ANNALES

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Tome 68, nº 6 (2018), p. 2445-2476.

<http://aif.cedram.org/item?id=AIF_2018__68_6_2445_0>

Ann. Inst. Fourier, Grenoble
68, 6 (2018) 2445-2476

ZEROES AND RATIONAL POINTS OF ANALYTIC FUNCTIONS

by Georges COMTE & Yosef YOMDIN (*)

Abstract. — For an analytic function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ on a neighbourhood of a closed disc $D \subset \mathbb{C}$, we give assumptions, in terms of the Taylor coefficients a_k of f, under which the number of intersection points of the graph Γ_f of $f_{|D}$ and algebraic curves of degree d is polynomially bounded in d. In particular, we show these assumptions are satisfied for random power series, for some explicit classes of lacunary series, and for solutions of algebraic differential equations with coefficients and initial conditions in $\mathbb{Q}$. As a consequence, for any function f in these families, Γ_f has less than $\beta \log^\alpha T$ rational points of height at most T, for some $\alpha, \beta > 0$.

Résumé. — Pour une function analytique $f(z) = \sum_{k=0}^{\infty} a_k z^k$ définie sur un voisinage d'un disque fermé $D \subset \mathbb{C}$, nous donnons des conditions, portant sur les coefficients de Taylor a_k de f, sous lesquelles le nombre de points d'intersection du graphe Γ_f de $f_{|D}$ avec les courbes algébriques de degré d est polynomialement borné en d. Nous montrons en particulier que ces conditions sont satisfaites pour les séries entières aléatoires, pour certaines classes explicites de séries lacunaires, et pour les solutions d'équations différentielles algébriques avec coefficients et conditions initiales rationnels. En conséquence, pour toute fonction f dans une de ces familles, Γ_f possède moins de $\beta \log^\alpha T$ points rationnels de hauteur au plus T, pour $\alpha, \beta > 0$.

1. Introduction

Let $D_R \subset \mathbb{C}$ be a closed disc, centred at the origin and of radius R, let $f : D_R \to \mathbb{C}$ be an analytic function on a neighbourhood of D_R, and for any $d \geqslant 1$, let us denote by $\mathcal{P}_d$ the subspace of polynomials of $\mathbb{C}[X, Y]$ of degree

Keywords: zeroes of analytic functions, rational points.
2010 *Mathematics Subject Classification:* 14G05, 30B10, 30B20, 30D15.
(*) This research was supported by the ISF, Grant No. 779/13, by FSMP and by ANR DEFIGEO. The authors would like to thank University of Savoie Mont Blanc and the Weizmann Institute for the hospitality during part of the research for this paper. The authors are very grateful to the anonymous referee for its careful reading and for many insightful suggestions.

at most d. Assuming that for any polynomial $P \in \mathbb{C}[X,Y]$, $P(z, f(z))$ is not identically zero, or in other words, that f is transcendental, the quantity

$$Z_d(f) = \sup_{P \in \mathcal{P}_d \setminus \{0\}} \#\{z \in D_R; P(z, f(z)) = 0\},$$

(where zeroes are counted with multiplicity) is finite.

The integer $Z_d(f)$ is also the maximum number of intersection points between the graph Γ_f of f and algebraic curves of degree at most d. Thus any bound of $Z_d(f)$ will be called in the sequel a Bézout bound for f, since such a bound is called a Bézout bound in the case of proper intersection of two algebraic curves. Moreover in the algebraic case a polynomial bound in the degrees involved holds.

Bounding $Z_d(f)$ is very closely related to the study of "Doubling", or "Bernstein - type" inequalities for restrictions of polynomials P as above to the graph Γ_f of f (see, for instance, [21, 37]). Through this connection, many methods and results in Potential Theory, on one side, and in Approximation Theory, on the other side, become relevant, starting with the basic results of [38, 39]. Still, we try to keep the references to the minimum.

This article studies the growth of $Z_d(f)$, as d goes to infinity; more specifically, we provide certain assumptions on the transcendental analytic function f, under which f has, like in the algebraic case, a Bézout bound polynomial in d.

