ANNALES DE L'INSTITUT FOURIER

M. S. NARASIMHAN

# The type and the Green's kernel of an open Riemann surface

*Annales de l'institut Fourier*, tome 10 (1960), p. 285-296
<http://www.numdam.org/item?id=AIF_1960__10__285_0>

Ann. Inst. Fourier, Grenoble
**10** (1960), 285-296.

# THE TYPE AND THE GREEN'S KERNEL OF AN OPEN RIEMANN SURFACE

by **M. S. NARASIMHAN**

Tata Institute of Fundamental Research, Bombay
et Centre National de la Recherche Scientifique, Paris.

## 1. — Introduction.

We give in this paper a new approach to the determination of the type and the construction of the Green's function of an open Riemann surface.

We first define an open Riemann surface to be of hyperbolic type if the completion of the pre-Hilbert space of $C^\infty$ functions with compact supports endowed with the Dirichlet scalar product is a space of currents. In this case we construct in a natural way an operator $\mathcal{G}$ and call the kernel in the sense of Schwartz of $\mathcal{G}$ the Green's kernel of the open Riemann surface. We then show that an open Riemann surface is of hyperbolic type if and only if it possesses the Green's function in the classical sense and that the Green's kernel is identical, upto a scalar factor, with the Green's function in the classical sense.

The invariance of the type of an open Riemann surface under quasi-conformal maps is derived as an immediate consequence of the definition of type.

## 2. — Some spaces of currents.

Let $\Omega$ be an open Riemann surface, that is, a non-compact connected complex analytic manifold of complex dimension one. We denote by $\overset{p}{\mathfrak{D}}(\Omega)$, $p = 0, 1, 2$, the space of $C^\infty$ forms of degree $p$ endowed with the topology of Schwartz [10, 11]. Let $\overset{p}{\mathfrak{D}}'(\Omega)$ denote the space of currents of degree $p$ endowed with the strong topology [10, 11]. Let further $\overset{p}{\mathcal{E}}(\Omega)$ denote the space of $C^\infty$ forms of degree $p$ and $\overset{p}{\mathcal{E}}'(\Omega)$ the space of currents of degree $p$ with compact supports, each endowed with the usual topology.

The operator $*$ is defined intrinsically on 1-forms in $\Omega$ [8]. The operator $*$ is defined on the currents of degree one by the formula :

$$\langle * T, \varphi \rangle = \langle T, - * \varphi \rangle, \qquad T \in \overset{1}{\mathfrak{D}}', \qquad \varphi \in \overset{1}{\mathfrak{D}},$$

$\langle , \rangle$ denoting the scalar product between $\overset{1}{\mathfrak{D}}'$ and $\overset{1}{\mathfrak{D}}$.

We denote by $L^2(\Omega)$ the Hilbert space of measurable square integrable 1-forms with the scalar product

$$(\omega_1, \omega_2) = \int_\Omega \omega_1 \wedge * \overline{\omega}_2, \qquad \omega_1, \omega_2 \in L^2.$$

Let further $BL(\Omega)$ be the pre-Hilbert space of currents T of degree 0 for which $dT \in L^2$ endowed with the scalar product

$$(T_1, T_2)_1 = (dT_1, dT_2), \qquad T_1, T_2 \in BL.$$

If $BL^\cdot(\Omega)$ denotes the quotient space of BL by the subspace of constants, $BL^\cdot$ is a *Hilbert space* with the induced scalar product [2, p. 308].

## 3. — The Laplacian $\tilde{\Delta}$.

On a Riemann surface the Laplacian is not defined intrinsically as an operator carrying functions into functions. However we can define an operator analogous to the Laplacian carrying functions into 2 forms.

We define an operator

$$\tilde{\Delta} : \overset{0}{\mathcal{D}}' \to \overset{2}{\mathcal{D}}'$$

by the formula

$$\tilde{\Delta} = d * d$$

where $d$ denotes the exterior derivation. $\tilde{\Delta}$ is an elliptic operator of type $(V_1, V_2)$ where $V_1$ denotes the trivial line bundle with $\mathbf{C}$ as the fibre and $V_2$ denotes the line bundle of 2-covectors [6].

