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RESTRICTIONS OF FOURIER TRANSFORMS TO CURVES(*)

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Introduction.

Given a smooth curve in $\mathbf{R}^n$ and a smooth measure σ on the curve one may ask for which a and b does the restriction estimate $\left\{\int |\hat{f}(x)|^b d\sigma(x)\right\}^{1/b} \leqslant C\|f\|_a$ $(f \in \mathcal{S})$ hold. Such an estimate implies that for f in $\widehat{L^a(\mathbf{R}^n)}$ the restriction of $\hat{f}$ to the curve "makes sense". We refer the reader to [1] and [2] for general information about restriction theorems. The object of this article is to extend the restriction theorem of Prestini [3] to the full range of exponents.

Since $\mathcal{F}L^a$ is an affinely invariant space (that is invariant under the group of affine motions) we will consider only affine invariants of the curve. For a discussion of these invariants the reader may consult Guggenheimer [4] pp. 170-173. For the sake of simplicity in laying out the basic idea of this paper we will restrict attention to the special case of the non-compact curve $x(t) = \left(t, \frac{1}{2}t^2, \frac{1}{6}t^3\right)$ in $\mathbf{R}^3$. This is essentially the unique curve for which the first and second affine curvatures vanish and the affine arc length measure is just dt.

THEOREM 1. – *Let* $1 \leqslant a < \frac{7}{6}$, *let* $a' = 6b$ $\left(\textit{so that } \frac{7}{6} < b \leqslant \infty\right)$ *and let* $x(t) = \left(t, \frac{1}{2}t^2, \frac{1}{6}t^3\right)$. Then

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$$\left\{ \int |\hat{f}(x(t))|^b \, dt \right\}^{1/b} \leqslant C_a \|f\|_a \quad \textit{for all } f \in \mathcal{S}(\mathbf{R}^3).$$

The same techniques also yield the corresponding result in higher dimensions.

THEOREM 1′. – *Let* $n \geqslant 2$, $1 \leqslant a < (n^2+n+2)/(n^2+n)$, $a' = \frac{1}{2} n(n+1) b$ *and let* $x(t) = \left(t, \frac{1}{2} t^2, \ldots, \frac{1}{n!} t^n\right)$. *Then*

$$\left\{ \int |\hat{f}(x(t))|^b \, dt \right\}^{1/b} \leqslant C_a \|f\|_a$$

for all $f \in \mathcal{S}(\mathbf{R}^n)$.

It is well known that the ranges of a and b in the above theorems are optimal at least for $n = 2$ and 3. The theorem is well known in case $n = 2$ (Zygmund [5]).

Our methods can also be used to establish a local result. For this we demand that the curve possess an affine arc length parametrization, that is a parametrization $x(t)$ such that

$$\det(x^{(1)}(t), x^{(2)}(t), \ldots, x^{(n)}(t)) = 1$$

for all t. Here $x^{(k)}$ denotes the *kth* derivative of x viewed as a column vector.

THEOREM 2. – *Let* $n \geqslant 2$ *and let* $x(t)$ *be a* $C^{(n)}$ *curve in* $\mathbf{R}^n$ *defined for* $\alpha < t < \beta$ *and such that* t *is the affine arc length. Then for* $1 \leqslant a < (n^2+n+2)/(n^2+n)$ *and* $a' \leqslant \frac{1}{2} n(n+1) b$ *we have* $\left\{ \int_{\alpha'}^{\beta'} |\hat{f}(x(t))|^b \, dt \right\}^{1/b} \leqslant C_{\alpha', \beta', a} \|f\|_a$ $(f \in \mathcal{S})$ *for every compact subinterval* $[\alpha', \beta']$ *of* (α, β).

Proofs of the theorems. – We now seek to prove Theorem 1. We will adopt the dual formulation of the problem. Thus we will prove that

$$\| (\varphi \cdot \sigma)^\wedge \|_q \leqslant C \|\varphi\|_p \tag{1}$$

for $1 \leqslant p < 7$ and $p^{-1} + 6q^{-1} = 1$. Here σ denotes the affine arc length measure on the curve and φ is a function in $L^p(\sigma)$. We will prove this result by induction on the exponent p. Therefore we shall *assume* that equation (1) holds in the range $1 \leqslant p \leqslant p_0$ for some fixed $p_0 < 7$.

