5R/4 \cap E) \).

Lemma A is proved in Section 3 in [13].

**Lemma B.** — Let \( E \) be a given closed set in the plane and let \( E_1 = E \cap \{z:|z-\rho|<R/2\} \). Let \( \rho > R \), and let \( \omega \) be the harmonic measure of the outer circle in \( \{z:|z-\rho|<R\}\setminus E \). We define

\[
B(r) = \max_{|z-\rho| = r} \omega(z).
\]

Then

\[
(3.5) \quad B(R/2) \leq (1-\alpha(R,E_1))B(R).
\]

**Proof** (Adapted from the first part of the proof of Theorem 1 in [13].) — We define \( \omega(z) = 0, \quad z \in E \cap \{z:|z-\rho|<\rho\} \). Let \( \omega_1 \) be the harmonic measure of \( E_1 \) in \( \{z:|z-\rho|<R\}\setminus E_1 \). If \( h(z) = \omega(z) - B(R)(1-\omega_1(z)) \), it is easy to check that \( h(z) \) is non-positive in \( \{z:|z-\rho|<R\}\setminus E \). Applying Lemma A, we obtain (3.5).

4. Applications of the estimates in Section 3.

We say that a closed set \( E \subset \mathbb{C} \) satisfies condition \( (K_4) \) if there exist positive numbers \( \delta \) and \( a \) in the interval \((0,1)\) such that for all real \( t \) with \( |t| \) sufficiently large, we have

\[
(4.1) \quad \text{cap} \{E \cap \{z:|z-\rho|<R\} \} \geq \delta R, \quad |t|^a \leq R \leq |t|.
\]
THEOREM 4. — Let $F$ be in the Nevanlinna class and assume that $F(0) = 0$. If the set $E = \{z : N(z, F) = 0\}$ satisfies condition $(K_1)$, then $\Re F \in h^1$, i.e.,

$$\sup_{0 < r < 1} \int_0^{2\pi} |\Re F(re^{i\theta})| \, d\theta < \infty.$$ 

Proof. — From (3.3) we see that $N(z, F)$ is uniformly bounded when $|z| \geq 1$. From (2.1) we see that it is sufficient to prove that

$$\int_{-\infty}^{\infty} N(it) < \infty.$$

Let $\omega$ be that harmonic measure of the outer circle in $\{z : |z - it| < |t|/2\} \setminus E$. Applying Lemma B with $\rho = |t|/2$, we see that for some $b > 0$, we have

$$B(R/2) \leq (1 - \alpha(R, E_1))B(R), \quad b < R < |t|/2.$$

It follows from condition $(K_1)$ that for all sufficiently large $|t|$, we have

$$\alpha(R, E_1) \geq \gamma > 0, \quad 2|t|^\alpha < R \leq |t|/2.$$

Putting $R_0 = 2|t|^\alpha$, we obtain

$$B(R_0) \leq (1 - \gamma)^p B(2^p R_0) \leq (1 - \gamma)^p,$$

where we can take $2^{p+1}|t|^\alpha \approx |t|$, i.e., $p \approx (1 - a) \log |t|/\log 2$. Thus, if $|t|$ is large, we have

$$(4.2) \quad \beta_E(it) \leq \omega(it) \leq \text{Const.} \cdot |t|^{-c},$$

where $c = (1 - a)\gamma/\log 2$.

Since $N(z)$ is bounded when $|z| > 1$, it follows from the maximum principle that

$N(it) \leq \text{Const.} \cdot \beta_E(it) \leq \text{Const.} \cdot |t|^{-c}, \quad |t| \geq 1.$

The Poisson integrals of $N$ in $\{\Re z > 0\}$ and $\{\Re z < 0\}$ are majorants of $N(z)$ in the respective halfplanes. We conclude that

$$N(z) \leq \text{Const.} \cdot |z|^{-c}, \quad |z| \geq 1,$$

provided that $0 < c < 1$. 
Repeating the previous argument, we see that

\[ N(it) \leq \text{Const. } |t|^{-c} \beta_E(it) \leq \text{Const. } |t|^{-2c}, \ |t| \geq 1. \]