We have the following elementary formulae:

1) $\langle T, \tilde{\Delta}\varphi \rangle = \langle \tilde{\Delta} T, \varphi \rangle$ for $T \in \overset{0}{\mathcal{D}}', \ \varphi \in \overset{0}{\mathcal{D}}$.

2) For $T \in \overset{0}{\mathcal{D}}', \ \varphi \in \overset{0}{\mathcal{D}}$ define:

$$(dT, d\varphi) = \langle dT, * d\bar{\varphi} \rangle.$$

We then have

$$(dT, d\varphi) = \langle -\tilde{\Delta} T, \bar{\varphi} \rangle.$$

## 4. — The type and the Green's kernel.

Let $\mathcal{H}_0(\Omega)$ denote the vector space $\overset{0}{\mathcal{D}}$ endowed with the Dirichlet scalar product

$$(\varphi, \psi)_1 = \int_\Omega d\varphi \wedge * d\bar{\psi}, \qquad \varphi, \psi \in \overset{0}{\mathcal{D}}.$$

Since $\Omega$ is non-compact and connected $\mathcal{H}_0(\Omega)$ is a separated pre-Hilbert space. Let $\mathcal{H}(\Omega)$ be the completion of $\mathcal{H}_0(\Omega)$.

DÉFINITION 1. — *An open Riemann surface $\Omega$ is said to be of* hyperbolic *type if the inclusion map*

$$i : \mathcal{H}_0(\Omega) \to \overset{0}{\mathcal{D}}'(\Omega)$$

*is continuous. Otherwise, it is said to be of* parabolic *type.*

Thus we define an open Riemann surface to be of hyperbolic type if the following condition is satisfiedd: if $\{\varphi_n\}$ is a sequence of $C^\infty$ functions with compact supports whose Dirichlet integrals tend to zero then the sequence $\{\varphi_n\}$ tends to zero in the sense of currents.

Let $\Omega$ be an open Riemann surface of hyperbolic type. Let

$$i' : \mathcal{H}(\Omega) \to \overset{0}{\mathfrak{D}}'(\Omega)$$

denote the canonical extension of the continuous inclusion

$$i : \mathcal{H}_0(\Omega) \to \overset{0}{\mathfrak{D}}'(\Omega).$$

The map $i'$ is an injection. In fact, if $T \in \mathcal{H}$ we have for every $\varphi \in \overset{0}{\mathfrak{D}}$

$$(T, \varphi)_{\mathcal{H}} = \langle -\tilde{\Delta} i'T, \bar{\varphi} \rangle.$$

Since $\overset{0}{\mathfrak{D}}$ is dense in $\mathcal{H}$ it follows from the above equality that $i'T = 0$ if and only if $T = 0$.

We identify $\mathcal{H}$ with a subspace of $\overset{0}{\mathfrak{D}}'$ by means of the injection $i'$. The completion of $\mathcal{H}_0$ is thus a space of currents. It is easily seen that $\mathcal{H}$ is contained in BL.

Since $\overset{0}{\mathfrak{D}}$ is dense in $\mathcal{H}$ the dual $\mathcal{H}'(\Omega)$ of $\mathcal{H}(\Omega)$ is canonically identified with a subspace of $\overset{2}{\mathfrak{D}}'(\Omega)$. We assert that $\tilde{\Delta}$ defines an isomorphism of $\mathcal{H}$ onto $\mathcal{H}'$. In fact let $\Lambda$ be the canonical isomorphism of $\mathcal{H}$ on the conjugate of its dual. Then for $T \in \mathcal{H}$, $\varphi \in \overset{0}{\mathfrak{D}}$ we have

$$\langle \Lambda T, \bar{\varphi} \rangle = (dT, d\varphi) = \langle -\tilde{\Delta} T, \bar{\varphi} \rangle$$

so that $\Lambda = -\tilde{\Delta}$. Hence $\tilde{\Delta} : \mathcal{H} \to \mathcal{H}'$ is an isomorphism. Let $\mathcal{G} : \mathcal{H}' \to \mathcal{H}$ be the inverse isomorphism.

