ANNALES

DE

L'INSTITUT FOURIER

Eleonora DI NEZZA & Charles FAVRE

Regularity of push-forward of Monge–Ampère measures

Tome 68, n° 7 (2018), p. 2965-2979.

<http://aif.cedram.org/item?id=AIF_2018__68_7_2965_0>

Ann. Inst. Fourier, Grenoble
68, 7 (2018) 2965-2979

REGULARITY OF PUSH-FORWARD OF MONGE–AMPÈRE MEASURES

by Eleonora DI NEZZA & Charles FAVRE

Dedicated to Jean-Pierre Demailly on the occasion of his 60th birthday

Abstract. — We prove that the image under any dominant meromorphic map of the Monge–Ampère measure of a Hölder continuous quasi-psh function still possesses a Hölder potential. We also discuss the case of lower regularity.

Résumé. — Nous démontrons que l'image par une application méromorphe dominante d'une mesure de Monge–Ampère d'une fonction quasi-psh et hölderienne possède aussi un potentiel hölderien. Nous discutons aussi le cas de régularité plus basse.

1. Introduction

Let (X, ω_X) be a compact Kähler manifold of dimension n normalized by the volume condition $\int_X \omega_X^n = 1$. We say that a potential $\varphi \in L^1(X)$ is ω_X-plurisubharmonic (ω_X-psh for short) if locally φ is the sum of a plurisubharmonic and a smooth function, and $\omega_X + \mathrm{dd}^c\varphi \geqslant 0$ in the weak sense of currents, where $\mathrm{d} = \partial + \bar\partial$ and $\mathrm{d}^c = \frac{i}{2\pi}(\bar\partial - \partial)$ so that $\mathrm{dd}^c = \frac{i}{\pi}\partial\bar\partial$. We denote by $\mathrm{PSH}(X, \omega_X)$ the set of all ω_X-psh functions on X. Recall from [13, Section 1] that the non-pluripolar Monge–Ampère measure of a function $\varphi \in \mathrm{PSH}(X, \omega_X)$ is a positive measure defined as the increasing limit

$$(\omega_X + \mathrm{dd}^c\varphi)^n = \lim_{j\to+\infty} \mathbb{1}_{\{\varphi > -j\}} \left(\omega_X + \mathrm{dd}^c \max\{\varphi, -j\}\right)^n$$

where the right hand side is defined using Bedford–Taylor intersection theory of bounded psh functions, see [2]. By construction this measure does not charge pluripolar sets.

Keywords: Kähler manifolds, meromorphic map, Monge–Ampère measures.
2010 *Mathematics Subject Classification:* 32Q15, 32W20, 32Uxx.

One of the main result of [13] states that if μ is a probability measure on X which does not charge pluripolar sets, then there exists a unique (up to a constant) ω_X-psh function φ such that $\int_X (\omega_X + dd^c\varphi)^n = 1$ and

$$\mu = (\omega_X + dd^c\varphi)^n. \tag{1.1}$$

We denote by $\mathcal{E}(X, \omega_X)$ the set of all ω_X-psh functions whose non-pluripolar Monge–Ampère measure has full mass so that any solution to (1.1) belongs to $\mathcal{E}(X, \omega_X)$.

In the same paper, Guedj and Zeriahi introduced for any $p > 0$ the subset $\mathcal{E}^p(X, \omega_X)$ of $\mathcal{E}(X, \omega_X)$ consisting of all ω_X-psh functions satisfying the integrability condition $\varphi \in L^p((\omega_X + dd^c\varphi)^n)$. Since ω_X-psh functions are bounded from above it follows that

$$\mathcal{E}^p(X, \omega_X) \subset \mathcal{E}^q(X, \omega_X), \text{ for all } p > q.$$

Observe also that any ω_X-psh function lying in L^∞ belongs to the intersection of all $\mathcal{E}^p(X, \omega_X)$.

We shall say that a probability measure which does not charge pluripolar sets $\mu = (\omega_X + dd^c\varphi)^n$ is a Monge–Ampère measure having Hölder, continuous, L^∞ or $\mathcal{E}^p$ potential for some $p > 0$ whenever φ is Hölder, continuous, L^∞ or belongs to the energy class $\mathcal{E}^p(X, \omega_X)$ respectively.

Let us now consider any dominant meromorphic map $f : X \dashrightarrow Y$ where (Y, ω_Y) is also a compact Kähler manifold of volume 1, and denote by m its complex dimension. Let Γ be a resolution of singularities of the graph of f. We obtain two surjective holomorphic maps $\pi_1 : \Gamma \to X$ and $\pi_2 : \Gamma \to Y$ where π_1 is bimeromorphic so that Γ is a modification of a compact Kähler manifold. By Hironaka's Chow lemma, see e.g. [17, Theorem 2.8] we may suppose that π_1 is a composition of blow-ups along smooth centers so that Γ is itself a compact Kähler manifold of complex dimension n. We fix any Kähler form ω_Γ on it.

One defines the push-forward under f of a measure μ not charging pluripolar sets as follows. Since π_1 is a bimeromorphism, there exist two closed analytic subsets $R \subset \Gamma$ and $V \subset X$ such that $\pi_1 : \Gamma \setminus R \to X \setminus V$ is a biholomorphism. One may thus set $\pi_1^*\mu$ to be the trivial extension through R of $(\pi_1)|_{\Gamma\setminus R}^*\mu$. This measure is again a probability measure which does not charge pluripolar sets.

We then define the probability measure $f_*\mu := (\pi_2)_*\pi_1^*\mu$. We observe that since f is dominant then π_2 is surjective hence the preimage of a pluripolar set in Y by π_2 is again pluripolar. By the preceding discussion, there exists $\psi \in \mathcal{E}(Y, \omega_Y)$ such that $f_*\mu = (\omega_Y + dd^c\psi)^m$.

Our main goal is to discuss the following question.

Problem 1.1. — *Suppose μ is a Monge–Ampère measure having Hölder, continuous, L^∞ or $\mathcal{E}^p$ potential. Is it true that $f_*\mu$ is also a Monge–Ampère measure of a potential lying in the same class of regularity?*

This problem is hard for Monge–Ampère measures having either continuous or L^∞ potentials since there is no known intrinsic characterization of these measures. For these classes of regularity even the case f is the identity map and $X = Y$ is still open (see for example [7, Question 15]).