Continuing in this way, we obtain

\[ N(it) \leq \text{Const. } |t|^{-qc}, \ |t| \geq 1, \]

where \( q \) is the integer determined by \( qc > 1 \) and \((q-1)c < 1\). (If \( qc = 1 \), we can decrease \( c \) slightly so that \( qc < 1, (q+1)c > 1 \)). Thus, we have

\[ \int_{-\infty}^{\infty} N(it) \, dt < \infty, \text{ and Theorem 4 is proved.} \]

As a second application of our ideas, we consider the class \( \text{L log L} \). We say that a closed set \( E \) in the complex plane satisfies condition \((K_2)\) if there exist positive numbers \( \delta \in (0,1) \) and \( q \) such that for all sufficiently large \( |t| \), we have

\[ \text{cap}(E \cap \{z:|z-it| \leq R\}) \geq \delta R, \ |t| (\log |t|)^{-q} \leq R \leq |t|/2. \] (4.3)

In the same way as in the proof of Theorem 4, we define for all sufficiently large \( |t| \)

\[ \gamma = \inf \alpha(R,E), 2|t| (\log |t|)^{-q} \leq R \leq |t|/2. \]

**Theorem 5.** – Let \( F \in H^1(U) \) and assume that \( F(0) = 0 \). If the set \( E = \{z:N(z,F) = 0\} \) satisfies condition \((K_2)\) with \( q\gamma > 2 \log 2 \), then \( \text{Re } F \in \text{L log L} \).

**Proof.** – Arguing in the same way as in the proof of Theorem 4 and choosing \( R_0 = 2|t| (\log |t|)^{-q} \), we have

\[ B(R_0) \leq (1 - \gamma)^p B(2^p R_0) \leq (1 - \gamma)^p, \]

where we can take \( 2^p R_0 \approx |t| \), i.e., \( p \approx (q/\log 2) \log \log |t| \). Thus, for \( |t| \) large,

\[ \beta_E(it) \leq \omega(it) \leq (1 - \gamma)^p \leq e^{-\gamma p} \approx (\log |t|)^{-q/\log 2} = (\log |t|)^{-2 - \varepsilon}, \]

where \( \varepsilon > 0 \). Theorem 5 now follows from Theorem 3.

We now point out an immediate consequence of Theorem 3 and some sharp estimates of Benedicks [3].
THEOREM 6. — Let \( p > 1 \) be a real number and put
\[
E = \bigcup_{m \neq 0} \left[ \text{sign} \left( m \right) |m|^p - d_m, \text{sign} \left( m \right) |m|^p + d_m \right],
\]
where \( \{d_m\}_{-\infty}^{\infty}, 0 < d_m < 1/2 \), is a sequence of positive numbers such that
\[
\log d_m \approx \log d_k, \ k \approx m,
\]
k, \( m \to \infty \) and \( k, m \to -\infty \). If \( F \in H^1(U) \) and \( N(w,F) = 0, \ w \in E \), a sufficient condition for \( \text{Re} \ F \in L \log L \) is that
\[
(4.4) \quad \sum \log \left( 1/d_m \right) \log m/m^2 < \infty.
\]

Remark. — It is clear that the set \( E \) can be chosen to be a very small subset of the imaginary axis.

Proof. — At the end of the proof of Theorem 5 in [3], Benedicks gives the estimate
\[
\beta_k(it) \leq \text{Const.} \left( \log p + (p-1) \log m + \log (1/d_m) + 1 \right)/m,
\]
\[
m^p \leq t \leq (m+1)^p, \ m = 1, 2, \ldots.
\]

This gives the convergence of \( \int_1^\infty \beta_k(iy) \log y \, dy/y \) provided that (4.4) holds. The argument as \( t \to -\infty \) is similar. Thus, Theorem 6 follows from Theorem 3.

5. \( H^p \)-classes and harmonic measure.