Consider the spaces $\mathcal{H} \cap \overset{0}{\mathcal{E}}$ and $\mathcal{H}' \cap \overset{2}{\mathcal{E}}$; $\mathcal{H} \cap \overset{0}{\mathcal{E}}$ will be endowed with the topology upper-bound of those of $\mathcal{H}$ and $\overset{0}{\mathcal{E}}$; the same for $\mathcal{H}' \cap \overset{2}{\mathcal{E}}$. Let $\mathcal{H}' + \overset{2}{\mathcal{E}}'$ (resp. $\mathcal{H} + \overset{0}{\mathcal{E}}'$) be the strong dual of $\mathcal{H} \cap \overset{0}{\mathcal{E}}$ (resp. $\mathcal{H}' \cap \overset{2}{\mathcal{E}}$). [An element of the dual of $\mathcal{H} \cap \overset{0}{\mathcal{E}}$ can be written in the form $f + T$, $f \in \mathcal{H}'$, $T \in \overset{2}{\mathcal{E}}'$. Hence the above notation. A similar remark applies to $\mathcal{H} + \overset{0}{\mathcal{E}}'$]. Since $\tilde{\Delta}$ is an elliptic operator we see exactly as in Lions [5, p. 36] that the operator $\mathcal{G}$ can be extended to an isomorphism, still denoted by $\mathcal{G}$, of $\mathcal{H}' + \overset{2}{\mathcal{E}}'$ onto $\mathcal{H} + \overset{0}{\mathcal{E}}'$ and $\tilde{\Delta}$ is its inverse.

$\mathcal{G} : \mathcal{H}' + \overset{2}{\mathcal{E}}' \to \mathcal{H} + \overset{0}{\mathcal{E}}'$ is called the Green's operator.

DÉFINITION 2. — *Let $\Omega$ be an open Riemann surface of hyperbolic type. The kernel in the sense of Schwartz of the operator $\mathcal{G}$ is called the Green's kernel of $\Omega$.*

The Green's kernel is a bilateral elementary kernel for $\tilde{\Delta}$. The green's kernel is very regular in the sense of Schwartz [11; 4, § 12; 5].

REMARK 1. — An open Riemann surface is of hyperbolic type if and only if the inclusion map $\mathcal{H}_0 \to L^2_{loc}$ is continuous, where $L^2_{loc}$ denotes the space of locally square summable functions endowed with the topology of convergence in $L^2$ on each compact set. [See 2, p. 321, Prop. 4. 1].

REMARK 2. — Let $\mathcal{H}_1$ denote the pre-Hilbert space of $C^1$ functions with compact supports endowed with the Dirichlet scalar product. We see by regularization and Remark 1. that $\Omega$ is hyperbolic if and only if the inclusion map $\mathcal{H}_1 \to L^2_{loc}$ is continuous.

## 5. — Some properties of the Green's operator.

In this section we prove some propositions concerning the Green's operator.

PROPOSITION 1. — *$\Omega$ is of hyperbolic type if and only if $\overset{2}{\mathfrak{D}} \subset \tilde{\Delta}(\mathrm{BL})$.*

*Proof.* If $\Omega$ is of hyperbolic type and $\psi \in \overset{2}{\mathfrak{D}}$, then $\mathcal{G}\psi \in \mathrm{BL}$ and $\tilde{\Delta}\mathcal{G}\psi = \psi$.

Suppose conversely that $\overset{2}{\mathfrak{D}} \subset \tilde{\Delta}(\mathrm{BL})$. Let $\{\varphi_n\}$, $\varphi_n \in \overset{0}{\mathfrak{D}}$ be a sequence converging to zero in $\mathcal{H}_0$. We shall show that $\langle \psi, \varphi_n \rangle \to 0$ for every $\psi \in \overset{2}{\mathfrak{D}}$. In fact let $T \in \mathrm{BL}$ be such that $\tilde{\Delta}\mathrm{T} = \bar{\psi}$. Then

$$\begin{aligned} \langle \psi, \varphi_n \rangle &= \langle \tilde{\Delta}\bar{\mathrm{T}}, \bar{\varphi}_n \rangle, \\ &= -(d\bar{\mathrm{T}}, d\bar{\varphi}_n). \end{aligned}$$

Since $d\bar{\mathrm{T}} \in L^2$ and $d\bar{\varphi}_n \to 0$ in $L^2$, we see that $\langle \psi, \varphi_n \rangle \to 0$.