Problem 1.2. — *Suppose μ is a probability measure on X not charging pluripolar sets and write $\mu = (\omega + \mathrm{dd}^c\varphi)^n = (\omega' + \mathrm{dd}^c\varphi')^n$ where ω, ω' are two Kähler forms of volume 1. Is it true that φ is continuous (resp. L^∞) iff φ' is?*

Remark. — A variant of Problem 1.1 has been recently investigated in [1, 18]. In particular, one can find in these papers a criterion on the singularities of an algebraic map $f : X \to Y$ which ensures that the push-forward of any continuous volume form remains continuous. We refer to these articles for the precise statements and for some far-reaching generalizations of them over any local fields.

Intrinsic characterizations of Monge–Ampère measures of Hölder functions are given by [4] and [9], and in the context of Hermitian compact manifolds by [15]. A characterization of Monge–Ampère measures of functions in the energy class $\mathcal{E}^p$ is also obtained in [13, Theorem C] so that Problem 1.2 has a positive answer for these two classes of regularity, see [5, Theorem 4.1]. Problem 1.1 remains though quite subtle. If we restrict our attention to the regularity in the $\mathcal{E}^p$ energy classes, then the answer is no in general. Suppose that $\pi : X \to \mathbb{P}^2$ is the blow-up at some point $p \in \mathbb{P}^2$, and let $E = \pi^{-1}(p)$. It was observed by the first author in [6, Proposition B] that there exists a probability measure $\mu = (\omega_X + \mathrm{dd}^c\varphi)^2$ with $\varphi \in \mathcal{E}^1(X, \omega_X)$ but $\pi_*\mu = (\omega_{FS} + \mathrm{dd}^c\psi)^2$ with $\psi \notin \mathcal{E}^1(\mathbb{P}^2, \omega_{FS})$, where ω_{FS} denotes the Fubini Study metric on $\mathbb{P}^2$ and ω_X is a Kähler form.

In this note we answer Problem 1.1 in two situations. We first treat the case μ is the Monge–Ampère of a Hölder function.

Theorem 1.3. — *Let $f : X \dashrightarrow Y$ be any dominant meromorphic map between two compact Kähler manifolds. If μ is a Monge–Ampère measure having a Hölder potential with Hölder exponent α, then $f_*\mu$ is a Monge–Ampère measure having a Hölder potential with Hölder exponent bounded by $C\alpha^{\dim(X)}$ for some constant $C > 0$ depending only on f.*

We expect that the technics developed in the paper of Kołodziej–Nguyen [15] in the present volume allows one to extend the previous result to arbitrary compact hermitian manifolds.

Next we treat the case the image of the map has dimension 1.

Theorem 1.4. — *Let $f : X \dashrightarrow Y$ be any dominant meromorphic map from a compact Kähler manifold to a compact Riemann surface. If μ is a Monge–Ampère measure having a Hölder, $\mathcal{C}^0$, L^∞, $\mathcal{E}^p$ potential respectively, then $f_*\mu$ is a Monge–Ampère measure having a potential lying in the same regularity class.*

Motivations for studying this question come from the analysis of degenerating measures on families of projective manifolds developed in [11]. Let us briefly recall the setting of that paper. Let $\mathcal{X}$ be a smooth connected complex manifold of dimension $n+1$, and $\pi\colon \mathcal{X} \to \mathbb{D}$ be a flat proper analytic map over the unit disk which is a submersion over the punctured disk and has connected fibers. We assume that $\mathcal{X}$ is Kähler so that each fiber $X_t = \pi^{-1}(t)$ is also Kähler.

A tame family of Monge–Ampère measures is by definition a family of positive measures $\{\mu_t\}_{t\in\mathbb{D}}$ each supported on X_t that can be written under the form

$$\mu_t = p_*(T|^n_{X_t}),$$

where T is a positive closed $(1,1)$-current having local Hölder continuous potentials and defined on a complex manifold $\mathcal{X}'$ which admits a proper bimeromorphic morphism $p\colon \mathcal{X}' \to \mathcal{X}$ which is an isomorphism over $X := \pi^{-1}(\mathbb{D}^*)$. It follows from [3, Corollary 1.6] that the family of measures $\mu'_t := T|^n_{X_t}$ in $\mathcal{X}'$ is continuous so that μ'_t converges to a positive measure μ'_0 supported on X'_0 as $t \to 0$. It follows that the convergence $\lim_{t\to 0}\mu_t = \mu_0$ also holds in $\mathcal{X}$.

As a corollary of the previous results we show the limiting measure μ_0 is of a very special kind:

Corollary 1.5. — *Let $\{\mu_t\}_{t\in\mathbb{D}}$ be any tame family of Monge–Ampère measures, so that $\mu_t \to \mu_0$ as $t \to 0$.*

Then there exist a finite collection of closed subvarieties $\{V_i\}_{i=0,\dots,N}$ of X_0 and for each index i a positive measure ν_i supported on V_i such that

$$\mu_0 = \sum_{i=1}^{N} \nu_i$$

and ν_i is a Monge–Ampère measure on V_i having a Hölder potential.

In the previous statement, it may happen that V_i is singular, in which case it is understood that the pull-back of ν_i to a (Kähler) resolution of V_i is a Monge–Ampère measure having a Hölder continuous potential.

Acknowledgement. We thank Ahmed Zeriahi for useful discussions on these problems.

2. Images of Monge–Ampère measures having a Hölder potential: proof of Theorem 1.3

As already mentioned, a dominant meromorphic map $f\colon X \dashrightarrow Y$ can be decomposed as $f = \pi_2 \circ \pi_1^{-1}$, where $\pi_1\colon \Gamma \to X$ is holomorphic and bimeromorphic and $\pi_2\colon \Gamma \to Y$ is a surjective holomorphic map. Recall that one can assume Γ to be Kähler, and that $f_*\mu := (\pi_2)_*\pi_1^*\mu$.

