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Harmonic measures on negatively curved manifolds

CENTRE
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HARMONIC MEASURES ON NEGATIVELY CURVED MANIFOLDS

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En l'honneur de Marcel Berger qui nous a initiés aux variétés pincées.

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ABSTRACT. — We prove that the harmonic measures on the spheres of a pinched Hadamard manifold admit uniform upper and lower bounds.

RÉSUMÉ. — Nous prouvons que les mesures harmoniques sur les sphères des variétés Hadamard pincées admettent des bornes supérieures et infériueures uniformes.

1. Introduction

Let X be a Hadamard manifold. This means that X is a complete simply connected Riemannian manifold of dimension $k \geqslant 2$ with non positive sectional curvature $K_X \leqslant 0$.

For a point x in X and a radius $R > 0$, we let $\sigma_{x,R}$ be the harmonic measure on the sphere $S(x,R)$. We refer to Section 3.1 for a precise definition of $\sigma_{x,R}$. The aim of these notes is to provide, under a pinching assumption, uniform upper and lower bounds for the harmonic measures $\sigma_{x,R}$.

THEOREM 1.1. — *Let $0 < a < b$ and $k \geqslant 2$. There exist positive constants M, N depending solely on a, b, k such that for any k-dimensional Hadamard manifold X with pinched curvature $-b^2 \leqslant K_X \leqslant -a^2$, any point x in X, any radius $R > 0$ and any angle $\theta \in [0, \pi/2]$, one has*

$$(1.1) \qquad \frac{1}{M}\,\theta^N \leqslant \sigma_{x,R}(C_x^\theta) \leqslant M\,\theta^{\frac{1}{N}}$$

where C_x^θ stands for any cone with vertex x and angle θ.

These inequalities (1.1) play a crucial role in the extension of the main result of [4] from rank one symmetric spaces to Hadamard manifolds.

Indeed, using (1.1), we prove in [3] that *any quasi-isometric map between pinched Hadamard manifolds is within a bounded distance from a unique harmonic map.* The key point in (1.1) for this application is the fact that the constants M and N do not depend on x nor R.

The proof of Theorem 1.1 relies on various technical tools of potential theory on pinched Hadamard manifolds: the Harnack inequality, the barrier functions constructed by Anderson and Schoen in [2], and upper and lower bounds for Green functions due to Ancona in [1]. Related estimates are available, like the one by Kifer–Ledrappier in [6, Theorem 3.1 and 4.1] where (1.1) is proven for the sphere at infinity, or by Ledrappier–Lim in [7, Proposition 3.9] where the Hölder regularity of the Martin kernel is proven. Our approach also gives a non probabilistic proof of the Kifer–Ledrappier estimates.

Here is the organization of this paper. In Section 2, we collect basic facts on Hadamard manifolds and their harmonic functions. In Section 3, we prove uniform estimates for the normal derivatives of Green functions. In Section 4 and 5 we successively prove the upper bound and the lower bound in (1.1). We postpone until Section 6 the proof of a few purely geometric estimates on Hadamard manifolds that are needed in the argument.

2.   Pinched Hadamard manifolds

In this chapter, we collect preliminary results on Hadamard manifolds and their harmonic functions.

Let X be a Hadamard manifold with dimension k. For instance, the Euclidean space $\mathbb{R}^k$ is a Hadamard manifold with zero curvature $K_X = 0$, and the real hyperbolic space $\mathbb{H}^k$ is a Hadamard manifold with constant curvature $K_X = -1$. We will say that X is pinched if there exist constants $a, b > 0$ such that

$$-b^2 \leqslant K_X \leqslant -a^2 < 0.$$

For instance, any non-compact rank one symmetric space is a pinched Hadamard manifold.

2.1. Laplacian and subharmonic functions

We introduce a few subharmonic functions on X, or on open subsets of X, which will play the role of *barriers* in the following chapters.

When o is a point in X, we denote by ρ_o the distance function defined by $\rho_o(x) = d(o, x)$ for x in X. When $F : X \to \mathbb{R}$ is a continuous function, we denote by ΔF its Laplacian seen as a distribution. In local coordinates $(x_1, \ldots, x_k)$ of X, we denote the coefficients of the metric tensor by g_{ij}, the volume density reads as $v := \sqrt{\det(g_{ij})}$, and, when F is a smooth function, its Laplacian is

$$\Delta F = v^{-1} \partial_{x_i} (v g^{ij} \partial_{x_j} F).$$

A real valued function F on X is *harmonic* if $\Delta F = 0$, *subharmonic* if $\Delta F \geqslant 0$ and *superharmonic* if $\Delta F \leqslant 0$.

LEMMA 2.1. — *Let X be a Hadamard manifold with dimension k, and $o \in X$. The Laplacian of the distance function ρ_o satisfies*

$$\Delta \rho_o \geqslant (k-1) \rho_o^{-1}.$$

If $K_X \leqslant -a^2 < 0$, one has

$$(2.1) \qquad \Delta \rho_o \geqslant (k-1)\, a\, \coth(a\, \rho_o).$$

If $-b^2 \leqslant K_X \leqslant 0$, one has

$$(2.2) \qquad \Delta \rho_o \leqslant (k-1)\, b\, \coth(b\, \rho_o).$$

These classical inequalities mean that the difference is a positive measure. See [2, Section 2] and [4, Lemma 3.2].

The following corollary provides useful barriers on X.

COROLLARY 2.2. — *Let X be a Hadamard manifold and $o \in X$.*
  (1) *If $K_X \leqslant -a^2 < 0$ and $0 < m_0 \leqslant (k-1)\, a$, the function $e^{-m_0 \rho_o}$ is superharmonic on X.*
  (2) *If $-b^2 \leqslant K_X \leqslant 0$ and $M_0 \geqslant (k-1)\, b \coth(b/4)$, the function $e^{-M_0 \rho_o}$ is subharmonic on $X \smallsetminus B(o, \frac{1}{4})$.*

Proof. — For any smooth function $f : [0, \infty[\to \mathbb{R}$, one has

$$\Delta(f \circ \rho_o) = f'' \circ \rho_o + (f' \circ \rho_o)\Delta\rho_o.$$

Therefore, one has for any $\tau > 0$:

$$\Delta(e^{-\tau \rho_o}) = \tau\, e^{-\tau \rho_o}\, (\tau - \Delta\rho_o).$$

(1). — Using (2.1), one gets:

$$\Delta(e^{-m_0 \rho_o}) \leqslant (m_0 - (k-1)\, a\, \coth(a\, \rho_o))\, m_0\, e^{-m_0 \rho_o} \leqslant 0.$$

(2). — Using (2.2), one gets outside the ball $B(o, \frac{1}{4})$:

$$\Delta(e^{-M_0 \rho_o}) \geqslant (M_0 - (k-1)\, b\, \coth(b\, \rho_o))\, M_0\, e^{-M_0 \rho_o} \geqslant 0. \qquad \square$$

2.2. Anderson–Schoen barrier

Another very useful barrier is the following function u introduced by Anderson and Schoen in [2].