To apply Theorem 1, we need a geometric criterion on the range of an analytic function \( F \) to decide whether \( F \in H^1(U) \). Our main tool is the following observation which we state as

THEOREM 7. — Let \( F : U \to F(U) \) be analytic with \( F(0) = 0 \), and assume that \( C \setminus F(U) \) has positive capacity. Let \( \omega_R \) be the harmonic measure of the outer circle in that component \( D_R \) of \( \{(z,F(z)) : z \in U, |F(z)| < R \} \) which contains \((0,0) = 0\). Then, for \( 0 < p < \infty \), \( F \in H^p(U) \) if and only if
\[
(5.1) \quad \int_0^\infty R^{p-1} \omega_R(0) \, dR < \infty.
\]
Remark 1. — Here we understand the range of $F$ to lie on a Riemann surface $\mathcal{R}$, and $\omega_R$ to be harmonic measure on $\mathcal{R}$. If $F$ is univalent, it is not necessary to use this terminology: $\omega_R$ will be the harmonic measure of the circle $\{w:|w|=R\}$ in that component of $F(U) \cap \{w:|w|<R\}$ which contains the origin. The rest of Theorem 7 will remain unchanged.

Remark 2. — As a corollary, we obtain the following result of Hayman and Weitsman [14]: Let $\omega'_R$ be the harmonic measure of the outer circle in $F(U) \cap \{w:|w|<R\}$. Then $F \in H^p(U)$ if

$$\int_0^\infty R^{p-1} \omega'_R(0) \, dR < \infty.$$  

This is immediate from Theorem 7 since we have $\omega_R(0) \leq \omega'_R(0)$.

Added in proof. — Conversely, if $F \in H^p(U)$, then (5.1') holds. An argument proving this when $F$ is the universal covering map of $U$ onto $V$, $V$ such that $C\setminus V$ has positive capacity, is given in Section 6 of [8a]. The general case follows via subordination.

Remark 3. — Theorem 7 is equivalent to a result of Burkholder (Theorem 2.2, p. 189 in [4]). In Section 6, we shall use Theorem 7 to discuss another result of Burkholder (cf. [5], p. 115-116).

Proof of Theorem 7. — Assume that (5.1) holds. We define $F_{\rho}(z) = F(\rho z)$, $0 < \rho < 1$. Let $R > 0$ be given and let $h_{\rho} = h_{\rho,R}$ be the harmonic function on $U$ which is 1 on $\{e^{i\theta}:|F_{\rho}(e^{i\theta})|>R\}$ and 0 on $\{e^{i\theta}:|F_{\rho}(e^{i\theta})|\leq R\}$. Let $\omega_{\rho,R}$ be the harmonic measure of the outer circle in that component $D_{\rho,R}$ of $\{(z,F_{\rho}(z)):z \in U,|F_{\rho}(z)|<R\}$ which contains $(0,0) = 0$.

We claim that for $(z,F_{\rho}(z)) \in D_{\rho,R}$, we have

$$h_{\rho}(z) \leq \omega_{\rho,R}(F_{\rho}(z)).$$

To prove this, we consider

$$E_{\rho,R} = \{z \in U:|F_{\rho}(z)|<R\}.$$ 

If $z \in \partial E_{\rho,R} \cap U$, we have $|F_{\rho}(z)| = R$ and

$$\omega_{\rho,R}(F_{\rho}(z)) = 1 \geq h_{\rho}(z).$$
If \( z \in \partial E_{p,R} \cap T \), we have \( |F_p(z)| \leq R \) and
\[
h_p(z) = 0 \leq \omega_{p,R}(F_p(z)).
\]
Hence (5.2) follows from the maximum principle. Since we have \( D_{p,R} \subset D_R \), we conclude that
\[
h_p(0) \leq \omega_{p,R}(0) \leq \omega_R(0).
\]
We have assumed that the complement of \( F(U) \) has positive capacity and thus \( F \) is in the Nevanlinna class (cf. R. Nevanlinna [16], p. 209). For almost all \( R \), we have
\[
(2\pi)^{-1} m\{e^{i\theta} : |F(e^{i\theta})| > R\} = \lim_{p \to 1} \omega_p(0) \leq \omega_R(0).
\]
Since we have (5.1), it is now clear that \( F \in H^p(U) \) because
\[
\|F\|^p_p = \int_0^\infty (2\pi)^{-1} m\{e^{i\theta} : |F(e^{i\theta})| > R\} \, dR^p \leq p \int_0^\infty \omega_R(0) R^{p-1} \, dR < \infty.
\]
This concludes the first part of the proof.