The asymptotic behaviour of $Z_d(f)$ as d goes to infinity was studied for different classes of functions, with different tools. In fact, $Z_d(f) < \infty$ generally holds when f is definable in an o-minimal structure expanding the real field[1]. But in this very general situation the behaviour of f with respect to algebraic curves of growing degree is difficult to predict. In the analytic case, considered in the present paper, the following observation is instructive: for any $\zeta \in \,]0,1[$ there is analytic function f such that for a sequence of degrees d going to infinity, $Z_d(f) \geqslant e^{d^\zeta}$ (see, for instance, [34, Example 7.5], [41] or [42], and inequality (5.4) of Remark 5.3 below). On the other hand, one also knows that for any analytic function f, $Z_d(f)$ is bounded from above by a polynomial in d of degree 2, for a certain sequence of degrees d going to infinity (see [22, Theorem 1.1]) and the asymptotic of this upper bound is best possible (see for instance [21, Corollary 2.6])

A polynomial in d Bézout bound for $f(z) = e^z$ was obtained in [43]. For entire functions f of positive order (under some additional conditions)

[1] Note that, on the opposite side, by [26], a polynomial Bézout bound for f does not imply that f is polynomial or even definable in some o-minimal structure.

such bounds were obtained in [21, 22][2] and in [13, 16]. For the Riemann zeta function, and for some other specific functions accurate polynomial bounds were obtained in [11, 30]. Very recently, for solutions of certain types of algebraic differential equations, a polynomial in d Bézout bound was obtained in [7].

Our approach to the problem of bounding $Z_d(f)$, for $f(z) = \sum_{k=0}^{\infty} a_k z^k$, is based on a detailed algebraic study of the Taylor coefficients a_k of f. It follows a long research line, starting with a classical work of Bautin ([2, 3]). Bautin's discovery was that for analytic families $f_\lambda(z) = \sum_{k=0}^{\infty} v_k(\lambda) z^k$, with λ, a parameter ranging in a finite dimensional space, and $v_k(\lambda)$, polynomials in λ, the number of zeroes can be bounded in terms of the polynomial ideals $I_r = \{v_0(\lambda), \dots, v_r(\lambda)\}$ (the Bautin ideals). This approach was further developed in many publications (see, for instance, [15, 17, 24, 37, 45] and references therein). In particular, the case of linear families (that is to say $v_k(\lambda)$ is linear in λ) where computations are reduced to linear algebra, was considered in [44]. In the present paper we concentrate on linear families of the form $P(z, f(z))$. Notice that some notions introduced below play important role also in Hermite–Padé approximations (see, for instance, [1, 33], and references therein, and Remark 3.10 below). We plan to present some instances of this important connection separately.

We will consider polynomials $P(z, y) = \sum_{0 \leqslant i,j \leqslant d} \lambda_{i,j} z^i y^j$. In this case the parameter vector $\lambda = (\lambda_{i,j})$ ranges in the space $\mathbb{C}^m$, with $m = (d+1)^2$ and we have

$$f_\lambda(z) = P(z, f(z)) = \sum_{k=0}^{\infty} v_k(\lambda) z^k,$$

with $v_k(\lambda)$ linear forms in the parameter λ. The specific coefficients of $v_k(\lambda)$ can be explicitly written through the Taylor coefficients a_k of f (see Section 3 below). For each k we can write

$$\begin{pmatrix} v_0(\lambda) \\ \vdots \\ v_k(\lambda) \end{pmatrix} = M_k \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix},$$

with a $(k+1) \times m$ matrix M_k. We define the Bautin index $b_d = b_d(f)$ as the minimal k for which the rank of M_k is equal to m, and call $M_{b_d} = M_{b_d}(f)$ the d-th Bautin matrix of f (see Definition 2.9). The Bautin index b_d is also the maximal, with respect to λ, multiplicity at 0 of $f_\lambda(z)$ (see

[2] For the first time instances of analytic functions (with lacunary Taylor series at the origin) having polynomial Bézout bound with prescribed growth were provided in [21, Corollary 6.2].

Proposition 2.3). Finally, we denote by $\delta_d = \delta_d(f) > 0$ the maximum of the absolute value of all non-zero minor determinants of size $m \times m$ of the Bautin matrix M_{b_d}. In this way we associate to each transcendental analytic function $f(z) = \sum_{k=0}^{\infty} a_k z^k$, two sequences $(b_d)_{d\geqslant 1}$, and $(\delta_d)_{d\geqslant 1}$. For hypertranscendental f (that is when all the derivatives $f^{(r)}$, $r \geqslant 0$, are algebraically independent) we also define a sequence of multiplicities $(\eta_d)_{d\geqslant 1}$ in the following way (see also Definition 3.13).

