M. ESSEN D. F. SHEA C. S. STANTON A value-distribution criterion for the class L logL and some related questions

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A VALUE-DISTRIBUTION CRITERION FOR THE CLASS L LOG L, AND SOME RELATED QUESTIONS

by M. ESSÉN (¹), D. F. SHEA (²) and C. S. STANTON

1. Introduction.

Let F belong to the Nevanlinna class of functions analytic in the unit disk U, so that

$$T(1,F) = \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |F(re^{i\theta})| \, d\theta < \infty \, .$$

In particular, $\lim_{r \to 1} F(re^{i\theta}) = F(e^{i\theta})$ exists a.e.. We shall say that Re $F \in L \log L$ if

 $\sup_{0 < r < 1} \int_0^{2\pi} |\operatorname{Re} F(re^{i\theta})| \log^+ |\operatorname{Re} F(re^{i\theta})| d\theta$ $= \int_0^{2\pi} |\operatorname{Re} F(e^{i\theta})| \log^+ |\operatorname{Re} F(e^{i\theta})| d\theta < \infty.$

The class L log L is closely related to the Hardy space $H^1(U)$, as is shown by the following classical results of Zygmund [22]:

THEOREM A. – If Re $F \in L \log L$, then $F \in H^1$.

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THEOREM B. – If $F \in H^1$ and $\operatorname{Re} F > 0$, then $\operatorname{Re} F \in L \log L$.

In this paper we prove some refinements of Theorem B. We state our basic results in terms of the usual Nevanlinna counting function

$$N(r,w) = N(r,w;F) = \int_0^r n(t,w) dt/t$$

where $n(t,w) = \sum_{|z_v| \leq t} 1$ and $\{z_v\} = f^{-1}(w)$. Our main result is

THEOREM 1. - Let $F \in H^{1}(U)$. The following are equivalent: (1.1 a) $\int_{-\infty}^{\infty} N(1, iv) \log^{+} |v| dv < \infty$. (1.1 b) Re $F \in L \log L$. (1.1 c) $\int_{0}^{2\pi} |\text{Re } F(e^{i\theta})| \log^{+} |F(e^{i\theta})| d\theta < \infty$.

Remark 1. – We note that if Re F > 0, then N(1,iv) = 0 for all $v \in \mathbf{R}$. Hence Theorem B follows from the equivalence of (1.1 a) and (1.1 b).

Remark 2. — We could replace the integration over the imaginary axis in (1.1 a) by integration over any vertical line, i.e., for any real u (1.1 a) is equivalent to

(1.1 a')
$$\int_{-\infty}^{\infty} N(1, u + iv) \log^+ |v| \, dv < \infty \, .$$

Once the Theorem has been proved, this follows immediately since N(1,u+iv;F) = N(1,iv;F-u) and $Re(F-u) \in L \log L$ if and only if $Re F \in L \log L$.

In Sections 3 through 6, we give some further refinements of Theorem B: from a geometrical condition on the range of F, we can deduce that $\text{Re } F \in L \log L$. To apply Theorem 1, we need a criterion to decide whether $F \in H^1(U)$. In this context, our main tool is a more general result which may have independent interest : in terms of harmonic measure, it gives a necessary and sufficient condition for $F \in H^{p}(U), 0 (cf. Theorem 7 in Section 5)). We also consider$ cases when the hypothesis $F \in H^1(U)$ in Theorem 1 is omitted. The material in Sections 5 and 6 overlaps with certain work of Burkholder in [4], [5].

The starting-point of our work was a study of the relation between the classical criterion of Zygmund and the following result of A. Baernstein. Let $f \in L^1(T)$ be a given real-valued function and consider

(1.2)
$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\phi}) \frac{e^{i\phi} + z}{e^{i\phi} - z} d\phi.$$

We have $F(e^{i\theta}) = f(e^{i\theta}) + i\tilde{f}(e^{i\theta})$ a.e. on T. Let g be the symmetric decreasing rearrangement of f and \tilde{g} the conjugate function of g with mean value zero. In particular, we have

$$||g||_1 = ||f||_1, \int_{T} |g| \log^+ |g| = \int_{T} |f| \log^+ |f|.$$

In [2], Baernstein proved that

(1.3)
$$\|\tilde{f}\|_{1} \leq \|\tilde{g}\|_{1}$$
.

Thus, when $f \in L^1(T)$ is given, (1.3) implies a sufficient condition for F to be in H¹, namely: $F \in H^1$ if $\|\tilde{g}\|_1 < \infty$. However, this consequence of (1.3) does not actually yield a new criterion for $F \in L^1$, in view of the following consequence of Theorem 1:

Corollary 1. $-\tilde{g} \in L^1(T) \Leftrightarrow g \in L \log L$.

Proof. – Assume that $\tilde{g} \in L^1(T)$. From the discussion in Section 6 in Baernstein [2], we see that the analytic function G associated to g by (1.2) maps U univalently onto a Steiner symmetric domain, i.e., a domain with the property that for all $u \in \mathbf{R}$,

$$G(U) \cap \{ \text{Re } w = u \} = \{ w = u + iv : |v| < b(u) \}.$$

If there exists u_0 such that $b(u_0) < \infty$, condition (1.1 a') will hold for $u = u_0$, and Theorem 1 implies that $g \in L \log L$ which is equivalent to $f \in L \log L$. But such a u_0 exists because if $b(u) = \infty$ for all $u \in \mathbf{R}$, G would map U onto the whole complex plane.

The converse assertion is simply Zygmund's Theorem A; Corollary 1 is proved.

Remark. - In a private discussion, Lennart Carleson has shown us a simple real-variable proof of Corollary 1.

In Section 2, we deduce Theorem 1 from a general identity :

THEOREM 2. – Let F be analytic in U and let Φ be subharmonic in C with $\Phi(F(0))$ finite. Then

(1.4)
$$(2\pi)^{-1} \int_0^{2\pi} \Phi(F(re^{i\theta})) d\theta = \int_C N(r,w) d\mu(w) + \Phi(F(0)),$$

 $0 < r < 1,$

where μ is the Riesz measure of Φ .

For each choice of Φ in Theorem 2, we get a formula connecting a mean of F over a circle in U with an integral of N(r,.) over the range of F. Examples of such formulas will be given in Section 2. The present proof of Theorem 1 avoids some lengthy estimates in the original proof of Essén and Shea, as announced in [8]; this simplification is made possible because of Stanton's proof of (1.4) in [19]. We apply (1.4) here with $\Phi(w) = |\text{Re } w| \log (1 + |w|^2)$, cf. (2.5) below.

If μ is a positive finite measure on C and $-\Phi$ is the logarithmic potential of μ , then (1.4) is a classical formula of Frostman, cf. [16, p. 177].