We first claim that if μ is the Monge–Ampère of a Hölder continuous function then $\pi_1^*\mu$ too. Let $\varphi \in \mathrm{PSH}(X, \omega_X)$ be the Hölder potential such that $\mu = (\omega_X + \mathrm{dd}^c\varphi)^n$. It then follows from Bedford and Taylor theory that $\pi_1^*\mu = (\pi_1^*\omega_X + \mathrm{dd}^c\pi_1^*\varphi)^n$. Since $\pi_1^*\omega_X$ is a semipositive smooth form, there exists a positive constant $C > 0$ such that $\pi_1^*\mu \leqslant (C\omega_\Gamma + \mathrm{dd}^c\pi^*\varphi)^n$ where ω_Γ is a Kähler form on Γ, and [4, Theorem 4.3] implies that $\tilde{\mu} := \pi_1^*\mu$ is the Monge–Ampère measure of a Hölder continuous $C\omega_\Gamma$-psh function. This proves the claim. We are then left to prove that $(\pi_2)_*\tilde{\mu}$ is the Monge–Ampère measure of a Hölder potential. This will be done in Lemma 2.4.

We first show that the push-forward of a smooth volume form has density in $L^{1+\varepsilon}$, for some constant $\varepsilon > 0$ depending only on f.

Proposition 2.1. — *Let $f\colon X \to Y$ be a surjective holomorphic map. Then $f_*\omega_X^n = g\omega_Y^m$ with $g \in L^{1+\varepsilon}(\omega_Y^n)$, for some $\varepsilon > 0$.*

This result is basically [19, Proposition 3.2] (see also [20, Section 2]). We give nevertheless a detailed proof for reader's convenience. Pick any coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$, and denote by $V(\mathcal{I}) = \mathrm{supp}(\mathcal{O}_X/\mathcal{I})$ the closed analytic subvariety of X associated to $\mathcal{I}$. Let $\{U_i\}_{i=1}^N$ be a finite open covering of X by balls and $\{V_i\}_i$ be a subcovering such that $\overline{V}_i \subset U_i$. The analytic sheaf $\mathcal{I}$ is globally generated on each U_i so that we can find holomorphic functions such that $\mathcal{I}|_{U_i} = (h_1^{(i)}, \ldots, h_k^{(i)}) \cdot \mathcal{O}_{U_i}$. Let $\{\rho_i\}$ be a partition of unity subordinate to $\overline{V_i}$. We then define

$$\Phi_{\mathcal{I}} := \sum_{i=1}^{N} \rho_i \left(\sum_{j=1}^{k} |h_j^{(i)}|^2 \right). \tag{2.1}$$

Then $\Phi_{\mathcal{I}}\colon X \to \mathbb{R}_+$ is a smooth function which vanishes exactly on $V(\mathcal{I})$. Observe that if $\Phi_{\mathcal{I}}$ and $\Phi'_{\mathcal{I}}$ are defined using two different coverings, then there exists $C > 0$ such that

$$\frac{1}{C}\Phi'_{\mathcal{I}} \leqslant \Phi_{\mathcal{I}} \leqslant C\Phi'_{\mathcal{I}}.$$

In the sequel we shall abuse notation and not write the dependence of $\Phi_{\mathcal{I}}$ in terms of the local generators of the ideal sheaf. The logarithm of the obtained function is then well-defined up to a bounded function so that all statements in the next Lemma make sense.

Lemma 2.2. — *Let $\mathcal{I}, \mathcal{J} \subseteq \mathcal{O}_X$ be two coherent ideal sheafs. The followings hold:*

1. *there exists $\varepsilon > 0$ such that $|\Phi_{\mathcal{I}}|^{-\varepsilon} \in L^1(X)$;*
2. *if $\mathcal{I} \subseteq \mathcal{J}$ then $\Phi_{\mathcal{J}} \geqslant c\Phi_{\mathcal{I}}$ for some positive $c > 0$;*
3. *if $V(\mathcal{J}) \subseteq V(\mathcal{I})$ then there exists $c, \theta > 0$ such that $\Phi_{\mathcal{J}} \geqslant c\,\Phi_{\mathcal{I}}^{\theta}$;*
4. *given $f\colon X \to Y$ a holomorphic surjective map and a coherent ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_Y$, then $\Phi_{f^*\mathcal{J}} = \Phi_{\mathcal{J}} \circ f$ (for a suitable choice of local generators of $\mathcal{J}$ and $f^*\mathcal{J}$).*

Proof. — Using a resolution of singularities of $\mathcal{I}$, one sees that the statement in (1) reduces to show that $|z_1|^{-\varepsilon}$ is locally integrable for some $\varepsilon > 0$, and this is the case if we choose ε small enough. The statements in (2) and (4) follow straightforward from the definition in (2.1). The statement in (3) is a consequence of Łojasiewicz theorem, see e.g. [16, Theorem 7.2]. □

Lemma 2.3. — *Let $f\colon X \to Y$ be a holomorphic surjective map and let $\mathcal{I} \subseteq \mathcal{O}_X$ be a coherent ideal sheaf. Then there exists a coherent ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_Y$, and constants $c, \theta > 0$ such that for any $y \in Y$ we have*

$$\inf_{x \in f^{-1}(y)} \Phi_{\mathcal{I}} \geqslant c\,\Phi_{\mathcal{J}}^{\theta}$$

Proof. — Let $\mathcal{J} \subseteq \mathcal{O}_Y$ be the coherent ideal sheaf of holomorphic functions vanishing on the set $f(V(\mathcal{I}))$ which is analytic since f is proper. Observe that $V(f^*\mathcal{J}) = f^{-1}(V(\mathcal{J})) \supset V(\mathcal{I})$, so that Lemma 2.2(3) and (4) insure that there exist $c, \theta > 0$ such that

$$\Phi_{\mathcal{I}} \geqslant c\Phi_{f^*\mathcal{J}}^{\theta} = c(\Phi_{\mathcal{J}} \circ f)^{\theta}.$$

Hence the conclusion. □

Proof of Proposition 2.1. — Recall that Sard's theorem implies the existence of a closed subvariety $S \subsetneq Y$ such that $f\colon X \setminus f^{-1}(S) \to Y \setminus S$ is a submersion.