Consider the Taylor expansion of $f(z) = \sum_{k=0}^{\infty} a_k(u)(z-u)^k$ at points u near the origin. Using these expansions we construct, as above, the square matrices $M_m(u)$, $m = (d+1)^2$. Assume that the determinants $\Delta_d(u)$ of $M_m(u)$ do not vanish identically in u, and let $\eta_d(f)$ be the multiplicity of zero of $\Delta_d(u)$ at $u = 0$. We then have

$$\Delta_d(u) = \alpha_d u^{\eta_d} + O(u^{\eta_d+1}), \text{ with } \alpha_d \neq 0.$$

The three sequences $(\delta_d), (b_d), (\eta_d)$ are defined algebraically, through a finite amount (depending on d) of the Taylor coefficients a_k of f, and they present natural transcendence measures of f. In the present paper we investigate some basic properties of these sequences, and their role in bounding $Z_d(f)$. All the three sequences can be defined for formal power series $f(z)$ with coefficients in an arbitrary field, although here we work only with complex (or real) analytic functions on D_R.

The following are our main results:

(1) We bound $Z_d(f)$ in terms of $b_d(f)$ and $\delta_d(f) > 0$ (Proposition 2.11). In particular, for $b_d(f) \leqslant R(d)$, and $\delta_d(f) \geqslant e^{-S(d)}$, with R, S polynomials in d, we have $Z_d(f) \leqslant T(d)$, with T also a polynomial in d.

(2) If the Taylor coefficients a_k of f are rational, we define $h_l(f)$ as the maximal denominator of a_k, $k = 0, 1, \ldots, l$. We bound from below δ_d in terms of b_d and h_{b_d} (Proposition 3.8). In particular, for $b_d(f) \leqslant R(d)$, and $h_l(f) \leqslant e^{S(l)}$, with R and S polynomials, we show that $\delta_d(f) \geqslant e^{-U(d)}$, and hence $Z_d(f) \leqslant T(d)$, with T, U also polynomials in d (Theorem 3.9). As an example, we show that this is the case for solution of algebraic ODE's with rational coefficients and rational initial values (Theorem 3.25). This gives another proof of one of results in [7].

(3) If the Taylor coefficients a_k at 0 of a hypertranscendental function f are rational and satisfy $h_l(f) \leqslant e^{S(l)}$, with S polynomial, and if there exists a polynomial R such that $\eta_d \leqslant R(d)$, $d \geqslant 1$, then we show that $Z_d(f) \leqslant T(d)$, with T also polynomial. (Theorem 3.14).

(4) We consider a class of lacunary series f, similar to the one considered in [20]. We show that for this class the Bautin index $b_d(f)$ can be explicitly estimated, as well as the Bautin matrices and the determinants $\delta_d(f)$. On this base we obtain examples of both polynomial and non-polynomial growth of $Z_d(f)$ (Theorem 3.20).

(5) Clearly, for a series f with random Taylor coefficients a_k, the square Bautin matrices $M_m(u),\ m = (d+1)^2$ are non-degenerate. Hence $b_d(f) = m = (d+1)^2$. We show that with probability one the determinants Δ_d of these matrices satisfy $\Delta_d \geqslant e^{-U(d)}$ (Theorem 4.2), and therefore $Z_d(f) \leqslant T(d)$, with T, U polynomials in d (Corollary 4.3).

(6) Analytic functions on compact domains with polynomial Bézout bounds have the following remarkable Diophantine property: the number of rational points of height $\leqslant T$ in their graph is bounded by a power of $\log T$. It means that they have few rational points of given height, since a sharp upper bound for this number, for analytic functions, is in $C_\epsilon T^\epsilon$, for any $\epsilon > 0$. Therefore our assumptions also provide families of transcendental sets with few rational points, as explained in Section 5 (see Theorem 5.2).

The paper is organized as follows. In Section 2 we consider general linear families f_λ, we introduce the definitions used throughout the paper, and we give a general bound for $Z_d(f)$ in terms of the Bautin index and the determinant δ. In Section 3 we prove the results (1) to (4). In Section 4 we prove (5). Finally in Section 5 we give direct Diophantine applications of our analytic Bézout theorems, as mentioned in (6).

2. Linear families of analytic functions

The beginning of this Section is based on [44].