We can extend Theorem 2 in a number of ways. For example, it holds when Φ is δ -subharmonic, i.e. the difference of two subharmonic functions, with μ a signed measure. An identity like (1.4) is true for fmeromorphic (in the disk or in the plane) provided Φ has sufficiently small growth at infinity. The theorem also can be extended to analytic functions mapping the polydisk or ball of \mathbb{C}^n into \mathbb{C} . Details of these extensions are given in Section 7.

2. Proofs of Theorems 1 and 2.

We need the following well-known facts on the Nevanlinna counting function (for further information and references, cf. Section 4 in Essén and Shea [7]) :

 $N(1,w,F) = \lim_{r \to 1^{-}} N(r,w,F)$ exists and is uniformly bounded except near F(0). The upper regularization N(w) = N(w,F) of N(1,w,F), defined by

$$N(w) = \limsup_{\zeta \to w} N(1,\zeta,F),$$

is subharmonic in $\mathbb{C}\setminus\{F(0)\}\$ and coincides with N(1,w,F) off a set of w-values of logarithmic capacity zero. The function $N(w) + \log |w - F(0)|$ can be defined at F(0) to be subharmonic in \mathbb{C} .

Throughout the paper, the H^p-norms are defined as in Duren [6], p. 35. The class h^p of harmonic functions in U is defined in [6], p. 2. The set of interior points of a set K is denoted by K^0 .

Proof of Theorem 2. – Let r be fixed, 0 < r < 1. Then there exists a compact set K such that supp $N(r, .) \subset K^0$ and we have

$$\Phi(\zeta) = \int_{K} \log |\zeta - w| \, d\mu(w) + h(\zeta) \, ,$$

where h is harmonic in K^0 . From Jensen's formula

$$(2\pi)^{-1} \int_0^{2\pi} \log |F(re^{i\theta}) - w| \, d\theta = N(r,w) + \log |F(0) - w|,$$

we deduce that

$$(2\pi)^{-1} \int_{0}^{2\pi} \Phi(F(re^{i\theta})) d\theta$$

= $\int_{K} d\mu(w) \int_{0}^{2\pi} \log |F(re^{i\theta}) - w| d\theta/(2\pi) + h(F(0))$
= $\int_{K} (N(r,w) + \log |F(0) - w|) d\mu(w) + h(F(0))$
= $\int_{K} N(r,w) d\mu(w) + \Phi(F(0)).$

The theorem is proved.

Next, we give a list of some special subharmonic functions Φ , their associated Riesz measures μ and the formulas which follow from (1.4). Most of these formulas are known; (2.5) is new and is basic to our proof of Theorem 1.

We write w = u + iv. δ_u and δ_v are Dirac measures supported by the v- and u-axis, respectively. (Formally, we should write $\delta_{\{u=0\}} \otimes 1$ and $1 \otimes \delta_{\{v=0\}}$). We put $k(t) = (2+t)(1+t)^{-2}$ and $C(t) = 2\pi t \log (1+t)$.

Ф]u	<i>v</i>	w	$ u \log\left(1+ u \right)$	$ u \log\left(1+ w ^2\right)$
$\Delta \Phi = 2\pi \ d\mu$	2δ"	2δ _v	w ⁻¹	k(u)	$2 \delta_u \log(1+v^2) + 4 u k(w ^2)$

Applying (1.4), we obtain

(2.1)
$$\int_{0}^{2\pi} |\operatorname{Re} F(re^{i\theta})| \, d\theta = 2 \int_{-\infty}^{\infty} N(r, iv) \, dv + 2\pi |\operatorname{Re} F(0)|,$$

(2.2)
$$\int_{0}^{2\pi} |\operatorname{Im} F(re^{i\theta})| \, d\theta = 2 \int_{-\infty}^{\infty} N(r,u) \, du + 2\pi |\operatorname{Im} F(0)|,$$

(2.3)
$$\int_0^{2\pi} |F(re^{i\theta})| \, d\theta = \iint_C N(r,w) \, du \, dv/|w| + 2\pi |F(0)|,$$

(2.4)
$$\int_{0}^{2\pi} (|\operatorname{Re} \mathbf{F}| \log (1 + |\operatorname{Re} \mathbf{F}|))(re^{i\theta}) d\theta = \iint_{C} \mathbf{N}(r, w) k(|u|) du dv + C(|\operatorname{Re} \mathbf{F}(0)|),$$

(2.5)
$$\int_{0}^{2\pi} (|\operatorname{Re} F| \log (1+|F|^{2}))(re^{i\theta}) d\theta$$
$$= 4 \iint_{C} N(r,w) |u| k(|w|^{2}) du dv$$
$$+ 2 \int_{-\infty}^{\infty} N(r,iv) \log (1+v^{2}) dv + 2\pi |\operatorname{Re} F(0)| \log (1+|F(0)|^{2}).$$

The equation $\Delta \Phi = 2\pi d\mu$ is interpreted in the distributional sense. This means that for all $\psi \in C_0^{\infty}(\mathbb{R}^2)$, we have

(2.6)
$$\iint_{\mathbf{C}} \Phi \, \Delta \psi = 2\pi \iint_{\mathbf{C}} \psi \, d\mu,$$

(cf. Lemmas 3.6 and 3.8 in Hayman and Kennedy [12]).

We illustrate the computation of these formulas by deriving the one needed for (2.5). We choose $\Phi(w) = |u| \log (1+|w|^2)$. Then if $u \neq 0$, $\Phi \in C^{\infty}$ near u + iv and $\Delta \Phi = 4|u|k(|w|^2)$.

Let $\psi \in C_0^{\infty}(\mathbb{R}^2)$. From Green's theorem, we deduce that

$$\iint_{\{u>0\}} \Phi \,\Delta \psi = \iint_{\{u>0\}} \psi(w) 4uk(|w|^2) \,du \,dv + \int_{-\infty}^{\infty} \psi(iv) \log (1+v^2) \,dv \,,$$
$$\iint_{\{u<0\}} \Phi \,\Delta \psi = \iint_{\{u<0\}} \psi(w) 4|u|k(|w|^2) \,du \,dv + \int_{-\infty}^{\infty} \psi(iv) \log (1+v^2) \,dv \,.$$

Adding these two formulas, and using (2.6), we obtain

$$2\pi \iint_{C} \psi \, d\mu = \iint_{C} \psi(w) 4 |u| k(|w|^2) \, du \, dv + 2 \int_{-\infty}^{\infty} \psi(iv) \log (1+v^2) \, dv \, ,$$

which is the fifth formula in the table above.

