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ON COMPATIBILITY OF THE ℓ-ADIC REALISATIONS OF AN ABELIAN MOTIVE

by Johan M. COMMELIN (*)

ABSTRACT. — In this article we introduce the notion of quasi-compatible system of Galois representations. The quasi-compatibility condition is a mild relaxation of the classical compatibility condition in the sense of Serre. The main theorem that we prove is the following: Let M be an abelian motive in the sense of Yves André. Then the ℓ-adic realisations of M form a quasi-compatible system of Galois representations. (In Theorem 5.1 we actually prove something stronger.) As an application, we deduce that the absolute rank of the ℓ-adic monodromy groups of M does not depend on ℓ. In particular, the Mumford–Tate conjecture for M does not depend on ℓ.

RÉSUMÉ. — Dans cet article, nous introduisons la notion de système quasi-compatible de représentations galoisiennes. La condition de quasi-compatibilité est un affaiblissement de la condition de compatibilité à la Serre. Le principal théorème que nous prouvons est le suivant: Soit M un motif abélien à la Yves André. Alors les réalisations ℓ-adiques de M forment un système quasi-compatible de représentations galoisiennes. Comme application, on en déduit que le rang absolu des groupes de monodromie ℓ-adiques de M ne dépend pas de ℓ. En particulier, la conjecture de Mumford–Tate pour M ne dépend pas de ℓ.

1. Introduction

1.1. MAIN RESULT. — The main result of this article is:

THEOREM 5.1. — *Let M be an abelian motive over a finitely generated subfield $K \subset \mathbb{C}$. Let E be a subfield of $\mathrm{End}(M)$, and let Λ be the set of finite places of E. Then the system $\mathrm{H}_\Lambda(M)$ is a quasi-compatible system of Galois representations.*

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To understand this result we need to explain what we mean by:

(i) the words “abelian motive”;
(ii) the notation $\mathrm{H}_{\Lambda}(M)$; and
(iii) the words “quasi-compatible system of Galois representations”.

1.2. Abelian motives. — In this text we use motives in the sense of André [1]. Alternatively we could have used the notion of absolute Hodge cycles. Let $K \subset \mathbb{C}$ be a finitely generated subfield of the complex numbers. An *abelian motive* over K is a summand of a Tate twist of the motive of an abelian variety over a finite K-algebra. In practice this means that an abelian motive M is a package consisting of a Hodge structure $\mathrm{H}_{\mathrm{B}}(M)$ and for each prime ℓ an ℓ-adic Galois representation $\mathrm{H}_{\ell}(M)$, that arise in a compatible way as summands of Tate twists of the cohomology of an abelian variety.

1.3. λ-adic realisations and the notation $\mathrm{H}_{\Lambda}(M)$. — Let M be an abelian motive over K, and let E be a subfield of $\mathrm{End}(M)$. Since M is finite-dimensional, the field E is a number field. Let Λ be the set of finite places of E. For each prime number ℓ, the field E acts $\mathbb{Q}_{\ell}$-linearly on the Galois representation $\mathrm{H}_{\ell}(M)$ by functoriality. Because $E_{\ell} = E \otimes \mathbb{Q}_{\ell} = \prod_{\lambda|\ell} E_{\lambda}$ we get a decomposition $\mathrm{H}_{\ell}(M) = \bigoplus_{\lambda|\ell} \mathrm{H}_{\lambda}(M)$ of Galois representations, where $\mathrm{H}_{\lambda}(M) = \mathrm{H}_{\ell}(M) \otimes_{E_{\ell}} E_{\lambda}$. We denote with $\mathrm{H}_{\Lambda}(M)$ the system of λ-adic Galois representations $\mathrm{H}_{\lambda}(M)$ as λ runs through Λ.

1.4. Quasi-compatible systems. — In Section 3 we develop a variation on the concept of compatible systems of Galois representations that has its origins in the work of Serre [27]. Besides the original work of Serre, we draw inspiration from Ribet [24], Larsen–Pink [15], and Chi [5]. The main feature of our variant is a certain robustness with respect to extension of the base field.

By this we mean the following. For the purpose of this introduction, let K be a number field; in Section 3 the field K is allowed to be any finitely generated field of characteristic 0. Let $\rho_{\ell} \colon \mathrm{Gal}(\bar{K}/K) \to \mathrm{GL}_n(\mathbb{Q}_{\ell})$ and $\rho_{\ell'} \colon \mathrm{Gal}(\bar{K}/K) \to \mathrm{GL}_n(\mathbb{Q}_{\ell'})$ be two Galois representations. Let v be a place of K such that the residue characteristic of v is neither ℓ nor ℓ'. Let $\bar{v}$ be an extension of v to $\bar{K}$. Assume that ρ_{ℓ} and $\rho_{\ell'}$ are unramified at $\bar{v}/v$. Let $F_{\bar{v}/v}$ be a Frobenius element with respect to v; that is, a lift of the inverse of the Frobenius automorphism of $\kappa(\bar{v})/\kappa(v)$ to the decomposition group $D_{\bar{v}/v} \subset \mathrm{Gal}(\bar{K}/K)$. Here $\kappa(\bar{v})$ and $\kappa(v)$ denote the residue field of $\bar{v}$ and the residue field of v respectively.

We can now contrast the usual compatibility condition with our condition. Recall that ρ_ℓ and $\rho_{\ell'}$ are called compatible at v if the characteristic polynomials of $\rho_\ell(F_{\bar{v}/v})$ and $\rho_{\ell'}(F_{\bar{v}/v})$ have coefficients in $\mathbb{Q}$ and are equal. Note that extensions of v to $\bar{K}$ are conjugate. Consequently, neither the condition that the representations are unramified nor the compatibility condition on the characteristic polynomials depends on the choice of $\bar{v}$.

Our variant replaces the compatibility condition on the characteristic polynomials of $\rho_\ell(F_{\bar{v}/v})$ and $\rho_{\ell'}(F_{\bar{v}/v})$ by the analogous condition for a power of $F_{\bar{v}/v}$ that is allowed to depend on v: We say that ρ_ℓ and $\rho_{\ell'}$ are *quasi-compatible at v* if there exists an integer n such that the characteristic polynomials of $\rho_\ell(F^n_{\bar{v}/v})$ and $\rho_{\ell'}(F^n_{\bar{v}/v})$ have coefficients in $\mathbb{Q}$ and are equal.

We may also take endomorphisms into account. Instead of only considering systems of $\mathbb{Q}_\ell$-linear Galois representations where ℓ runs over the finite places of $\mathbb{Q}$, we may consider systems of E_λ-linear Galois representations where λ runs over the finite places of a number field E. This was already suggested by Serre [27], and Ribet pursued this further in [24].

The quasi-compatibility condition mentioned above must then be adapted as follows. Let $\rho_\lambda : \mathrm{Gal}(\bar{K}/K) \to \mathrm{GL}_n(E_\lambda)$ and $\rho_{\lambda'} : \mathrm{Gal}(\bar{K}/K) \to \mathrm{GL}_n(E_{\lambda'})$ be two Galois representations. Assume that the residue characteristic of v is different from the residue characteristics of λ and λ', and assume that ρ_λ and $\rho_{\lambda'}$ are unramified at v. We say that ρ_λ and $\rho_{\lambda'}$ are quasi-compatible at v if there is a positive integer n such that the characteristic polynomials of $\rho_\lambda(F^n_{\bar{v}/v})$ and $\rho_{\lambda'}(F^n_{\bar{v}/v})$ have coefficients in E and are equal.

The system $\mathrm{H}_\Lambda(M)$ mentioned above is *quasi-compatible* if for all $\lambda, \lambda' \in \Lambda$ there is a cofinite subset $U \subset \mathrm{Spec}(\mathcal{O}_K)$ such that $\mathrm{H}_\lambda(M)$ and $\mathrm{H}_{\lambda'}(M)$ are quasi-compatible at all places $v \in U$.

1.5. Related work. — In [17], Laskar obtained related results. Though he does not state this explicitly, his results imply that for a large class of abelian motives the λ-adic realisations form a compatible system of Galois representations in the sense of Serre. The contribution of the main result in this paper is that M may be an arbitrary abelian motive; although we need to weaken the concept of compatibility to quasi-compatibility to achieve this. See Remark 5.9 for more details.

1.6. Organisation of the paper. — Every section starts with a paragraph labeled "Readme". These paragraphs highlight the important parts of their section, or describe the role of the section in the text as a whole. We hope that these paragraphs aid in navigating the text.

In Section 2 we recall the definition of abelian motives in the sense of André [1]. We also recall useful properties of abelian motives. This section does not contain new results. In Section 3 we give the main definition of this paper, namely the notion of a quasi-compatible system of Galois representations. In Section 4 we recall results showing that abelian varieties and so-called CM motives give rise to such quasi-compatible systems. These results are known over number fields, and we make the rather straightforward generalisation to finitely generated fields. Section 5 is the heart of this paper, as it proves the main result. See below for an outline of its contents. Finally, Section 6 and Section 7 are appendices. In the former we show that quasi-compatible systems share some of the familiar properties of compatible systems in the sense of Serre. The latter appendix shows that for abelian motives the Mumford–Tate conjecture does not depend on the prime number ℓ that occurs in its statement.

1.7. Outline of the main proof. — Shimura showed that if $M = \mathrm{H}^1(A)$, with A an abelian variety, then the system $\mathrm{H}_\Lambda(M)$ is an E-rational compatible system in the sense of Serre. In Theorem 4.1 we recall this result of Shimura in the setting of quasi-compatible systems of Galois representations.

In Section 5 we prove the main result of this paper, namely that $\mathrm{H}_\Lambda(M)$ is a quasi-compatible system of Galois representations for every abelian motive M. Roughly speaking, the proof works by constructing a family of abelian motives over a certain Shimura variety such that M is a fibre of this family of motives. Verifying the quasi-compatibility condition for M may then be reduced to verifying the quasi-compatibilty condition at a CM point on the Shimura variety. At such a CM point we can prove the result by reducing to the case of abelian varieties mentioned above. To make this work, we need a recent result of Kisin [14]: Let $\mathscr{S}$ be an integral model of a Shimura variety of Hodge type over the ring of integers $\mathcal{O}_K$ of a p-adic field K, satisfying some additional technical conditions. Then every point in the special fibre of $\mathscr{S}$ is *isogenous* to a point that lifts to a CM point of the generic fibre $\mathscr{S}_K$. We refer to Section 5.6 and Section 5.7 for details.

1.8. Terminology and notation. — We say that a field is a *finitely generated field* if it is finitely generated over its prime field. A motive M over a field $K \subset \mathbb{C}$ is called *geometrically irreducible* if $M_{\mathbb{C}}$ is irreducible. If G is a semiabelian variety, then we denote with $\mathrm{End}^0(G)$ the $\mathbb{Q}$-algebra $\mathrm{End}(G) \otimes \mathbb{Q}$. If X is a scheme, then X^{cl} denotes the set of closed points of X.

If E is a field, V a finite-dimensional vector space over E, and g an endomorphism of V, then we denote with $\mathrm{c.p.}_E(g|V)$ the characteristic polynomial of g. If there is no confusion possible, then we may drop E or V from the notation, and write $\mathrm{c.p.}(g|V)$ or simply $\mathrm{c.p.}(g)$.

Let E be a number field. Recall that E is called *totally real* (TR) if for all complex embeddings $\sigma\colon E \hookrightarrow \mathbb{C}$ the image $\sigma(E)$ is contained in $\mathbb{R}$. The field E is called a *complex multiplication* field (CM) if it is a quadratic extension of a totally real field, typically denoted E^0, and if all complex embeddings $\sigma\colon E \hookrightarrow \mathbb{C}$ have an image that is not contained in $\mathbb{R}$.

Let C be a Tannakian category, and let V be an object of C. With $\langle V\rangle^{\otimes}$ we denote the smallest full Tannakian subcategory of C that contains V. This means that it is the smallest full subcategory of C that contains V and that is closed under directs sums, tensor products, duals, and subquotients.

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2. Abelian motives

Readme. — We briefly review the definition of abelian motives in the sense of André [1], and we recall some of their useful properties.

2.1. — Let $K \subset \mathbb{C}$ be a field, and let $\bar{K}$ be the algebraic closure of K in $\mathbb{C}$. Let X be a smooth projective variety over K. For every prime number ℓ, let $\mathrm{H}^i_\ell(X)$ denote the Galois representation $\mathrm{H}^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$. Write $\mathrm{H}^i_{\mathrm{B}}(X)$ for the Hodge structure $\mathrm{H}^i_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{Q})$.

In this text a *motive* over K shall mean a motive in the sense of André [1]. To be precise, our category of base pieces is the category of smooth projective varieties over K, and our reference cohomology is Betti cohomology, $\mathrm{H}_{\mathrm{B}}(\,\cdot\,)$. The resulting notion of motive does not depend on the chosen reference cohomology, see [1, Prop. 2.3]. We denote the category of motives over K with Mot_K. If X is a smooth projective variety over K, then we write $\mathrm{H}^i(X)$ for the motive in degree i associated with X. The cohomology functors mentioned in Section 2.1 induce realisation functors on the category of motives over K. Let M be a motive over K. For every prime ℓ, we write $\mathrm{H}_\ell(M)$ for the ℓ-adic realisation; it is a finite-dimensional $\mathbb{Q}_\ell$-vector space equipped with a continuous representation of $\mathrm{Gal}(\bar{K}/K)$. Similarly, we write $\mathrm{H}_\mathrm{B}(M)$ for the Betti realisation; it is a polarisable $\mathbb{Q}$-Hodge structure.