PROPOSITION 2. — *Suppose that $\Omega$ is of hyperbolic type. Let $\Omega'$ be a subdomain of $\Omega$. Then $\Omega'$ is hyperbolic and there* *exists a continuous linear map* $u \to u^{\sim}$ *of* $\mathcal{H}(\Omega')$ *into* $\mathcal{H}(\Omega)$ *such that* $u^{\sim} = u$ *in* $\Omega'$ *and* $u^{\sim} = 0$ *in* $\complement\,\Omega'$. *One has*

$$(du,\ du)_{L^2(\Omega')} = (du^{\sim},\ du^{\sim})_{L^2(\Omega)}.$$

*Proof.* For $\varphi \in \overset{0}{\mathcal{D}}(\Omega')$ let $\varphi^{\sim} \in \overset{0}{\mathcal{D}}(\Omega)$ be the function obtained by extending $\varphi$ by zero outside $\Omega'$. The map $j : \varphi \to \varphi^{\sim}$ is an isometry of $\mathcal{H}_0(\Omega')$ into $\mathcal{H}_0(\Omega)$. The inclusion map $\mathcal{H}_0(\Omega') \to \overset{0}{\mathcal{D}}{}'(\Omega')$ is the composition of the map $j$, the continuous inclusion $\mathcal{H}_0(\Omega) \to \overset{0}{\mathcal{D}}{}'(\Omega)$ and the restriction map $r$: $\overset{0}{\mathcal{D}}{}'(\Omega) \to \overset{0}{\mathcal{D}}{}'(\Omega')$ and is hence continuous. Since the map $j$ can be extended into an isometry (still denoted by $j$) the second part of the proposition follows.

We identify $\mathcal{H}(\Omega')$ with a subspace of $\mathcal{H}(\Omega)$ by means of the isometry $j$.

PROPOSITION 3. — *Let* $\Omega$ *be of hyperbolic type. Let* $\{\Omega_k\}$, $k = 1, 2, \ldots$ *be an increasing sequence of sub-domains of* $\Omega$ *such that* $\bigcup_k \Omega_k = \Omega$. *Let* $\mathcal{G}_k$ *(resp.* $\mathcal{G}$*) be the Green's operator of* $\Omega_k$ *(resp.* $\Omega$*). Let* $T \in \mathcal{H}'(\Omega)$ *and let* $T_k$ *be the restriction of* $T$ *to* $\Omega_k$. *Then* $\mathcal{G}_k T_k^{\sim} \to \mathcal{G}T$ *in* $\mathcal{H}(\Omega)$.

*Proof.* $\mathcal{H}(\Omega_k)$ is a closed subspace of $\mathcal{H}(\Omega)$ and $\mathcal{H}(\Omega)$ is the closure of $\bigcup_k \mathcal{H}(\Omega_k)$. If we verify that $\mathcal{G}_k T_k^{\sim}$ is the orthogonal projection of $\mathcal{G}T$ into the closed subspace $\mathcal{H}(\Omega_k)$ it would follow, from a known theroem on projections in a Hilbert space, that $\mathcal{G}_k T_k^{\sim} \to \mathcal{G}T$ in $\mathcal{H}(\Omega)$. Now for every $\varphi \in \overset{0}{\mathcal{D}}(\Omega_k)$ we have

$$\begin{aligned}(\mathcal{G}_k T_k,\ \varphi)_{\mathcal{H}(\Omega_k)} &= \langle -\tilde{\Delta}\mathcal{G}_k T_k,\ \bar{\varphi}\rangle \\ &= -\langle T_k,\ \bar{\varphi}\rangle \\ &= -\langle T,\ \bar{\varphi}^{\sim}\rangle.\end{aligned}$$

On the other hand if $P_k$ denotes the projection operator on $\mathcal{H}(\Omega_k)$ we have, for $\varphi \in \overset{0}{\mathcal{D}}(\Omega_k)$,

$$\begin{aligned}(P_k\mathcal{G}T,\ \varphi)_{\mathcal{H}(\Omega)_k} &= (d\mathcal{G}T,\ d\bar{\varphi}^{\sim})_{L^2(\Omega)} \\ &= \langle -\tilde{\Delta}\mathcal{G}T,\ \bar{\varphi}^{\sim}\rangle \\ &= -\langle T,\ \bar{\varphi}^{\sim}\rangle.\end{aligned}$$

Hence $\mathcal{G}_k T_k^{\sim} = P_k\mathcal{G}T$.