We denote by ∂X the visual boundary of the Hadamard manifold X. For each point w in X, this boundary is naturally identified with the set of geodesic rays $w\xi$ starting at w. For $0 < \theta \leqslant \frac{\pi}{2}$ we denote by $C_{w\xi}^{\theta}$ the closed cone with axis $w\xi$ and angle θ: it is the union of all the geodesic rays $w\eta$ with vertex w and whose angle with the ray $w\xi$ is at most θ. Two geodesic rays with vertex w are said to be *opposite* if their union is a geodesic i.e. if their angle is equal to π.

LEMMA 2.3. — *Let X be a Hadamard manifold with $-b^2 \leqslant K_X \leqslant -a^2 < 0$. There exist constants $0 < \varepsilon_0 \leqslant 1$ and $C_0 > 1$ such that for every two opposite geodesic rays $w\xi^+$ and $w\xi^-$ with the same vertex $w \in X$, there exists a positive superharmonic function u on X such that:*

$$(2.3) \qquad u(x) \geqslant 1 \qquad\qquad \text{for all } x \text{ in the cone } C_{w\xi^+}^{\pi/2}$$

$$(2.4) \qquad u(x) \leqslant C_0\, e^{-\varepsilon_0 d(w,x)} \qquad \text{for all } x \text{ on the ray } w\xi^-.$$

This function $u = u_{w\xi^+}$ will be called the *Anderson–Schoen barrier* for the ray $w\xi^+$. The constants C_0 and ε_0 depend only on a, b and on the dimension of X.

Proof. — See [2, Proof of Theorem 3.1]. We briefly sketch the construction of the function u. As shown in Figure 2.1, let o be the point at distance 1 from w on the geodesic ray $w\xi^-$.
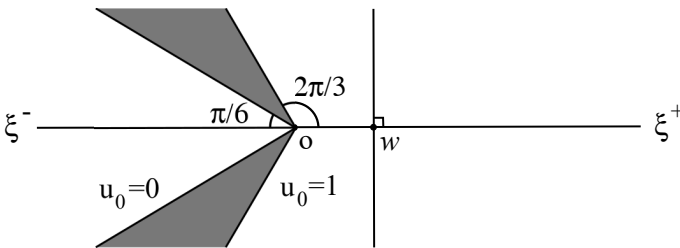

Figure 2.1. Construction of the Anderson–Schoen barrier.

Choose a non negative continous function u_0 on $X \smallsetminus o$ which is constant on each ray with vertex o, and which is equal to 1 on the cone $C_{o\xi^+}^{2\pi/3}$ and is equal to 0 on the cone $C_{o\xi^-}^{\pi/6}$ as in Figure 2.1. Then, consider a function

u_1 obtained by smoothly averaging u_0 on balls of radius 1. This function u_1 is defined as

$$u_1(x) = \frac{\int_X \chi(d(x,y)) u_0(y) \, \mathrm{d}y}{\int_X \chi(d(x,y)) \, \mathrm{d}y},$$

where $\mathrm{d}y$ is the Riemannian measure on X and $\chi \in \mathcal{C}^\infty(\mathbb{R})$ is a positive even function whose support is $[-1,1]$. Since χ is even this function u_1 is of class $\mathcal{C}^2$. Moreover, this function u_1 has the expected behavior (2.3) and (2.4) and its second covariant derivative decays exponentially at infinity. Therefore, using the same computation as in Corollary 2.2, one can find explicit constants $\varepsilon_0 > 0$ and $C_0' > 0$ depending only on a, b and $\dim X$ such that the function $u := u_1 + C_0' e^{-\varepsilon_0 \rho_w}$ is superharmonic. This is the required function u. $\qquad\square$

2.3. Harnack-Yau inequality

We state without proof a version of Harnack inequality due to Yau in [9].

LEMMA 2.4. — *Let X be a Hadamard manifold with $-b^2 \leqslant K_X \leqslant 0$. Then, there exists a constant $C_1 = C_1(k, b)$ such that for every open set $\Omega \subset X$ and every positive harmonic function $u : \Omega \to]0, \infty[$, one has*

$$(2.5) \qquad \|D_x \log u\| \leqslant C_1 \quad \text{for all } x \text{ in } X \text{ with } d(x, \partial\Omega) \geqslant 1.$$

This lemma is true for any complete Riemannian manifold whose Ricci curvature is bounded below. A short proof has been written by Peter Li and Jiaping Wang in [8, Lemma 2.1].

3. Green functions

In this chapter, we collect various estimates for the Green functions on Hadamard manifolds. We also explain why these Green functions are useful to estimate harmonic measures.

3.1. Harmonic measures

We first recall the definition of harmonic measures.

Since X is a Hadamard manifold, each exponential map $\exp_x : T_x X \to X$ is a $\mathcal{C}^\infty$-diffeomorphism ($x \in X$). In particular, any sphere $S(x, R)$ with

$R > 0$ is a $\mathcal{C}^\infty$-submanifold of X. Solving the Dirichlet problem on the ball $B(x, R)$ gives rise to a family of Borel probability measures $\sigma^y_{x,R}$ on $S(x, R)$ indexed by $y \in B(x, R)$. These measures are called *harmonic measures*. Indeed, for every continuous function f on the sphere $S(x, R)$, there exists a unique continuous function h_f on the closed ball $B(x, R)$ such that

$$(3.1) \qquad \Delta h_f = 0 \quad \text{in} \quad \mathring{B}(x, R) \qquad \text{and} \qquad h_f = f \quad \text{on} \quad S(x, R).$$

The map $f \mapsto h_f(y)$ is then a probability measure $\sigma^y_{x,R}$ on $S(x, R)$. This probability measure is defined by the equality

$$(3.2) \qquad h_f(y) = \int_{S(x,R)} f(\eta) \, d\sigma^y_{x,R}(\eta),$$

valid for any $f \in C^0(S(x, R))$.

The harmonic measures that occur in Theorem 1.1 correspond to $y = x$, namely one has $\sigma_{x,R} := \sigma^x_{x,R}$. Our aim is to prove the bound (1.1) for these measures $\sigma_{x,R}$.

Remark 3.1. — When X is the hyperbolic space $\mathbb{H}^k$, or more generally when X is a rank one symmetric space, the harmonic measure $\sigma_{x,R}$ is a multiple of the Riemannian measure $A_{x,R}$ on the sphere $S(x, R)$. But, for a general Hadamard manifold X, these two measures need not be proportional.

Remark 3.2. — When solving the Dirichlet problem on the visual compactification $\overline{X} = X \cup \partial X$, one gets a family of probability measures $(\sigma^y_\infty)_{y \in X}$ on ∂X which are also called harmonic measures. See [2, Theorem 3.1].

3.2. Estimating the Green functions

Before begining the proof of Theorem 1.1, we recall the definition and a few estimates for the Green functions.

For any closed ball $B(x, R)$ in X and any point y in the interior $\mathring{B}(x, R)$, one denotes by $G^y_{x,R}$ the corresponding Green function. It is the unique function on the ball $B(x, R)$ which is continuous outside y and such that

$$(3.3) \quad \Delta G^y_{x,R} = -\delta_y \quad \text{in} \quad \mathring{B}(x, R) \qquad \text{and} \qquad G^y_{x,R} = 0 \quad \text{on} \quad S(x, R).$$

When X is a pinched Hadamard manifold, one denotes by G^y_∞ the Green function on X corresponding to the point $y \in X$. It is the unique function on X which is continuous outside y and such that

$$\Delta G^y_\infty = -\delta_y \quad \text{and} \quad \lim_{z \to \infty} G^y_\infty(z) = 0.$$

These Green functions $G^y_{x,R}$ and G^y_∞ are non negative.