2.2. — The category Mot_K is a *semisimple* neutral Tannakian category and therefore the motivic Galois group of a motive is a reductive algebraic group. We further mention that Künneth projectors exist in Mot_K. If $K = \mathbb{C}$, then we know that the Betti realisation functor is fully faithful on the Tannakian subcategory generated by motives of abelian varieties, see Theorem 2.6.

2.3. — Let M be a motive over $\mathbb{C}$, and let ℓ be a prime number. There is an isomorphism of $\mathbb{Q}_\ell$-vector spaces $\mathrm{H}_\mathrm{B}(M) \otimes_\mathbb{Q} \mathbb{Q}_\ell \cong \mathrm{H}_\ell(M)$ that is functorial in M. If M is defined over K, then there is an isomorphism of $\mathbb{Q}_\ell$-vector spaces $\mathrm{H}_\ell(M) \cong \mathrm{H}_\ell(M_\mathbb{C})$; and therefore

$$\mathrm{H}_\mathrm{B}(M) \otimes_\mathbb{Q} \mathbb{Q}_\ell \cong \mathrm{H}_\ell(M).$$

This isomorphism was proven for varieties by Artin in [2, Exposé XI]. The generalisation to motives follows from the fact that the isomorphism is compatible with cycle class maps.

2.4. — Let V be a $\mathbb{Q}$-Hodge structure. The *Mumford–Tate group* $\mathrm{G}_\mathrm{B}(V)$ of V is the linear algebraic group over $\mathbb{Q}$ associated with the Tannakian category $\langle V\rangle^\otimes$ generated by V with the forgetful functor $\mathbb{Q}\mathrm{HS} \to \mathrm{Vect}_\mathbb{Q}$ as fibre functor. If V is polarisable, then the Tannakian category $\langle V\rangle^\otimes$ is semisimple; which implies that $\mathrm{G}_\mathrm{B}(V)$ is reductive.

For an alternative description, recall that the Hodge structure on V is determined by a homomorphism of algebraic groups $\mathbb{S} \to \mathrm{GL}(V \otimes_\mathbb{Q} \mathbb{R})$, where $\mathbb{S}$ is the Deligne torus $\mathrm{Res}_\mathbb{R}^\mathbb{C} \mathbb{G}_\mathrm{m}$. The Mumford–Tate group is the smallest algebraic subgroup G of $\mathrm{GL}(V)$ such that $G_\mathbb{R}$ contains the image of $\mathbb{S}$. Since $\mathbb{S}$ is connected, so is $\mathrm{G}_\mathrm{B}(V)$.

If M is a motive over a field $K \subset \mathbb{C}$, then we write $\mathrm{G}_\mathrm{B}(M)$ for $\mathrm{G}_\mathrm{B}(\mathrm{H}_\mathrm{B}(M))$.

2.5. — Let $K \subset \mathbb{C}$ be a field. An *abelian motive* over K is an object of the Tannakian subcategory of motives over K generated by the motives of abelian varieties over K and the motives $\mathrm{H}(\mathrm{Spec}(L))$ for finite field extensions L/K. Recall that $\mathrm{H}(A) \cong \bigwedge^\star \mathrm{H}^1(A)$ for every abelian variety A over K, and thus we have $\langle \mathrm{H}(A)\rangle^\otimes = \langle \mathrm{H}^1(A)\rangle^\otimes$. If A is a non-trivial abelian variety, then the class of any effective non-zero divisor realises $\mathbb{1}(-1)$ as a subobject of $\mathrm{H}^2(A)$, and therefore $\mathbb{1}(-1) \in \langle \mathrm{H}^1(A) \rangle^{\otimes}$. In particular $\mathbb{1}(-1)$ is an abelian motive.

Let M be an abelian motive over K. By definition there are abelian varieties $(A_i)_{i=1}^k$ and field extensions $(L_j)_{j=1}^l$ such that M is contained in the Tannakian subcategory generated by the $\mathrm{H}(A_i)$ and $\mathrm{H}(\mathrm{Spec}(L_j))$. Put $A = \prod_{i=1}^k A_i$, so that $\mathrm{H}^1(A) \cong \bigoplus_{i=1}^k \mathrm{H}^1(A_i)$, and let L be a common overfield of the fields L_j. It follows that M is contained in $\langle \mathrm{H}^1(A) \oplus \mathrm{H}(\mathrm{Spec}(L)) \rangle^{\otimes}$. Upon replacing L by its normal closure, we see that for every abelian motive M over K there is a finite field extension L/K, and an abelian variety A over L, such that the motive M_L is contained in $\langle \mathrm{H}^1(A) \rangle^{\otimes}$.

Theorem 2.6. — *The Betti realisation functor* $\mathrm{H}_{\mathrm{B}}(\,\cdot\,)$ *is fully faithful on the subcategory of abelian motives over* $\mathbb{C}$.

Proof. — See [1, Thm. 0.6.2]. □

2.7. — In the rest of this section we focus on so-called CM motives. They will play a crucial rôle in the proof of our main result. An important tool in understanding abelian CM motives is the half-twist construction that we describe in Section 2.10.

Definition 2.8. — *A motive M over a field $K \subset \mathbb{C}$ is called a* CM motive *if* $\mathrm{H}_{\mathrm{B}}(M)$ *is a* CM *Hodge structure, i.e., if the group* $\mathrm{G}_{\mathrm{B}}(M)$ *is commutative.*

2.9. — Let E be a CM field. Let $\Sigma(E)$ be the set of complex embeddings of E. The complex conjugation on E induces an involution $\sigma \mapsto \sigma^{\dagger}$ on $\Sigma(E)$. If T is a subset of $\Sigma(E)$, then we denote with $T^{\dagger}$ the image of T under this involution. Recall that a CM type $\Phi \subset \Sigma(E)$ is a subset such that $\Phi \cup \Phi^{\dagger} = \Sigma(E)$ and $\Phi \cap \Phi^{\dagger} = \varnothing$. Each CM type Φ defines a Hodge structure E_{Φ} on E of type $\{(0,1),(1,0)\}$, via

$$E_{\Phi} \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathbb{C}^{\Sigma(E)}, \qquad E_{\Phi}^{0,1} \cong \mathbb{C}^{\Phi^{\dagger}}, \qquad E_{\Phi}^{1,0} \cong \mathbb{C}^{\Phi}.$$

2.10. Half-twists. — The idea of half-twists originates from [12], though we use the description in [18, §7]. Let V be a Hodge structure of weight n. The level of V, denoted m, is by definition $\max\{p-q \mid V^{p,q} \neq 0\}$. Suppose that $\mathrm{End}(V)$ contains a CM field E. Let $\Sigma(E)$ denote the set of complex embeddings $E \hookrightarrow \mathbb{C}$. Let $T \subset \Sigma(E)$ be the embeddings through which E acts on $\bigoplus_{p \geqslant \lceil n/2 \rceil} V^{p,q}$. Assume that $T \cap T^{\dagger} = \varnothing$. Note that if $\dim_E(V) = 1$, then the condition $T \cap T^{\dagger} = \varnothing$ is certainly satisfied.

Let $\Phi \subset \Sigma(E)$ be a CM type, and let E_{Φ} be the associated Hodge structure on E. If $T \cap \Phi = \varnothing$ and $m \geqslant 1$, then the Hodge structure $W = E_{\Phi} \otimes_E V$

has weight $n+1$ and level $m-1$. In that case we call W a *half-twist* of V. Note that under our assumption $T \cap T^{\dagger} = \varnothing$ we can certainly find a CM type with $T \cap \Phi = \varnothing$, so that there exist half-twists of V. For each CM type Φ with $T \cap \Phi = \varnothing$, there is a complex abelian variety A_{Φ} well-defined up to isogeny, with $\mathrm{H}^1_{\mathrm{B}}(A_{\Phi}) \cong E_{\Phi}$. By construction we have $E \subset \mathrm{End}(\mathrm{H}^1_{\mathrm{B}}(A_{\Phi}))$ and $E \subset \mathrm{End}(W)$. Note that $V \cong \underline{\mathrm{Hom}}_E(\mathrm{H}^1_{\mathrm{B}}(A_{\Phi}), W)$. In the next paragraph we will see that this construction generalises to abelian motives.

2.11. — Let M be an abelian motive over $K \subset \mathbb{C}$. Assume that M is pure of weight n, and assume that $\mathrm{End}(M)$ contains a CM field E. Note that $\mathrm{H}_{\mathrm{B}}(M)$ is a Hodge structure of weight n. Let $T \subset \Sigma(E)$ be the set of embeddings through which E acts on $\bigoplus_{p \geqslant \lceil n/2 \rceil} \mathrm{H}_{\mathrm{B}}(M)^{p,q}$. Assume that $T \cap T^{\dagger} = \varnothing$. Then there exists a finitely generated extension L/K, an abelian variety A over L, and a motive N over L, such that $E \subset \mathrm{End}(\mathrm{H}^1(A))$, and $E \subset \mathrm{End}(N)$, and such that $M_L \cong \underline{\mathrm{Hom}}_E(\mathrm{H}^1(A), N)$. Indeed, choose a CM type $\Phi \subset \Sigma(E)$ such that $T \cap \Phi = \varnothing$. Put $A = A_{\Phi}$, and $N = \mathrm{H}^1(A_{\Phi}) \otimes_E M_{\mathbb{C}}$. Then $M_{\mathbb{C}} \cong \underline{\mathrm{Hom}}_E(\mathrm{H}^1(A), N)$, by Theorem 2.6 and the construction above. The abelian variety A and the motive N are defined over some finitely generated extension of K, and so is the isomorphism $M_{\mathbb{C}} \cong \underline{\mathrm{Hom}}_E(\mathrm{H}^1(A), N)$.

3. Quasi-compatible systems of Galois representations

README. — In this section we develop the notion of quasi-compatible systems of Galois representations, a variant on Serre's compatible systems of Galois representations [27]. We follow Serre's suggestion of developing an E-rational version, where E is a number field; which has also been done by Ribet [24] and Chi [5]. The main benefit of the variant that we develop is that we relax the compatibility condition, thereby gaining a certain robustness with respect to extensions of the base field and residue fields. We will need this property in a crucial way in the proof of Theorem 5.1.

3.1. — Let κ be a finite field with q elements, and let $\bar{\kappa}$ be an algebraic closure of κ. We denote with $F_{\bar{\kappa}/\kappa}$ the geometric Frobenius element in $\mathrm{Gal}(\bar{\kappa}/\kappa)$, i.e., the inverse of $x \mapsto x^q$.

3.2. — Let K be a number field. Let v be a finite place of K, and let K_v denote the completion of K at v. Let $\bar{K}_v$ be an algebraic closure of K_v. Let $\bar{\kappa}/\kappa$ be the extension of residue fields corresponding with $\bar{K}_v/K_v$. The inertia group, denoted I_v, is the kernel of the natural surjection $\mathrm{Gal}(\bar{K}_v/K_v) \twoheadrightarrow \mathrm{Gal}(\bar{\kappa}/\kappa)$. The inverse image of $F_{\bar{\kappa}/\kappa}$ in $\mathrm{Gal}(\bar{K}_v/K_v)$

is called the *Frobenius coset* of v. An element $\alpha \in \mathrm{Gal}(\bar{K}/K)$ is called a *Frobenius element with respect to* v if there exists an embedding $\bar{K} \hookrightarrow \bar{K}_v$ that extends the composite embedding $K \hookrightarrow K_v \hookrightarrow \bar{K}_v$ such that α is the restriction of an element of the Frobenius coset of v.

3.3. — Let K be a finitely generated field. A *model* of K is an integral scheme X of finite type over $\mathrm{Spec}(\mathbb{Z})$ together with an isomorphism between K and the function field of X. Remark that if K is a number field, and $R \subset K$ is an order, then $\mathrm{Spec}(R)$ is naturally a model of K. The only model of a number field K that is normal and proper over $\mathrm{Spec}(\mathbb{Z})$ is $\mathrm{Spec}(\mathcal{O}_K)$.

3.4. — Let X be a model of K, and recall that we denote the set of closed points of X with X^{cl}. Let $x \in X^{\mathrm{cl}}$ be a closed point. Let K_x be the function field of the Henselisation of X at x; and let $\kappa(x)$ be the residue field at x. We denote with I_x the kernel of $\mathrm{Gal}(\bar{K}_x/K_x) \twoheadrightarrow \mathrm{Gal}(\bar{\kappa}(x)/\kappa(x))$. Every embedding $\bar{K} \hookrightarrow \bar{K}_x$ that extends the composite embedding $K \hookrightarrow K_x \hookrightarrow \bar{K}_x$ induces an inclusion $\mathrm{Gal}(\bar{K}_x/K_x) \hookrightarrow \mathrm{Gal}(\bar{K}/K)$.