We rewrite (2.5) in the following way:

 $I_1(r) = 4I_2(r) + 2I_3(r) + 2\pi |\operatorname{Re} F(0)| \log (1 + |F(0)|^2).$

Let $I_j = \sup I_j(r), 0 < r < 1, j = 1, 2, 3.$

LEMMA 1. – Let F be analytic in U. Then $F \in H^1(U)$ if and only if Re $F \in h^1$ and I_2 is finite.

Proof. – Let $F \in H^1(U)$. From the inequality $k(t) < t^{-1}$, t > 0, it follows that $|u|k(|w|^2) \le |u||w|^{-2} \le |w|^{-1}$ and I_2 must be finite since we have (2.3). Trivially, we have $\operatorname{Re} F \in h^1$.

Conversely, if I_2 is finite, we use the subharmonicity of N(r,w) in $C \setminus \{F(0)\}$ to deduce that

$$\begin{split} \int_{|u|>2|F(0)|} \mathbf{N}(r,u) \, du &\leq 4 \int_{-\infty}^{\infty} (\pi u^2)^{-1} \, du \iint_{|\zeta-u|<|u|/2} \mathbf{N}(r,\zeta) \, d\xi \, d\eta \\ &\leq (4/\pi) \iint_{\mathbf{D}_+} \mathbf{N}(r,\zeta) (\xi^2 - 3\eta^2)^{1/2} \, d\xi \, d\eta/|\zeta|^2 \\ &\leq (4/\pi) \iint_{\mathbf{C}} \mathbf{N}(r,\zeta) |\xi| \, d\xi \, d\eta/|\zeta|^2 = \mathbf{I}_4(r) \,, \end{split}$$

where $D_{+} = \{\zeta = \xi + i\eta : \xi^2 \ge 3\eta^2\}$.

If I_2 is finite, $\sup_{0 < r < 1} I_4(r)$ will also be finite. It is now clear from (2.2) that $\operatorname{Im} F \in h^1$. Since $\operatorname{Re} F \in h^1$, we must have $F \in H^1(U)$ and the lemma is proved.

Proof of Theorem 1. - The proof will show that

$$(1.1 a) \rightarrow (1.1 b) \rightarrow (1.1 c) \rightarrow (1.1 a).$$

To prove $(a) \rightarrow (b)$, we first note that it follows from (a) that I_3 is finite and from Lemma 1 that I_2 is finite. Thus, by (2.5), I_1 is finite and we have proved (b).

To prove $(b) \rightarrow (c)$, we consider the following simple chain of inequalities:

$$|u|\log(1+u^2+v^2) \leq 2|u|\log(1+|u|+|v|) \leq 2(|u|\log(1+|u|)+|v|)$$

If we now choose u = Re F, v = Im F and integrate with respect to θ , we obtain

$$I_1 \leq 2 \int_0^{2\pi} (|\operatorname{Re} F| \log (1 + |\operatorname{Re} F|) + |\operatorname{Im} F|)(e^{i\theta}) \, d\theta < \infty,$$

and have proved (c).

The implication $(c) \rightarrow (a)$ is an immediate consequence of (2.5).

Remark. – It is easy to prove Zygmund's Theorem A that (1.1 b) implies $F \in H^1(U)$, using these methods. This is immediate from (2.3), (2.4) and the inequality $k(|u|) \ge (|u|+1)^{-1} \ge (2|w|)^{-1}$ (valid for $|w|\ge 1$), which yield

$$\frac{1}{2\pi} \int_0^{2\pi} |\mathbf{F}(re^{i\theta})| \, d\theta \leq \frac{1}{\pi} \int_0^{2\pi} |\operatorname{Re} \mathbf{F}| \log (1 + |\operatorname{Re} \mathbf{F}|) (re^{i\theta}) \, d\theta + C_1$$

for $C_1 = T(1,F) + |F(0)| + 1$.

In the opposite direction, (2.3) and (2.5) together with $|u|k(|w|^2) \leq |w|^{-1}$ imply

$$\frac{1}{2\pi} \int_0^{2\pi} |\operatorname{Re} \mathbf{F}| \log (1+|\mathbf{F}|^2)^{1/2} (re^{i\theta}) \, d\theta \leq \frac{1}{\pi} \int_0^{2\pi} |\mathbf{F}(re^{i\theta})| \, d\theta + C_2 \,,$$
$$C_2 = \frac{1}{\pi} \int_{-\infty}^{\infty} \mathcal{N}(1,iv) \log (1+v^2)^{1/2} \, dv + |\operatorname{Re} \mathbf{F}(0)| \log (1+|\mathbf{F}(0)|) \,.$$

With heavy restrictions on F(U), such as Re F > 0, inequalities of this type are classical (cf. [22], p. 256).

Let us finally give some further examples of formulas which are immediate consequences of Theorem 2. Successively choosing $\Phi(w)$ as $\log^+ |w|$, $\log(1+|w|^2)$, $|w|^p$ with p > 0, $|u|^p$ with p > 1 and as |w|A (arg w) where

$$A(\varphi) = \begin{cases} (1/2)\varphi \sin \varphi, |\varphi| \leq \pi/2, \\ (1/2)(\pi - \varphi) \sin \varphi - \cos \varphi, \pi/2 \leq \varphi \leq 3\pi/2, \end{cases}$$

we obtain

$$(2.7) \int_{0}^{2\pi} \log^{+} |F(re^{i\theta})| \, d\theta = \int_{0}^{2\pi} N(r, e^{i\phi}) \, d\phi + 2\pi \log^{+} |F(0)|,$$

$$(2.8) \int_{0}^{2\pi} \log (1 + |F(re^{i\theta})|^{2}) \, d\theta$$

$$= 4 \iint_{C} N(r, w) (1 + |w|^{2})^{-2} \, du \, dv + 2\pi \log (1 + |F(0)|^{2})$$

$$(2.9) \int_{0}^{2\pi} |F(re^{i\theta})|^{p} \, d\theta = p^{2} \iint_{C} N(r, w) |w|^{p-2} \, du \, dv$$

$$+ 2\pi |F(0)|^{p}, p > 0.$$

(2.10)
$$\int_{0}^{2\pi} |\operatorname{Re} \operatorname{F}(re^{i\theta})|^{p} d\theta$$
$$= p(p-1) \iint_{C} \operatorname{N}(r,w) |u|^{p-2} du dv + 2\pi |\operatorname{Re} \operatorname{F}(0)|^{p}, p > 1,$$

(2.11)
$$\int_{0}^{2\pi} (|F|A(\arg F))(re^{i\theta}) d\theta$$
$$= \iint_{C} N(r,w)|u| |w|^{-2} du dv + 2\pi |F(0)|A(\arg F(0)).$$

Remarks. – Equation (2.7) is Cartan's identity (see Hayman [11], p. 8). Equation (2.8) is a version of a classical formula for the Ahlfors characteristic (see (3.1), p. 173, in Nevanlinna [16]). Equations (2.9), (2.1) and (2.2) are classical (see e.g. Lehto [15], pp. 12, 14). Baernstein derives (2.1), (2.2), (2.9) and (2.10) from Cartan's identity in [2].