We now state various classical estimates for Green functions on Hadamard manifolds.

The first lemma provides a uniform estimate for a fixed radius R_0.

LEMMA 3.3. — Let X be a Hadamard manifold with $-b^2 \leqslant K_X \leqslant 0$. For each $R_0 \geqslant 1$, there exist constants $C_2 > c_2 > 0$ such that, for any x in X:

(3.4) $$c_2 \leqslant G^x_{x,R_0}(z) \leqslant C_2 \qquad \text{for all} \quad z \in S(x, \tfrac{1}{2}).$$

Proof. — This is a special case of [5, Theorem 11.4]. □

The second lemma, due to Ancona in [1], provides estimates for the Green functions which are uniform in the radius R under pinching conditions.

LEMMA 3.4. — Let X be a Hadamard manifold with $-b^2 \leqslant K_X \leqslant -a^2 < 0$.

(a) There exist constants $C'_2 > c'_2 > 0$ such that for any $R > 1$, x in X and y in $\mathring{B}(x, R)$ with $d(x, y) \leqslant R - 1$, one has:

(3.5) $$c'_2 \leqslant G^y_{x,R}(z) \leqslant C'_2 \qquad \text{for all} \quad z \in S(y, \tfrac{1}{2}).$$

Similarly, one has for any $y \in X$:

$$c'_2 \leqslant G^y_\infty(z) \leqslant C'_2 \qquad \text{for all} \quad z \in S(y, \tfrac{1}{2}).$$

(b) One can also choose constants c''_2, C''_2 such that, for any y in X:

$$c''_2 \, e^{-M_0 d(y,z)} \leqslant G^y_\infty(z) \leqslant C''_2 \, e^{-m_0 d(y,z)} \qquad \text{for all} \quad z \in X \smallsetminus B(y, \tfrac{1}{2}),$$

where $m_0 := (k - 1)a$ and $M_0 := (k - 1)\, b \coth(b/4)$.

Here and in the sequel of this chapter, the constants c_i and C_i only depend on b and $k = \dim X$, while c'_i and C'_i only depend on a, b and k ($i = 2, 3, 4$).

Proof of Lemma 3.4.

(a). — For the lower bound: by the maximum principle, one has $G^y_{y,1} \leqslant G^y_{x,R}$; then, use (3.4). For the upper bound: the maximum principle yields $G^y_{x,R} \leqslant G^y_\infty$ and the bounds for G^y_∞ are in [1, Proposition 7].

(b). — According to point $a)$ and Corollary 2.2, both functions

$$z \mapsto G^y_\infty(z) - c'_2 e^{-M_0 d(y,z)} \quad \text{and} \quad z \mapsto C'_2 e^{m_0} e^{-m_0 d(y,z)} - G^y_\infty(z)$$

are positive on the sphere $S(y, \tfrac{1}{2})$, go to zero at infinity, and are superharmonic on $X \smallsetminus B(y, \tfrac{1}{2})$. Therefore, by the maximum principle, they are positive on $X \smallsetminus B(y, \tfrac{1}{2})$. See [1, Remark 2.1, p. 505] for more details. □

3.3. Bounding above the gradient of the Green functions

We explain why we are interested in bounding the gradient of the Green functions, and prove such an upper bound.

Combining Equalities (3.1) and (3.3) with the Green formula, one gets the equality

$$(3.6) \qquad h_f(y) = \int_{S(x,R)} f(\eta) \, \frac{\partial G^y_{x,R}}{\partial n}(\eta) \, \mathrm{d}A_{x,R}(\eta),$$

where $\frac{\partial G}{\partial n} := \mathrm{grad}\, G.\vec{n}$ denotes the derivative of G in the direction of the inward normal vector $\vec{n}$ to the sphere $S(x,R)$, and where $A_{x,R}$ denotes the Riemannian measure on this sphere. Comparing with Formula (3.2), we get the following expression for the harmonic measure:

$$(3.7) \qquad \sigma^y_{x,R} = \frac{\partial G^y_{x,R}}{\partial n} \, A_{x,R} \, .$$

The following two lemmas will provide a uniform upper bound for this normal derivative when y and η are not too close.

The first lemma gives uniform estimates for a fixed radius R_0.

LEMMA 3.5. — *Let X be a Hadamard manifold with $-b^2 \leqslant K_X \leqslant 0$. For each $R_0 > 0$, there exists $C_3 > 0$ such that, for any $x \in X$ and $\eta \in S(x, R_0)$:*

$$(3.8) \qquad \frac{\partial G^x_{x,R_0}}{\partial n}(\eta) \leqslant C_3 \, .$$

The second lemma gives estimates which are uniform in the radius R under a pinching condition.

LEMMA 3.6. — *Let X be a Hadamard manifold with $-b^2 \leqslant K_X \leqslant -a^2 < 0$. There exists $C'_3 > 0$ such that for $R \geqslant 1$, $x \in X$, $y \in \mathring{B}(x,R)$, $\eta \in S(x,R)$:*

$$(3.9) \qquad \frac{\partial G^y_{x,R}}{\partial n}(\eta) \leqslant C'_3 \qquad \text{as soon as} \quad d(y,\eta) \geqslant 1.$$

Proof of Lemmas 3.5 and 3.6. — The proofs of these two lemmas are the same, except that they rely either on Lemma 3.3 or on Lemma 3.4. We will only prove Lemma 3.6.

The strategy is to construct an explicit superharmonic function F on the ball $B(x,R)$ such that $F(\eta) = 0$, such that $F \geqslant G^y_{x,R}$ in a neighborhood of η, and whose normal derivative at η is uniformly bounded.

As shown in Figure 3.1.A, we introduce the point y_0 on the ray $x\eta$ such that $d(x, y_0) = R + \frac{1}{3}$. By construction one has $d(\eta, y_0) = \frac{1}{3}$.
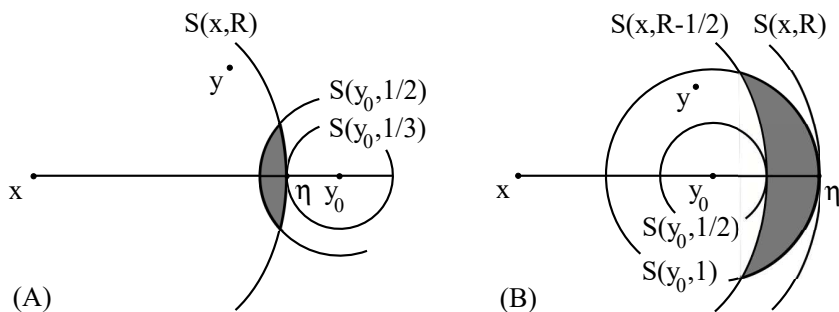
Figure 3.1. Proofs of uniform majoration and minoration for $\frac{\partial G^y_{x,R}}{\partial n}(\eta)$.