Like in Section 3.2, the inverse image of $F_{\bar{\kappa}(x)/\kappa(x)}$ in $\mathrm{Gal}(\bar{K}_x/K_x)$ is called the Frobenius coset of x. An element $\alpha \in \mathrm{Gal}(\bar{K}/K)$ is called a *Frobenius element with respect to* x if there exists an embedding $\bar{K} \hookrightarrow \bar{K}_x$ that extends the composite embedding $K \hookrightarrow K_x \hookrightarrow \bar{K}_x$ such that α is the restriction of an element of the Frobenius coset of x.

Definition 3.5. — *Let K be a field, let E be a number field, and let λ be a place of E. A* λ-adic Galois representation *of K is a representation of* $\mathrm{Gal}(\bar{K}/K)$ *on a finite-dimensional E_λ-vector space that is continuous for the λ-adic topology.*

Let $\rho_\lambda \colon \mathrm{Gal}(\bar{K}/K) \to \mathrm{GL}(V_\lambda)$ *be a λ-adic Galois representation of K. We denote with* $\mathrm{G}_\lambda(\rho_\lambda)$ *or* $\mathrm{G}_\lambda(V_\lambda)$ *the Zariski closure of the image of* $\mathrm{Gal}(\bar{K}/K)$ *in* $\mathrm{GL}(V_\lambda)$*. In particular, if $E = \mathbb{Q}$ and $\lambda = \ell$, then we denote this group with* $\mathrm{G}_\ell(\rho_\ell)$ *or* $\mathrm{G}_\ell(V_\ell)$*.*

3.6. — For the rest of Section 3, we fix the following notation: Let K be a finitely generated field, let E be a number field, and let Λ be a set of finite places of E. Fix $\lambda \in \Lambda$, and let $\rho = \rho_\lambda$ be a λ-adic Galois representation of K.

Let X be a model of K, and let $x \in X^{\mathrm{cl}}$ be a closed point. We use the notation introduced in Section 3.4. We say that ρ is *unramified at* x if there is an embedding $\bar{K} \hookrightarrow \bar{K}_x$ for which $\rho(I_x) = \{1\}$. If this is true for one embedding, then it is true for all embeddings.

Let F_x be a Frobenius element with respect to x. If ρ is unramified at x, then the element $F_{x,\rho} = \rho(F_x)$ is well-defined up to conjugation. For

$n \in \mathbb{Z}$, we write $P_{x,\rho,n}(t)$ for the characteristic polynomial c.p.$(F^n_{x,\rho})$. Note that $P_{x,\rho,n}(t)$ is well-defined, since conjugate endomorphisms have the same characteristic polynomial.

3.7. — In the following definitions, one recovers the notions of Serre [27] by demanding $n = 1$ everywhere. By not making this demand we gain a certain flexibility that will turn out to be crucial for our proof of Theorem 5.1.

Definition 3.8. — *Let $x \in X^{\mathrm{cl}}$ be a closed point of some model X of K. The representation ρ is said to be* quasi-E-rational at x *if ρ is unramified at x, and $P_{x,\rho,n}(t) \in E[t]$, for some $n \geqslant 1$.*

Definition 3.9. — *Let λ_1 and λ_2 be two finite places of E. For $i = 1, 2$, let ρ_i be a λ_i-adic Galois representation of K.*

(1) *Let X be a model of K, and let $x \in X^{\mathrm{cl}}$ be a closed point. Then ρ_1 and ρ_2 are said to be* quasi-compatible at x *if ρ_1 and ρ_2 are both quasi-E-rational at x, and if there is an integer n such that $P_{x,\rho_1,n}(t) = P_{x,\rho_2,n}(t)$ as polynomials in $E[t]$.*
(2) *Let X be a model of K. The representations ρ_1 and ρ_2 are* quasi-compatible with respect to X *if there is a non-empty open subset $U \subset X$, such that ρ_1 and ρ_2 are quasi-compatible at x for all $x \in U^{\mathrm{cl}}$.*
(3) *Let X be a model of K. The representations ρ_1 and ρ_2 are* strongly quasi-compatible with respect to X *if ρ_1 and ρ_2 are quasi-compatible at all points $x \in X^{\mathrm{cl}}$ that satisfy the following condition:*

    The places λ_1 and λ_2 have a residue characteristic that is different from the residue characteristic of x, and ρ_1 and ρ_2 are unramified at x.
(4) *The representations ρ_1 and ρ_2 are* (strongly) quasi-compatible *if they are (strongly) quasi-compatible with respect to every model of K.*

Remark 3.10. — Let λ_1, λ_2, ρ_1, and ρ_2 be as in the above definition.

(1) If there is one model X of K such that ρ_1 and ρ_2 are quasi-compatible with respect to X, then ρ_1 and ρ_2 are quasi-compatible with respect to every model of K, since all models of K are birational to each other.
(2) It is *not* known whether the notion of strong quasi-compatibility is stable under birational equivalence: if ρ_1 and ρ_2 are quasi-compatible with respect to some model X of K, there is no *a priori* reason to expect that ρ_1 and ρ_2 are strongly quasi-compatible with respect

to X. However, by definition there exists a non-empty open subset $U \subset X$ such that ρ_1 and ρ_2 are strongly quasi-compatible with respect to U; and of course U is birational to X.

(3) It is *not* known whether strong quasi-compatibility is an equivalence relation: Let ρ_1, ρ_2, and ρ_3 be respectively λ_1-adic, λ_2-adic, and λ_3-adic Galois representations of K. Suppose that ρ_1 and ρ_2 are strongly quasi-compatible and suppose that ρ_2 and ρ_3 are strongly quasi-compatible. Then it is *not* known whether ρ_1 and ρ_3 are strongly quasi-compatible. This remark also holds for strong quasi-compatibility with respect to a specific model of K.

Definition 3.11. — *A* system of Galois representations *of K is a triple $(E, \Lambda, (\rho_\lambda)_{\lambda\in\Lambda})$, where E is a number field; Λ is a set of finite places of E; and the ρ_λ are λ-adic Galois representations of K.*

3.12. — In what follows, we often denote a system of Galois representations $(E, \Lambda, (\rho_\lambda)_{\lambda\in\Lambda})$ with ρ_Λ, leaving the number field E implicit. In contexts where there are multiple number fields the notation will make clear which number field is meant, e.g., by denoting the set of finite places of a number field E' with Λ', etc.

Definition 3.13. — *Let ρ_Λ be a system of Galois representations of K.*

(1) *Let X be a model of K. The system ρ_Λ is* (strongly) quasi-compatible with respect to X *if for all $\lambda_1, \lambda_2 \in \Lambda$ the representations ρ_{λ_1} and ρ_{λ_2} are (strongly) quasi-compatible with respect to X.*

(2) *The system ρ_Λ is called* (strongly) quasi-compatible *if for all $\lambda_1, \lambda_2 \in \Lambda$ the representations ρ_{λ_1} and ρ_{λ_2} are (strongly) quasi-compatible.*

Remark 3.14. — The first two points of Remark 3.10 apply mutatis mutandis to the above definition: compatibility is stable under birational equivalence, but for strong compatibility we do not know this.

Lemma 3.15. — *Let ρ_Λ be a system of Galois representations of K. Let L be a finitely generated extension of K, and fix an embedding $\bar{K} \hookrightarrow \bar{L}$ that extends $K \subset L$. Let ρ'_Λ denote the system of Galois representations of L obtained by restricting the system ρ_Λ to $\mathrm{Gal}(\bar{L}/L)$.*

(1) *The system ρ_Λ is quasi-compatible if and only if the system ρ'_Λ is quasi-compatible.*

(2) *If the system ρ'_Λ is strongly quasi-compatible, then the system ρ_Λ is strongly quasi-compatible.*

Proof. — Without loss of generality we may and do assume that $\Lambda = \{\lambda_1, \lambda_2\}$. Let X be a model of K. Let Y be an X-scheme that is a model of L. Let $x \in X^{\mathrm{cl}}$ be a closed point whose residue characteristic is different from the residue characteristic of λ_1 and λ_2.

For the remainder of the proof, we may and do assume that ρ_{λ_1} and ρ_{λ_2} are both unramified at x. Then ρ'_{λ_1} and ρ'_{λ_2} are both unramified at all points $y \in Y_x^{\mathrm{cl}}$. If $y \in Y_x^{\mathrm{cl}}$ is a closed point, and k denotes the residue extension degree $[\kappa(y) : \kappa(x)]$, then F_{y,ρ'_λ} and F_{x,ρ_λ}^k are conjugate for all $\lambda \in \Lambda$. This leads to the following conclusions:

(i) For every point $y \in Y_x^{\mathrm{cl}}$, if ρ'_{λ_1} and ρ'_{λ_2} are quasi-compatible at y, then ρ_{λ_1} and ρ_{λ_2} are quasi-compatible at x; and

(ii) if ρ_{λ_1} and ρ_{λ_2} are quasi-compatible at x, then ρ'_{λ_1} and ρ'_{λ_2} are quasi-compatible at all points $y \in Y_x^{\mathrm{cl}}$.

Together, these two conclusions complete the proof. □

Note that I cannot prove the converse implication in point (ii), for the following reason. Let $y \in Y^{\mathrm{cl}}$ be a closed point whose residue characteristic is different from the residue characteristic of λ_1 and λ_2. If ρ'_{λ_1} and ρ'_{λ_2} are unramified at y, but ρ_{λ_1} and ρ_{λ_2} are not unramified at the image x of y in X, then I do not see how to prove that ρ'_{λ_1} and ρ'_{λ_2} are quasi-compatible at y.

3.16. — Let ρ_Λ be a system of Galois representations over K. Let $E' \subset E$ be a subfield, and let Λ' be the set of places λ' of E' satisfying the following condition:

For all places λ of E with $\lambda | \lambda'$, we have $\lambda \in \Lambda$.

For each $\lambda' \in \Lambda'$, the representation $\rho_{\lambda'} = \bigoplus_{\lambda|\lambda'} \rho_\lambda$ is naturally a λ'-adic Galois representation of K. We thus obtain a system of Galois representations $\rho_{\Lambda'}$.

Lemma 3.17. — *We use the notation of the preceding paragraph. If ρ_Λ is a (strongly) quasi-compatible system of Galois representations, then $\rho_{\Lambda'}$ is a (strongly) quasi-compatible system of Galois representations.*

Proof. — To see this, we may assume that $\Lambda' = \{\lambda'_1, \lambda'_2\}$ and Λ is the set of all places λ of E that lie above a place $\lambda' \in \Lambda'$. Let X be a model of K. Let $x \in X^{\mathrm{cl}}$ be a closed point whose residue characteristic is different from the residue characteristic of λ'_1 and λ'_2. Assume that $\rho_{\lambda'_1}$ and $\rho_{\lambda'_2}$ are both unramified at x. Suppose that for all $\lambda_1, \lambda_2 \in \Lambda$, the representations ρ_{λ_1} and ρ_{λ_2} are quasi-compatible at x. If ρ_Λ is a strongly quasi-compatible system, then this is automatic. If ρ_Λ is merely a quasi-compatible system, then this is true for $x \in U^{\mathrm{cl}}$, for some non-empty open subset $U \subset X$.

Since we assumed that Λ is a finite set, there exists an integer $n \geqslant 1$ such that $P(t) = P_{x,\rho_\lambda,n}(t)$ does not depend on $\lambda \in \Lambda$. We may then compute

$$P_{x,\rho_{\lambda'},n}(t) = \prod_{\lambda|\lambda'} \mathrm{Nm}^{E_\lambda}_{E'_{\lambda'}} P_{x,\rho_\lambda,n}(t) = \mathrm{Nm}^{E}_{E'} P(t).$$

See [11] for a very general definition of the norm map Nm. We conclude that $P_{x,\rho_{\lambda'},n}(t)$ is a polynomial in $E'[t]$ that does not depend on $\lambda' \in \Lambda'$. □

3.18. — A counterpart to the preceding lemma is as follows. Let ρ_Λ be a system of Galois representations over K. Let $E \subset \tilde{E}$ be a finite extension, and let $\tilde{\Lambda}$ be the set of finite places $\tilde{\lambda}$ of $\tilde{E}$ that lie above the places $\lambda \in \Lambda$.

Let $\lambda \in \Lambda$ be a finite place of E. Write $\tilde{E}_\lambda$ for $\tilde{E} \otimes_E E_\lambda$ and recall that $\tilde{E}_\lambda = \prod_{\tilde{\lambda}|\lambda} \tilde{E}_{\tilde{\lambda}}$. Consider the representation $\tilde{\rho}_\lambda = \rho_\lambda \otimes_{E_\lambda} \tilde{E}_\lambda$, and observe that it naturally decomposes as $\tilde{\rho}_\lambda = \bigoplus_{\tilde{\lambda}|\lambda} \tilde{\rho}_{\tilde{\lambda}}$, where $\tilde{\rho}_{\tilde{\lambda}} = \rho_\lambda \otimes_{E_\lambda} \tilde{E}_{\tilde{\lambda}}$. We assemble these Galois representations $\tilde{\rho}_{\tilde{\lambda}}$ in a system of Galois representations that we denote with $\tilde{\rho}_{\tilde{\Lambda}}$ or $\rho_\Lambda \otimes_E \tilde{E}$.