3. The class L log L and estimates of harmonic measure.

What more can we say about the connection between the closed set E on which N(z) = N(z,F) vanishes, and the integrability condition (1.1 b)? From now on, we assume that F(0) = 0.

We need an idea of M. Benedicks [3], developed to study positive harmonic functions vanishing on the boundary in sets of the form $C \setminus E_0$, where E_0 is a subset of the imaginary axis. Our set E is not necessarily restricted in this way.

Following Benedicks, we introduce a function β_E which measures how « thin » the set E is at infinity near the imaginary axis. If $z \neq 0$, let K_z be the open square in the plane with centre at z, sides parallel to the axis and with side length |z|. Let $\Omega_z = K_z \setminus E$. In Ω_z , we consider the harmonic function V^z which has boundary values 1 on ∂K_z and 0 on $E \cap K_z$. We define $\beta_E(z) = V^z(z)$.

THEOREM 3. – Let $F \in H^1(U)$ and assume that F(0) = 0. A sufficient condition for Re F to be in L log L is that

(3.1)
$$\int_{|y|>1} \beta_{\mathrm{E}}(iy) \log |y| \, dy/y < \infty \, .$$

In Section 4, we shall give examples of conditions on the omitted set E which ensure that (3.1) holds.

In the proof of Theorem 3, we need

LEMMA 2. – Assume that $F \in H^p(U)$ for some p > 0 and that F(0) = 0. Then

(3.2)
$$N(z,F) \leq C_p ||F||_p^p |z|^{-p}, z \neq 0,$$

where $C_p = p^{-1}$, $0 , and <math>C_p \le 4$, p > 1.

Proof. – For any F in the Nevanlinna class with F(0) = 0, it follows from Jensen's theorem that we have

(3.3)
$$N(w,F) \leq (2\pi)^{-1} \int_0^{2\pi} \log(1+|F(e^{i\theta})||w|^{-1}) d\theta$$

When 0 , (3.2) is an immediate consequence of (3.3) and the inequality

$$\log(1+u) \leq u^p/p, \ 0 0.$$

When p > 2, we use the fact that N(z) is subharmonic in $\mathbb{C} \setminus \{0\}$ to deduce that, if $\rho = 2/p$,

$$\begin{split} \mathbf{N}(z) &\leq (\pi \rho^2 |z|^2)^{-1} \iint_{|w-z| < \rho |z|} \mathbf{N}(w) \, du \, dv \\ &\leq |z|^{-p} \rho^{-2} (1-\rho)^{2-p} \iint_{\mathbf{C}} \mathbf{N}(w) |w|^{p-2} \, du \, dv / \pi \\ &\leq \mathbf{C}_p \|\mathbf{F}\|_p^p |z|^{-p}, \qquad \mathbf{C}_p = (1-2/p)^{2-p} / 2 \,. \end{split}$$

In the last step, we used (2.9). The argument is similar when $1 , with <math>\rho = 1$ and $C_p = (2/p^2)2^{2-p}$.

Proof of Theorem 3. - Using the maximum principle, we deduce from Lemma 2 that

$$N(\zeta) \leq 2 \|F\|_1 V^{iy}(\zeta)/|y|, \ \zeta \in \Omega_{iy},$$

$$N(iy) \leq 2 \|F\|_1 \beta_F(iy)/|y|, \ y \neq 0.$$

Hence, Theorem 3 is an immediate consequence of Theorem 1.

In the study of the function $\beta_{\rm E}(z)$, we need two lemmas of Hayman and Pommerenke [13].

LEMMA A. – Let E_1 be a compact subset of $\{z:|z| \leq R/2\}$ and let ω_{E_1} be the harmonic measure of E_1 in $\{z:|z| < R\} \setminus E_1$. Then

(3.4)
$$\omega_{\mathbf{E}_1}(z) \ge \alpha(\mathbf{R},\mathbf{E}_1), \ |z| \le \mathbf{R}/2,$$

where $\alpha(R,E_1) = \log (5/4) / \log (5R/4 \operatorname{cap} E_1)$.

Lemma A is proved in Section 3 in [13].

LEMMA B. – Let E be a given closed set in the plane and let $E_1 = E \cap \{z : |z - it| \le R/2\}$. Let $\rho > R$, and let ω be the harmonic measure of the outer circle in $\{z : |z - it| < \rho\} \setminus E$. We define $B(r) = \max_{|z - it| = r} \omega(z)$. Then

(3.5)
$$B(R/2) \leq (1 - \alpha(R, E_1))B(R)$$
.

Proof (Adapted from the first part of the proof of Theorem 1 in [13].). – We define $\omega(z) = 0$, $z \in E \cap \{z: |z-it| < \rho\}$. Let ω_1 be the harmonic measure of E_1 in $\{z: |z-it| < R\} \setminus E_1$. If $h(z) = \omega(z) - B(R)(1-\omega_1(z))$, it is easy to check that h(z) is non-positive in $\{z: |z-it| < R\} \setminus E$. Applying Lemma A, we obtain (3.5).

4. Applications of the estimates in Section 3.

We say that a closed set $E \subset C$ satisfies condition (K_1) if there exist positive numbers δ and a in the interval (0,1) such that for all real twith |t| sufficiently large, we have

(4.1)
$$\operatorname{cap} \{ E \cap \{ z : |z - it| \leq R \} \} \geq \delta R, \ |t|^a \leq R \leq |t|.$$

THEOREM 4. – Let F be in the Nevanlinna class and assume that F(0) = 0. If the set $E = \{z: N(z,F)=0\}$ satisfies condition (K_1) , then Re $F \in h^1$, i.e.,

$$\sup_{0< r<1} \int_0^{2\pi} |\operatorname{Re} F(re^{i\theta})| \, d\theta < \infty \, .$$

Proof. – From (3.3) we see that N(z,F) is uniformly bounded when $|z| \ge 1$. From (2.1) we see that it is sufficient to prove that $\int_{-\infty}^{\infty} N(it) < \infty$.