Let us denote by D_R the disc $\{z \in \mathbb{C}, |z| \leqslant R\}$, let $\psi : (D_R, 0) \to (\mathbb{C}^n, 0)$ be an analytic curve, and let

$$\psi(t) = \sum_{k=1}^{\infty} a_k z^k,\ a_k \in \mathbb{C}^n, \tag{2.1}$$

be the Taylor expansion of ψ at $0 \in \mathbb{C}$. We assume that the series (2.1) converges in a neighbourhood of the disk $D_R = \{z \in \mathbb{C}, |z| \leqslant R\}$, and that $\|\psi(t)\| \leqslant A$ for any $z \in D_R$.

For $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{C}^m$, let

$$Q_\lambda = \sum_{i=1}^{m} \lambda_i Q_i,$$

with $Q_j : (\Omega, 0) \to \mathbb{C}$, $j = 1, \dots, m$, analytic functions in a neighbourhood Ω of the polydisc of $\mathbb{C}^n$ of radius A, and bounded there by B.

In what follows we are interested in linear families of analytic functions of the form

$$f_\lambda(z) = Q_\lambda(\psi(z)) = \sum_{i=1}^{m} \lambda_i Q_i(\psi(z)). \tag{2.2}$$

Write

$$Q_i(\psi(z)) = \sum_{k=0}^{\infty} c_k^i z^k. \tag{2.3}$$

By our assumptions $Q_i(\psi(z)) \leqslant B$ on D_R, $i = 1, \dots, m$. Hence, by Cauchy's estimates, we have

$$|c_k^i| \leqslant \frac{B}{R^k}. \tag{2.4}$$

For the Taylor development of the linear family f_λ we have

$$f_\lambda(z) = \sum_{k=0}^{\infty} v_k(\lambda) z^k, \quad v_k(\lambda) = \sum_{i=1}^{m} c_k^i \lambda_i. \tag{2.5}$$

Thus the coefficients $v_k(\lambda)$ of the series (2.5) are linear forms in λ and by (2.4), for any k

$$|v_k(\lambda)| \leqslant \frac{mB|\lambda|}{R^k}, \tag{2.6}$$

where $|\lambda| = \max\{|\lambda_1|, \dots, |\lambda_m|\}$.

Now we associate to the family f_λ an integer $b = b(f_\lambda)$. Let, for $i \in \mathbb{N}$, $L_i \subseteq \mathbb{C}^m$ be the linear subspace of $\mathbb{C}^m$ defined by the equations $v_0(\lambda) = \dots = v_i(\lambda) = 0$. We have

$$L_0 \supseteq L_1 \supseteq \dots L_i \supseteq \dots.$$

Hence on a certain step b this sequence stabilizes

$$L_{b-1} \supsetneq L_b = L_{b+1} = \dots = L.$$

Note that $\lambda \in L_b \iff v_k(\lambda) = 0$, for $k = 0, 1, \dots$

Definition 2.1. — *We call this number b the* Bautin index *of the family f_λ (see* [2, 3]*).*

Remark 2.2. — Of course the dimension of the subspaces $L_i \subset \mathbb{C}^m$, $i \in \mathbb{N}$, is at most m and if $z \mapsto f_\lambda(z)$ is not identically zero for all λ, $\dim(L_b) \leqslant m-1$. More accurately, if we assume that for $\lambda \neq 0$ the function $f_\lambda(t)$ does not vanish identically, then we have $L_b = \{0\}$. But the dimension of L_i can drop at most by one at each step, thus necessarily in this situation $b \geqslant m-1$. This will be the case if $f(z)$ is a transcendental function for the family of polynomials of degree at most d, that is for $\psi(z) = (z, f(z))$ and $Q_{i,j} = X^i Y^j$ with $i, j \leqslant d$, a classical case mainly considered in the following sections.

On the other hand the Bautin index may be as big as wished. In general one cannot explicitly find the Bautin index of f_λ, since the moments, when the dimension of L_i drops, are usually difficult to determine.

A first characterization of the Bautin index of the family f_λ is the following.