We first prove that $f_*\omega_X^n$ is absolutely continuous w.r.t. ω_Y^m. We need to check that $\omega_Y^m(E) = 0$ implies $f_*\omega_X^n(E) = 0$ for any Borel subset $E \subset Y$. As S and $f^{-1}(S)$ have volume zero one may assume that f is a submersion in which case the claim follows from Fubini's theorem.

Radon–Nikodym theorem now guarantees that $f_*\omega_X^n = g\omega_Y^m$ for some $0 \leqslant g \in L^1(Y)$. We want to show that the integral

$$\int_Y g^{1+\varepsilon}\omega_Y^m = \int_Y g^\varepsilon f_*\omega_X^n = \int_X (f^*g)^\varepsilon\, \omega_X^n$$

is finite for some $\varepsilon > 0$ small enough. Consider the smooth function $\phi(x) := \frac{f^*\omega_Y^m \wedge \omega_X^{n-m}}{\omega_X^n}(x)$, and set $\tilde{\phi}(y) := \inf_{x \in f^{-1}(y)} \phi(x)$ so that $\phi \geqslant f^*\tilde{\phi}$. We claim that for any $y \in Y$

$$g(y) \leqslant \frac{c}{\tilde{\phi}(y)}, \tag{2.2}$$

for some constant $c > 0$. Let χ be a test function (i.e. a non negative smooth function) on Y, then

$$\begin{aligned}\int_Y \chi\, g\omega_Y^n = \int_X f^*\chi\, \omega_X^n &= \int_X \frac{f^*\chi}{\phi}\, f^*\omega_Y^m \wedge \omega_X^{n-m}\\ &\leqslant \int_X f^*\left(\frac{\chi}{\tilde{\phi}}\omega_Y^m\right) \wedge \omega_X^{n-m}\\ &\overset{\text{Fubini}}{=} C(f)\int_Y \frac{\chi}{\tilde{\phi}}\omega_Y^m\end{aligned}$$

where $c := C(f) = \int_{f^{-1}(y)} \omega_X^{n-m}$ is the volume of a fiber over a generic point $y \in Y$. The claim is thus proved. Lemma 2.2 (1) and (4) combined with Lemma 2.3 then insure that there exists $\varepsilon > 0$ such that $(f^*g)^\varepsilon \in L^1(\omega_X^n)$. ☐

Theorem 1.3 is reduced to the following result which relies in an essential way on Proposition 2.1.

Proposition 2.4. — *Suppose $f\colon X \to Y$ is a surjective holomorphic map between compact Kähler manifolds. If μ is a positive measure on X with Hölder continuous potentials, then $f_*\mu$ is a positive measure on Y with Hölder potentials.*

Observe that by multiplying ω_X by a suitable positive constant we may assume that $f^*\omega_Y \leqslant \omega_X$. The volume normalization is no longer satisfied but a positive multiple of μ is still the Monge–Ampère measure of a ω_X-psh Hölder continuous function. Write $f_*\mu = (\omega_Y + dd^c\psi)^m$ with $\psi \in \mathrm{PSH}(Y, \omega_Y)$.

We claim that there exists $C > 0$, and $\varepsilon > 0$ such that for all $u \in \mathrm{PSH}(Y, \omega_Y)$ with $\int_X u\,\omega_X^n = 0$

$$\int_Y \exp(-\varepsilon u)\,\mathrm{d}(f_*\mu) \leqslant C. \tag{2.3}$$

Indeed, for any $u \in \mathrm{PSH}(Y, \omega_Y)$ we have that

$$\int_Y e^{-\varepsilon u}\,\mathrm{d}(f_*\mu) = \int_X e^{-\varepsilon(u\circ f)}\,\mathrm{d}\mu.$$

Now the integral $\int_X e^{-\varepsilon(u\circ f)}\,\mathrm{d}\mu$ is uniformly bounded by [10, Theorem 1.1] since:

- μ has Hölder continuous potentials;
- $f^*\omega_Y \leqslant \omega_X$ hence $u \circ f \in \mathrm{PSH}(X, \omega_X)$;
- and the set of functions in $\mathrm{PSH}(X, \omega_X)$ such that $\int_X u\,\omega_X^n = 0$ is compact by [12, Proposition 2.6].

This proves our claim. Using the terminology of [9] this means that $f_*\mu$ is *moderate*. It is worth mentioning that if [7, Question 16] holds true then the conclusion of Proposition 2.4 would follow immediately since any moderate measure would have a Hölder continuous potential. To get around this problem we use the characterization of measures with Hölder potentials given by Dinh and Nguyen.

Proof of Proposition 2.4. — By [9, Lemma 3.3], $f_*\mu$ is the Monge–Ampère measure of a Hölder potential if and only if there exist $\tilde{c} > 1$ and $\tilde{\beta} \in (0, 1)$ such that

$$\int_Y |u - v|\,\mathrm{d}f_*\mu \leqslant \tilde{c}\max\left(\|u - v\|_{L^1(\omega_Y^n)}, \|u - v\|^{\tilde{\beta}}_{L^1(\omega_Y^n)}\right) \tag{2.4}$$

for all $u, v \in \mathrm{PSH}(Y, \omega_Y)$. By assumption on μ we know there exist $c > 1$ and $\beta \in (0, 1)$ such that $\int_Y |u - v|\, df_*\mu = \int_X |f^*u - f^*v|\,\mathrm{d}\mu$, and

$$\int_X |f^*u - f^*v|\,\mathrm{d}\mu \leqslant c\max\left(\|f^*u - f^*v\|_{L^1(\omega_X^n)}, \|f^*u - f^*v\|^{\beta}_{L^1(\omega_X^n)}\right). \tag{2.5}$$

Also, Proposition 2.1 gives

$$\int_X |f^*u - f^*v|\omega_X^n = \int_Y |u - v|\, g\,\omega_Y^n \leqslant \|g\|_{L^{1+\varepsilon}(\omega_Y^n)}\|u - v\|_{L^p(\omega_Y^n)} \tag{2.6}$$

where p is the conjugate exponent of $1 + \varepsilon$. Set $C_g := \|g\|_{L^{1+\varepsilon}(\omega_Y^n)} < +\infty$. Up to replace C_g with $C_g + 1$ we can assume that $C_g \geqslant 1$.