We let $M_0 = (k-1)\, b \coth(b/4)$, and we define

$$F(z) = C_4(e^{-M_0/3} - e^{-M_0\, d(y_0,z)})$$

for some constant C_4 that we will soon determine. We first notice that, according to Corollary 2.2,

(3.10) F is a positive superharmonic function on the ball $B(x, R)$.

Moreover, since the point $y \in B(x, R)$ satisfies $d(y, \eta) \geqslant 1$, since the angle between the geodesic segments $[\eta y]$ and $[\eta y_0]$ is obtuse, and since $K_X \leqslant 0$, one must have $d(y, y_0) \geqslant 1$. We now give a uniform bound for the Green function $G^y_{x,R}$ on the sphere $S(y_0, \frac{1}{2})$. By the maximum principle, one has

(3.11) $G^y_{x,R}(z) \leqslant G^y_{x,R+1}(z)$ for all z in $B(x, R)$.

Moreover, using the bound (3.5) in Lemma 3.4, one gets

(3.12) $G^y_{x,R+1}(z) \leqslant C'_2$ for all z in $S(y, \frac{1}{2})$.

Combining (3.11) and (3.12) with the maximum principle, one infers that

$$G^y_{x,R}(z) \leqslant C'_2 \qquad \text{for all } z \text{ in } B(x, R) \smallsetminus B(y, \tfrac{1}{2}).$$

In particular, one has

(3.13) $G^y_{x,R} \leqslant F$ on $S(y_0, \tfrac{1}{2}) \cap B(x, R)$

for the choice of the constant

$$C_4 := \frac{C'_2}{e^{-M_0/3} - e^{-M_0/2}}.$$

Combining (3.10), (3.13) and the maximum principle it follows that, on the grey zone of Figure 3.1.A:

$$G^y_{x,R} \leqslant F \quad \text{on} \quad B(y_0, \tfrac{1}{2}) \cap B(x, R).$$

Therefore, one has the following inequality between the normal derivatives of these functions at the point η:

$$\frac{\partial G^y_{x,R}}{\partial n}(\eta) \leqslant \frac{\partial F}{\partial n}(\eta) = \frac{C'_2 M_0}{1 - e^{-M_0/6}}.$$

This proves the bound (3.9).   □

3.4. Bounding below the gradient of the Green functions

We will also need a lower bound for the gradient of the Green functions.

The following two lemmas provide a uniform lower bound for the normal derivative when y is not too far from η and not too close to the sphere.

The first lemma gives uniform estimates for a fixed radius R_0.

LEMMA 3.7. — *Let X be a Hadamard manifold with $-b^2 \leqslant K_X \leqslant 0$. For each $R_0 \geqslant 1$, there exists $c_3 > 0$ such that for $x \in X$, $\eta \in S(x, R_0)$, one has*

$$(3.14) \qquad\qquad \frac{\partial G^x_{x,R_0}}{\partial n}(\eta) \geqslant c_3 \,.$$

The second lemma gives estimates which are uniform in the radius R under a pinching condition.

LEMMA 3.8. — *Let X be a Hadamard manifold with $-b^2 \leqslant K_X \leqslant -a^2 < 0$. There exists $c'_3 > 0$ such that for $R \geqslant 1$, $x \in X$, $y \in B(x, R-1)$, $\eta \in S(x, R)$:*

$$(3.15) \qquad\qquad \frac{\partial G^y_{x,R}}{\partial n}(\eta) \geqslant c'_3 \qquad \text{as soon as} \quad d(y, \eta) \leqslant 4.$$

Proof of Lemmas 3.7 and 3.8. — The proofs of these two lemmas are the same, except that they rely either on Lemma 3.3 or on Lemma 3.4. We will only prove Lemma 3.8.

The strategy is as in Section 3.3. We construct a subharmonic function f on the ball $B(x, R)$ such that $f(\eta) = 0$, such that $f \leqslant G^y_{x,R}$ in a small ball tangent at η to the sphere $S(x, R)$, and whose normal derivative at η is uniformly bounded below.

As shown in Figure 3.1.B, we introduce the point y_0 on the ray $x\eta$ such that $d(x, y_0) = R - 1$. By construction one has $d(\eta, y_0) = 1$. We let again $M_0 = (k-1)\, b \coth(b/4)$ and define

$$f(z) = c_4(e^{-M_0\, d(y_0, z)} - e^{-M_0}),$$

for a constant c_4 that we will soon determine. We first notice that, according to Lemma 2.2,

(3.16) f is subharmonic outside $B(y_0, \frac{1}{2})$ and $f \equiv 0$ on $S(y_0, 1)$.

We now give a uniform lower bound for the Green function $G^y_{x,R}(w)$ for all points w in $S(x, R - \frac{1}{2}) \cap B(y_0, 1)$. Since $d(x, y) \leqslant R - 1$, we observe that it follows from Lemma 3.4 that, for all z in $S(y, \frac{1}{2})$:

$$G^y_{x,R}(z) \geqslant c'_2.$$

For $w \in S(x, R - \frac{1}{2}) \cap B(y_0, 1)$, pick $z \in S(y, \frac{1}{2})$ on the segment $[yw]$. Since the segment $[zw]$ is included in the ball $B(x, R - \frac{1}{2})$ and has length at most 6, it follows from Harnack inequality (2.5), applied to X with the distance $2d$, that:

$$G^y_{x,R}(w) \geqslant c'_2 \, e^{-12 \, C_1}.$$

This means that

(3.17) $G^y_{x,R} \geqslant f$ on $S(x, R - \frac{1}{2}) \cap B(y_0, 1)$

for the choice of the constant

$$c_4 := \frac{c'_2 \, e^{-12 \, C_1}}{e^{-M_0/2} - e^{-M_0}}.$$

Combining (3.16), (3.17) and the maximum principle, one gets the bound on the grey zone of Figure 3.1.B

$$G^y_{x,R} \geqslant f \quad \text{on} \quad B(y_0, 1) \smallsetminus B(x, R - \frac{1}{2}).$$

Therefore, one has the following inequality between the normal derivatives at the point η:

$$\frac{\partial G^y_{x,R}}{\partial n}(\eta) \geqslant \frac{\partial f}{\partial n}(\eta) = \frac{c'_2 \, M_0 \, e^{-12 \, C_1}}{e^{M_0/2} - 1}.$$

This proves the bound (3.9). $\square$

4. Upper bound for the harmonic measures

The aim of this chapter is to prove the upper bound in (1.1).

We recall that X is a k-dimensional Hadamard manifold satisfying the pinching condition $-b^2 \leqslant K_X \leqslant -a^2 < 0$. Let x be a point in X. We will denote by ξ a point on the sphere $S(x, R)$, by $x\xi$ the ray with vertex x that contains ξ, and by $C^\theta_{x\xi}$ the cone with axis $x\xi$ and angle θ. We want to bound

(4.1) $\sigma^x_{x,R}(C^\theta_{x\xi}) \leqslant M\theta^{1/N}$

where the constants M and N depend only on a, b and k. It is not restrictive to assume that $b = 1$. We will distinguish three cases, letting

$$\theta_R := 10^{-3} e^{-(R-2)} :$$

- Bounded radius: $R \leqslant 2$.
- Large angle: $R \geqslant 2$ and $\theta \geqslant \theta_R$.
- Small angle: $R \geqslant 2$ and $\theta \leqslant \theta_R$.