Let ω be that harmonic measure of the outer circle in $\{z:|z-it| < |t|/2\} \setminus E$. Applying Lemma B with $\rho = |t|/2$, we see that for some b > 0, we have

$$B(R/2) \leq (1 - \alpha(R, E_1))B(R), b < R < |t|/2.$$

It follows from condition (K_1) that for all sufficiently large |t|, we have

$$\alpha(\mathbf{R},\mathbf{E}_1) \ge \gamma > 0, \quad 2|t|^a < \mathbf{R} \le |t|/2.$$

Putting $R_0 = 2|t|^a$, we obtain

$$\mathbf{B}(\mathbf{R}_0) \leq (1-\gamma)^p \mathbf{B}(2^p \mathbf{R}_0) \leq (1-\gamma)^p,$$

where we can take $2^{p+1}|t|^a \approx |t|$, i.e., $p \approx (1-a) \log |t|/\log 2$. Thus, if |t| is large, we have

(4.2)
$$\beta_{\rm E}(it) \leq \omega(it) \leq {\rm Const.} |t|^{-c},$$

where $c = (1-a)\gamma/\log 2$.

Since N(z) is bounded when |z| > 1, it follows from the maximum principle that

$$N(it) \leq Const. \ \beta_E(it) \leq Const. \ |t|^{-c}, |t| \geq 1.$$

The Poisson integrals of N in $\{\text{Re } z > 0\}$ and $\{\text{Re } z < 0\}$ are majorants of N(z) in the respective halfplanes. We conclude that

$$N(z) \leq Const. |z|^{-c}, |z| \geq 1,$$

provided that 0 < c < 1.

Repeating the previous argument, we see that

$$N(it) \leq Const. |t|^{-c}\beta_E(it) \leq Const. |t|^{-2c}, |t| \geq 1.$$

Continuing in this way, we obtain

$$N(it) \leq Const. |t|^{-qc}, |t| \geq 1$$
,

where q is the integer determined by qc > 1 and (q-1)c < 1. (If qc = 1, we can decrease c slightly so that qc < 1, (q+1)c > 1). Thus, we have $\int_{-\infty}^{\infty} N(it) dt < \infty$, and Theorem 4 is proved.

As a second application of our ideas, we consider the class L log L. We say that a closed set E in the complex plane satisfies condition (K_2) if there exist positive numbers $\delta \in (0,1)$ and q such that for all sufficiently large |t|, we have

$$(4.3) \quad \operatorname{cap}\left(\mathbb{E} \cap \{z : |z - it| \leq \mathbb{R}\}\right) \geq \delta \mathbb{R}, |t| (\log |t|)^{-q} \leq \mathbb{R} \leq |t|/2.$$

In the same way as in the proof of Theorem 4, we define for all sufficiently large |t|

$$\gamma = \inf \alpha(\mathbf{R}, \mathbf{E}_1), \ 2|t|(\log |t|)^{-q} \leq \mathbf{R} \leq |t|/2.$$

THEOREM 5. - Let $F \in H^1(U)$ and assume that F(0) = 0. If the set $E = \{z: N(z;F)=0\}$ satisfies condition (K_2) with $q\gamma > 2 \log 2$, then Re $F \in L \log L$.

Proof. – Arguing in the same way as in the proof of Theorem 4 and choosing $R_0 = 2|t| (\log |t|)^{-q}$, we have

$$\mathbf{B}(\mathbf{R}_0) \leq (1-\gamma)^p \mathbf{B}(2^p \mathbf{R}_0) \leq (1-\gamma)^p,$$

where we can take $2^{p}R_{0} \approx |t|$, i.e., $p \approx (q/\log 2) \log \log |t|$. Thus, for |t| large,

$$\beta_{\mathrm{E}}(it) \leq \omega(it) \leq (1-\gamma)^{p} \leq e^{-\gamma p} \approx (\log |t|)^{-q/\log 2} = (\log |t|)^{-(2+\varepsilon)},$$

where $\varepsilon > 0$. Theorem 5 now follows from Theorem 3.

We now point out an immediate consequence of Theorem 3 and some sharp estimates of Benedicks [3].

THEOREM 6. – Let $p \ge 1$ be a real number and put

$$\mathbf{E} = \bigcup_{m \neq 0} \left[\operatorname{sign}(m) |m|^p - d_m, \operatorname{sign}(m) |m|^p + d_m \right],$$

where $\{d_m\}_{-\infty}^{\infty}$, $0 < d_m < 1/2$, is a sequence of positive numbers such that

$$\log d_m \approx \log d_k, \ k \approx m,$$

 $k, m \to \infty$ and $k, m \to -\infty$. If $F \in H^1(U)$ and $N(w,F) = 0, w \in E, a$ sufficient condition for $Re F \in L \log L$ is that

(4.4)
$$\Sigma \log (1/d_m) \log m/m^2 < \infty.$$

Remark. — It is clear that the set E can be chosen to be a very small subset of the imaginary axis.

Proof. - At the end of the proof of Theorem 5 in [3], Benedicks gives the estimate

$$\beta_{\rm E}(it) \leq {\rm Const.} (\log p + (p-1)\log m + \log (1/d_m) + 1)/m,$$

 $m^p \leq t \leq (m+1)^p, m = 1, 2, \dots$

This gives the convergence of $\int_{1}^{\infty} \beta_{\rm E}(iy) \log y \, dy/y$ provided that (4.4) holds. The argument as $t \to -\infty$ is similar. Thus, Theorem 6 follows from Theorem 3.

5. H^p-classes and harmonic measure.

To apply Theorem 1, we need a geometric criterion on the range of an analytic function F to decide whether $F \in H^1(U)$. Our main tool is the following observation which we state as

THEOREM 7. – Let $F: U \to F(U)$ be analytic with F(0) = 0, and assume that $C \setminus F(U)$ has positive capacity. Let ω_R be the harmonic measure of the outer circle in that component D_R of $\{(z,F(z)): z \in U, |F(z)| < R\}$ which contains (0,0) = 0. Then, for $0 , <math>F \in H^p(U)$ if and only if

(5.1)
$$\int_0^\infty \mathbf{R}^{p-1}\omega_{\mathbf{R}}(0)\,d\mathbf{R}\,<\,\infty\,.$$

Remark 1. – Here we understand the range of F to lie on a Riemann surface \mathscr{R} , and ω_R to be harmonic measure on \mathscr{R} . If F is univalent, it is not necessary to use this terminology: ω_R will be the harmonic measure of the circle $\{w:|w|=R\}$ in that component of $F(U) \cap \{w:|w|<R\}$ which contains the origin. The rest of Theorem 7 will remain unchanged.

Remark 2. — As a corollary, we obtain the following result of Hayman and Weitsman [14]: Let ω'_{R} be the harmonic measure of the outer circle in $F(U) \cap \{w:|w| < R\}$. Then $F \in H^{p}(U)$ if

(5.1')
$$\int_0^\infty \mathbf{R}^{p-1} \omega'_{\mathbf{R}}(0) \, d\mathbf{R} < \infty \, .$$

This is immediate from Theorem 7 since we have $\omega_R(0) \leq \omega'_R(0)$.

Added in proof. – Conversely, if $F \in {}^{\theta p}(U)$, then (5.1') holds. An argument proving this when F is the universal covering map of U onto V, V such that C\V has positive capacity, is given in Section 6 of [8a]. The general case follows via subordination.