Denote by $m_u := \int_Y u\,\omega_Y^n$ and observe that $u' := u - m_u$, $v' := v - m_v$ satisfy $\int_X u'\,\omega_X^n = 0 = \int_X v'\,\omega_X^n$. Then the triangle inequality gives

$$\begin{aligned}\|u-v\|_{L^p(\omega_Y^n)} &= \left(\int_Y |(u'-v')+(m_u-m_v)|^p \omega_Y^n\right)^{1/p} \\ &\leqslant \|u'-v'\|_{L^p(\omega_Y^n)} + |m_u - m_v| \\ &\leqslant \|u'-v'\|_{L^p(\omega_Y^n)} + \|u-v\|_{L^1(\omega_Y^n)}.\end{aligned} \tag{2.7}$$

At this point, we make use of [9, Proposition 3.2] (that holds for normalized potentials) to replace the L^p-norm with the L^1-norm. We then infer the existence of a constant $c' > 1$ such that

$$\|u'-v'\|_{L^p(\omega_Y^n)} \leqslant c' \max(1, -\log\|u'-v'\|_{L^1(\omega_Y^n)})^{\frac{p-1}{p}} \|u'-v'\|_{L^1(\omega_Y^n)}^{\frac{1}{p}}.$$

When $t := \|u'-v'\|_{L^1(\omega_Y^n)} \geqslant 1/e$ we clearly have

$$\|u'-v'\|_{L^p(\omega_Y^n)} \leqslant c'\|u'-v'\|_{L^1(\omega_Y^n)}^{\frac{1}{p}},$$

whereas for any integer $N \in \mathbb{N}^*$, there exists a constant $c_N > 0$ such that $-\log t \leqslant c_N t^{-1/N}$ when $t \leqslant 1/e$, hence

$$\|u'-v'\|_{L^p(\omega_Y^n)} \leqslant c''\|u'-v'\|_{L^1(\omega_Y^n)}^{\frac{1}{p}\left(1-\frac{p-1}{N}\right)}.$$

As $\|u'-v'\|_{L^1(\omega_Y^n)} \leqslant 2\|u-v\|_{L^1(\omega_Y^n)}$, combining (2.5), (2.6) and (2.7) we get

$$\|f^*u - f^*v\|_{L^1(\mu)} \leqslant C \max\left(\|u-v\|_{L^1(\omega_Y^n)}^{\tilde{\beta}}, \|u-v\|_{L^1(\omega_Y^n)}\right),$$

with $\tilde{\beta} = \frac{\beta}{p}\left(1 - \frac{p-1}{N}\right)$. By [9, Lemma 3.3] $f_*\mu = (\omega_Y + dd^c\psi)^n$ where ψ is a Hölder continuous function.

To get a bound on the Hölder regularity of ψ, one argues as follows. First if $\mu = (\omega + dd^c\varphi)^n$ with φ a α-Hölder potential, and $\pi\colon \Gamma \to X$ is a proper modification, then $\pi^*\mu$ is dominated by a Monge–Ampère measure with α-Hölder potential, and [4, Proposition 3.3 (ii)] is satisfied with $b = 2\alpha/(\alpha + 2n)$ by [4, Theorem 4.3 (iii)]. Hence, following the proof of [4, Theorem], we see that $\pi^*\mu$ is a Monge–Ampère measure of a α_1-Hölder continuous potential with $\alpha_1 < b/(n+1)$ (see Remark below for more details about the latter statement).

By [9, Proposition 4.1], (2.5) holds with $\beta = \alpha_1^n/(2+\alpha_1^n)$, and (2.4) is then satisfied for any $\tilde{\beta} < \beta/p$ so that $f_*\mu$ is a Monge–Ampère measure with $\tilde{\alpha}$-Hölder potential for any $\tilde{\alpha} < 2\tilde{\beta}/(m+1)$, see the discussion on [9, p. 83]. Combining all these estimates we see that any

$$\tilde{\alpha} < \frac{\alpha^n}{p(m+1)(\alpha/2+n)^n(n+1)^n}$$

works where p is the conjugate of the larger constant $\varepsilon > 0$ for which Proposition 2.1 holds. □

Remark. — We borrow notations from the proof of [4, Theorem A]. Fix $\alpha_1 < b/(n+1)$ and choose $\varepsilon > 0$ such that $\alpha_1 \leqslant \alpha \leqslant \alpha_0 \leqslant b - \alpha_0(n+\varepsilon)$. By the previous arguments we know that condition (*ii*) in [4, Proposition 3.3] holds, i.e. for any $\phi \in \mathrm{PSH}(\Gamma, \omega_\Gamma)$, we have $\|\rho_\delta\phi - \phi\|_{L^1(\pi^*\mu)} = O(\delta^b)$, where $b = 2\alpha/(\alpha + 2n)$. In particular, this gives

$$\pi^*\mu(E_0) \leqslant c_1\delta^{b-\alpha_0}.$$

Let $g \in L^1(\pi^*\mu)$ be defined as $g = 0$ on E_0 and $g = c$ on $\Gamma \setminus E_0$ where c is a positive constant such that $\pi^*\mu(\Gamma) = \int_\Gamma g \,\mathrm{d}(\pi^*\mu)$. An easy computation gives that $c = \pi^*\mu(\Gamma)/\pi^*\mu(\Gamma \setminus E_0)$. Let $v \in \mathrm{PSH}(\Gamma, \omega_\Gamma)$ be the bounded solution of the Monge–Ampère equation $(\omega_\Gamma + \mathrm{dd}^c v)^n = g \cdot \pi^*\mu$. Observe that

$$\|1 - g\|_{L^1(\pi^*\mu)} = \int_{E_0} \mathrm{d}\pi^*\mu + \int_{\Gamma\setminus E_0} |1 - c| \,\mathrm{d}\pi^*\mu = 2\int_{E_0} \mathrm{d}\pi^*\mu \leqslant 2c_1\delta^{b-\alpha_0}.$$

Since $\pi^*\mu = (\omega_\Gamma + \mathrm{dd}^c\tilde{\varphi})^n$ satisfies the $\mathcal{H}(\infty)$ property we can still apply [8, Theorem 1.1] and get

$$\|\tilde{\varphi} - v\|_{L^\infty} \leqslant c_3\delta^{\frac{b-\alpha_0}{n+\varepsilon}}.$$

The exact same arguments as in [4, Theorem A] then insure that the Hölder exponent of $\tilde{\varphi}$ is α_1.