Conversely, let us assume that \( F \in H^p(U) \) for some \( p > 0 \). We shall also assume that \( F \) is continuous on \( U \cup T \). If this is not the case, we argue as in the first part of the proof. Let \( NF \) be the nontangential maximal function of \( F \) (let the opening angle of the associated Stolz domain be \( 2\pi/3 \) (cf. Petersen [17], p. 8)). Let \( H = H_R \) be the harmonic function on \( U \) which is 1 on \( \{e^{i\theta} : NF(e^{i\theta}) \geq R\} \) and 0 on \( \{e^{i\theta} : NF(e^{i\theta}) < R\} \). If \( |F(z_0)| = R \), where \( z_0 = r_0 e^{i\alpha} = (1-\delta)e^{i\alpha} \) with \( \delta \in (0,1) \), we have
\[
NF(e^{i\theta}) \geq R, \quad |\theta - \alpha| < \delta,
\]
and it follows that
\[
H(z_0) \geq (2\pi)^{-1} \int_{|\varphi - \alpha| < \delta} (1 - r_0^2) (1 + r_0^2 - 2r_0 \cos (\varphi - \alpha))^{-1} \, d\varphi
\geq \pi^{-1} \int_0^\delta \delta (\delta^2 + t^2)^{-1} \, dt = 1/4.
\]
Let \( E_R = \{z \in U : |F(z)| < R\} \). We claim that
\[
(5.3) \quad \omega_R(z, F(z)) \leq 4H(z), \quad z \in E_R.
\]
Again, we use the maximum principle. If \( z \in \partial E_R \cap U \), we have \(|F(z)| = R\) and \(4H(z) \geq 1\). Thus, (5.3) holds in this case. If \( z \in \partial E_R \cap T \), we have either \(NF(z) \geq R\) and \(H(z) = 1\) or \(|F(z)| \leq NF(z) < R\) and consequently \(a_R(z,F(z)) = 0 \leq 4H(z)\). In both cases, (5.3) is true. In a standard way, we conclude that

\[
\omega_R(0) \leq (2/\pi)m\{e^{i\theta}:NF(e^{i\theta}) \geq R\},
\]

\[
\int_0^\infty \omega_R(0) dR^p \leq (2/\pi)\|NF\|^p_p \leq \text{Const.} \|F\|^{\mu p}_p.
\]

In the last step, we used a result of Hardy and Littlewood (cf. Theorem IV.40, p. 186 in Tsuji [21]). This concludes the proof of Theorem 7.


All examples \( F_\phi \) discussed below satisfy condition (1.1 a), while \( F_\phi \) may or may not be in \( H^1(U) \). In case \( F_\phi \in H^1(U) \), these examples may be considered to yield variants of Zygmund’s Theorem B, mentioned in the Introduction, by means of an obvious subordination argument.

A simple first example is \( F_0(z) = 2z(1-z^2)^{-1} \) which maps \( U \) onto \( C \{w = iv:|v| \geq 1\} \). Consequently, (1.1 a) is true for \( F_0 \). On the other hand, \( F_0 \) is not in \( H^1(U) \).

We proceed to construct a class of univalent functions \( F = F_\phi \) which are such that \( F(U) \) avoids a neighborhood of the imaginary axis near infinity. The function \( F \) will be or will not be in \( H^1(U) \) depending on the size of this neighborhood. Let

\[
D = D(\Phi) = \{z = re^{i\theta}:|\theta| - \pi/2 \leq \Phi(r), r \geq 2\},
\]

where the function \( \Phi \) will be in one of the following two classes: We say that \( \Phi : [2, \infty) \to [0, \pi/3] \) is in \( Q_1 \) if \( \Phi \) is continuous, \( \Phi(r) \to 0, r \to \infty \), and \( \Phi(2) = 0 \).

We say that \( \Phi : [2, \infty) \to [0, \pi/3] \) is in \( Q_2 \) if \( \Phi \in Q_1 \) and \( \Phi \) is differentiable with \( \Phi' \in L^\infty \) and with \( \int_2^\infty r\Phi'(r)^2 dr < \infty \).

Let \( F = F_\phi \) map \( U \) onto \( C \setminus D \) in such a way that \( F(0) = 0 \). We also introduce \( J = J(\Phi) = \int_2^\infty \Phi(r) dr/r \).
**PROPOSITION.** — If $\Phi \in Q_2$ and $J(\Phi)$ is finite, $F$ will not be in $H^1(U)$. If $\Phi \in Q_1$ and $J(\Phi)$ is infinite, with

$$
\left(\frac{2}{\pi} \int_2^R \frac{\Phi(t)}{t} \right) \frac{dR}{R} < \infty,
$$

then $F \in H^1(U)$. 