Proposition 2.3. — *Let us assume that for $\lambda \neq 0$ the function $z \mapsto f_\lambda(z)$ is not identically zero, and let us denote by μ the maximal multiplicity, with respect to the parameters $\lambda \neq 0$, of the Taylor series at the origin of $f_\lambda(z)$. Then $\mu = b$.*

Proof. — There exists a parameter λ such that $f_\lambda(z)$ has multiplicity μ, that is such that

$$f_\lambda(z) = v_\mu(\lambda) z^\mu + v_{\mu+1}(\lambda) z^{\mu+1} + \dots,$$

with $v_\mu(\lambda) \neq 0$. Therefore $\lambda \in L_{\mu-1} \setminus L_\mu$. It follows that $L_\mu \subsetneqq L_{\mu-1}$, and $b \geqslant \mu$. On the other hand, since no parameter $\lambda \neq 0$ can cancel $v_0, \dots, v_\mu$ in the same time, $L_\mu = \{0\} = L_{\mu+1} = \dots$, and thus $b \leqslant \mu$. □

Remark 2.4. — The system $v_0(\lambda) = \dots = v_{m-2}(\lambda) = 0$, with m parameters, always having a non-zero solution, one sees that the maximal multiplicity of f_λ is at least $m-1$. Proposition 2.3 then implies that $b \geqslant m-1$, that was already noted in Remark 2.2.

Remark 2.5. — Under the assumption that for $\lambda \neq 0$ the function $z \mapsto f_\lambda(z)$ is not identically zero, and counting zeroes with multiplicity, from Proposition 2.3 one observes that

$$\lim_{r \to 0} \max_{\lambda \neq 0} \#\{z \in D_r, f_\lambda(z) = 0\} \geqslant b.$$

We give hereafter in Theorem 2.7 more accurate relations between the Bautin index and the number of zeroes of the family f_λ, showing in particular that the above inequality is an equality.

Since L is defined by $v_0(\lambda) = \cdots = v_b(\lambda) = 0$, any linear form $\ell(\lambda)$, which vanishes on L, can be expressed as a linear combination of $v_0, \ldots, v_b$. Now a basis $(v_{i_1}, \ldots, v_{i_\sigma})$, $i_1, \ldots, i_\sigma \in \{0, \ldots, b\}$, of the space of linear forms vanishing on L being chosen among the elements of the family $(v_0, \ldots, v_b)$, there exists a constant $\tilde{c} > 0$, depending on this basis, such that for any ℓ with $\ell_{|L} \equiv 0$, we have

$$\ell(\lambda) = \sum_{j=1}^{\sigma} \mu_j v_{i_j}(\lambda), \ \mu_j \in \mathbb{C}, \ \text{ and } \ |\mu_j| \leqslant \tilde{c}\|\ell\|, j = 1, \ldots, \sigma. \tag{2.7}$$

where for $\ell(\lambda) = \sum_{i=1}^{m} \alpha_i \lambda_i$, $\|\ell\| = \max_i |\alpha_i|$.

Notation 2.6. — We denote by $c = c(f_\lambda) > 0$ the minimum of the constants $\tilde{c}$ satisfying (2.7).

An effective estimation of c is difficult, in general. However, if the Bautin index $b(f_\lambda)$ is known, $c(f_\lambda)$ can be estimated via a finite computation in terms of the Taylor coefficients of ψ and of Q_i (see Proposition 2.11 below).

Now let a family f_λ be given, and let the Bautin index $b(f_\lambda)$ and the constant $c(f_\lambda)$ of this family be defined as above.

Theorem 2.7. — *Zeroes being counted with multiplicity, we have the following uniform bounds with respect to the parameter λ, when $f_\lambda \not\equiv 0$.*

(1) *The maximal number of zeroes of f_λ in the disk $D_{\frac{R}{4}}$ is at most*

$$\begin{cases} 5b\log\left(4 + 2c(b+1)\frac{B}{R^b}\right) & \text{if } R \leqslant 1, \\ 5b\log\left(4 + 2c(b+1)B\right) & \text{if } R \geqslant 1, \end{cases}$$

(2) *This maximal number of zeroes of f_λ is at most b in D_ρ, where*

$$\rho = \frac{R}{e^{10b+2}\max(2, c(b+1)B\max(\frac{1}{R},1)^b)}.$$

Proof. — For any $\lambda \in \mathbb{C}^m$, and for any $j \geqslant b+1$, by the definition of c and from the bound (2.4) (see Theorem 1.1 of [44] and the last inequality in its proof), we have

$$|v_j(\lambda)|R^j \leqslant c(b+1)B\max\left(\frac{1}{R}, 1\right)^b \max_{i=0,\ldots,b}|v_i(\lambda)|R^i. \tag{2.8}$$

Then, the bounds to prove on the number of zeroes are consequences of (2.8) and [37, Lemma 2.2.1, Theorem 2.1.3]. □

Remark 2.8. — Note that under the assumption that for $\lambda \neq 0$ the function $z \mapsto f_\lambda(z)$ is not identically zero, and counting zeroes with multiplicity, from Theorem 2.7 and Remark 2.5 one has

$$\lim_{r\to 0}\max_{\lambda\neq 0}\#\{z\in D_r, f_\lambda(z)=0\}=b.$$

This infinitesimal maximal number of zeroes is called the *cyclicity of Q_λ on ψ* (see [44]).