Without loss of generality, we may assume that $\theta \leqslant 10^{-3}$.

4.1. Upper bound for a bounded radius

We prove (4.1) when $R \leqslant 2$.

More precisely, when $R \leqslant 2$, we will prove the upper bound (4.1) under the weaker pinching condition $-1 \leqslant K_X \leqslant 0$. This allows us to multiply the metric by a ratio $2/R$, while preserving this pinching condition. Hence we can assume that the radius is $R_0 = 2$. Using the expression (3.7) for the density of the harmonic measure, the bound (3.8) for this density and the bound (6.7) for the volume $A_{x,R_0}(C_{x\xi}^\theta)$, we get

$$\sigma_{x,R_0}^x (C_{x\xi}^\theta) = \int_{C_{x\xi}^\theta} \frac{\partial G_{x,R_0}^x}{\partial n}(\eta) \, dA_{x,R_0}(\eta) \leqslant C_3 \, A_{x,R_0}(C_{x\xi}^\theta) \leqslant C_3 \, V_k \, \theta^{k-1}.$$

This proves (4.1) when $R \leqslant 2$.

4.2. Upper bound for a large angle

We prove (4.1) when $R \geqslant 2$ and $\theta \geqslant \theta_R$.

As shown in Figure 4.1.A, we introduce the point w on the ray $x\xi$ such that $d(x, w) = r$, where r is given by

$$(4.2) \qquad\qquad\qquad \theta = 10^{-3} e^{-r}.$$

Since $\theta_R \leqslant \theta \leqslant 10^{-3}$, one has $0 \leqslant r \leqslant R - 2$. In particular, the point w is at distance at least 2 from every point η of the sphere $S(x, R)$. According to Lemma 6.1, and since $4 \, e^r \, \theta \leqslant \frac{\pi}{2}$, one has

$$C_{x\xi}^\theta \cap S(x, R) \subset C_{w\xi}^{\pi/2} \cap S(x, R) \,.$$

We now introduce the Anderson–Schoen barrier $u = u_{w\xi}$ for the ray $w\xi$, as constructed in Lemma 2.3. Since u is superharmonic, with $u \geqslant 0$ everywhere
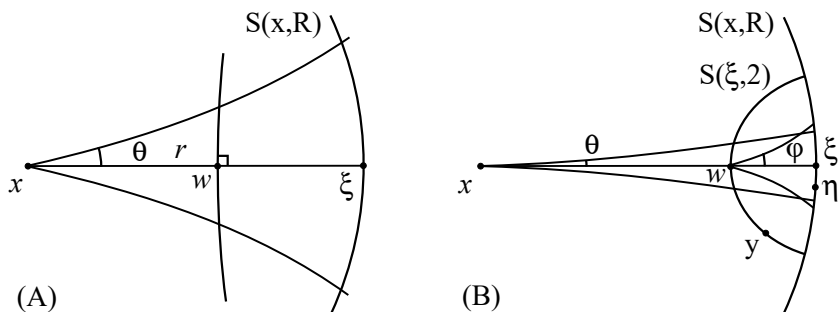
Figure 4.1. Majoration of $\sigma^x_{x,R}(C^\theta_{x\xi})$ for a large angle θ, and for a small angle θ.

and $u \geqslant 1$ on the cone $C^{\pi/2}_{w\xi}$, one infers from the maximum principle that one has for all y in $\mathring{B}(x, R)$:

$$\sigma^y_{x,R}(C^\theta_{x\xi}) \leqslant \sigma^y_{x,R}(C^{\pi/2}_{w\xi}) \leqslant u(y).$$

Applying this equality with $y = x$, remembering the exponential decay (2.4) of the Anderson–Schoen barrier on the ray wx and using (4.2), one gets:

$$\sigma^x_{x,R}(C^\theta_{x\xi}) \leqslant u(x) \leqslant C_0 e^{-\varepsilon_0 r} \leqslant 10^{3\varepsilon_0} C_0 \theta^{\varepsilon_0}.$$

This proves (4.1) when $R \geqslant 2$ and $\theta \geqslant \theta_R$.

4.3. Upper bound for a small angle

We prove (4.1) when $R \geqslant 2$ and $\theta \leqslant \theta_R$. The argument will combine both the arguments used in Sections 4.1 and 4.2.

As shown in Figure 4.1.B, we introduce the point w on the ray $x\xi$ such that $d(x, w) = R - 2$, and the angle φ given by

(4.3) $$\varphi := 4\, e^{R-2}\, \theta.$$

Since $\theta < \theta_R$, one has $\varphi \leqslant 1/100$. According to Lemma 6.1, one has

$$C^\theta_{x\xi} \cap S(x, R) \subset C^\varphi_{w\xi} \cap S(x, R).$$

First step. — We estimate the measure of the cone $C^\varphi_{w\xi}$ for the *harmonic measure $\sigma^y_{x,R}$ at a point y within a bounded distance from ξ*.

LEMMA 4.1. — *Let X be a Hadamard manifold with $-1 \leqslant K_X \leqslant -a^2 < 0$. Keep the above notation $x \in X$, $\xi \in S(x, R)$, $w \in [x\xi]$ with $d(w, \xi) = 2$ and $\varphi \leqslant 1/100$ as in Figure 4.1.B. Then, there exists a constant $C_5 > 0$*

depending only on a and $k = \dim X$ such that for all y in $\mathring{B}(x, R) \cap S(\xi, 2)$ one has

$$(4.4) \qquad \sigma^y_{x,R}(C^\varphi_{w\xi}) \leqslant C_5\, \varphi^{k-1}\,.$$

Proof of Lemma 4.1. — One uses again the expression (3.7) for the density of the harmonic measure. Since $\varphi \leqslant 1/100$, it follows from Lemma 6.3.a that, for all η in $C^\varphi_{w\xi} \cap S(x, R)$, one has $d(\xi, \eta) \leqslant 1$ hence $d(y, \eta) \geqslant 1$. Therefore the bound (3.9) is valid at the point η. Hence one estimates

$$\sigma^y_{x,R}(C^\varphi_{w\xi} = \int_{C^\varphi_{w\xi}} \frac{\partial G^y_{x,R}}{\partial n}(\eta)\,\mathrm{d}A_{x,R}(\eta) \leqslant C'_3\, A_{x,R}(C^\varphi_{w\xi}) \leqslant C'_3\, V'_k\, \varphi^{k-1}\,,$$

thanks to the bound (6.9) for the volume $A_{x,R}(C^\varphi_{w\xi})$.   □

Second step. — We will need again the Anderson–Schoen barrier $u = u_{w\xi}$ associated to the geodesic ray $w\xi$ (see Lemma 2.3). Since u is superharmonic, with $u \geqslant 0$ everywhere and $u \geqslant 1$ on the sphere $S(\xi, 2) \subset C^{\pi/2}_{w\xi}$, it follows from (4.4) and the maximum principle that one has, for every point y in $\mathring{B}(x, R) \smallsetminus \mathring{B}(\xi, 2)$:

$$\sigma^y_{x,R}(C^\theta_{x\xi}) \leqslant \sigma^y_{x,R}(C^\varphi_{w\xi}) \leqslant C_5\, \varphi^{k-1}\, u(y) \leqslant C_5\, \varphi^{\varepsilon_0}\, u(y),$$

where we have used the constant $\varepsilon_0 \leqslant 1$ from Lemma 2.3. Applying this equality with $y = x$, remembering again the exponential decay (2.4) of the Anderson–Schoen barrier on the ray wx and using (4.3), one finally gets:

$$\sigma^x_{x,R}(C^\theta_{x\xi}) \leqslant C_5\, \varphi^{\varepsilon_0}\, u(x) \leqslant C_0 C_5\, \varphi^{\varepsilon_0}\, e^{-\varepsilon_0(R-2)} \leqslant C_0 C_5\, 4^{\varepsilon_0}\, \theta^{\varepsilon_0}.$$

This proves (4.1) when $R \geqslant 2$ and $\theta \leqslant \theta_R$.