Lemma 3.19. — *We use the notation of the preceding paragraph. If ρ_Λ is a (strongly) quasi-compatible system of Galois representations, then $\tilde{\rho}_{\tilde{\Lambda}}$ is a (strongly) quasi-compatible system of Galois representations.*

Proof. — Let X be a model of K and let $x \in X^{\mathrm{cl}}$ be a closed point. Let $\tilde{\lambda} \in \tilde{\Lambda}$ be a place that lies above $\lambda \in \Lambda$, and let $n \geqslant 1$ be an integer. Then $P_{x,\rho_\lambda,n} = P_{x,\tilde{\rho}_{\tilde{\lambda}},n}$. □

Lemma 3.20. — *Let ρ_Λ and ρ'_Λ be two systems of Galois representations over K. Then one may naturally form the following systems of Galois representations:*

(a) *the dual:* $\check{\rho}_\Lambda = (E, \Lambda, (\check{\rho})_{\lambda\in\Lambda})$;
(b) *the direct sum:* $\rho_\Lambda \oplus \rho'_\Lambda = (E, \Lambda, (\rho_\lambda \oplus \rho'_\lambda)_{\lambda\in\Lambda})$;
(c) *the tensor product:* $\rho_\Lambda \otimes \rho'_\Lambda = (E, \Lambda, (\rho_\lambda \otimes \rho'_\lambda)_{\lambda\in\Lambda})$;
(d) *the internal Hom:* $\underline{\mathrm{Hom}}(\rho_\Lambda, \rho'_\Lambda) = (E, \Lambda, (\underline{\mathrm{Hom}}(\rho_\lambda, \rho'_\lambda))_{\lambda\in\Lambda})$.

If ρ_Λ and ρ'_Λ are systems of Galois representations over K that are both quasi-compatible, then the constructions (a) *through* (d) *form a quasi-compatible system of Galois representations.*

Proof. — It follows immediately from the following lemma. □

Lemma 3.21. — *Let V and V' be finite-dimensional vector spaces over a field E, of dimension n respectively n'. Let g and g' be endomorphisms of V and V' respectively.*

(1) *There exist polynomials with integral coeffecients that depend only on n and n' that express the coefficients of the characteristic polynomial* $\mathrm{c.p.}(g \oplus g' | V \oplus V')$ *in terms of the coefficients of* $\mathrm{c.p.}(g|V)$ *and* $\mathrm{c.p.}(g'|V')$.

(2) *There exist polynomials with integral coeffecients that depend only on n and n' that express the coefficients of* $\mathrm{c.p.}(g \otimes g' | V \otimes V')$ *in terms of the coefficients of* $\mathrm{c.p.}(g|V)$ *and* $\mathrm{c.p.}(g'|V')$.

Proof. — Write f for $\mathrm{c.p.}(g|V)$ and f' for $\mathrm{c.p.}(g'|V')$. For point 1, note that $\mathrm{c.p.}(g \oplus g' | V \oplus V) = f \cdot f'$. For point 2, put $f = \prod_{i=1}^{n}(x - \alpha_i)$ and $f' = \prod_{j=1}^{n'}(x - \alpha'_j)$ in $\bar{E}[x]$, and note that

$$\begin{aligned} \mathrm{c.p.}(g \otimes g' | V \otimes V') &= \prod_{i,j}(x - \alpha_i \alpha'_j) \\ &= \prod_{i=1}^{n} \alpha_i^{n'} \prod_{j=1}^{n'} (x/\alpha_i - \alpha'_j) \\ &= \mathrm{res}_y(f(y), f'(x/y) \cdot y^{n'}), \end{aligned}$$

where $\mathrm{res}_y(\,\cdot\,,\cdot\,)$ denotes the resultant of the polynomials in y. ☐

4. Examples of quasi-compatible systems

Readme. — In this section we show that abelian varieties and abelian CM motives give rise to strongly quasi-compatible systems of Galois representations, in respectively Theorem 4.1 and Theorem 4.9. These results are known over number fields. We recall their proofs and generalise the results to finitely generated fields.

Let M be a motive over a finitely generated field $K \subset \mathbb{C}$. Let $E \subset \mathrm{End}(M)$ be a number field, and let Λ be the set of finite places of E. Let ℓ be a prime number. Then $\mathrm{H}_\ell(M)$ is a module over $E_\ell = E \otimes \mathbb{Q}_\ell \cong \prod_{\lambda|\ell} E_\lambda$. Correspondingly, the Galois representation $\mathrm{H}_\ell(M)$ decomposes as $\mathrm{H}_\ell(M) \cong \bigoplus_{\lambda|\ell} \mathrm{H}_\lambda(M)$, with $\mathrm{H}_\lambda(M) = \mathrm{H}_\ell(M) \otimes_{E_\ell} E_\lambda$. The λ-adic representations $\mathrm{H}_\lambda(M)$, with $\lambda \in \Lambda$, form a system of Galois representations that we denote with $\mathrm{H}_\Lambda(M)$. It is expected that $\mathrm{H}_\Lambda(M)$ is a quasi-compatible system of Galois representations, and even a compatible system in the sense of Serre. Indeed, this assertion is implied by the Tate conjecture.

The following theorem is a slightly weaker version of a result proven by Shimura in [30, §11.10.1]. We present the proof by Shimura in modern notation, and with a bit more detail. The proof is given in Section 4.7, and relies on Proposition 4.2, which is Proposition 11.9 of [30]. For similar discussions, see [5], [24, §II], [20], and [21].

THEOREM 4.1 ([30, §11.10.1]). — *Let A be an abelian variety over a finitely generated field K, and let $E \subset \mathrm{End}^0(A)$ be a number field. Let Λ be the set of finite places of E whose residue characteristic is different from $\mathrm{char}(K)$. Then $\mathrm{H}^1_\Lambda(A)$ is a strongly quasi-compatible system of Galois representations.*

Proof. — See Section 4.7. □

PROPOSITION 4.2 ([30, Prop. 11.9]). — *Let E be a number field. Let $\mathscr{L}$ be a set of prime numbers, and let Λ be the set of finite places of E that lie above a prime number in $\mathscr{L}$. For every prime number $\ell \in \mathscr{L}$, let H_ℓ be a finitely generated E_ℓ-module. Recall that $E_\ell = E \otimes \mathbb{Q}_\ell \cong \prod_{\lambda|\ell} E_\lambda$. Write H_λ for $\mathrm{H}_\ell \otimes_{E_\ell} E_\lambda$, so that $\mathrm{H}_\ell \cong \bigoplus_{\lambda|\ell} \mathrm{H}_\lambda$.*

Let R be a finite-dimensional commutative semisimple E-algebra; and suppose that, for every prime number $\ell \in \mathscr{L}$, we are given E-algebra homomorphisms $R \to \mathrm{End}_{E_\ell}(\mathrm{H}_\ell)$. Assume that for every $r \in R$ the characteristic polynomial $\mathrm{c.p.}_{\mathbb{Q}_\ell}(r|\mathrm{H}_\ell)$ has coefficients in $\mathbb{Q}$ and is independent of $\ell \in \mathscr{L}$. Under these assumptions, for every $r \in R$ the characteristic polynomial $\mathrm{c.p.}_{E_\lambda}(r|\mathrm{H}_\lambda)$ has coefficients in E and is independent of $\lambda \in \Lambda$.

Proof. — The assumptions on R imply that R is a finite product of finite field extensions K_i/E. Let ϵ_i be the idempotent of R that is 1 on K_i and 0 elsewhere. For $r \in R$, observe that

$$\mathrm{c.p.}_{\mathbb{Q}_\ell}(r|\mathrm{H}_\ell) = \prod_i \mathrm{c.p.}_{\mathbb{Q}_\ell}(\epsilon_i r|\epsilon_i \mathrm{H}_\ell), \quad \mathrm{c.p.}_{E_\lambda}(r|\mathrm{H}_\lambda) = \prod_i \mathrm{c.p.}_{E_\lambda}(\epsilon_i r|\epsilon_i \mathrm{H}_\lambda).$$

We conclude that we only need to prove the lemma for $R = K_i$, and $\mathrm{H}_\ell = \epsilon_i \mathrm{H}_\ell$, i.e., that we can reduce to the case where R is a field.

Suppose R is a finite field extension of E, and choose an element $\pi \in R$ that generates R as a field. Let $f^\pi_{\mathbb{Q}}$ be the minimum polynomial of π over $\mathbb{Q}$. Observe that $\mathrm{c.p.}_{\mathbb{Q}_\ell}(\pi|\mathrm{H}_\ell)$ is a divisor of a power of $f^\pi_{\mathbb{Q}}$ in $\mathbb{Q}_\ell[t]$. Since both are elements of $\mathbb{Q}[t]$ and $f^\pi_{\mathbb{Q}}$ is irreducible, we conclude that $\mathrm{c.p.}_{\mathbb{Q}_\ell}(\pi|\mathrm{H}_\ell)$ is equal to $(f^\pi_{\mathbb{Q}})^d$, for some positive integer d. Since π is semisimple, it follows that $\mathrm{H}_\ell \cong \mathbb{Q}_\ell[\pi]^d$ as $\mathbb{Q}_\ell[\pi]$-modules. Let H be the R-vector space R^d. By construction $\mathrm{H}_\ell \cong \mathrm{H} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ as $(R \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)$-modules. Because $R \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong R \otimes_E E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$, this implies that $\mathrm{H}_\lambda \cong \mathrm{H} \otimes_E E_\lambda$ as $(R \otimes_E E_\lambda)$-modules. For all $r \in R$, we have $\mathrm{c.p.}_{E_\lambda}(r|\mathrm{H}_\lambda) = \mathrm{c.p.}_E(r|\mathrm{H})$, and therefore $\mathrm{c.p.}_{E_\lambda}(r|\mathrm{H}_\lambda)$ has coefficients in E and is independent of $\lambda \in \Lambda$. □

Corollary 4.3 ([24, Thm. II.2.1.1]). — *Let A be an abelian variety over a finitely generated field K, and fix a prime number $\ell \neq \mathrm{char}(K)$. Let $E \subset \mathrm{End}^0(A)$ be a number field. Then $\mathrm{H}^1_\ell(A)$ is a free E_ℓ-module.*

Lemma 4.4. — *Let A be an abelian variety over a finite field κ of characteristic p. Note that $\mathrm{Spec}(\kappa)$ is the only model of κ, and let x denote the single point of $\mathrm{Spec}(\kappa)$. Let E be a number field inside $\mathrm{End}^0(A)$, and let Λ be the set of finite places of E whose residue characteristic is different from p. Then $P_{x,\rho_\lambda,1}(t)$ has coefficients in E and is independent of $\lambda \in \Lambda$. In particular, $\mathrm{H}^1_\Lambda(A)$ is a quasi-compatible system of Galois representations.*

Proof. — Let $E[F_A]$ be the subalgebra of $\mathrm{End}^0(A)$ generated by E and F_A, where F_A is the Frobenius automorphism of A over κ. Note that $E[F_A]$ may naturally be viewed as the subalgebra of $\mathrm{End}(\mathrm{H}^1_\ell(A))$ generated by E and F_{x,ρ_ℓ}. This algebra is semisimple by work of Weil. For every $r \in E[F_x]$ the characteristic polynomial $\mathrm{c.p.}(r|\mathrm{H}^1_\ell(A))$ has coefficients in $\mathbb{Q}$, and is independent of ℓ, by Theorem 2.2 of [13]. It follows from Proposition 4.2 that $P_{x,\rho_\lambda,1}(t)$ has coefficients in E and is independent of $\lambda \in \Lambda$. $\square$

Lemma 4.5. — *Let $T \hookrightarrow G \xrightarrow{\alpha} A$ be a semiabelian variety over some field K, and let E be a number field inside $\mathrm{End}^0(G)$. Then E naturally maps to $\mathrm{End}^0(A)$ and $\mathrm{End}^0(T)$.*

Proof. — Let f be an endomorphism of G. Consider the composition $g\colon T \hookrightarrow G \xrightarrow{f} G \twoheadrightarrow A$. The image of g is affine, since it is a quotient of T, and it is projective, since it is a closed subgroup of A. It is also connected and reduced, and therefore factors via $0 \in A(K)$. We conclude that $f(T) \subset T$, which proves the result. $\square$

Lemma 4.6. — *Let X be the spectrum of a discrete valuation ring. Let η (resp. x) denote the generic (resp. special) point of X. Let A be a semistable abelian variety over η. Let E be a number field inside $\mathrm{End}^0(A)$. Let λ be a finite place of E such that the residue characteristics of λ and x are different. Then A has good reduction at x if and only if $\mathrm{H}^1_\lambda(A)$ is unramified at x.*

Proof. — This is a slight generalisation of the criterion of Néron–Ogg–Shafarevic [29, Thm. 1]. It is clear that if A has good reduction at x, then $\mathrm{H}^1_\lambda(A)$ is unramified at x. We focus on the converse implication. Let ℓ be the residue characteristic of λ. By [29, Thm. 1] it suffices to show that $\mathrm{H}^1_\ell(A)$ is unramified at x. Let $\mathrm{H}^1_\ell(A)^I$ denote the subspace of $\mathrm{H}^1_\ell(A)$ that is invariant under inertia. Let G be the Néron model of A over X. Recall that $\mathrm{H}^1_\ell(A)^I \cong \mathrm{H}^1_\ell(G_x)$, by [29, Lem. 2]. It follows from the definition of the Néron model that E embeds into $\mathrm{End}(G) \otimes \mathbb{Q}$. Hence E embeds into $\mathrm{End}(G_x) \otimes \mathbb{Q}$, and we claim that $\mathrm{H}^1_\ell(A)^I \cong \mathrm{H}^1_\ell(G_x)$ is a free E_ℓ-module. Recall that with E_ℓ we mean $E \otimes \mathbb{Q}_\ell \cong \prod_{\lambda | \ell} E_\lambda$. Before proving the claim, let us see why it is sufficient for proving the lemma. By Corollary 4.3 we know that $\mathrm{H}^1_\ell(A)$ is a free E_ℓ-module. Thus $\mathrm{H}^1_\ell(A)/\mathrm{H}^1_\ell(A)^I$ is a free E_ℓ-module. We conclude that $\mathrm{H}^1_\lambda(A)$ is unramified at x, if and only if $\mathrm{H}^1_\ell(A)$ is unramified at x.