Remark 3. — Theorem 7 is equivalent to a result of Burkholder (Theorem 2.2, p. 189 in [4]). In Section 6, we shall use Theorem 7 to discuss another result of Burkholder (cf. [5], p. 115-116).

Proof of Theorem 7. – Assume that (5.1) holds. We define $F_{\rho}(z) = F(\rho z)$, $0 < \rho < 1$. Let R > 0 be given and let $h_{\rho} = h_{\rho,R}$ be the harmonic function on U which is 1 on $\{e^{i\theta}: |F_{\rho}(e^{i\theta})| > R\}$ and 0 on $\{e^{i\theta}: |F_{\rho}(e^{i\theta})| \le R\}$. Let $\omega_{\rho,R}$ be the harmonic measure of the outer circle in that component $D_{\rho,R}$ of $\{(z,F_{\rho}(z)):z \in U, |F_{\rho}(z)| < R\}$ which contains (0,0) = 0.

We claim that for $(z, F_{\rho}(z)) \in D_{\rho,R}$, we have

(5.2) $h_{\mathfrak{o}}(z) \leq \omega_{\mathfrak{o},\mathsf{R}}(\mathsf{F}_{\mathfrak{o}}(z)).$

To prove this, we consider

 $E_{o,R} = \{z \in U : |F_o(z)| < R\}.$

If $z \in \partial E_{\rho,R} \cap U$, we have $|F_{\rho}(z)| = R$ and

 $\omega_{\rho,\mathbf{R}}(\mathbf{F}_{\rho}(z)) = 1 \ge h_{\rho}(z).$

If $z \in \partial E_{o,R} \cap T$, we have $|F_o(z)| \leq R$ and

$$h_{\rho}(z) = 0 \leq \omega_{\rho,R}(F_{\rho}(z)).$$

Hence (5.2) follows from the maximum principle. Since we have $D_{o,R} \subset D_R$, we conclude that

$$h_{o}(0) \leq \omega_{o,R}(0) \leq \omega_{R}(0)$$
.

We have assumed that the complement of F(U) has positive capacity and thus F is in the Nevanlinna class (cf. R. Nevanlinna [16], p. 209). For almost all R, we have

$$(2\pi)^{-1}m\{e^{i\theta}:|\mathbf{F}(e^{i\theta})|>\mathbf{R}\} = \lim_{\rho\to 1_{-}}h_{\rho}(0) \leq \omega_{\mathbf{R}}(0).$$

Since we have (5.1), it is now clear that $F \in H^{p}(U)$ because

$$||\mathbf{F}||_p^p = \int_0^\infty (2\pi)^{-1} m\{e^{i\theta}: |\mathbf{F}(e^{i\theta})| > \mathbf{R}\} d\mathbf{R}^p \leq p \int_0^\infty \omega_{\mathbf{R}}(0) \mathbf{R}^{p-1} d\mathbf{R} < \infty.$$

This concludes the first part of the proof.

Conversely, let us assume that $F \in H^{p}(U)$ for some p > 0. We shall also assume that F is continuous on $U \cup T$. If this is not the case, we argue as in the first part of the proof. Let NF be the nontangential maximal function of F (let the opening angle of the associated Stolz domain be $2\pi/3$ (cf. Petersen [17], p. 8)). Let $H = H_R$ be the harmonic function on U which is 1 on $\{e^{i\theta}: NF(e^{i\theta}) \ge R\}$ and 0 on $\{e^{i\theta}: NF(e^{i\theta}) < R\}$. If $|F(z_0)| = R$, where $z_0 = r_0 e^{i\alpha} = (1-\delta)e^{i\alpha}$ with $\delta \in (0,1)$, we have

$$NF(e^{i\theta}) \ge R, |\theta - \alpha| < \delta,$$

and it follows that

$$H(z_0) \ge (2\pi)^{-1} \int_{|\varphi-\alpha| < \delta} (1 - r_0^2) (1 + r_0^2 - 2r_0 \cos(\varphi - \alpha))^{-1} d\varphi$$
$$\ge \pi^{-1} \int_0^{\delta} \delta(\delta^2 + t^2)^{-1} dt = 1/4.$$

Let $E_R = \{z \in U : |F(z)| < R\}$. We claim that

(5.3)
$$\omega_{\mathsf{R}}(z, F(z)) \leq 4H(z), \quad z \in \mathsf{E}_{\mathsf{R}}.$$

Again, we use the maximum principle. If $z \in \partial E_R \cap U$, we have |F(z)| = R and $4H(z) \ge 1$. Thus, (5.3) holds in this case. If $z \in \partial E_R \cap T$, we have either $NF(z) \ge R$ and H(z) = 1 or $|F(z)| \le NF(z) < R$ and consequently $\omega_R(z,F(z)) = 0 \le 4H(z)$. In both cases, (5.3) is true. In a standard way, we conclude that

$$\omega_{\mathbf{R}}(0) \leq (2/\pi) m\{e^{i\theta} : \mathrm{NF}(e^{i\theta}) \geq \mathbf{R}\},$$
$$\int_{0}^{\infty} \omega_{\mathbf{R}}(0) d\mathbf{R}^{p} \leq (2/\pi) \|\mathrm{NF}\|_{p}^{p} \leq \mathrm{Const.} \|\mathrm{F}\|_{\mathrm{H}^{p}}^{p}$$

In the last step, we used a result of Hardy and Littlewood (cf. Theorem IV.40, p. 186 in Tsuji [21]). This concludes the proof of Theorem 7.

6. Examples.

All examples F_{Φ} discussed below satisfy condition (1.1 *a*), while F_{Φ} may or may not be in $H^{1}(U)$. In case $F_{\Phi} \in H^{1}(U)$, these examples may be considered to yield variants of Zygmund's Theorem B, mentioned in the Introduction, by means of an obvious subordination argument.

A simple first example is $F_0(z) = 2z(1-z^2)^{-1}$ which maps U onto $\mathbb{C} \setminus \{w = iv : |v| \ge 1\}$. Consequently, (1.1 *a*) is true for F_0 . On the other hand, F_0 is not in $H^1(U)$.

We proceed to construct a class of univalent functions $F = F_{\Phi}$ which are such that F(U) avoids a neighborhood of the imaginary axis near infinity. The function F will be or will not be in $H^{1}(U)$ depending on the size of this neighborhood. Let

$$\mathbf{D} = \mathbf{D}(\Phi) = \{z = re^{i\theta} : ||\theta| - \pi/2| \leq \Phi(r), r \geq 2\},\$$

where the function Φ will be in one of the following two classes: We say that $\Phi: [2,\infty) \to [0,\pi/3]$ is in Q_1 if Φ is continuous, $\Phi(r) \to 0$, $r \to \infty$, and $\Phi(2) = 0$.