5. Lower bound for the harmonic measures

The aim of this chapter is to prove the lower bound in (1.1).

The structure of this chapter is very similar to the structure of Section 4. We recall that X is a k-dimensional Hadamard manifold satisfying the pinching condition $-b^2 \leqslant K_X \leqslant -a^2 < 0$, x is a point on X, ξ is a point on the sphere $S(x, R)$ and $C^\theta_{x\xi}$ is the cone with axis $x\xi$ and angle θ. We want to prove that

$$(5.1) \qquad \sigma^x_{x,R}(C^\theta_{x\xi}) \geqslant \frac{1}{M}\theta^N\,,$$

where the constants M and N depend only on a, b and k. It is not restrictive to assume that $b = 1$. Fix a length $l_0 \geqslant 2$ such that

$$(5.2) \qquad \frac{1}{2} \geqslant C_0\, e^{-\varepsilon_0(l_0-1)}\,.$$

We will distinguish three cases, letting

$$\theta'_R = 2\pi e^{-a(R-l_0)} \ :$$

- Bounded radius: $R \leqslant l_0$.
- Large angle: $R \geqslant l_0$ and $\theta \geqslant \theta'_R$.
- Small angle: $R \geqslant l_0$ and $\theta \leqslant \theta'_R$.

5.1. Lower bound for a bounded radius

We prove (5.1) when $R \leqslant l_0$.

As in Section 4.1, when $R \leqslant l_0$, we will prove the lower bound (5.1) under the weaker pinching condition $-1 \leqslant K_X \leqslant 0$. This allows us to multiply the distance by a ratio l_0/R, while preserving this pinching condition. Hence we can assume that the radius is $R_0 = l_0$. Using the expression (3.7) for the density of the harmonic measure, the bound (3.14) for this density, and the bound (6.7) for the volume $A_{x,R_0}(C^{\theta}_{x\xi})$, one estimates

$$\sigma^x_{x,R_0}(C^{\theta}_{x\xi}) = \int_{C^{\theta}_{x\xi}} \frac{\partial G^x_{x,R_0}}{\partial n}(\eta)\, \mathrm{d}A_{x,R_0}(\eta) \geqslant c_3\, A_{x,R_0}(C^{\theta}_{x\xi}) \geqslant c_3\, v_k\, \theta^{k-1}.$$

This proves (5.1) when $R \leqslant l_0$.

5.2. Lower bound for a large angle

We prove (5.1) when $R \geqslant l_0$ and $\theta \geqslant \theta'_R$.

As shown in Figure 5.1.A, we introduce the point w on the ray $x\xi$ such that $d(x,w) = r$, where r is given by

$$(5.3) \qquad\qquad \theta = 2\pi\, e^{-ar}.$$

Since $\theta'_R \leqslant \theta \leqslant \pi/2$, one has $0 \leqslant r \leqslant R - l_0$.

In particular the point w is at distance at least l_0 from every point η on the sphere $S(x,R)$. Since $\frac{1}{4}\, e^{ar}\, \theta \geqslant \pi/2$, it follows from Lemma 6.1 that

$$C^{\theta}_{x\xi} \cap S(x,R) \supset C^{\pi/2}_{w\xi} \cap S(x,R)\,.$$

First step. — We first estimate the measure of the cone $C^{\theta}_{x\xi}$ for the harmonic measure $\sigma^v_{x,R}$ at a point v suitably chosen on the ray $x\xi$.

Here we need the Anderson–Schoen barrier $u = u_{wx}$ for the ray wx i.e. the ray opposite to $w\xi$. Since u is superharmonic, with $u \geqslant 0$ everywhere
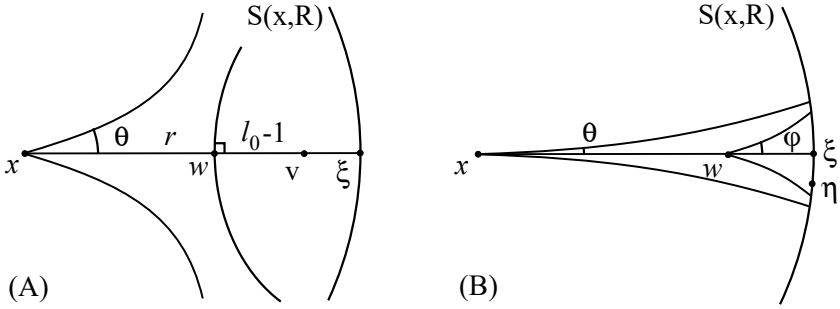
Figure 5.1. *Minoration of* $\sigma^x_{x,R}(C^\theta_{x\xi})$ *for a large angle* θ, *and for a small angle* θ.

and $u \geqslant 1$ on the cone $C^{\pi/2}_{wx}$, it follows from the maximum principle that, for all y in $\mathring{B}(x, R)$:

$$\sigma^y_{x,R}(C^\theta_{x\xi}) \geqslant \sigma^y_{x,R}(C^{\pi/2}_{w\xi}) = 1 - \sigma^y_{x,R}(C^{\pi/2}_{wx}) \geqslant 1 - u(y).$$

Applying this estimate to the point $y = v$ on the ray $w\xi$ with $d(w, v) = l_0 - 1$ and remembering the exponential decay (2.4) of the Anderson–Schoen barrier on the ray $w\xi$, one gets using (5.2) that:

$$(5.4) \qquad \sigma^v_{x,R}(C^\theta_{x\xi}) \geqslant 1 - u(v) \geqslant 1 - C_0 e^{-\varepsilon_0(l_0-1)} \geqslant \frac{1}{2}.$$

Second step. — We now apply the Harnack inequality (2.5) to the positive harmonic function $y \mapsto \sigma^y_{x,R}(C^\theta_{x\xi})$ on the ball $\mathring{B}(x, R)$. Since the segment $[xv]$ stays at a distance at least 1 from the sphere $S(x, R)$ and has length bounded by $r + l_0$, it follows from (5.3) and (5.4) that:

$$\sigma^x_{x,R}(C^\theta_{x\xi}) \geqslant \sigma^v_{x,R}(C^\theta_{x\xi})e^{-C_1 l_0 - C_1 r} \geqslant \frac{1}{2}e^{-C_1 l_0}\left(\frac{\theta}{2\pi}\right)^{C_1/a}.$$

This proves (5.1) when $R \geqslant l_0$ and $\theta \geqslant \theta'_R$.