3. Over a one-dimensional base: proof of Theorem 1.4

In this section we treat Problem 1.1 in the case the base is a Riemann surface.

We start with the case of a surjective holomorphic map $f\colon X \to Y$ from a Kähler compact manifold to a compact Riemann surface.

Let $\mu = (\omega_X + \mathrm{dd}^c\varphi)^n$ be a Monge–Ampère measure of a continuous ω_X-psh function φ. Suppose v_k, v is a family of ω_X-psh functions such that $v_k \to v$ in L^1, then

$$\begin{aligned}\int_X v_k \,\mathrm{d}\mu &= \int_X v_k\,(\omega_X + \mathrm{dd}^c\varphi)^n \\ &= \int_X v_k\,\omega_X^n + \sum_{j=0}^{n-1}\int_X \varphi\,\mathrm{dd}^c v_k \wedge \omega_X^j \wedge (\omega_X + \mathrm{dd}^c\varphi)^{n-j-1} \to \int_X v\,\mathrm{d}\mu\end{aligned}$$

by [3, Corollary 1.6 (a)]. Observe that in the last equality we used the fact that

$$(\omega_X + \mathrm{dd}^c\varphi)^n - \omega_X^n = \sum_{j=0}^{n-1} \mathrm{dd}^c\varphi \wedge \omega_X^j \wedge (\omega_X + \mathrm{dd}^c\varphi)^{n-j-1}$$

and Stokes' theorem.

Normalize the Kähler form on Y such that $\int \omega_Y = 1$, and pick any sequence $y_k \to y_\infty \in Y$. Let w_k be the solutions of the equations $\Delta w_k = \delta_{y_k} - \omega_Y$ with $\sup w_k = 0$ so that $w_k(y) - \log|y - y_k|$ is continuous in local coordinates near y_k. Write $f_*\mu = \omega_Y + \mathrm{dd}^c\psi$ so that

$$\int_Y w_k\,\mathrm{d}(f_*\mu) = \int_Y w_k\,\omega_Y + \int_Y \psi\,\Delta w_k = \psi(y_k) + \int_Y (w_k - \psi)\omega_Y.$$

Since $w_k \to w_\infty$ in L^p_{loc} for all $p < \infty$, Proposition 2.1 implies that $f^*w_k \to f^*w_\infty$ in the L^1 topology, so that the argument above gives $\int_Y w_k\,\mathrm{d}(f_*\mu) = \int_X f^*w_k\,\mathrm{d}\mu \to \int_X f^*w_\infty\,\mathrm{d}\mu = \int_Y w_\infty\,\mathrm{d}(f_*\mu)$ We then conclude that $\psi(y_k) \to \psi(y_\infty)$. Hence ψ is continuous.

Suppose then that μ is locally the Monge–Ampère of a bounded psh function, and pick any subharmonic function u defined in a neighborhood of a point $y \in Y$. Then f^*u is again psh in a neighborhood of $f^{-1}(y)$, and the standard Chern–Levine–Nirenberg inequalities imply that $f^*u \in L^1(\mu)$ so that $u \in L^1(f_*\mu)$ with a norm depending only on the L^1-norm of u. It follows that $f_*\mu$ is locally the laplacian of a bounded subharmonic function.

Finally, assume $\mu = (\omega_X + \mathrm{dd}^c\varphi)^n$ for some $\varphi \in \mathcal{E}^p(X, \omega_X)$. By [13, Theorem C] this is equivalent to have that $\mathcal{E}^p(X, \omega_X) \subset L^p(\mu)$. Write as usual $f_*\mu = (\omega_Y + \mathrm{dd}^c\psi)$ with $\psi \in \mathcal{E}(Y, \omega_Y)$.

We claim that $u \in \mathcal{E}^p(Y, \omega_Y)$ implies $f^*u \in \mathcal{E}^p(X, \omega_X)$. Indeed, without loss of generality we can assume that $\Omega := \omega_X - f^*\omega_Y$ is a Kähler form and by the multilinearity of the non-pluripolar product we have

$$\begin{aligned}\int_X |f^*u|^p(\omega_X + \mathrm{dd}^c f^*u)^n &= \int_X |f^*u|^p(f^*\omega_Y + \Omega + \mathrm{dd}^c f^*u)^n \\ &= \int_X |f^*u|^p\left(\Omega^n + (f^*\omega_Y + \mathrm{dd}^c f^*u) \wedge \Omega^{n-1}\right)\end{aligned}$$

where the last identity follows from the fact that $(f^*\omega_Y + \mathrm{dd}^c f^*u)^j = 0$ for $j > 1$. The term $\int_X |f^*u|^p\Omega^n$ is bounded thanks to the integrability properties of quasi-plurisubharmonic functions w.r.t. volume forms [14, Theorem 1.47]; while the term

$$\int_X |f^*u|^p(f^*\omega_Y + \mathrm{dd}^c f^*u) \wedge \Omega^{n-1} = C(f)\int_Y |u|^p\,(\omega_Y + \mathrm{dd}^c u)$$

is finite since $u \in \mathcal{E}^p(Y, \omega_Y)$. This proves the claim.

Now, given any $u \in \mathcal{E}^p(Y, \omega_Y)$ we have

$$\int_Y |u|^p \, \mathrm{d}(f_* \mu) = \int_X |f^* u|^p \mathrm{d}\mu < +\infty$$

since $f^* u \in \mathcal{E}^p(X, \omega_X) \subset L^p(\mu)$. The conclusion follows from [13, Theorem C].