Because of the special geometry of the situation there is a 1-parameter group of affine motions of $\mathbf{R}^3$ given by

$$\alpha_s(x,y,z) = \left(x+s, y+sx+\frac{1}{2}s^2, z+sy+\frac{1}{2}s^2x+\frac{1}{6}s^3\right)$$

which fix our curve and act on it by translation of the parameter t. The orbits of this action are curves affinely equivalent to the initial one. In fact let us parametrize the curves by y and z (taking $x = 0$) so that the corresponding curve is

$$t \longrightarrow \left(t, y+\frac{1}{2}t^2, z+ty+\frac{1}{6}t^3\right).$$

By affine equivalence our induction hypothesis applies equally well to each of the orbits. For a function $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^3)$ we introduce the auxiliary function F by defining

$$\mathrm{F}(y,z;t) = f\left(t, y+\frac{1}{2}t^2, z+ty+\frac{1}{6}t^3\right). \tag{2}$$

By disintegrating the function f on the family of orbits and applying the induction hypothesis on each orbit we have

LEMMA 1. – *Let* $1 \leqslant p < p_0$ *and* $p^{-1} + 6q^{-1} = 1$. *Then*

$$\|\hat{f}\|_q \leqslant \mathrm{C}_p \|\mathrm{F}\|_{\mathrm{L}^1(\mathrm{L}^p)}.$$

Here the mixed norm space is $\mathrm{L}^1(\mathbf{R}^2_{y,z}, \mathrm{L}^p(\mathbf{R}_t))$.

A simple change of variable and an application of the Plancherel Theorem also yield $\|\hat{f}\|_2 = \|f\|_2 = \|\mathrm{F}\|_{\mathrm{L}^2(\mathrm{L}^2)}$. Thus by a routine interpolation argument (Benedeck and Panzone [6]) we have

LEMMA 2. – *For* (a^{-1}, b^{-1}) *in the triangle with vertices* $(1,1)$, $(1, p_0^{-1})$, $\left(\frac{1}{2}, \frac{1}{2}\right)$ *and* c *defined by* $5a^{-1} + b^{-1} + 6c^{-1} = 6$ *we have* $\|\hat{f}\|_c \leqslant \mathrm{C}_{a,b} \|\mathrm{F}\|_{\mathrm{L}^a(\mathrm{L}^b)}$.

This lemma may be viewed as a substitute for the Hausdorff-Young Theorem.

We now follow the method of Prestini. Let φ be a function on $\mathbf{R}$ satisfying $|\varphi| \leqslant \mathrm{I}_{\mathrm{E}}$ and $\mathrm{meas}(\mathrm{E}) = m$. We consider

$(\varphi \cdot \sigma) * (\varphi \cdot \sigma) * (\varphi \cdot \sigma)$ and scale the resulting measure by a factor of $\frac{1}{3}$ so as to adapt it to the original curve. The scaled measure is given by a locally integrable density f such that

$$((\varphi \cdot \sigma)^\wedge (3u))^3 = \hat{f}(u) \tag{3}$$

and

$$f\left(\frac{1}{3}(x(t_1) + x(t_2) + x(t_3))\right) = cv^{-1} \varphi(t_1)\varphi(t_2)\varphi(t_3)$$

where v stands for the Vandermonde $|(t_2 - t_3)(t_3 - t_1)(t_1 - t_2)|$ and c is an absolute constant. A calculation now leads to

$$\| F \|_{L^a(L^b)} = c \left\{ \int v^{-(a-1)} \left\{ \Phi(h_1, h_2) \right\}^{ab^{-1}} dh_1 dh_2 \right\}^{a^{-1}} \tag{4}$$

with v the Vandermonde $|h_1 h_2 (h_1 - h_2)|$ and

$$\Phi(h_1, h_2) = \int |\varphi(t)\varphi(t + h_1)\varphi(t + h_2)|^b \, dt.$$

Clearly $|\Phi(h_1, h_2)| \leqslant m$ and $\int |\Phi(h_1, h_2)| \, dh_1 dh_2 \leqslant m^3$. Combining these estimates gives

$$\| \Phi^{ab^{-1}} \|_{L_{s,1}} \leqslant C_{s,a,b} \, m^{ab^{-1} + 2s^{-1}} \tag{5}$$

where $L_{s,1}$ denotes the Lorentz space $L_{s,1}(dh_1 dh_2)$ (see Stein and Weiss [7] or Hunt [8]) and where $1 \leqslant ba^{-1} < s < \infty$. On the other hand routine calculations show that $v^{-(a-1)}$ lies in the dual Lorentz space $L_{s',\infty}(dh_1 dh_2)$ for $2 = 3(a - 1)s'$ and $1 < a < \frac{5}{3}$. Thus we obtain from (4) and (5) that for