We say that $\Phi: [2,\infty) \to [0,\pi/3]$ is in Q_2 if $\Phi \in Q_1$ and Φ is differentiable with $\Phi' \in L^{\infty}$ and with $\int_2^{\infty} r \Phi'(r)^2 dr < \infty$.

Let $F = F_{\Phi}$ map U onto C\D in such a way that F(0) = 0. We also introduce $J = J(\Phi) = \int_{2}^{\infty} \Phi(r) dr/r$.

PROPOSITION. – If $\Phi \in Q_2$ and $J(\Phi)$ is finite, F will not be in $H^1(U)$. If $\Phi \in Q_1$ and $J(\Phi)$ is infinite, with

(6.1)
$$\int_{2}^{\infty} \left\{ \exp\left(-\frac{2}{\pi} \int_{2}^{R} \Phi(t) \frac{dt}{t}\right) \right\} \frac{dR}{R} < \infty ,$$

then $F \in H^1(U)$.

To prove the Proposition, we consider the harmonic measure, ω_R , of the outer circle in $F(U) \cap \{z: |z| < R\}$. From Haliste ([9], formulas (2.1) and (2.3)), we see that if $\Phi \in Q_1$ and R is large enough, we have

$$\begin{split} \omega_{\rm R}(0) &\leq (4/\pi) \exp\left(4\pi - \pi \int_{2}^{\rm R} (\pi - 2\Phi(t))^{-1} dt/t\right) \\ &\leq ({\rm C}_{0}/{\rm R}) \exp\left(-\frac{2}{\pi} \int_{2}^{\rm R} \Phi(t) dt/t\right), \quad {\rm C}_{0} = 8e^{4\pi}. \end{split}$$

Now, (6.1) implies $\int_{\sigma}^{\infty} \omega_{R}(0) dR < \infty$ and thus $F \in H^{1}(U)$, by Theorem 7.

From Theorem 2.1 in Haliste [9], we see that if $\Phi \in Q_2$ and R is large enough, we have

(6.2)
$$\omega_{\mathbf{R}}(0) \ge C_1 \exp\left(-\pi \int_2^{\mathbf{R}} (\pi - 2\Phi(t))^{-1} dt/t - \pi \int_2^{\mathbf{R}} \{t\Phi'(t)^2/(\pi - 2\Phi(t))\} dt/3\right),$$

where $C_1 = (1/9) \exp(-8\pi(1+4||\Phi'||_{\infty}^2/3))$.

It follows that if $\Phi \in Q_2$ and J is finite, we have

$$\omega_{\rm R}(0) \approx \exp\left(-\int_2^{\rm R} (1-2\Phi(t)/\pi)^{-1} dt/t\right) \approx \exp\left(-2J/\pi\right)/{\rm R}$$

Thus, we see that $\int_{2}^{\infty} \omega_{R}(0) dR = \infty$. Applying Theorem 7, we see that $F \notin H^{1}(U)$, and we have proved the first part of the Proposition.

Let us in particular take $\Phi(r) = (\log r)^{-a}$, when $r \ge 3$. The associated domain is essentially of the form

$$\{z = x + iy: |x| \leq |y| (\log |y|)^{-a}, |y| \geq 3\}.$$

When a > 1, the argument above applies and F_{Φ} is not in $H^{1}(U)$. On the other hand, (1.1 a) is clearly true.

When 0 < a < 1, (6.1) holds and consequently $F_{\Phi} \in H^{1}(U)$. It follows from Theorem 1 that Re $F_{\Phi} \in L \log L$.

If $\Phi(r) = C (\log r)^{-1}$, $r \ge 3$, we have $\omega_R(0) \approx R^{-1} (\log R)^{-2C/\pi}$ when R is large and it follows that

(6.3) $F_{\phi} \notin H^{1}(U), C \leq \pi/2, F_{\phi} \in H^{1}(U), C > \pi/2.$

This illustrates the second part of the Proposition. In particular, it follows from Theorem 1 that Re $F_{\phi} \in L \log L$ if $C > \pi/2$.

This last example is related to a problem considered by Burkholder (cf. [5], p. 115-116). Let $S_{\delta} = \{x+iy:x>1, |y| < \delta x \log x\}$ and let F_{δ} be a univalent analytic function mapping U onto S_{δ} . Burkholder uses his theorem on "generalized subordination" to prove that

(6.4)
$$F_{\delta} \in H^{1}(U), \ \delta < 2/\pi, \ F_{\delta} \notin H^{1}(U), \ \delta > 2/\pi.$$

Using our notation with $D(\Phi) \cap {\text{Re } z > 0} = S_{\delta}$, we have

$$\Phi(r) = (\delta \log r)^{-1} + O(\log \log r / (\log r)^2), r \to \infty,$$

and it follows from (6.3) that

 $F_{\delta} \in H^{1}(U), \ \delta < 2/\pi, \ F_{\delta} \notin H^{1}(U), \ \delta \ge 2/\pi.$

Thus, we obtain Burkholder's result (6.4), as well as the boundary case $\delta = 2/\pi$.

Remark. – Using estimates of harmonic measure in "strip domains", K. Haliste has in [10] given still another method to treat Burkholder's problem, including the boundary case.

The following observation is due to Haliste. Let $T_{\delta} = \{re^{i\theta}: r > 1, |\theta| > p^{-1} \arctan(\delta p \log r)\}$ and let G_{δ} be a univalent analytic function mapping U onto T. Then

 $G_{\delta} \in H^{p}(U), \quad \delta < 2/\pi, \quad G_{\delta} \notin H^{p}(U), \quad \delta \ge 2/\pi.$

This result also follows in a simple way from our Theorem 7.

Let us now return to the more general regions $D(\Phi)$ considered earlier. If a function F is such that

(6.5)
$$F(U) \subset C - D(\Phi)$$

with $J(\Phi)$ finite, then we cannot expect $F \in H^1(U)$. If however we require (6.5) to hold with $D(\Phi)$ replaced by a somewhat larger set, we can achieve $F \in H^1(U)$ and thus will be able to apply Theorem 1. Our last example is of this type.

Let Φ be in Q_1 with $J(\Phi)$ finite and let Ω be a collection of intervals contained in $(-\infty, -2] \cup [2, \infty)$ which is such that for all sufficiently large R and for a constant c > 1, we have

$$\int_{\Omega(\mathbf{R})} dt/t \ge (c\pi/2) \log \log \mathbf{R}, \qquad \int_{\Omega(-\mathbf{R})} dt/t \ge (c\pi/2) \log \log \mathbf{R}.$$

Here $\Omega(R) = \Omega \cap [2,R]$ and $\Omega(-R) = \Omega \cap [-R,-2]$.