5.3. Lower bound for a small angle

We prove (5.1) when $R \geqslant l_0$ and $\theta \leqslant \theta'_R$. The argument will be similar to those in Section 4.3.

As shown in Figure 5.1.B, we introduce the point w on the ray $x\xi$ such that $d(x, w) = R - 2$. Let φ be the angle given by

$$(5.5) \qquad\qquad \varphi := 10^{-3}\, e^{a(R-l_0)}\, \theta.$$

Since $\theta \leqslant \theta'_R$, one has $\varphi \leqslant \frac{1}{100}$. Moreover, the definition of φ ensures that $\frac{1}{4}e^{a(R-2)}\theta \geqslant \varphi$, so that Lemma 6.1 ensures that:

$$C^\theta_{x\xi} \cap S(x,R) \supset C^\varphi_{w\xi} \cap S(x,R).$$

First step. — We estimate the measure of the cone $C^\varphi_{w\xi}$ for the *harmonic measure* $\sigma^w_{x,R}$ *seen from the point* w.

LEMMA 5.1. — *Let X be a Hadamard manifold with $-1 \leqslant K_X \leqslant -a^2 < 0$. Keep the above notation $x \in X$, $\xi \in S(x,R)$, $w \in [x\xi]$ with $d(w,\xi) = 2$ and $\varphi \leqslant 1/100$ as in Figure 5.1.B. Then, there exists a constant $c_5 > 0$ depending only on a and $k = \dim X$ such that*

$$(5.6) \qquad\qquad \sigma^w_{x,R}(C^\varphi_{w\xi}) \geqslant c_5\, \varphi^{k-1}\,.$$

Proof of Lemma 5.1. — Once again, we use the expression (3.7) for the density of the harmonic measure. Since $\varphi \leqslant 1/100$, it follows from Lemma 6.3.a that one has $d(\xi,\eta) \leqslant 1$ for any η in $C^\varphi_{w\xi} \cap S(x,R)$,. Therefore one has $2 \leqslant d(w,\eta) \leqslant 3$ and the bound (3.15) with $y = w$ is valid for this density. Thus

$$\sigma^w_{x,R}(C^\varphi_{w\xi}) = \int_{C^\varphi_{w\xi}} \frac{\partial G^w_{x,R}}{\partial n}(\eta)\, \mathrm{d}A_{x,R}(\eta) \geqslant c'_3\, \mathrm{A}_{x,R}(C^\varphi_{w\xi}) \geqslant c'_3\, v'_k\, \varphi^{k-1}\,,$$

thanks to the bound (6.9) for the volume $\mathrm{A}_{x,R}(C^\varphi_{w\xi})$. $\square$

Second step. — We apply again the Harnack inequality (2.5) to the positive harmonic function $y \mapsto \sigma^y_{x,R}(C^\varphi_{w\xi})$ on the ball $\mathring{B}(x,R)$. Since the segment $[xw]$ stays at distance at least 1 from the sphere $S(x,R)$ and has length smaller than R, this gives, using (5.6):

$$\sigma^x_{x,R}(C^\theta_{x\xi}) \geqslant \sigma^x_{x,R}(C^\varphi_{w\xi}) \geqslant \sigma^w_{x,R}(C^\varphi_{w\xi})e^{-C_1 R} \geqslant c_5\, \varphi^{k-1}e^{-C_1 R}\,.$$

Increasing C_1, one may assume $C_1/a \geqslant k$. Hence one gets, using also (5.5):

$$\sigma^x_{x,R}(C^\theta_{x\xi}) \geqslant c'_5\, \varphi^{C_1/a}\left(\frac{\theta}{\varphi}\right)^{C_1/a} = c'_5\, \theta^{C_1/a}\,,$$

with $c'_5 := 10^{-3C_1/a}e^{-C_1 l_0}c_5$. This proves (5.1) when $R \geqslant l_0$ and $\theta \leqslant \theta'_R$.

6. Geometry of Hadamard manifold

This last chapter is self-contained. We collect here two basic geometric estimates in Hadamard manifolds that were used in the previous chapters.

6.1. Geometry of triangles

We first compare the angles in a triangle.

We will denote by $\mathbb{H}^2(-a^2)$ the real hyperbolic plane with curvature $-a^2$.

LEMMA 6.1. — *Let X be a Hadamard manifold with $-b^2 \leqslant K_X \leqslant -a^2 < 0$. Let r, R, L be the side lengths of a geodesic triangle in X and let θ, φ be the two angles as in Figure 6.1. Assume that $0 \leqslant \varphi \leqslant \pi/2$ and $bL \geqslant 2$. Then one has the following angle estimates*

$$(6.1) \qquad \frac{1}{4}\,e^{ar} \leqslant \frac{\varphi}{\theta} \leqslant 4\,e^{br}.$$
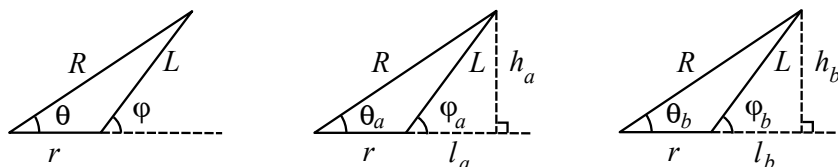
Figure 6.1. A triangle in X and its comparison triangles in $\mathbb{H}^2(-a^2)$ and $\mathbb{H}^2(-b^2)$.

Proof. — The proof relies on comparison triangles in the hyperbolic planes $\mathbb{H}^2(-a^2)$ and $\mathbb{H}^2(-b^2)$ i.e. the triangles with same side lengths r, R and L.

We denote by θ_a and φ_a the angles and by h_a, l_a the lengths seen in $\mathbb{H}^2(-a^2)$ as in Figure 1. We use similar notations in $\mathbb{H}^2(-b^2)$. The pinching assumption tells us that

$$(6.2) \qquad \theta_b \leqslant \theta \leqslant \theta_a \quad \text{and} \quad \varphi_a \leqslant \varphi \leqslant \varphi_b.$$

We will use the following identities for the right triangle in $\mathbb{H}^2(-a^2)$ with side lengths L, l_a and h_a:

$$(6.3) \quad \sinh(aL)\sin\varphi_a = \sinh(ah_a) \quad \text{and} \quad \cosh(aL) = \cosh(al_a)\cosh(ah_a).$$

Taking the ratio of these two equalities and repeating this computation for the right triangle with side lengths R, $r + l_a$ and h_a, one gets

$$(6.4) \qquad \frac{\sin\varphi_a}{\sin\theta_a} = \frac{\tanh(aR)}{\tanh(aL)}\frac{\cosh(ar + al_a)}{\cosh(al_a)}.$$

We will also use the easy inequalities, valid for any $t \geqslant 0$ and $0 \leqslant \alpha \leqslant \frac{\pi}{2}$:

$$(6.5) \qquad \frac{1}{2}e^t \leqslant \cosh(t) \leqslant e^t \quad \text{and} \quad \frac{2}{\pi}\alpha \leqslant \sin\alpha \leqslant \alpha.$$