To prove the Proposition, we consider the harmonic measure, $\omega_R$, of the outer circle in $F(U) \cap \{z: |z| < R\}$. From Haliste ([9], formulas (2.1) and (2.3)), we see that if $\Phi \in Q_1$ and $R$ is large enough, we have

$$
\omega_R(0) \leq \frac{4}{\pi} \exp \left(4\pi - \pi \int_2^R (\pi - 2\Phi(t))^{-1} \frac{dt}{t} \right) \leq \frac{C_0}{R} \exp \left(-\frac{2}{\pi} \int_2^R \Phi(t) \frac{dt}{t} \right), \quad C_0 = 8e^{4\pi}.
$$

Now, (6.1) implies $\int_0^\infty \omega_R(0) dR < \infty$ and thus $F \in H^1(U)$, by Theorem 7.

From Theorem 2.1 in Haliste [9], we see that if $\Phi \in Q_2$ and $R$ is large enough, we have

$$
\omega_R(0) \geq C_1 \exp \left(-\pi \int_2^R (\pi - 2\Phi(t))^{-1} \frac{dt}{t} \right. \\
\left. - \pi \int_2^R \{r\Phi'(t)^2/(\pi - 2\Phi(t))\} \frac{dt}{3} \right),
$$

where $C_1 = (1/9) \exp (-8\pi(1 + 4\|\Phi\|_\infty^2/3))$.

It follows that if $\Phi \in Q_2$ and $J$ is finite, we have

$$
\omega_R(0) \approx \exp \left(-\int_2^R \frac{1}{\pi} \frac{1}{2\Phi(t)\pi}^{-1} \frac{dt}{t} \right) \approx \exp \left(-2J/\pi\right)/R.
$$

Thus, we see that $\int_2^\infty \omega_R(0) dR = \infty$. Applying Theorem 7, we see that $F \notin H^1(U)$, and we have proved the first part of the Proposition.

Let us in particular take $\Phi(r) = (\log r)^{-a}$, when $r \geq 3$. The associated domain is essentially of the form

$$
\{z = x + iy: |x| \leq |y| (\log |y|)^{-a}, |y| \geq 3\}.
$$
When \( a > 1 \), the argument above applies and \( F_\phi \) is not in \( H^1(U) \). On the other hand, (1.1a) is clearly true.

When \( 0 < a < 1 \), (6.1) holds and consequently \( F_\phi \in H^1(U) \). It follows from Theorem 1 that \( \text{Re } F_\phi \in L \log L \).

If \( \Phi(r) = C (\log r)^{-1}, \ r \geq 3 \), we have \( \omega_r(0) \approx R^{-1} (\log R)^{-2C/\pi} \) when \( R \) is large and it follows that

\[(6.3) \quad F_\phi \notin H^1(U), \ C \leq \pi/2, \ F_\phi \in H^1(U), \ C > \pi/2.\]

This illustrates the second part of the Proposition. In particular, it follows from Theorem 1 that \( \text{Re } F_\phi \in L \log L \) if \( C > \pi/2 \).

This last example is related to a problem considered by Burkholder (cf. [5], p. 115-116). Let \( S_\delta = \{x+iy : x>1, |y|<\delta \log x\} \) and let \( F_\delta \) be a univalent analytic function mapping \( U \) onto \( S_\delta \). Burkholder uses his theorem on "generalized subordination" to prove that

\[(6.4) \quad F_\delta \in H^1(U), \ \delta < 2/\pi, \ F_\delta \notin H^1(U), \ \delta > 2/\pi.\]

Using our notation with \( D(\Phi) \cap \{\text{Re } z > 0\} = S_\delta \), we have

\( \Phi(r) = (\delta \log r)^{-1} + O(\log \log r/(\log r)^2), \ r \to \infty, \)

and it follows from (6.3) that

\( F_\delta \in H^1(U), \ \delta < 2/\pi, \ F_\delta \notin H^1(U), \ \delta > 2/\pi. \)

Thus, we obtain Burkholder's result (6.4), as well as the boundary case \( \delta = 2/\pi. \)

Remark. — Using estimates of harmonic measure in "strip domains", K. Haliste has in [10] given still another method to treat Burkholder's problem, including the boundary case.