Let $\Omega_0(R)$ be the one of the two sets $\Omega(R)$ and $\Omega(-R)$ which has the smallest logarithmic length. Let F map U univalently onto the infinite covering surface over $C \setminus (D(\Phi) \cup \Omega)$ in such a way that F(0) = 0. From standard estimates of harmonic measure (cf. Tsuji [21], p. 116), we see that

$$\omega_{\mathsf{R}}(0) \leq \omega_{\mathsf{R}}'(0) \leq \text{Const.} \exp\left(-\left(\int_{2}^{\mathsf{R}/2} + \int_{\Omega_{0}(\mathsf{R}/2)}\right)(1 - 2\Phi(t)/\pi)^{-1} dt/t\right).$$

Thus, for R large, we have

$$\omega_{\mathbf{R}}(0) \leq \text{Const. } \mathbf{R}^{-1} (\log \mathbf{R})^{-c},$$

and consequently $\int_{2}^{\infty} \omega_{R}(0) dR < \infty$. From Theorem 7, we see that $F \in H^{1}(U)$. Applying Theorem 1, we conclude that $Re F \in L \log L$.

Finally, we observe that the function $G(z) = iw/\log^2 (1+w)$ with w = (1+z)/(1-z) is in H¹(U), but Re G \notin L log L. Thus by Theorem 1 the integral (1.1 *a*) diverges, and in fact N(1,*iv*) > $(v \log^2 v)^{-1}$ is easy to see, for large v.

7. Extensions of Theorem 2.

Theorem 2 can be extended to meromorphic functions f, provided the subharmonic function Φ is not very large at infinity. We put $M(r,\Phi) = \sup_{\alpha} \Phi(re^{i\theta})$. We have

THEOREM 8. – Suppose f is meromorphic in $\{z:|z| < R\}$, where $0 < R \le \infty$, and that f does not have a pole at the origin. Let Φ be subharmonic in C, with $\Phi(f(0))$ finite, and suppose that for some $\tau \in (0,1)$

(7.1)
$$\Phi(w) \leq O(|w|^{r}), \quad w \to \infty.$$

Then, for each r such that f does not have a pole on the circle $\{z:|z|=r\}$, we have

(7.2)
$$\frac{1}{2\pi} \int_0^{2\pi} \Phi(f(re^{i\theta})) d\theta = \int_C (N(r,w) - N(r,\infty)) d\mu(w) + \Phi(f(0)),$$

where μ is the Riesz measure of Φ and 0 < r < R.

Here the main case of interest is that of Φ small at ∞ , in the sense that (7.1) holds for all $\tau > 0$; in this case (7.2) is finite for every r < R. (Compare the Φ in (2.7)-(2.9).)

Proof. – Our assumption (7.1) implies that the following representation for Φ holds on the entire plane (cf. Hayman and Kennedy [12], pp. 141, 146):

(7.3)
$$\Phi(z) = \int_{\{|w|<1\}} \log |z-w| \, d\mu(w) + \int_{\{|w|\ge1\}} \log |zw^{-1}-1| \, d\mu(w) + c,$$

where c is a real constant, and $\int_{|w| \ge 1} d\mu(w)/|w| < \infty$. Put $z = f(re^{i\theta})$ in (7.3) and integrate $d\theta$, as in the proof of Theorem 2, using Jensen's theorem on f - w or $w^{-1}f - 1$ according as |w| < 1 or $|w| \ge 1$. Using (7.3) again, with z = f(0), to evaluate c, we obtain (7.2).

We can also extend our results to functions mapping the polydisk or ball of C^n to C. Let U^n be the unit polydisk in C:

$$\mathbf{U}^{n} = \{z \in \mathbf{C}^{n} : |z_{j}| < 1, j = 1, \dots, n\}.$$

Uⁿ has distinguished boundary

$$T^n = \{z \in C^n : |z_1| = \cdots = |z_n| = 1\}.$$

For an *n*-tuple $\varphi = (\varphi_1, \ldots, \varphi_n)$, $\varphi_j \in [0, 2\pi]$, we define a function f_{φ} on the unit disc by

$$f_{\mathbf{\Phi}}(\zeta) = f(\zeta e^{i\varphi_1}, \ldots, \zeta e^{i\varphi_n}).$$

We define a counting function for $w \in \mathbf{C}$ by

$$N_f(r,w) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} N(r,w;f_{\varphi}) \, d\varphi_1 \, \ldots \, d\varphi_n.$$

Here $N(r,w;f_{\varphi})$ is the usual one-dimensional counting function for the function f_{φ} . Jensen's formula is ([18], p. 326):

$$N_f(r,w) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \log |f(re^{i\varphi_1},\ldots,re^{i\varphi_n}) - w| d\varphi_1 \ldots d\varphi_n$$
$$-\log |f(0) - w|.$$

Now consider the unit ball B^n in C^n :

$$\mathbf{B}^{n} = \{ z \in \mathbf{C}^{n} : \Sigma |z_{j}|^{2} < 1 \}.$$

The boundary of B^n is the unit sphere S^{2n-1} . For $z \in S^{2n-1}$ we define a function f_z on the unit disc by $f_z(\zeta) = f(\zeta z)$. For the ball, the counting function is

$$N_f(r,w) = \frac{1}{C_n} \int_{\mathbb{S}^{2n-1}} N(r,w;f_z) \, d\sigma(z) \, .$$

Here the volume element $d\sigma$ is Lebesgue measure on S^{2n-1} and C_n is the volume of S^{2n-1} , i.e. $C_n = \frac{2\pi^n}{(n-1)!}$.

In this setting, Jensen's formula is ([20], p. 404) :

$$N_f(r,w) = \frac{1}{C_n} \int_{S^{2n-1}} \log |f(rz) - w| \, d\sigma(z) - \log |f(0) - w| \, .$$

Using these versions of Jensen's formula as in the proof of Theorem 2, we get

THEOREM 9. – Suppose Φ is subharmonic in the complex plane with Riesz measure μ . If f is holomorphic in the unit polydisk Uⁿ, then

$$\left(\frac{1}{2\pi}\right)^n\int_{\mathbb{T}^n}\Phi(f(re^{i\varphi_1},\ldots,re^{i\varphi_n})\,d\varphi_1\,\ldots\,d\varphi_n=\int_{\mathbb{C}}\mathcal{N}(r,w)\,d\mu(w)+\Phi(f(0))\,.$$

If f is holomorphic in the unit ball B^n , then

$$\frac{1}{C_n}\int_{S^{2n-1}}\Phi(f(rz))\,d\sigma(z)=\int_{C}N(r,w)\,d\mu(w)\,+\,\Phi(f(0))\,.$$

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