Consider now any dominant meromorphic map $f \colon X \dashrightarrow Y$ from a Kähler compact manifold to a compact Riemann surface. As above we decompose f such that $f_* \mu = (\pi_2)_* \pi_1^* \mu$ for any positive measure μ on X.

Assume that μ has continuous potentials. If we write $\mu = (\omega_X + \mathrm{dd}^c \varphi)^n$ then $\pi_1^* \mu = (\pi_1^* \omega_X + \mathrm{dd}^c \varphi \circ \pi)^n \leqslant (C \omega_\Gamma + \mathrm{dd}^c \varphi \circ \pi)^n := \hat{\mu}$ where $\hat{\mu}$ has a continuous potential. This implies $f_* \mu \leqslant (\pi_2)_* \hat{\mu}$. Observe that by the previous arguments the measure $(\pi_2)_* \hat{\mu}$ has continuous potential. It follows that locally $f_* \mu = \Delta v \leqslant \Delta u$ where u, v are subharmonic functions. It follows that v is the sum of a continuous function and the opposite of a subharmonic (hence u.s.c.) function. Since it is also u.s.c we conclude to its continuity.

When μ has bounded potentials, the same argument applies noting that subharmonic functions are always bounded from above which implies v to be bounded.

Finally, we consider the case where μ is the Monge–Ampère measure of $\varphi \in \mathcal{E}^p(X, \omega_X)$. We first observe that given $v \in \mathcal{E}^p(\Gamma, \omega_\Gamma)$ we have $(\pi_1)_* v \in \mathcal{E}^p(X, \omega_X)$. Indeed,

$$\begin{aligned} \int_X |v \circ \pi^{-1}|^p (\omega_X + \mathrm{dd}^c v \circ \pi^{-1})^n &= \int_\Gamma |v|^p (\pi_1^* \omega_X + \mathrm{dd}^c v)^n \\ &\leqslant \int_\Gamma |v|^p (C \omega_\Gamma + \mathrm{dd}^c v)^n < +\infty. \end{aligned}$$

This and the previous arguments give that if $u \in \mathcal{E}^p(Y, \omega_Y)$ then $f_* u = (\pi_1)_* \pi_2^* u \in \mathcal{E}^p(X, \omega_X)$, hence

$$\int_Y |u|^p \mathrm{d} f_* \mu = \int_X |u \circ f|^p \mathrm{d}\mu < +\infty.$$

It follows from [13, Theorem C] that $f_* \mu$ is the Monge–Ampère measure of a function in $\mathcal{E}^p(Y, \omega_Y)$.

4. The case of submersions

In this section we let $(X,\omega_X),(Y,\omega_Y)$ be two compact Kähler manifolds of dimension n and m, respectively and normalized such that $\int_X \omega_X^n = 1 = \int_Y \omega_Y^n$.

Proposition 4.1. — *Let $f\colon X \to Y$ be a submersion. Then, $u \in \mathcal{E}^p(Y,\omega_Y)$ implies $f^*u \in \mathcal{E}^p(X,\omega_X)$. In particular, if a probability measure μ is the Monge–Ampère of a function in $\mathcal{E}^p$ then also $f_*\mu$ has also a potential in $\mathcal{E}^p$.*

Proof. — Since f is a submersion we can assume that there is a finite number of open neighbourhoods U_i such that $X \subset \bigcup_{j=0}^N U_j$, $f|_{U_j}(z,w) = z$ where $z = (z_1,\dots,z_m)$ and $w = (z_{m+1},\dots,z_n)$. Moreover we can assume that on each U_j we have

$$\omega_X \leqslant C_j \frac{i}{2}\left(\mathrm{d}z\wedge \mathrm{d}\bar z + \mathrm{d}w \wedge \mathrm{d}\bar w\right), \qquad \frac{i}{2}\mathrm{d}z\wedge \mathrm{d}\bar z \leqslant A_j f^*\omega_Y$$

where $A_j, C_j > 1$ and $\mathrm{d}z\wedge\mathrm{d}\bar z, \mathrm{d}w\wedge\mathrm{d}\bar w$ are short notations for $\sum_{j=1}^m \mathrm{d}z_j\wedge d\bar z_j$ and $\sum_{k=m+1}^n \mathrm{d}z_k \wedge \mathrm{d}\bar z_k$, respectively. We then write

$$\begin{aligned}
&\int_X |f^*u|^p(\omega_X + \mathrm{dd}^c f^*u)^n \\
&\quad\leqslant \sum_{j=1}^N \int_{U_j} |f^*u|^p \left(C_j\frac{i}{2}\mathrm{d}z\wedge\mathrm{d}\bar z + C_j\frac{i}{2}\mathrm{d}w\wedge\mathrm{d}\bar w + \mathrm{dd}^c f^*u\right)^n \\
&\quad\leqslant \sum_{j=1}^N \int_{U_j} |f^*u|^p \left(A_j' f^*\omega_Y + C_j\frac{i}{2}\mathrm{d}w\wedge\mathrm{d}\bar w + \mathrm{dd}^c f^*u\right)^n \\
&\quad= \sum_{j=1}^N\sum_{\ell=0}^n \int_{U_j} |f^*u|^p \left(A_j' f^*\omega_Y + \mathrm{dd}^c f^*u\right)^\ell \wedge \left(C_j\frac{i}{2}\mathrm{d}w\wedge\mathrm{d}\bar w\right)^{n-\ell} \\
&\quad= \sum_{j=1}^N \int_{U_j} |f^*u|^p \left(A_j' f^*\omega_Y + \mathrm{dd}^c f^*u\right)^m \wedge \left(C_j\frac{i}{2}\mathrm{d}w\wedge\mathrm{d}\bar w\right)^{n-m}.
\end{aligned}$$

The above integral is then finite because by assumption $u \in \mathcal{E}^p(Y, A\omega_Y)$ for any $A \geqslant 1$.