We will now prove the claim that $\mathrm{H}^1_\ell(A)^I \cong \mathrm{H}^1_\ell(G_x)$ is a free E_ℓ-module. Since A is semistable, the special fibre G_x is a semiabelian variety $T \hookrightarrow G_x \to B$. The semiabelian variety G_x is a special case of a 1-motive, and thus we have a short exact sequence

$$0 \to \mathrm{H}^1_\ell(B) \to \mathrm{H}^1_\ell(G_x) \to \mathrm{H}^1_\ell(T) \to 0.$$

We also have $\mathrm{H}^1_\ell(T) \cong \mathrm{Hom}(T, \mathbb{G}_{\mathrm{m}}) \otimes \mathbb{Q}_\ell(-1)$, see [7, variante 10.1.10]. By Lemma 4.5, the action of E on G_x gives an action of E on both T and B. Since $\mathrm{Hom}(T, \mathbb{G}_{\mathrm{m}}) \otimes \mathbb{Q}$ is a free E-module, we know that $\mathrm{H}^1_\ell(T)$ is a free E_ℓ-module. By Corollary 4.3 we also know that $\mathrm{H}^1_\ell(B)$ is free as E_ℓ-module. Therefore, $\mathrm{H}^1_\ell(G_x) \cong \mathrm{H}^1_\ell(A)^I$ is free as E_ℓ-module. □

4.7. Proof of Theorem 4.1. — Let X be a model of K, and let $x \in X^{\mathrm{cl}}$ be a closed point. Let $\Lambda^{(x)}$ be the set of places $\lambda \in \Lambda$ that have a residue characteristic ℓ that is different from the residue characteristic of x. If there is a $\lambda \in \Lambda^{(x)}$ such that $\mathrm{H}^1_\lambda(A)$ is unramified at x, then A has good reduction at x, by Lemma 4.6. Assume that A has good reduction at x. We denote this reduction with A_x. It follows from Lemma 4.4 that $P_{x,\rho_\lambda,1}(t)$ has coefficients in E and is independent of $\lambda \in \Lambda^{(x)}$. □

Proposition 4.8. — *Let M be an abelian motive of weight n over a finitely generated field $K \subset \mathbb{C}$. Let $E \subset \mathrm{End}(M)$ be a* cm *field such that $\dim_E(M) = 1$, and let Λ be the set of finite places of E. Then the system $\mathrm{H}_\Lambda(M)$ is a strongly quasi-compatible system of Galois representations.*

Proof. — Let X be a model of K, and let $x \in X^{\mathrm{cl}}$ be a closed point. Let $\Lambda^{(x)}$ be the set of finite places of E whose residue characteristic is different from the residue characteristic of x. Fix $\lambda \in \Lambda^{(x)}$. Let m be the level of M, that is $\max\{p - q \mid \mathrm{H}_{\mathrm{B}}(M)^{p,q} \neq 0\}$. We apply induction to m, and use half-twists as described in Section 2.10.

Suppose that $m = 0$; in particular n is even. By Lemma 3.15 we may replace K by a finite field extension and therefore we may assume that M is a Tate motive: $M \cong \mathbb{1}(-\frac{n}{2}) \otimes E$. The Frobenius element F_x acts on $\mathrm{H}_\lambda(M)$ as multiplication by $\#\kappa(x)^{n/2}$. Thus $\mathrm{H}_\Lambda(M)$ is a strongly quasi-compatible system.

Suppose that $m \geqslant 1$. Let $T \subset \Sigma(E)$ be the set of embeddings through which E acts on $\bigoplus_{p \geqslant \lceil n/2 \rceil} \mathrm{H}_{\mathrm{B}}(M)^{p,q}$. Since $\dim_E(M) = 1$ we know that $T \cap T^\dagger = \varnothing$. It follows from the discussion in Section 2.10 and Section 2.11 that there exists a finitely generated extension L/K, an abelian variety A over L, and a motive N over L such that $M_L \cong \underline{\mathrm{Hom}}_E(\mathrm{H}^1(A), N)$, and such that the level of N is $m-1$, and $\dim_E(N) = 1$. By Theorem 4.1 we know that $\mathrm{H}^1_\Lambda(A)$ is a strongly quasi-compatible system, and by induction we may assume that $\mathrm{H}^1_\Lambda(N)$ is a strongly quasi-compatible system. It follows from Lemma 3.20 that $\mathrm{H}_\Lambda(M_L) \cong \underline{\mathrm{Hom}}_E(\mathrm{H}^1_\Lambda(A), \mathrm{H}^1_\Lambda(N))$ is a quasi-compatible system of Galois representations over L, and we will now argue that it is even a strongly quasi-compatible system.

We may assume that A is semistable over L, possibly after replacing L with a finite field extension. Since A is a semistable CM abelian variety, we know that A has good reduction everywhere, and thus $\mathrm{H}^1_\lambda(A)$ is unramified at x. Hence $\mathrm{H}_\lambda(M_L)$ is unramified at x if and only if $\mathrm{H}^1_\lambda(N)$ is unramified at x. Finally, Lemma 3.15 shows that $\mathrm{H}_\Lambda(M)$ is also a strongly quasi-compatible system of Galois representations over K. □

Theorem 4.9 (See also [25, Cor. I.6.5.7]). — *Let M be an abelian* CM *motive over a finitely generated field $K \subset \mathbb{C}$. Let E be a subfield of* $\mathrm{End}(M)$, *and let Λ be the set of finite places of E. Then the system* $\mathrm{H}_\Lambda(M)$ *is a strongly quasi-compatible system of Galois representations.*

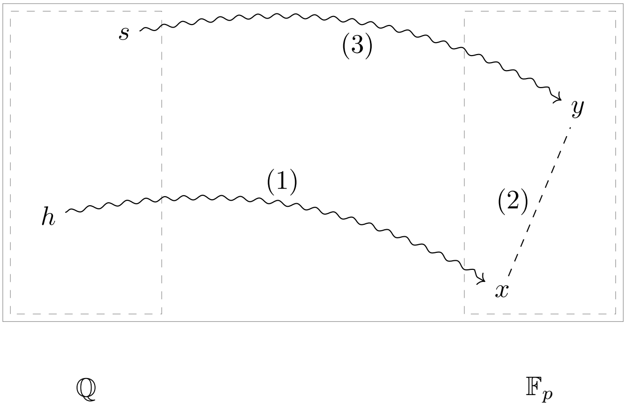
Proof. — By Lemma 3.15 we may replace K by a finitely generated extension and thus we may and do assume that M decomposes into a sum of geometrically isotypical components $M = M_1 \oplus \ldots \oplus M_r$. Observe that $E \subset \mathrm{End}(M_i)$ for $i = 1, \ldots, r$. By Lemma 3.20 we see that it suffices to show that $\mathrm{H}_\Lambda(M_i)$ is a strongly quasi-compatible system for $i = 1, \ldots, r$. Therefore we may assume that $M \cong (M')^{\oplus k}$, where M' is a geometrically irreducible CM-motive. If M' is a Tate motive, then the result is trivially true. Hence, let us assume that $E' = \mathrm{End}(M')$ is a CM field. Notice that $\dim_{E'}(M') = 1$. By assumption E acts on $(M')^{\oplus k}$, and thus we get a specific embedding $E \subset \mathrm{Mat}_k(E')$. We may find a field $\tilde{E} \subset \mathrm{Mat}_k(E')$ that contains E, and such that $[\tilde{E} : E'] = k$. Then $M = M' \otimes_{E'} \tilde{E}$ as motives with E-action. Let $\tilde{\Lambda}$ be the set of finite places of $\tilde{E}$. By Proposition 4.8, the system $\mathrm{H}_{\Lambda'}(M')$ is a strongly quasi-compatible (quasi-E'-rational) system of Galois representations, and by Lemma 3.19 we find that $\mathrm{H}_{\tilde{\Lambda}}(M) = \mathrm{H}_{\Lambda'}(M') \otimes_{E'} \tilde{E}$ is a strongly quasi-compatible (quasi-$\tilde{E}$-rational) system. We conclude that $\mathrm{H}_\Lambda(M)$ is a strongly quasi-compatible (quasi-E-rational) system of Galois representations by Lemma 3.17. □

5. Deformations of abelian motives

README. — In this section we prove the main result of this article, which is the following theorem.

THEOREM 5.1. — *Let M be an abelian motive over a finitely generated field $K \subset \mathbb{C}$. Let E be a subfield of* $\mathrm{End}(M)$*, and let Λ be the set of finite places of E. Then the system $\mathrm{H}_\Lambda(M)$ is a quasi-compatible system of Galois representations.*

5.2. — The proof of this theorem relies heavily on the fact that an abelian motive can be placed naturally as fibre in a family of abelian motives over a certain Shimura variety of Hodge type. We summarise this result in Lemma 5.4. Its proof uses the rather technical Construction 5.3. Once we have the family of motives in place, the rest of the section is devoted to the proof of the main theorem. The following picture aims to capture the intuition of the proof.



The picture is a cartoon of an integral model of a Shimura variety, and the motive M fits into a family $\mathcal{M}$ over the generic fibre, such that $M \cong \mathcal{M}_h$. The Zariski closure X of the point h in this integral model is a model for the field K. We give a rough sketch of the strategy for the proof that explains the three steps in the picture:

(1) We have a system of Galois representations $\mathrm{H}_\Lambda(\mathcal{M}_h)$ and we want to show that it is quasi-compatible at $x \in X^{\mathrm{cl}}$;
(2) we replace x by an isogenous point y in the sense of Kisin [14]; in such a way that
(3) we may assume that y lifts to a special point s.

The upshot is that we have to show that the system $\mathrm{H}_\Lambda(\mathcal{M}_s)$ is quasi-compatible at y. We will see that this follows from Theorem 4.9.

5.3. Construction. — Fix an integer $g \in \mathbb{Z}_{\geqslant 0}$. Let $(G, X) \hookrightarrow (\mathrm{GSp}_{2g}, \mathfrak{H}^{\pm})$ be a morphism of Shimura data, and let $h \in X$ be a morphism $\mathbb{S} \to G_{\mathbb{R}}$. In this paragraph we will construct an abelian scheme over an integral model of the Shimura variety $\mathrm{Sh}_{\mathcal{K}}(G, X)$, where $\mathcal{K}$ is a certain compact open subgroup of $G(\mathbb{A}_{\mathrm{f}})$. Along the way, we make two choices, labeled *(i)* and *(ii)* so that we may refer to them later on.

Let $\mathcal{G}$ denote the Zariski closure of G in $\mathrm{GSp}_{2g}/\mathbb{Z}$. Note that $G_{\mathbb{Z}_p}$ is reductive for almost all primes p. For each integer $n \geqslant 3$, let $\mathcal{K}_{(n)}$ (resp. $\mathcal{K}'_{(n)}$) denote the principal congruence subgroup of $\mathcal{G}(\hat{\mathbb{Z}})$ (resp. $\mathrm{GSp}_{2g}(\hat{\mathbb{Z}})$) consisting of elements congruent to 1 modulo n. Observe that we have $\mathcal{K}_{(n)} = \mathcal{K}'_{(n)} \cap G(\mathbb{A}_{\mathrm{f}})$. This gives a morphism of Shimura varieties

$$\mathrm{Sh}_{\mathcal{K}_{(n)}}(G, X) \to \mathrm{Sh}_{\mathcal{K}'_{(n)}}(\mathrm{GSp}_{2g}, \mathfrak{H}^{\pm}).$$

By applying Lemma 3.3 of [19] with $p = 6$ we can choose n in such a way that it is coprime with p and such that this morphism of Shimura varieties is a closed immersion. In [19], Noot assumes that p is prime, but he does not use this fact in his proof.