PROPOSITION 4. — *Assume that* $\Omega$ *is of hyperbolic type. With the same notations as in Proposition* 3, *let* T *be an element of* $\overset{2}{\mathcal{E}}'(\Omega)$ *such that the support of* T *is contained in* $\Omega_1$. *Then* $\mathcal{G}_k T^{\sim} \to \mathcal{G}T$ *in* $\overset{0}{\mathcal{D}}'(\Omega)$ *and the convergence is uniform on every compact set contained in the complement of the support of* T ($\mathcal{G}_k T^{\sim}$ *denotes the extension of* $\mathcal{G}_k T$ *to* $\Omega$ *by zero outside* $\Omega_k$; $\mathcal{G}_k T^{\sim}$ *and* $\mathcal{G}T$ *are functions in the complement of the support of* T).

*Proof.* Let $S_k = \mathcal{G}_k T^{\sim}$. To prove that $S_k \to \mathcal{G}T$ in $\overset{0}{\mathcal{D}}'(\Omega)$ it is sufficient to prove that, for every $\psi \in \overset{2}{\mathcal{D}}(\Omega)$, $\langle S_k, \psi\rangle$ tends to $\langle \mathcal{G}T, \psi\rangle$. Now $T \in \overset{2}{\mathcal{E}}'(\Omega_k)$ and $\psi \in \overset{2}{\mathcal{D}}(\Omega_k)$ for all sufficiently large $k$, say for $k \geqslant k_0$. In $\Omega_{k_0}$, we have

$$\tilde{\Delta}(\mathcal{G}\psi - \mathcal{G}_k\psi) = \psi - \psi = 0.$$

Since $\tilde{\Delta}$ is an elliptic operator, $\overset{0}{\mathcal{D}}'$ and $\overset{0}{\mathcal{E}}$ induce the same topology on the space of solutions of the equations $\tilde{\Delta}f = 0$ [6, p. 331; 11, ch. v, Th. XII]. By Proposition 3, $\mathcal{G}_k\psi \to \mathcal{G}\psi$ in $\overset{0}{\mathcal{D}}'(\Omega_{k_0})$. Hence $\overset{0}{\mathcal{G}_k}\psi \to \mathcal{G}\psi$ in $\mathcal{E}(\Omega_{k_0})$. Now

$$\langle S_k, \psi\rangle = \langle \mathcal{G}_k T, \psi\rangle = \langle T, \mathcal{G}_k\psi\rangle.$$

Since $T \in \overset{2}{\mathcal{E}}'(\Omega_{k_0})$ and $\mathcal{G}_k\psi \to \mathcal{G}\psi$ in $\overset{0}{\mathcal{E}}(\Omega_{k_0})$, we see that $\langle T, \mathcal{G}_k\psi\rangle \to \langle T, \mathcal{G}\psi\rangle$. Hence $\langle S_k, \psi\rangle$ tends to $\langle T, \mathcal{G}\psi\rangle = \langle \mathcal{G}T, \psi\rangle$.

The second part follows from the property of elliptic equations used above.

PROPOSITION 5. — *Assume that* $\Omega$ *is hyperbolic.* Let $\Omega_0$ *be a relatively compact sub-domain of* $\Omega$ *bounded by a finite number of disjoint analytic Jordan curves. Let* $\mathcal{G}_0$ *be the Greens' operator of* $\Omega_0$. *Let* $p \in \Omega_0$ *and let* $g_p$ *be the Green's function in the classical sense of* $\Omega_0$ *with « pole »* p [7, 8]. *Then we have*

$$\mathcal{G}_0\delta_p = -\frac{1}{2\pi}g_p$$

*where* $\delta_p$ *is the Dirac measure at* $p$.