The following observation is due to Haliste. Let

\( T_\delta = \{re^{i\theta} : r > 1, |\theta| > p^{-1} \arctan (\delta p \log r)\} \)

and let \( G_\delta \) be a univalent analytic function mapping \( U \) onto \( T \). Then

\[ G_\delta \in H^p(U), \ \delta < 2/\pi, \ G_\delta \notin H^p(U), \ \delta > 2/\pi. \]

This result also follows in a simple way from our Theorem 7.

Let us now return to the more general regions \( D(\Phi) \) considered earlier. If a function \( F \) is such that

\[(6.5) \quad F(U) \subset C - D(\Phi) \]
with $J(\Phi)$ finite, then we cannot expect $F \in H^1(U)$. If however we require (6.5) to hold with $D(\Phi)$ replaced by a somewhat larger set, we can achieve $F \in H^1(U)$ and thus will be able to apply Theorem 1. Our last example is of this type.

Let $\Phi$ be in $Q_1$ with $J(\Phi)$ finite and let $\Omega$ be a collection of intervals contained in $(-\infty, -2] \cup [2, \infty)$ which is such that for all sufficiently large $R$ and for a constant $c > 1$, we have

$$\int_{\Omega(R)} dt/t \geq (c\pi/2) \log \log R,$$

$$\int_{\Omega(-R)} dt/t \geq (c\pi/2) \log \log R.$$ 

Here, $\Omega(R) = \Omega \cap [2, R]$ and $\Omega(-R) = \Omega \cap [-R, -2]$. Let $\Omega_0(R)$ be the one of the two sets $\Omega(R)$ and $\Omega(-R)$ which has the smallest logarithmic length. Let $F$ map $U$ univalently onto the infinite covering surface over $C \setminus (D(\Phi) \cup \Omega)$ in such a way that $F(0) = 0$. From standard estimates of harmonic measure (cf. Tsuji [21], p. 116), we see that

$$\int_2^{R/2} dt/t \leq \text{Const. exp} \left( -\left( \int_2^{R/2} + \int_{\Omega_0(R/2)} \right) (1 - 2\Phi(t)/\pi)^{-1} dt/t \right).$$

Thus, for $R$ large, we have

$$\omega_R(0) \leq \text{Const. } R^{-1} (\log R)^{-c},$$

and consequently $\int_2^\infty \omega_R(0) \, dR < \infty$. From Theorem 7, we see that $F \in H^1(U)$. Applying Theorem 1, we conclude that $\text{Re } F \in L \log L$.

Finally, we observe that the function $G(z) = iw/\log^2 (1 + w)$ with $w = (1 + z)/(1 - z)$ is in $H^1(U)$, but $\text{Re } G \notin L \log L$. Thus by Theorem 1 the integral (1.1 a) diverges, and in fact $N(1, iv) > (v \log^2 v)^{-1}$ is easy to see, for large $v$.

### 7. Extensions of Theorem 2.

Theorem 2 can be extended to meromorphic functions $f$, provided the subharmonic function $\Phi$ is not very large at infinity. We put $M(r, \Phi) = \sup_{\theta} \Phi(re^{i\theta})$. We have
Theorem 8. — Suppose \( f \) is meromorphic in \( \{ z : |z| < R \} \), where \( 0 < R \leq \infty \), and that \( f \) does not have a pole at the origin. Let \( \Phi \) be subharmonic in \( \mathbb{C} \), with \( \Phi(f(0)) \) finite, and suppose that for some \( \tau \in (0,1) \)

\[ \Phi(w) \leq O(|w|^\tau), \quad w \to \infty. \]  

Then, for each \( r \) such that \( f \) does not have a pole on the circle \( \{ z : |z| = r \} \), we have

\[ \frac{1}{2\pi} \int_0^{2\pi} \Phi(f(re^{i\theta})) \, d\theta = \int_{|w|=1} \left( N(r,w) - N(r,\infty) \right) d\mu(w) + \Phi(f(0)), \]

where \( \mu \) is the Riesz measure of \( \Phi \) and \( 0 < r < R \).