(i). — Fix such an integer n, and write $\mathcal{K}$ for $\mathcal{K}_{(n)}$. Since $n > 3$, the subgroup $\mathcal{K}'_{(n)}$ is neat, hence $\mathcal{K}$ is neat, and therefore $\mathrm{Sh}_{\mathcal{K}}(G, X)$ is smooth. As is common, we denote with $\mathscr{A}_{g,1,n}/\mathbb{Z}[1/n]$ the moduli space of principally polarised abelian varieties of dimension g with a level-n structure. Recall that $\mathscr{A}_{g,1,n}$ is smooth over $\mathbb{Z}[1/n]$. Let $F' \subset \mathbb{C}$ be the reflex field of (G, X). We have a closed immersion of Shimura varieties

$$\mathrm{Sh}_{\mathcal{K}}(G, X) \hookrightarrow \mathrm{Sh}_{\mathcal{K}'_{(n)}}(\mathrm{GSp}_{2g}, \mathfrak{H}^{\pm}) \cong \mathscr{A}_{g,1,n,\mathbb{C}}$$

that is defined over F'. Let $\mathscr{S}_{\mathcal{K}}(G, X)$ be the Zariski closure of $\mathrm{Sh}_{\mathcal{K}}(G, X)$ in $\mathscr{A}_{g,1,n}$ over $\mathcal{O}_{F'}[1/n]$. Recall that $\mathrm{Sh}_{\mathcal{K}}(G, X)$ is smooth. Hence there exists an integer multiple N_0 of n such that $\mathscr{S}_{\mathcal{K}}(G, X)_{\mathcal{O}_{F'}[1/N_0]}$ is smooth.

For a prime number p, let $\mathcal{K}_p$ be $\mathcal{K} \cap G(\mathbb{Q}_p)$, and let $\mathcal{K}^p$ be $\mathcal{K} \cap G(\mathbb{A}_{\mathrm{f}}^p)$. Since $\mathcal{K} = \mathcal{K}_{(n)}$ is a principal congruence subgroup we have $\mathcal{K} = \mathcal{K}_p\mathcal{K}^p$. The group $\mathcal{K}_p$ is called *hyperspecial* if there is a reductive model $\mathcal{G}'/\mathbb{Z}_{(p)}$ of $G/\mathbb{Q}$ such that $\mathcal{K}_p = \mathcal{G}'(\mathbb{Z}_p)$. By Section 3.2 of [31] such a hyperspecial subgroup is a maximal compact open subgroup of $G(\mathbb{Q}_p)$. Observe that $\mathcal{K}_p = \mathcal{G}(\mathbb{Z}_p)$ and recall that $\mathcal{G}_{\mathbb{Z}_{(p)}}/\mathbb{Z}_{(p)}$ is reductive for almost all prime numbers p. Thus the set of primes for which $\mathcal{K}_p$ is not hyperspecial is finite. Write N_1 for the product of those prime numbers, and let N be the integer $N_0 \cdot N_1$. By construction $\mathcal{K} = \mathcal{K}_p\mathcal{K}^p$ and $\mathcal{K}_p$ is hyperspecial for almost all p, so that we may apply results by Kisin [14] in Sections 5.6 and 5.7.

The point $h \in X$ is a complex point of $\mathscr{S}_{\mathcal{K}}(G, X)$. After replacing F' by a finite extension $F \subset \mathbb{C}$ we may assume that the generic fibre of the irreducible component $\mathscr{S} \subset \mathscr{S}_{\mathcal{K}}(G, X)_{\mathcal{O}_F[1/N]}$ that contains the point h is geometrically irreducible.

(ii). — Choose such a field $F \subset \mathbb{C}$. In the following paragraphs we will consider the closed immersion of Shimura varieties $\mathscr{S} \hookrightarrow \mathscr{A}_{g,1,n}$ as a morphism of schemes over $\mathcal{O}_F[1/N]$.

Lemma 5.4. — *Let M be an abelian motive over a finitely generated field $K \subset \mathbb{C}$. There exist*

- *finitely generated fields $F \subset L \subset \mathbb{C}$, with $K \subset L$;*
- *a smooth irreducible component $\mathscr{S}$ of an integral model of a Shimura variety over F, such that the generic fibre $\mathscr{S}_F$ is geometrically irreducible;*
- *an abelian scheme $f\colon \mathcal{A} \to \mathscr{S}$;*
- *an idempotent motivated cycle γ on $\mathcal{A}/\mathscr{S}$;*
- *a family of abelian motives $\mathcal{M}/\mathscr{S}_L$, such that $\mathcal{M}/\mathscr{S}_{\mathbb{C}} \cong \mathrm{Im}(\gamma)(m)$, for some $m \in \mathbb{Z}$;*
- *an isomorphism $M_L \cong \mathcal{M}_h$, for some point $h \in \mathscr{S}(L)$.*

Proof. — Since M is an abelian motive, there exist a principally polarised complex abelian variety A, a motivated projector γ_0 on A, and an integer m such that $M_{\mathbb{C}} = (A, \gamma_0, m)$. Write V for $\mathrm{H_B}(M)$. Observe that $\mathrm{G_B}(V)$ is naturally a quotient of $\mathrm{G_B}(A)$. Write G for $\mathrm{G_B}(A)$, and let $h\colon \mathbb{S} \to G_{\mathbb{R}}$ be the map that defines the Hodge structure on $\mathrm{H_B}(A)$. Let X be the $G(\mathbb{R})$-orbit of h in $\mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$. Let g be $\dim(A)$. The pair (G, X) is a Shimura datum, and by construction we get a morphism of Shimura data $(G, X) \hookrightarrow (\mathrm{GSp}_{2g}, \mathfrak{H}^{\pm})$. Now run Construction 5.3, choosing

(i) an integer $n > 3$;
(ii) a number field $F \subset \mathbb{C}$;

and producing a closed immersion of Shimura varieties $\mathscr{S} \hookrightarrow \mathscr{A}_{g,1,n}$ over $\mathcal{O}_F[1/N]$, for some integer N.

It follows from Construction 5.3, that the Hodge structure V gives rise to a variation of Hodge structure $\mathcal{V}$ on $\mathscr{S}_{\mathbb{C}}$ such that the fibre of $\mathcal{V}$ above h is V, and such that h is a Hodge generic point of $\mathscr{S}_{\mathbb{C}}$ with respect to the variation $\mathcal{V}$. The embedding of Shimura varieties $\mathscr{S} \hookrightarrow \mathscr{A}_{g,1,n}$ gives a natural abelian scheme $f\colon \mathcal{A} \to \mathscr{S}$. The point h is also a Hodge generic point with respect to f because f is induced by an embedding of Shimura varieties. Observe that $A = \mathcal{A}_h$.

The motivated projector γ_0 acts on $\mathrm{H}^*_{\mathrm{B}}(A)$, and $V = \mathrm{Im}(\gamma_0)(m)$. Since h is a Hodge generic point of $\mathscr{S}_{\mathbb{C}}$, the projector γ_0 spreads out to a projector γ on the variation of Hodge structure $\bigoplus_i \mathrm{R}^i f_{\mathbb{C},*}\mathbb{Q}$, and $\mathcal{V}_{\mathscr{S}_{\mathbb{C}}} \cong \mathrm{Im}(\gamma)(m)$.

By Theorem 2.6, the projector γ is motivated, and thus we obtain a family of abelian motives $\mathcal{M}/\mathscr{S}_{\mathbb{C}}$ whose Betti realisation is $\mathcal{V}_{\mathscr{S}_{\mathbb{C}}}$. In particular $\mathcal{M}_h \cong M_{\mathbb{C}}$. Finally, the point h, the projector γ, and the family of motives $\mathcal{M}$ are all defined over a finitely generated subfield $L \subset \mathbb{C}$ that contains the fields F and K. $\square$

5.5. — We will now start the proof of Theorem 5.1. We retain the assumptions and notation of Construction 5.3 and Lemma 5.4. Write S for $\mathscr{S}_L$. Recall that $\mathcal{V}$ is the m-th Tate twist of the image of γ in $\bigoplus_i \mathrm{R}^i f_{\mathbb{C},*}\mathbb{Q}$; it is the variation of Hodge structure that is the Betti realisation of $\mathcal{M}/S(\mathbb{C})$. Because h is a Hodge generic point, the field E is a subfield of $\mathrm{End}(\mathcal{V})$. Let $(e_i)_i$ be a basis of E as $\mathbb{Q}$-vector space.

Let ℓ be a prime number. By Theorem 2.6, the projector γ on $\bigoplus_i \mathrm{R}^i f_{\mathbb{C},*}\mathbb{Q}$ induces a projector on $\bigoplus_i \mathrm{R}^i f_{S,*}\mathbb{Q}_\ell$ over S that spreads out to a projector γ_ℓ on $\bigoplus_i \mathrm{R}^i f_*\mathbb{Q}_\ell$ over the entirety of $\mathscr{S}$. Let $\mathcal{V}_\ell$ denote $\mathrm{Im}(\gamma_\ell)(m)$. Note that $\mathcal{V}_{\ell,S}$ is the ℓ-adic realisation of $\mathcal{M}/S$.

By Theorem 2.6 we see that $E_\ell = E \otimes \mathbb{Q}_\ell$ is a subalgebra of $\mathrm{End}(\mathcal{V}_{\ell,S})$. Since S is the generic fibre of $\mathscr{S}$, we see that $E_\ell \subset \mathrm{End}(\mathcal{V}_\ell)$. This has two implications, namely

(i) we obtain classes $e_{i,\ell} \in \mathrm{End}(\mathcal{V}_\ell)$ that form a $\mathbb{Q}_\ell$-basis for E_ℓ; and
(ii) because $E_\ell = E \otimes \mathbb{Q}_\ell \cong \prod_{\lambda|\ell} E_\lambda$, the lisse ℓ-adic sheaf $\mathcal{V}_\ell$ decomposes as a sum $\bigoplus_{\lambda|\ell} \mathcal{V}_\lambda$ of lisse λ-adic sheaves, where $\mathcal{V}_\lambda = \mathcal{V}_\ell \otimes_{E_\ell} E_\lambda$.

5.6. — Let p be a prime number that does not divide N, so that $\mathcal{K}$ decomposes as $\mathcal{K}_p\mathcal{K}^p$, and $\mathcal{K}_p$ is hyperspecial. Let $\mathbb{F}_q/\mathbb{F}_p$ be a finite field. Let $x \in \mathscr{S}(\mathbb{F}_q)$ be a point. Kisin defines the *isogeny class of x* in [14, §1.4.14]. It is a subset of $\mathscr{S}(\bar{\mathbb{F}}_q)$.

Let y be a point in $\mathscr{S}(\bar{\mathbb{F}}_q)$ that is isogenous to x. Proposition 1.4.15 of [14] implies that there is an isomorphism of Galois representations $\mathrm{H}^*_\ell(\mathcal{A}_x) \cong \mathrm{H}^*_\ell(\mathcal{A}_y)$ such that $\gamma_{\ell,x} \in \mathrm{End}(\mathrm{H}^*_\ell(\mathcal{A}_x))$ is mapped to $\gamma_{\ell,y} \in \mathrm{End}(\mathrm{H}^*_\ell(\mathcal{A}_y))$, and such that $e_{i,\ell,x}$ is mapped to $e_{i,\ell,y}$. This implies that $\mathcal{V}_{\ell,x} \cong \mathcal{V}_{\ell,y}$ as $E_\ell[\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)]$-modules. We conclude that $\mathcal{V}_{\lambda,x} \cong \mathcal{V}_{\lambda,y}$ as λ-adic Galois representations.

5.7. — We need one more key result by Kisin [14]. Theorem 2.2.3 of [14] states that for every point $x \in \mathscr{S}(\bar{\mathbb{F}}_q)$, there is a point $y \in \mathscr{S}(\bar{\mathbb{F}}_q)$ that is isogenous to x and such that y is the reduction of a special point in S.

5.8. — We are now set for the attack on Theorem 5.1. Let λ_1 and λ_2 be two finite places of E. Let ℓ_1 and ℓ_2 be the residue characteristics of λ_1 respectively λ_2. Let X be the Zariski closure of h in $\mathscr{S}$. Note that X is a model for the residue field of h. Let $U \subset X$ be the Zariski open locus of points $x \in X$ such that the residue characteristic p of x does not divide $N \cdot \ell_1 \cdot \ell_2$. To prove Theorem 5.1, it suffices to show that $\mathrm{H}_{\lambda_1}(M)$ and $\mathrm{H}_{\lambda_2}(M)$ are quasi-compatible at all points $x \in U^{\mathrm{cl}}$. Fix a point $x \in U^{\mathrm{cl}}$. Observe that by construction the representations $\mathrm{H}_{\lambda_1}(M)$ and $\mathrm{H}_{\lambda_2}(M)$ are unramified at x. Let $\mathbb{F}_q$ be the residue field of x. We want to show that $\mathcal{V}_{\lambda_1,x}$ and $\mathcal{V}_{\lambda_2,x}$ are quasi-compatible. This means that we have to show that the characteristic polynomials of the Frobenius automorphisms of $\mathcal{V}_{\lambda_1,x}$ and $\mathcal{V}_{\lambda_2,x}$ are equal, possibly after replacing the Frobenius automorphism by some power. Equivalently, we may pass to a finite extension of $\mathbb{F}_q$. This is what we will now do.

As mentioned in Section 5.7, Theorem 2.2.3 of [14] shows that there exists a point $y \in \mathscr{S}(\bar{\mathbb{F}}_q)$ such that y is isogenous to x and such that y is the reduction of a special point $s \in S$. The point y is defined over a finite extension of $\mathbb{F}_q$. As explained in the preceding paragraph, we may replace $\mathbb{F}_q$ with a finite extension. Thus we may and do assume that y is $\mathbb{F}_q$-rational. By our remarks in Section 5.6 it suffices to show that $\mathcal{V}_{\lambda_1,y}$ and $\mathcal{V}_{\lambda_2,y}$ are quasi-compatible. In other words, it suffices to show that $\mathrm{H}_{\lambda_1}(\mathcal{M}_s)$ and $\mathrm{H}_{\lambda_2}(\mathcal{M}_s)$ are quasi-compatible at y. Recall that s is a special point in S. Therefore $\mathcal{M}_s$ is an abelian CM motive, and we conclude by Theorem 4.9 that $\mathrm{H}_{\lambda_1}(\mathcal{M}_s)$ and $\mathrm{H}_{\lambda_2}(\mathcal{M}_s)$ are quasi-compatible at y. This completes the proof of Theorem 5.1.