*Proof.* — We first remark that $\mathcal{G}_0\delta_p$ is the only element T of $\mathcal{H}(\Omega_0) + \overset{0}{\mathcal{E}}'(\Omega_0)$ which satisfies the equation $\tilde{\Delta}T = \delta_p$.

One knows that $\tilde{\Delta}\left(-\frac{1}{2\pi} g_p\right) = \delta_p$. The lemma will be proved if we show that $g_p \in \mathcal{H}(\Omega_0) + \overset{0}{\mathcal{E}}'(\Omega_0)$.

$g_p$ is a $C^\infty$ function in $\Omega_0$ except at $p$. Since $g_p$ attains the boundary value zero on the boundary [8, § 28. 3] the reflection principle shows that $g_p$ is $C^\infty$ in $\overline{\Omega}_0$ except at $p$. Let $\varphi \in \overset{0}{\mathcal{D}}(\Omega_0)$ equal to 1 in a neighbourhood of $p$. $\varphi g_p$ is a current of degree 0, with compact support. $(1 - \varphi)\, g_p$ is $C^\infty$ in $\overline{\Omega}_0$ and hence has a finite Dirichlet integral; moreover $(1 - \varphi)\, g_p$ vanishes on the boundary. It is known that such a function belongs to $\mathcal{H}_0(\Omega_0)$ [4, § 2.4; 8, § 32. 1]. Since $g_p = \varphi g_p + (1 - \varphi)\, g_p$ we have $g_p \in \mathcal{H}(\Omega_0) + \overset{0}{\mathcal{E}}'(\Omega_0)$.

Remark 3. — Another method to prove Proposition 5 is to show directly, without using $g_p$, that $\mathcal{G}_0\delta_p$ attains the boundary value zero (« Regularity at the boundary »). This may be shown as in [1, ch. VII, § 4] or [4, § 12.3].

## 6. — The potential with respect to the Green's function.

The proposition proved in this section is more or less classical.

Proposition 6. — *Let $\Omega$ be an open Riemann surface which has the Green's function $g(p, q)$ in the classical sense* [7, ch. VI, § 2]. *Then for $\psi \in \overset{2}{\mathcal{D}}(\Omega)$ the function*

$$h(p) = \int_\Omega g(p, q) \wedge \psi$$

*belongs to* BL.

Before proving the proposition we prove the following

Lemma. — *Let $\Omega_0$ be a relatively compact sub-domain of $\Omega$ bounded by a finite number of disjoint analytic Jordan curves. Let $g_0(p, q)$ be the Green's function of $\Omega_0$ and $\psi \in \overset{2}{\mathcal{D}}(\Omega_0)$. Then*

$$h_0(p) = \int_{\Omega_0} g_0(p, q) \wedge \psi$$

*is $C^\infty$ in $\overline{\Omega}_0$. $h_0(p)$ is zero on the boundary and one has*

$$(dh_0,\, dh_0)_{L^2(\Omega_0)} = \langle -\tilde{\Delta} h_0,\, \overline{h_0}\rangle.$$

*Proof.* — Let K be the support of $\psi$ and $\Omega'$ a relatively compact sub-domain of $\Omega_0$ containing K. By Harnack's principle there exists, for $q_0 \in$ K, a constant $k$ such *that* $g(p, q) \leqslant kg(p, q_0)$ for each $q \in$ K and $p \in \complement \Omega'$. Hence

$$|h_0(p)| \leqslant k \left(\int_{\Omega_0} |\psi|\right) g(p, q_0) \qquad \text{for} \qquad p \in \complement \Omega'.$$

Using the symmetry of the Green's function we see that

$$|h_0(p)| \leqslant k' g(q_0, p) \qquad \text{for} \qquad p \in \complement \Omega',$$

where $k'$ is a positive constant. Since $g(q_0, p)$ attains the boundary value zero we see that $h_0(p)$ is continuous in $\overline{\Omega}_0$ and is zero on the boundary. By the reflection principle $h_0$ is $C^\infty$ in $\overline{\Omega}_0$ and an application of the Green's formula yields the equality $(dh_0, dh_0)_{L^2(\Omega_0)} = \langle -\tilde{\Delta} h_0, \overline{h}_0 \rangle$.