The last statement follows from the same arguments in the last part of the proof in the previous section. □

5. Tame families of Monge–Ampère measures: proof of Corollary 1.5

Recall the setting from the introduction: $\mathcal{X}$ is a smooth connected complex manifold of dimension $n+1$, and $\pi\colon \mathcal{X} \to \mathbb{D}$ is a flat proper analytic map over the unit disk which is a submersion over the punctured disk and has connected fibers. We let $p\colon \mathcal{X}' \to \mathcal{X}$ be a proper bi-meromorphic map from a smooth complex manifold $\mathcal{X}'$ which is an isomorphism over $\pi^{-1}(\mathbb{D}^*)$.

We let T be any closed positive $(1,1)$-current on $\mathcal{X}'$ admitting local Hölder continuous potentials. Observe that by e.g. [3, Corollary 1.6] we have

$$\mu'_t = \mathrm{dd}^c \log|\pi \circ p - t| \wedge T^n \to \mu'_0 := \mathrm{dd}^c \log|\pi \circ p| \wedge T^n.$$

Let us now analyze the structure of the positive measure $\mu_0 := p_* \mu'_0$. First observe that μ'_0 can be decomposed as a finite sum of positive measures $\mu'_E := (T|_E)^n$ where the sum is taken over all irreducible components E of $\mathcal{X}'_0$. Each of these measures is locally the Monge–Ampère of a Hölder continuous psh function.

Write $V := (E)$. Since E is irreducible, V is also an irreducible (possibly singular) subvariety of dimension ℓ. To conclude the proof it remains to show that $p_*(\mu'_E)$ is the Monge–Ampère measure of Hölder continuous function that is locally the sum of a smooth and psh function. More precisely, one needs to show that $p_*(\mu'_E)$ does not charge any proper algebraic subset of V, and given any resolution of singularities $\varpi\colon V' \to V$ the pull-back measure $\varpi^*(p_*(\mu'_E))$ can be locally written as $(\mathrm{dd}^c u)^\ell$ where u is a Hölder psh function on V'.

This follows from Theorem 1.3 applied to any resolution of singularities V' of V and to any E' which admits a birational morphism $E' \to E$ such that the map $E' \to V'$ induced by p is also a morphism.

BIBLIOGRAPHY

[1] A. Aizenbud & N. Avni, "Representation Growth and Rational Singularities of the Moduli Space of Local Systems", *Invent. Math.* **204** (2016), no. 1, p. 245-316.

[2] E. Bedford & B. A. Taylor, "The Dirichlet problem for the complex Monge-Ampère equation", *Invent. Math.* **37** (1976), p. 1-44.

[3] J.-P. Demailly, "Monge–Ampère operators, Lelong numbers and intersection theory", in *Complex analysis and geometry*, The University Series in Mathematics, Plenum Press, 1993.

[4] J.-P. Demailly, S. Dinew, V. Guedj, P. H. Hiep, S. Kołodziej & A. Zeriahi, "Hölder continuous solutions to Monge-Ampère equations", *J. Eur. Math. Soc.* **16** (2014), no. 4, p. 619-647.

[5] E. Di Nezza, "Stability of Monge–Ampère energy classes", *J. Geom. Anal.* **25** (2014), no. 4, p. 2565-2589.

[6] ———, "Finite Pluricomplex energy measures", *Potential Anal.* **44** (2015), no. 1, p. 155-167.

[7] S. Dinew, V. Guedj & A. Zeriahi, "Open problems in pluripotential theory", *Complex Var. Elliptic Equ.* **61** (2016), no. 7, p. 902-930.

[8] S. Dinew & Z. Zhang, "On stability and continuity of bounded solutions of degenerate complex Monge–Ampère equations over compact Kähler manifolds", *Adv. Math.* **225** (2010), no. 1, p. 367-388.

[9] T. C. Dinh & V. A. Nguyen, "Characterization of Monge-Ampère measures with Hölder continuous potential", *J. Funct. Anal.* **266** (2014), p. 67-84.

[10] T. C. Dinh, V. A. Nguyen & N. Sibony, "Exponential estimates for plurisubharmonic functions and stochastic dynamics", *J. Differ. Geom.* **84** (2010), p. 465-488.

[11] C. Favre, "Degeneration of endomorphisms of the complex projective space in the hybrid space", *J. Inst. Math. Jussieu* (2018), 43 p., published online.

[12] V. Guedj & A. Zeriahi, "Intrinsic capacities on compact Kähler manifolds", *J. Geom. Anal.* **15** (2005), no. 4, p. 607-639.

[13] ———, "The weighted Monge-Ampère energy of quasiplurisubharmonic functions", *J. Funct. Anal.* **250** (2007), no. 2, p. 442-482.

[14] ———, *Degenerate Complex Monge–Ampère Equations*, EMS Tracts in Mathematics, vol. 26, Société Mathématique de France, 2017, xxiv+472 pages.

[15] S. Kołodziej & N. C. Nguyen, "Hölder continuous solutions of the Monge-Ampère equation on compact hermitian manifolds", *Ann. Inst. Fourier* **68** (2018), no. 7, p. 2951-2964.

[16] M. Lejeune-Jalabert, B. Teissier & J.-J. Risler, "Clôture intégrale des idéaux et équisingularité", *Ann. Fac. Sci. Toulouse, Math.* **17** (2008), no. 4, p. 781-859.

[17] T. Peternell, "Modifications", in *Several complex variables VII: Sheaf theoretic methods in complex analysis*, Encyclopaedia of Mathematical Sciences, vol. 74, Springer, 1994, p. 285-317.

[18] A. Reiser, "Pushforwards of Measures on Real Varieties under Maps with Rational Singularities", `https://arxiv.org/abs/1807.00079v1`, 2018.

[19] J. Song & G. Tian, "Canonical measures and Kähler-Ricci flow", *J. Am. Math. Soc.* **25** (2012), no. 2, p. 303-353.

[20] V. Tosatti, "Adiabatic limits of Ricci-flat Kähler metrics", *J. Differ. Geom.* **84** (2010), no. 2, p. 427-453.

Eleonora DI NEZZA
Sorbonne Université
4 Place Jussieu
75005 Paris (France)
eleonora.dinezza@imj-prg.fr

Charles FAVRE
CMLS, École polytechnique, CNRS, Université Paris-Saclay
91128 Palaiseau Cedex (France)
charles.favre@polytechnique.edu