Remark 5.9. — Laskar [17] has obtained similar results. Let M be an abelian motive over a number field K. Let $E \subset \mathrm{End}(M)$ be a number field, and let Λ be the set of finite places of E. Laskar needs the following condition: Assume that $\mathrm{G}_{\mathrm{B}}(M)^{\mathrm{ad}}$ does not have a simple factor whose Dynkin diagram has type D_k or, more precisely, type $D_k^{\mathbb{H}}$ in the sense of Table 1.3.8 of [8]. Then Theorem 1.1 of [17] implies that the system $\mathrm{H}_{\Lambda}(M)$ is a compatible system in the sense of Serre after replacing K by a finite extension. If $\mathrm{G}_{\mathrm{B}}(M)^{\mathrm{ad}}$ does have a simple factor whose Dynkin diagram has type $D_k^{\mathbb{H}}$, then Laskar also obtains results, but I do not see how to translate them into our terminology. See [17] for more details.

6. Properties of quasi-compatible systems

Readme. — We establish some properties of quasi-compatible systems:

(1) (Definition 6.1) We recover the notion of a Frobenius torus, just as in the classical concept of a compatible system in the sense of Serre.
(2) (Theorem 6.4) Let K be a finitely generated field. Let E be a number field, and let Λ be a set of finite places of E. Let ρ_Λ and ρ'_Λ be two quasi-E-rational quasi-compatible systems of semisimple Galois representations of K. If there is a place $\lambda \in \Lambda$ such that $\rho_\lambda \cong \rho'_\lambda$ as λ-adic Galois representations, then ρ_Λ and ρ'_Λ are isomorphic as quasi-E-rational systems of Galois representations of K.
(3) (Proposition 6.5) We show that under reasonable conditions, we can recover the field E as subring of $\mathrm{End}_{\mathrm{Gal}(\bar{K}/K),E_\lambda}(\rho_\lambda)$ for some $\lambda \in \Lambda$.
(4) (Lemma 6.14) Let ρ_Λ be a quasi-compatible system of semisimple Galois representations. We prove that the absolute rank of $\mathrm{G}_\lambda(\rho_\lambda)$ is independent of λ.

Definition 6.1 (See also [5, §3]). — *Let K be a finitely generated field, let X be a model of K, and let $x \in X^{\mathrm{cl}}$ be a closed point. Let E be a number field, and let λ be a finite place of E. Let ρ be a semisimple λ-adic Galois representation of K. Assume that ρ is unramified at x. The algebraic subgroup $H_n \subset \mathrm{G}_\lambda(\rho)$ generated by $F^n_{x,\rho}$ is well-defined up to conjugation. Note that H_n is a finite-index subgroup of H_1, and therefore the identity component of H_n does not depend on n. We denote this identity component with $T_x(\rho)$.*

If there is an integer $n > 0$ such that $F^n_{x,\rho}$ is semisimple, then we call $T_x(\rho)$ the Frobenius torus *at x. In this case the algebraic group $T_x(\rho)$ is indeed an algebraic torus, which means that $T_x(\rho)_{\bar{E}_\lambda} \cong \mathbb{G}^k_{\mathrm{m}}$, for some $k \geqslant 0$.*

Remark 6.2. — For the remainder of this section we fix the following notation: Let K be a finitely generated field. Let E be a number field, and let Λ be the set of finite places of E whose residue characteristic is different from $\mathrm{char}(K)$.

Now fix $\lambda \in \Lambda$, and let $\rho = \rho_\lambda$ be a λ-adic Galois representation of K. Let $x \in X^{\mathrm{cl}}$ be a closed point of some model X of K. Assume that there is an integer $n > 0$ such that $F^n_{x,\rho}$ is semisimple. Also assume that ρ is quasi-E-rational.

Fix an integer $n > 0$ such that $F^n_{x,\rho}$ is semisimple and generates the Frobenius torus $T_x(\rho)$, and such that c.p.$(F^n_{x,\rho})$ has coefficients in E. Let $(\alpha_i)_i$ be the roots of c.p.$(F^n_{x,\rho})$ in some algebraic closure $\bar{E}$ of E. Let $\Gamma \subset \bar{E}^\star$

be the subgroup generated by the α_i; it is a free abelian group with an action of $\mathrm{Gal}(\bar{E}/E)$. Let $\bar{E}_\lambda$ be an algebraic closure of E_λ, and fix an embedding $\bar{E} \hookrightarrow \bar{E}_\lambda$ that extends $E \hookrightarrow E_\lambda$. As $\mathrm{Gal}(\bar{E}_\lambda/E_\lambda)$-module, Γ may be canonically identified with the character lattice $\mathrm{Hom}(T_x(\rho)_{\bar{E}_\lambda}, \mathbb{G}_{\mathrm{m},\bar{E}_\lambda})$. Let T be the algebraic torus over E whose character lattice is Γ. By construction we have $T_{E_\lambda} \cong T_x(\rho)$.

The upshot of this computation is that we may view $T_x(\rho)$ in a canonical way as an algebraic torus over E, if ρ is a quasi-E-rational Galois representation.

Proposition 6.3. — *Fix $\lambda \in \Lambda$. For $i = 1, 2$, let ρ_i be a λ-adic Galois representation of K. If ρ_1 and ρ_2 are semisimple, quasi-compatible, and $\mathrm{G}_\lambda(\rho_1 \oplus \rho_2)$ is connected, then $\rho_1 \cong \rho_2$.*

Proof. — See Sections 6.6 through 6.13. □

Theorem 6.4. — *Let ρ_Λ and ρ'_Λ be two quasi-compatible systems of semisimple Galois representations. Assume that $\mathrm{G}_\lambda(\rho_\lambda \oplus \rho'_\lambda)$ is connected for all $\lambda \in \Lambda$. If there is a $\lambda \in \Lambda$ such that $\rho_\lambda \cong \rho'_\lambda$, then $\rho_\Lambda \cong \rho'_\Lambda$.*

Proof. — This is an immediate consequence of Proposition 6.3. □

Proposition 6.5. — *Let $\mathscr{L}$ be the set of prime numbers different from $\mathrm{char}(K)$. Let ρ_Λ be a quasi-compatible system of semisimple Galois representations of K. Let $\rho_{\mathscr{L}}$ be the quasi-compatible system of Galois representations obtained by restricting to $\mathbb{Q} \subset E$, as in Section 3.16; in other words, $\rho_\ell = \bigoplus_{\lambda|\ell} \rho_\lambda$. Assume that $\mathrm{G}_\ell(\rho_\ell)$ is connected for all $\ell \in \mathscr{L}$. Fix $\lambda_0 \in \Lambda$. Define the field $E' \subset E$ to be the subfield of E generated by elements $e \in E$ that satisfy the following condition:*

> *There exists a model X of K, a point $x \in X^{\mathrm{cl}}$, and an integer $n \geqslant 1$, such that $P_{x,\rho_{\lambda_0},n}(t) \in E[t]$ and e is a coefficient of $P_{x,\rho_{\lambda_0},n}(t)$.*

Let ℓ be a prime number that splits completely in $E/\mathbb{Q}$. If the endomorphism algebra $\mathrm{End}_{\mathrm{Gal}(\bar{K}/K),\mathbb{Q}_\ell}(\rho_\ell)$ is isomorphic to $E \otimes \mathbb{Q}_\ell$, then $E = E'$.

Proof. — We restrict our attention to a finite subset of Λ, namely $\Lambda_0 = \{\lambda_0\} \cup \{\lambda|\ell\}$. Let $U \subset X$ be an open subset such that for all $\lambda_1, \lambda_2 \in \Lambda_0$ the representations ρ_{λ_1} and ρ_{λ_2} are quasi-compatible at all $x \in U^{\mathrm{cl}}$. For each $x \in U^{\mathrm{cl}}$, let n_x be an integer such that $P_x(t) = P_{x,\rho_\lambda,n_x}(t) \in E[t]$ does not depend on $\lambda \in \Lambda_0$.

Let λ' be a place of E' above ℓ. Let λ_1 and λ_2 be two places of E that lie above λ'. We view ρ_{λ_1} and ρ_{λ_2} as λ'-adic representations. Since ℓ

splits completely in $E/\mathbb{Q}$, the embeddings $\mathbb{Q}_\ell \hookrightarrow E'_{\lambda'} \hookrightarrow E_{\lambda_i}$ are isomorphisms. By definition of E' we have $P_x(t) \in E'[t]$. Therefore ρ_{λ_1} and ρ_{λ_2} are quasi-compatible λ'-adic representations; hence they are isomorphic by Proposition 6.3. Let $\rho_{\lambda'}$ be the λ'-adic Galois representation $\bigoplus_{\lambda|\lambda'} \rho_\lambda$, as in Section 3.16. We conclude that $\mathrm{End}_{\mathrm{Gal}(\bar{K}/K),E'_{\lambda'}}(\rho_{\lambda'}) \cong \mathrm{Mat}_{[E:E']}(E'_{\lambda'})$, which implies $[E : E'] = 1$. □

6.6. — Let X be a model of K. There is a good notion of density for subsets of X^{cl}. This is described by Serre in [26] and [28], and by Pink in appendix B of [23]. For the convenience of the reader, we list some features of these densities. Most of the following list is a reproduction of the statement of Proposition B.7 of [23]. Let $T \subset X^{\mathrm{cl}}$ be a subset. If T has a density, we denote it with $\mu_X(T)$.

(1) If $T \subset X^{\mathrm{cl}}$ has a density, then $0 \leqslant \mu_X(T) \leqslant 1$.
(2) The set X^{cl} has density 1.
(3) If T is contained in a proper closed subset of X, then T has density 0.
(4) If $T_1 \subset T \subset T_2 \subset X^{\mathrm{cl}}$ such that $\mu_X(T_1)$ and $\mu_X(T_2)$ exist and are equal, then $\mu_X(T)$ exists and is equal to $\mu_X(T_1) = \mu_X(T_2)$.
(5) If $T_1, T_2 \subset X^{\mathrm{cl}}$ are two subsets, and three of the following densities exist, then so does the fourth, and we have
$$\mu_X(T_1 \cup T_2) + \mu_X(T_1 \cap T_2) = \mu_X(T_1) + \mu_X(T_2).$$
(6) If $u\colon X \to X'$ is a birational morphism, then T has a density if and only if $u(T)$ has a density, and if this is the case, then $\mu_X(T) = \mu_{X'}(u(T))$.

6.7. — Chebotarev's density theorem generalises to this setting. Let $Y \to X$ be a finite étale Galois covering of integral schemes of finite type over $\mathrm{Spec}(\mathbb{Z})$. Denote the Galois group with G. For each point $y \in Y^{\mathrm{cl}}$ with image $x \in X^{\mathrm{cl}}$ the inverse of the Frobenius automorphism of $\kappa(y)/\kappa(x)$ determines an element $F_y \in G$. The conjugacy class of F_y only depends on x, and we denote it with $\mathcal{F}_x$.