*Proof of Proposition* 6. — Let $\{\Omega_k\}$, $k = 1, 2, \ldots$ be an exhaustion of $\Omega$ by relatively compact sub-domains $\Omega_k$ bounded by a finite number of analytic Jordan curves [8, p. 25]. We may assume that the support of $\psi$ is contained in $\Omega_1$. Let $g_k(p, q)$ be the Green's function of $\Omega_k$. Let

$$\begin{aligned} h_k(p) &= \int g_k(p, q) \wedge \psi(q), \\ h(p) &= \int g(p, q) \wedge \psi(q). \end{aligned}$$

By the lemma,

$$\begin{aligned} (dh_k, dh_k)_{L^2(\Omega_k)} &= \langle -\tilde{\Delta} h_k, \overline{h}_k \rangle, \\ &= 2\pi \langle \psi, \overline{h}_k \rangle, \\ &= 2\pi \iint_{\Omega_k \times \Omega_k} g_k(p, q) \psi \otimes \overline{\psi}. \end{aligned}$$

Since $g_k(p, q)$ tends increasingly to $g(p, q)$ we have, for $\psi$, $\psi' \in \overset{2}{\mathcal{D}}(\Omega)$,

$$\iint g_k \psi \otimes \psi' \to \iint g \psi \otimes \psi'.$$

Hence, if $h_k^{\sim}$ denotes the extension of $h_k$ to $\Omega$ by zero outside $\Omega_k$, $h_k^{\sim} \to h$ in $\overset{0}{\mathcal{D}}{}'(\Omega)$ and $\|dh_k^{\sim}\|_{L^2(\Omega)} \leqslant C$, C being a constant independant of $k$. Since $dh_k^{\sim}$ is bounded in $L^2(\Omega)$, there exists a weakly convergent subsequence $\{dh_{k_n}\}$ converging say to

$T \in L^2(\Omega)$. Since $dh_{k_n} \to T$ weakly in $L^2$, $dh_{k_n} \to T$ in $\overset{1}{\mathfrak{D}}'(\Omega)$. Since $h_k^{\sim} \to h$ in $\overset{0}{\mathfrak{D}}'(\Omega)$, $dh_k^{\sim} \to dh$ in $\overset{1}{\mathfrak{D}}'(\Omega)$. Hence $dh_{k_n} \to dh$ in $\overset{1}{\mathfrak{D}}'(\Omega)$. Consequently $T = dh$. Since $T \in L^2(\Omega)$, $dh \in L^2(\Omega)$, that is $h \in \mathrm{BL}$.

Remark 4. — $h \in \mathcal{H}(\Omega)$.

### 7. — Green's kernel and the Green's function. Type

Theorem. — *An open Riemann surface $\Omega$ is of hyperbolic type (in the sense of Definition 1) if and only if $\Omega$ possesses the Green's function in the classical sense and in this case the Green's kernel is equal to the Green's function in the classical sense multiplied by $-1/2\pi$.*

*Proof.* — Suppose that $\Omega$ is hyperbolic. Let $\{\Omega_k\}$ be an exhaustion of $\Omega$ by relatively compact subdomains $\Omega_k$ bounded by a finite number of disjoint analytic Jordan curves. Let $p \in \Omega$. We may suppose that $p \in \Omega_1$. Let $g_{k,p}$ be the Green's function of $\Omega_k$ with pole at $p$. By Proposition 5, $\mathcal{G}_k \delta_p = -\frac{1}{2\pi} g_{k,p}$ where $\mathcal{G}_k$ denotes the Green's operator of $\Omega_k$. By Proposition 4, $\mathcal{G}_k \delta_p^{\sim} \to \mathcal{G}\delta_p$ in $\overset{0}{\mathfrak{D}}'(\Omega)$, the convergence being uniform on compact sets not containing $p$. Hence $-\frac{1}{2\pi} g_{k,p}^{\sim} \to \mathcal{G}\delta_p$ uniformly on compact sets not containing $p$. Hence $\Omega$ possesses a Green's function $g_p$ with pole at $p$ in the classical sense, and one has $\mathcal{G}\delta_p = -\frac{1}{2\pi} g_p$, $\mathcal{G}$ denoting the Green's operator of $\Omega$. It follows that the Green's kernel is equal to the Green's function multiplied by $-1/2\pi$.