Here the main case of interest is that of \( \Phi \) small at \( \infty \), in the sense that (7.1) holds for all \( \tau > 0 \); in this case (7.2) is finite for every \( r < R \). (Compare the \( \Phi \) in (2.7)-(2.9).)

Proof. — Our assumption (7.1) implies that the following representation for \( \Phi \) holds on the entire plane (cf. Hayman and Kennedy [12], pp. 141, 146):

\[ \Phi(z) = \int_{|w|<1} \log |z-w| \, d\mu(w) + \int_{|w|>1} \log |zw^{-1}-1| \, d\mu(w) + c, \]

where \( c \) is a real constant, and \( \int_{|w|>1} d\mu(w)/|w| < \infty \). Put \( z = f(re^{i\theta}) \) in (7.3) and integrate \( d\theta \), as in the proof of Theorem 2, using Jensen’s theorem on \( f - w \) or \( w^{-1}f - 1 \) according as \( |w| < 1 \) or \( |w| \geq 1 \). Using (7.3) again, with \( z = f(0) \), to evaluate \( c \), we obtain (7.2).

We can also extend our results to functions mapping the polydisk or ball of \( \mathbb{C}^n \) to \( \mathbb{C} \). Let \( U^n \) be the unit polydisk in \( \mathbb{C} \):

\[ U^n = \{ z \in \mathbb{C}^n : |z_j| < 1, j = 1, \ldots, n \}. \]

\( U^n \) has distinguished boundary

\[ T^n = \{ z \in \mathbb{C}^n : |z_1| = \cdots = |z_n| = 1 \}. \]

For an \( n \)-tuple \( \varphi = (\varphi_1, \ldots, \varphi_n) \), \( \varphi_j \in [0,2\pi] \), we define a function \( f_\varphi \) on the unit disc by

\[ f_\varphi(z) = f(\zeta z^{\varphi_1}, \ldots, \zeta z^{\varphi_n}). \]
We define a counting function for $w \in \mathbb{C}$ by

$$N_f(r,w) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} N(r,w; f_\phi) \, d\phi_1 \ldots d\phi_n.$$  

Here $N(r,w; f_\phi)$ is the usual one-dimensional counting function for the function $f_\phi$. Jensen's formula is ([18], p. 326):

$$N_f(r,w) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \log |f(re^{i\phi_1}, \ldots, re^{i\phi_n}) - w| \, d\phi_1 \ldots d\phi_n$$

$$- \log |f(0) - w|.$$

Now consider the unit ball $B^n$ in $\mathbb{C}^n$:

$$B^n = \{ z \in \mathbb{C}^n : |z|^2 < 1 \}.$$

The boundary of $B^n$ is the unit sphere $S^{2n-1}$. For $z \in S^{2n-1}$ we define a function $f_z$ on the unit disc by $f_z(\zeta) = f(\zeta z)$. For the ball, the counting function is

$$N_f(r,w) = \frac{1}{C_n} \int_{S^{2n-1}} N(r,w; f_\phi) \, d\sigma(z).$$

Here the volume element $d\sigma$ is Lebesgue measure on $S^{2n-1}$ and $C_n$ is the volume of $S^{2n-1}$, i.e. $C_n = \frac{2\pi^n}{(n-1)!}$.

In this setting, Jensen's formula is ([20], p. 404):

$$N_f(r,w) = \frac{1}{C_n} \int_{S^{2n-1}} \log |f(\zeta z) - w| \, d\sigma(z) - \log |f(0) - w|.$$  

Using these versions of Jensen's formula as in the proof of Theorem 2, we get

**Theorem 9.** Suppose $\Phi$ is subharmonic in the complex plane with Riesz measure $\mu$. If $f$ is holomorphic in the unit polydisk $U^n$, then

$$\left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} \Phi(f(re^{i\phi_1}, \ldots, re^{i\phi_n})) \, d\phi_1 \ldots d\phi_n = \int_{C} N(r,w) \, d\mu(w) + \Phi(f(0)).$$

If $f$ is holomorphic in the unit ball $B^n$, then

$$\frac{1}{C_n} \int_{S^{2n-1}} \Phi(f(\zeta z)) \, d\sigma(z) = \int_{C} N(r,w) \, d\mu(w) + \Phi(f(0)).$$
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