Theorem 6.8. — *Let $Y \to X$ be a finite étale Galois covering of integral schemes of finite type over* $\mathrm{Spec}(\mathbb{Z})$ *with group G. For every conjugacy class $C \subset G$, the set $\{x \in X^{\mathrm{cl}} \mid \mathcal{F}_x = C\}$ has density $\frac{\#C}{\#G}$.*

Proof. — See [23, Prop. B.9]. □

Theorem 6.9. — *Fix $\lambda \in \Lambda$, and let ρ be a semisimple λ-adic Galois representation of K. Assume that* $\mathrm{G}_\lambda(\rho)$ *is connected. There is a non-empty*

Zariski open subset $U \subset \mathrm{G}_\lambda(\rho)$ *such that for every model* X *of* K*, and every closed point* $x \in X^{\mathrm{cl}}$ *the following statement holds: if* ρ *is unramified at* x*, and for some* $n \geqslant 1$ *the Frobenius element* $F^n_{x,\rho}$ *is conjugate to an element of* $U(E_\lambda)$*, then* $T_x(\rho)$ *is a maximal torus of* $\mathrm{G}_\lambda(\rho)$*.*

Proof. — See [5, Thm. 3.7]. The statement in [5] is for abelian varieties, but the proof is completely general and is not even limited to Frobenius elements. □

Corollary 6.10 ([5, 3.8]). — *Fix* $\lambda \in \Lambda$*, and let* ρ *be a semisimple* λ*-adic Galois representation of* K*. Assume that* $\mathrm{G}_\lambda(\rho)$ *is connected. Let* X *be a model of* K*. Let* $\Sigma \subset X^{\mathrm{cl}}$ *be the set of points* $x \in X^{\mathrm{cl}}$ *for which* ρ *is unramified at* x *and* $T_x(\rho)$ *is a maximal torus of* $\mathrm{G}_\lambda(\rho)$*. Then* Σ *has density* 1*.*

Lemma 6.11. — *Let* E *be a field of characteristic* 0*. Let* G *be a reductive group over* E*. Let* ρ_1 *and* ρ_2 *be two finite-dimensional semisimple representations of* G*. Let* $S \subset G(E)$ *be a subset that is Zariski-dense in* G*. Assume that for all* $g \in S$ *we have* $\mathrm{tr}(\rho_1(g)) = \mathrm{tr}(\rho_2(g))$*. Then* $\rho_1 \cong \rho_2$ *as representations of* G*.*

Proof. — Note that $\mathrm{tr} \circ \rho_i$ is a separated morphism of schemes $G \to \mathbb{A}^1_E$. Therefore we have $\mathrm{tr}(\rho_1(g)) = \mathrm{tr}(\rho_2(g))$ for all $g \in G(E)$. By linearity, we find that $\mathrm{tr}(\rho_1(\alpha)) = \mathrm{tr}(\rho_2(\alpha))$ for all α in the group algebra $E[G(E)]$. By Proposition 3 in [4, §12, no. 1], we conclude that $\rho_1 \cong \rho_2$ as representations of $G(E)$, hence as representations of G. □

Lemma 6.12. — *Fix* $\lambda \in \Lambda$*. For* $i = 1, 2$*, let* ρ_i *be a semisimple* λ*-adic Galois representation of* K*. Write* ρ *for* $\rho_1 \oplus \rho_2$*. Assume that* $\mathrm{G}_\lambda(\rho)$ *is connected. If there is a model* X *of* K*, and a point* $x \in X^{\mathrm{cl}}$ *such that* ρ *is unramified at* x*, and* $T_x(\rho)$ *is a maximal torus, and* $P_{x,\rho_1,n}(t) = P_{x,\rho_2,n}(t)$ *for some* $n \geqslant 1$*, then* $\rho_1 \cong \rho_2$ *as* λ*-adic Galois representations.*

Proof. — Write T for $T_x(\rho)$. Observe that $P_{x,\rho_1,kn}(t) = P_{x,\rho_2,kn}(t)$ for all $k \geqslant 1$. Let H_n be the algebraic subgroup of $\mathrm{G}_\lambda(\rho)$ that is generated by $F^n_{x,\rho}$. Recall that T is the identity component of H_n. Note that for some $k \geqslant 1$, we have $F^{kn}_{x,\rho} \in T(E_\lambda)$. Replace n by kn, so that we may assume that $F^n_{x,\rho}$ generates T as algebraic group.

The set $\{F^{kn}_{x,\rho} \mid k \geqslant 1\}$ is a Zariski dense subset of T. Since $P_{x,\rho_1,kn}(t) = P_{x,\rho_2,kn}(t)$, for all $k \geqslant 1$, Lemma 6.11 implies that $\rho_1|_T \cong \rho_2|_T$. Note that $\mathrm{G}_\lambda(\rho)$ is reductive, since ρ is semisimple by assumption. Because T is a maximal torus of $\mathrm{G}_\lambda(\rho)$ and $\mathrm{G}_\lambda(\rho)$ is connected and reductive, we find that $\rho_1 \cong \rho_2$ as representations of $\mathrm{G}_\lambda(\rho)$, and hence as λ-adic Galois representations of K. □

6.13. Proof of Proposition 6.3. — Let X be a model of K. By Corollary 6.10, the subset of points $x \in X^{\mathrm{cl}}$ for which $T_x(\rho_1 \oplus \rho_2)$ is a maximal torus is a subset with density 1. By definition of quasi-compatibility, and Section 6.6, the subset of points $x \in X^{\mathrm{cl}}$ at which ρ_1 and ρ_2 are quasi-compatible is also a subset with density 1. Once again by properties listed in Section 6.6, these subsets have non-empty intersection: there exists a point $x \in X^{\mathrm{cl}}$ such that $T_x(\rho_1 \oplus \rho_2)$ is a maximal torus and ρ_1 and ρ_2 are quasi-compatible at x. Now Proposition 6.3 follows from Lemma 6.12. □

Lemma 6.14. — *Let ρ_Λ be a quasi-compatible system of semisimple Galois representations of K. Assume $\mathrm{G}_\lambda(\rho_\lambda)$ is connected, for all $\lambda \in \Lambda$. Then the absolute rank of $\mathrm{G}_\lambda(\rho_\lambda)$ is independent of λ.*

Proof. — It suffices to assume that $\Lambda = \{\lambda_1, \lambda_2\}$. Let X be a model of K. For $i = 1, 2$, let $\Sigma_i \subset X^{\mathrm{cl}}$ be the set of points $x \in X^{\mathrm{cl}}$ for which ρ_{λ_i} is unramified at x and $T_x(\rho_{\lambda_i})$ is a maximal torus of $\mathrm{G}_{\lambda_i}(\rho_{\lambda_i})$. Then Σ_i has density 1, by Corollary 6.10. Let $U \subset X$ be an open subset such that ρ_{λ_1} and ρ_{λ_2} are compatible at all $x \in U^{\mathrm{cl}}$.

Put $\Sigma = \Sigma_1 \cap \Sigma_2 \cap U^{\mathrm{cl}}$; by item (5) of Section 6.6 we know that Σ is non-empty. Fix a closed point $x \in \Sigma$. Since ρ_{λ_1} and ρ_{λ_2} are quasi-E-rational and quasi-compatible at x, there exists a torus T over E such that $T_{E_{\lambda_i}} \cong T_x(\rho_{\lambda_i})$, see Remark 6.2. The tori $T_x(\rho_{\lambda_i}) \subset \mathrm{G}_{\lambda_i}(\rho_{\lambda_i})$ are maximal tori, by assumption. We conclude that $\mathrm{G}_{\lambda_1}(\rho_{\lambda_1})$ and $\mathrm{G}_{\lambda_2}(\rho_{\lambda_2})$ have the same absolute rank. □

Remark 6.15. — One property of compatible systems in the sense of Serre that does not carry over to quasi-compatible systems is the independence of the component group. For example, consider the following system of Galois representations of $\mathbb{Q}$: for $\ell = 2$, and $\ell \equiv 1 \pmod 4$, let ρ_ℓ be the trivial representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on $\mathbb{Q}_\ell$; for $\ell \equiv 3 \pmod 4$, let $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ act on $\mathbb{Q}_\ell$ via $\mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q})$, where the non-trivial element of $\mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q})$ acts as multiplication with -1.

This system of representations is unramified away from 2, and is quasi-compatible, because $\rho_\ell(F_p^2) = 1$, for all $\ell \neq p \neq 2$. However, the component group $\mathrm{G}_\ell(\rho_\ell)$ depends on $\ell \pmod 4$.

7. Remark on the Mumford–Tate conjecture

Readme. — In this section we recall the Mumford–Tate conjecture. A priori, this conjecture depends on the choice of a prime number ℓ. We show that for abelian motives this conjecture does not depend on ℓ, see Corollary 7.5. For abelian varieties, this result is proven in [16, Thm. 4.3].

Conjecture 7.1 (Mumford–Tate). — *Let M be a motive over a finitely generated subfield of $\mathbb{C}$. Let ℓ be a prime number. Under the comparison isomorphism $\mathrm{H_B}(M) \otimes \mathbb{Q}_\ell \cong \mathrm{H}_\ell(M)$, see Section 2.3, we have*

$$\mathrm{MTC}_\ell(M)\colon \quad \mathrm{G_B}(M) \otimes \mathbb{Q}_\ell = \mathrm{G}_\ell^\circ(M).$$

Lemma 7.2. — *Let $K \subset L$ be finitely generated subfields of $\mathbb{C}$. Let M be a motive over K. Then $\mathrm{G_B}(M_L) = \mathrm{G_B}(M)$ and $\mathrm{G}_\ell^\circ(M_L) = \mathrm{G}_\ell^\circ(M)$. In particular $\mathrm{MTC}_\ell(M) \iff \mathrm{MTC}_\ell(M_L)$.*

Proof. — See [18, Prop. 1.3]. □

Proposition 7.3. — *Let M be an abelian motive over a finitely generated field $K \subset \mathbb{C}$. Let $\mathrm{Z_B}(M)$ be the centre of the Mumford–Tate group $\mathrm{G_B}(M)$, and let $\mathrm{Z}_\ell(M)$ be the centre of $\mathrm{G}_\ell^\circ(M)$. Then $\mathrm{Z}_\ell(M) \subset \mathrm{Z_B}(M) \otimes \mathbb{Q}_\ell$, and $\mathrm{Z}_\ell(M)^\circ = \mathrm{Z_B}(M)^\circ \otimes \mathbb{Q}_\ell$.*

Proof. — The result is true for abelian varieties, see [33, Thm. 1.3.1] or [32, Cor. 2.11]. We use this result in the diagram below.

Fix a prime number ℓ. By Lemma 7.2, we may replace K by a finitely generated field extension, and therefore we may assume that there is an abelian variety A over K such that $M \in \langle \mathrm{H}(A) \rangle^\otimes$ and such that $\mathrm{G}_\ell(A)$ is connected. By definition of abelian motive, there is an abelian variety A such that M is contained in the Tannakian subcategory of motives generated by $\mathrm{H}(A)$.

We have a surjection $\mathrm{G_B}(A) \twoheadrightarrow \mathrm{G_B}(M)$. Since $\mathrm{G_B}(A)$ is reductive, $\mathrm{Z_B}(M)$ is the image of $\mathrm{Z_B}(A)$ under this map. The same is true on the ℓ-adic side. Note that $\mathrm{G}_\ell(A)$ is reductive, by Satz 3 in [9, §5]; see also [10]. Thus we obtain a commutative diagram with solid arrows

$$\begin{tikzcd}
\mathrm{Z}_\ell(A) \arrow[r, two heads] \arrow[d, hook] & \mathrm{Z}_\ell(M) \arrow[r, hook] \arrow[d, hook, dotted] & \mathrm{G}_\ell(M) \arrow[d, hook] \\
\mathrm{Z_B}(A) \otimes \mathbb{Q}_\ell \arrow[r, two heads] & \mathrm{Z_B}(M) \otimes \mathbb{Q}_\ell \arrow[r, hook] & \mathrm{G_B}(M) \otimes \mathbb{Q}_\ell
\end{tikzcd}$$

which shows that the dotted arrow exists and is an inclusion. The vertical arrow on the left exists and is an inclusion by the result on abelian varieties mentioned at the beginning of the proof. The vertical arrow on the right exists and is an inclusion, by Theorem 2.6.

Finally, observe that $\mathrm{Z_B}(M)$ and $\mathrm{Z}_\ell(M)$ have the same rank. Indeed, as remarked at the beginning of the proof, we know that $\mathrm{Z}_\ell(A)^\circ \hookrightarrow \mathrm{Z_B}(A)^\circ \otimes$

$\mathbb{Q}_\ell$ is an isomorphism. The commutative diagram above shows that the inclusion $\mathrm{Z}_\ell(M)^\circ \hookrightarrow \mathrm{Z}_\mathrm{B}(M)^\circ \otimes \mathbb{Q}_\ell$ must be an isomorphism. □

Proposition 7.4. — *Let M be an abelian motive over a finitely generated subfield $K \subset \mathbb{C}$. Let ℓ be a prime number. If $\mathrm{G}_\mathrm{B}(M)$ and $\mathrm{G}_\ell(M)$ have the same absolute rank, then $\mathrm{MTC}(M)_\ell$ is true.*

Proof. — We apply the Borel–de Siebenthal theorem, see [3]; or [22]: since $\mathrm{G}_\ell^\circ(M) \subset \mathrm{G}_\mathrm{B}(M) \otimes \mathbb{Q}_\ell$ has maximal rank, it is equal to the connected component of the centraliser of its centre. By Proposition 7.3, we know that the centre of $\mathrm{G}_\ell^\circ(M)$ is contained in the centre of $\mathrm{G}_\mathrm{B}(M) \otimes \mathbb{Q}_\ell$. Hence $\mathrm{G}_\ell^\circ(M) = \mathrm{G}_\mathrm{B}(M) \otimes \mathbb{Q}_\ell$. □

Corollary 7.5. — *Let M be an abelian motive over a finitely generated subfield $K \subset \mathbb{C}$. The Mumford–Tate conjecture is independent of the choice of the prime number ℓ.*

Proof. — Note: For abelian varieties, a proof of this result can be found in [16, Thm. 4.3].

Let $\mathscr{L}$ be a finite set of prime numbers, and assume that $\mathrm{MTC}_\ell(M)$ holds for at least one prime $\ell \in \mathscr{L}$. Without loss of generality, we may and do assume that the groups $\mathrm{G}_\ell(M)$ are connected for all $\ell \in \mathscr{L}$, by Lemma 7.2. By Theorem 5.1 the Galois representations $\mathrm{H}_\ell(M)$ form a quasi-compatible system, and by Lemma 6.14 the rank of the groups $\mathrm{G}_\ell(M)$ does not depend on ℓ. Since the Mumford–Tate conjecture is true for one $\ell \in \mathscr{L}$, the groups $\mathrm{G}_\mathrm{B}(M)$ and $\mathrm{G}_\ell(M)$ have the same absolute rank for all $\ell \in \mathscr{L}$. The result follows from Proposition 7.4. □

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