Suppose conversely that $\Omega$ has the Green's function $g(p, q)$ in the classical sense. Let $\psi \in \overset{2}{\mathfrak{D}}(\Omega)$. By Proposition 6

$$h(p) = -\frac{1}{2\pi} \int_{\Omega} g(p, q) \wedge \psi(q)$$

belongs to BL and one has $\tilde{\Delta} h = \psi$. By Proposition 1, $\Omega$ is of hyperbolic type.

REMARK 5. — The first part of the theorem has been proved for plane domains by Deny-Lions [2, ch. II, Th. 2.1, p. 350]. We may also refer to Weyl [12, § 7, § 8].

REMARK 6. — Another proof of the theorem may be given using the notion of the harmonic measure of the ideal boundary and Remark 4.

## 8. — The Invariance of type under quasi-conformal maps.

We shall show that the type of a Riemann surface is invariant under quasi-conformal maps. This result has been proved by Pfluger [9].

Let $\Omega_1$ and $\Omega_2$ be two open Riemann surfaces. Let $\Phi$ : $\Omega_1 \to \Omega_2$ be a $(C^\infty)$ diffeomorphism which is quasi-conformal [8, § 43.4]. Let $\varphi \in \overset{0}{\mathfrak{D}}(\Omega_2)$ and write $\varphi' = \varphi \circ \Phi$. It is easily proved [3, p. 5] that there exists a constant $k > 0$ independent of $\varphi$ such that

$$\frac{1}{k}(d\varphi,\ d\varphi)_{L^2(\Omega_2)} \leqslant (d\varphi',\ d\varphi')_{L^2(\Omega_1)} \leqslant k(d\varphi,\ d\varphi)_{L^2(\Omega_2)}.$$

That is, $\Phi$ induces an isomorphism of $\mathcal{H}_0(\Omega_2)$ onto $\mathcal{H}_0(\Omega_1)$. On the other hand $\Phi$, being a diffeomorphism, induces an isomorphism of $\overset{0}{\mathfrak{D}}{}'(\Omega_2)$ onto $\overset{0}{\mathfrak{D}}{}'(\Omega_1)$. Hence $\mathcal{H}_0(\Omega_1) \to \overset{0}{\mathfrak{D}}{}'(\Omega_1)$ is continuous if and only if $\mathcal{H}_0(\Omega_2) \to \overset{0}{\mathfrak{D}}{}'(\Omega_2)$ is continuous. Hence the type is invariant under $\Phi$.

REMARK 7. — In the above proof we assumed $\Phi$ to be $C^\infty$. If we use Remark 2, it is sufficient to assume $\Phi$ to be $C^1$.

## BIBLIOGRAPHY


[1] R. COURANT and D. HILBERT, *Methoden der Mathematischen Physik II*, Berlin, 1937.

[2] J. DENY and J. L. LIONS, Les espaces de type de Beppo-Levi, *Annales de l'Institut Fourier*, 5, (1953-1954), pp. 305-370.

[3] Mme J. LELONG-FERRAND, *Représentation conforme et transformations à intégrale de Dirichlet bornée*, Paris, 1955.

[4] J. L. LIONS, Lectures on elliptic partial differential equations, *Tata Institute of Fundamental Research*, Bombay, 1957.

[5] J. L. LIONS, Problèmes aux limites en théorie des distributions, *Acta. Math.* 94, (1955), pp. 13-153.

[6] B. MALGRANGE, Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution, *Annales de l'Institut Fourier*, VI (1955-1956), pp. 221-354.

[7] R. NEVANLINNA, *Uniformisierung*, Berlin, 1953.

[8] A. PFLUGER, *Theorie der Riemannschen Flächen*, Berlin, 1957.

[9] A. PFLUGER, Sur une propriété de l'application quasi-conforme d'une surface de Riemann ouverte, *C.R. Acad. Sci* (Paris), 227 (1948), 25-26.

[10] G. de RHAM, *Variétés différentiables*, Paris, 1955.

[11] L. SCHWARTZ, *Théorie des distributions I*, Paris, 1957.

[12] H. WEYL, The method of orthogonal projection in potential theory, *Duke Math. J.*, 7 (1940), 411-444.