$$1 < a < \frac{5}{3}, \quad 5a^{-1} - 2b^{-1} < 3, \quad a \leqslant b,$$

we have,

$$\| F \|_{L^a(L^b)} \leqslant C_{a,b} \, m^{5a^{-1} + b^{-1} - 3}.$$

We are now in a position to apply lemma 2 for (a^{-1}, b^{-1}) in the quadrilateral defined by $a^{-1} \geqslant b^{-1}, a^{-1} > \frac{3}{5}$, $5a^{-1} - 2b^{-1} < 3$ and $(p_0 - 2)a^{-1} + p_0 b^{-1} \geqslant p_0 - 1$. Thus there exists a number a_0 depending only on p_0 with $a_0 < \frac{5}{3}$ so that lemma 2 can be applied in case $a_0 < a < \frac{5}{3}$ and b is given by

$$(p_0 - 2) a^{-1} + p_0 b^{-1} = p_0 - 1 .$$

We conclude that for suitable c_0 we have $\|\hat{f}\|_c \leqslant C_c m^{3-6c^{-1}}$ for all c in the range $30p_0(13p_0 - 1)^{-1} < c < c_0$. Thus by (3)

$$\| (\varphi \cdot \sigma)^\wedge \|_q \leqslant C_q m^{1-6q^{-1}}$$

for all q in the range $90p_0(13p_0 - 1)^{-1} < q < 3c_0$. Routine interpolation arguments now yield $\| \varphi \cdot \sigma^\wedge \|_q \leqslant C_p \| \varphi \|_p$ for $p^{-1} + 6q^{-1} = 1$ and $1 \leqslant p < 15p_0(2p_0 + 1)^{-1}$. This completes the induction step.

The induction starts trivially with $p_0 = 1$. One step of the induction yields the result for $1 \leqslant p < 5$ — that is the result of Prestini and with the same proof. With two steps we have the result for $1 \leqslant p < 75/11$ and it is clear that for any p with $1 \leqslant p < 7$ the result for that p will follow after only finitely many steps.

The proof of theorem $1'$ is entirely analogous.

We will leave the detailed proof of theorem 2 to the reader. Some comments however are in order. First of all in general there is no group action preserving the initial curve. Thus a typical curve of our family will be defined by

$$t \longrightarrow n^{-1} \sum_{k=1}^{n} x(t + h_k)$$

the family of curves being indexed by the $(n-1)$-dimensional manifold of $(h_1, \ldots, h_n)$ satisfying $\sum_{k=1}^{n} h_k = 0$. The inductive nature of the proof then leads in general to further curves of the form

$$y(t) = \sum_{k=1}^{K} \alpha_k x(t + \ell_k) \tag{6}$$

where $\alpha_k > 0$, $\sum_{k=1}^{K} \alpha_k = 1$ and the ℓ's are sums of the h's.

Let t_0 be a fixed point $\alpha < t_0 < \beta$. It will be necessary to establish uniform estimates for the curve (6) on an interval

$t_0 - \epsilon < t < t_0 + \epsilon$ and for $|\ell_k| < \epsilon$. Towards this we select convex neighbourhoods V_k of $x^{(k)}(t_0)$ such that

$$2 \geqslant \det(v_1, \ldots, v_n) \geqslant \frac{1}{2}$$

for $v_k \in V_k\,(1 \leqslant k \leqslant n)$. It now follows from the fact that the initial curve is $C^{(n)}$ that there exists a number $\epsilon > 0$ such that

$$2 \geqslant \det(y^{(1)}(\tau_1), \ldots, y^{(n)}(\tau_n)) \geqslant \frac{1}{2}$$

for $|\tau_k - t_0| < \epsilon$ and $|\ell_k| < \epsilon$. (In particular it follows that the measure dt is uniformly equivalent to the affine arc length measure of (6) for $|t - t_0| < \epsilon$, $|\ell_k| < \epsilon$). The vital estimate is a lower bound on the absolute value of the Jacobian J of the barycentre map $(t_1, \ldots, t_n) \longrightarrow \frac{1}{n} \sum_{k=1}^{n} y(t_k)$. Up to a constant factor this is $|\det(y^{(1)}(t_1), \ldots, y^{(1)}(t_n))|$ and by a generalization of the mean-value theorem (Polya, Szegö [9]. Vol. II, part V, Chap. 1, No. 95) this is equivalent to

$$\left(\prod_{1 \leqslant i < j \leqslant n} |t_i - t_j|\right) |\det(y^{(1)}(\tau_1), \ldots, y^{(n)}(\tau_n)|$$

for suitable $\tau_1, \ldots, \tau_n$. This now yields the uniform estimate

$$|J| \geqslant c_n \prod_{1 \leqslant i < j \leqslant n} |t_i - t_j| \quad (c_n > 0)$$

for $|t_0 - t_k| < \epsilon(1 \leqslant k \leqslant n)$, $|\ell_k| < \epsilon$. This completes our comments on Theorem 2.

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