We first prove the lower bound in (6.1). We notice that (6.2) ensures that the angle φ_a is acute, so that the angle θ_a is also acute. Using (6.2), (6.4), (6.5) and the bound $L \leqslant R$, we obtain

$$\frac{\varphi}{\theta} \geqslant \frac{\varphi_a}{\theta_a} \geqslant \frac{2}{\pi} \frac{\sin \varphi_a}{\sin \theta_a} = \frac{2}{\pi} \frac{\tanh(aR)}{\tanh(aL)} \frac{\cosh(ar + al_a)}{\cosh(al_a)} \geqslant \frac{e^{ar}}{\pi} \geqslant \frac{1}{4} e^{ar}.$$

We now prove the upper bound in (6.1) when the angle φ_b is acute. The computation is similar, using also the assumption $bL \geqslant 2$:

$$\frac{\varphi}{\theta} \leqslant \frac{\varphi_b}{\theta_b} \leqslant \frac{\pi}{2} \frac{\sin \varphi_b}{\sin \theta_b} = \frac{\pi}{2} \frac{\tanh(bR)}{\tanh(bL)} \frac{\cosh(br + bl_b)}{\cosh(bl_b)} \leqslant \frac{\pi e^{br}}{\tanh(2)} \leqslant 4 \, e^{br}.$$

Finally, we prove the upper bound in (6.1) when the angle φ_b is obtuse. We notice that since the angle φ is acute, one has $r \leqslant R$ and therefore

$$(6.6) \qquad\qquad 2h_b \geqslant R + L - r \geqslant L \,,$$

so that the hypothesis ensures that $bh_b \geqslant 1$. Similar computations using equalities analogous to (6.3) now yield

$$\frac{\varphi}{\theta} \leqslant \frac{\pi}{2\theta_b} \leqslant \frac{\pi}{2} \frac{1}{\sin \theta_b} = \frac{\pi}{2} \frac{\tanh(bR)}{\tanh(bh_b)} \cosh(br - bl_b) \leqslant \frac{\pi e^{br}}{2 \tanh(1)} \leqslant 4 \, e^{br}.$$

Note that the constants in (6.1) are not optimal. $\qquad\qquad\qquad\square$

6.2. Volume estimates

We now estimate the Riemannian measures on spheres.

As before, for x in X and $R > 0$, we denote by $A_{x,R}$ the Riemannian measure of the sphere $S(x, R)$.

The first lemma provides volume estimates for a fixed radius $R_0 \geqslant 1$.

LEMMA 6.2. — Let X be a Hadamard manifold with $-1 \leqslant K_X \leqslant 0$ and fix $R_0 \geqslant 1$. There exist constants $V_k > v_k > 0$ depending only on $k = \dim X$ and R_0 and such that, for every x in X, ξ in $S(x, R)$ and $\varphi \leqslant \frac{\pi}{2}$, one has:

$$(6.7) \qquad\qquad v_k \, \varphi^{k-1} \leqslant A_{x,R_0}(C_{x\xi}^{\varphi}) \leqslant V_k \, \varphi^{k-1}.$$

Proof. — Let $x \in X$. Because of the pinching assumption, the exponential map $\exp_x : T_x X \to X$ is a diffeomorphism and its restriction to the ball $B(0, R_0) \subset T_x X$ induces a diffeomorphism $\Phi_x : B(0, R_0) \to B(x, R_0)$ whose derivatives are uniformly bounded

$$(6.8) \qquad\qquad \|D\Phi_x\| \leqslant e^{R_0} \quad \text{and} \quad \|D\Phi_x^{-1}\| \leqslant 1 \,.$$

The bounds (6.7) follow. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The second lemma provides volume estimates which are uniform in R.

LEMMA 6.3. — *Let X be a Hadamard manifold with $-1 \leqslant K_X \leqslant 0$. Let $R \geqslant 2$, $x \in X$, $\xi \in S(x, R)$, $w \in [x\xi]$ with $d(w, \xi) = 2$ and $\varphi \leqslant \varphi_0 := 1/100$ as in Figure 6.2.*

(a) *One has the inclusion $C_{w\xi}^{\varphi} \cap S(x, R) \subset B(\xi, 1)$.*

(b) *There exist constants $V_k' > v_k' > 0$ depending only on $k = \dim X$ such that*

$$(6.9) \qquad v_k' \, \varphi^{k-1} \leqslant \mathrm{A}_{x,R}(C_{w\xi}^{\varphi}) \leqslant V_k' \, \varphi^{k-1}.$$
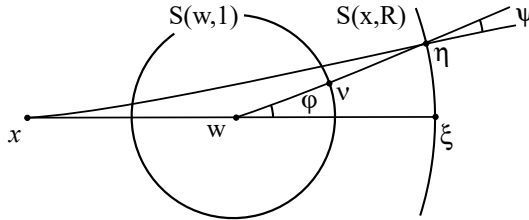

Figure 6.2. Estimation of the volume $A_{x,R}(C_{w\xi}^{\varphi})$.

Proof.

(a). — Let η be a point on $S(x, R)$ such that the angle φ between $w\xi$ and $w\eta$ is bounded by $1/100$. The triangle (w, ξ, η) also satisfies the following properties:

$$d(w, \xi) = 2, \; d(w, \eta) \geqslant 2, \; \text{and the angle between } \xi w \text{ and } \xi \eta \text{ is acute.}$$

Since $-1 \leqslant K_X \leqslant 0$, the comparison triangle (w', ξ', η') in $\mathbb{H}^2$ satisfies the same properties. A direct computation in $\mathbb{H}^2$ gives then $d(\eta', \xi') \leqslant 1$. Therefore one also has $d(\eta, \xi) \leqslant 1$.

(b). — As shown in Figure 6.2, since X is a Hadamard manifold, the intersection $S(x, R) \cap C_{w\xi}^{\varphi_0}$ is a hypersurface that can be parametrized in polar coordinates with origin w: there exists a $\mathcal{C}^\infty$ diffeomorphism

$$\Psi_w \, : \, S(w, 1) \cap C_{w\xi}^{\varphi_0} \longrightarrow S(x, R) \cap C_{w\xi}^{\varphi_0}$$
$$\nu \longmapsto \eta = \Psi_w(\nu) = \exp_w(\rho_\nu \exp_w^{-1} \nu),$$

where ρ is a $\mathcal{C}^\infty$ function on $S(w, 1) \cap C_{w\xi}^{\varphi_0}$ with values in the interval $[2, 3]$.

Since X is a Hadamard manifold, at every point of this hypersurface $S(x, R) \cap C_{w\xi}^{\varphi_0}$ the angle ψ between the normal vector to $S(x, R)$ and the

radial vector seen from w is at most φ_0. Therefore, using Jacobi fields, one checks that the derivatives of Ψ_w and its inverse are uniformly bounded

$$\|D\Psi_w\| \leqslant \frac{e^2}{\cos(\varphi_0)} \leqslant 10 \qquad \text{and} \qquad \|D\Psi_w^{-1}\| \leqslant 1 \,.$$

Therefore, one has

$$\mathrm{A}_{w,1}(C_{w\xi}^{\varphi}) \leqslant \mathrm{A}_{x,R}(C_{w\xi}^{\varphi}) \leqslant 10^{k-1}\mathrm{A}_{w,1}(C_{w\xi}^{\varphi})$$

The bounds (6.9) now follow from (6.7). $\hspace{2cm}\square$

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