In what follows we develop explicit bounds on the number of zeroes of f_λ in $D_{\frac{R}{4}}$ in terms of the coefficients c^i_k, with $i=1,\dots,m,\ k=0,\dots,b$, i.e., ultimately, in terms of the Taylor coefficients of ψ and of Q_i up to the order b. By Theorem 2.7, this amounts bounding the constant $c(f_\lambda)$ introduced in Notation 2.6. Note also that such a bound is given by a bound on the coefficients $\mu_j,\dots,\mu_\sigma$ of the system (2.7), since in this system one can consider only linear forms ℓ of norm 1. Let us denote the dimension of the stabilized subspace $L=L_b$ by $s=m-\sigma$ ($\leqslant m-1$). All the information we need, as we will see, is encoded in the rank σ matrix $M=(c^i_k),\ k=0,\dots,b,\ i=1,\dots,m$, with $b+1$ lines and m columns. With our notation, $M=M_b$ is defined by

$$\begin{pmatrix} v_0(\lambda)\\ \vdots\\ v_b(\lambda)\end{pmatrix}=M_b\begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_m\end{pmatrix}.$$

Definition 2.9. — *With the above notation, the $(b+1)\times m$ matrix $M=M_b$ is called the* Bautin matrix of the family f_λ. *The matrix M_b is the matrix of the linear map Λ sending elements of the vector space $\mathcal{Q}$ spanned by the analytic functions $Q_1,\dots,Q_m$ (assumed to be linearly independent) to the space of b-jets at the origin of analytic functions. The map Λ is the composition of the linear maps $\widetilde{\Lambda}:\mathcal{Q}\to\mathbb{C}\{z\}$ sending Q_λ to f_λ with the linear map $j^b_0:\mathbb{C}\{z\}\to\mathbb{C}^{b+1}$ of b-jets at the origin, b being the first order of jets at the origin such that* $\dim(j^b_0(\widetilde{\Lambda}(\mathcal{Q}))=\dim(\widetilde{\Lambda}(\mathcal{Q}))$.

Notation 2.10. — We will denote by $\delta>0$ the maximum of the absolute value of all non-zero minor determinants of size $\sigma\times\sigma$ of the Bautin matrix M.

Proposition 2.11. — *Let f_λ be given as above. Then*

$$c(f_\lambda)\leqslant\sigma\frac{(B\sqrt{\sigma})^{\sigma-1}}{\delta R^{\beta(\sigma-1)}},\quad \text{where } \beta=\begin{cases} b & \text{if } R\leqslant 1\\ \frac{\sigma}{2} & \text{if } R\geqslant 1.\end{cases}$$

In turn, in the disk $D_{\frac{R}{4}}$, the maximal number $Z(f_\lambda)$ of zeroes of the family f_λ, with respect to the parameter λ, satisfies

$$Z(f_\lambda) \leqslant \begin{cases} 5b \log\left(4 + 2(b+1)\sqrt{\sigma}\frac{(B\sqrt{\sigma})^\sigma}{\delta R^{b\sigma}}\right) & \textit{if } R \leqslant 1, \\ 5b \log\left(4 + 2(b+1)\sqrt{\sigma}\frac{(B\sqrt{\sigma})^\sigma}{\delta R^{\frac{\sigma}{2}(\sigma-1)}}\right) & \textit{if } R \geqslant 1, \end{cases}$$

Proof. — Let $\hat{M}$ be a submatrix of M of size $\sigma \times \sigma$, with the absolute value of the determinant equal to δ, according to notation 2.10. $\hat{M}$ is obtained from M by the choice of the lines $i_1 < \cdots < i_\sigma$ of M (corresponding to the choice $(v_{i_1}, \dots, v_{i_\sigma})$ for a basis of the space of linear forms cancelling on L), and the choice of certain σ columns, say, the first σ columns in M. When $\|\ell\| \leqslant 1$, (2.7) gives the linear system

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_\sigma \end{pmatrix} = {}^t\hat{M} \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_\sigma \end{pmatrix}$$

with $|\alpha_j| \leqslant 1$, $j = 1, \dots, \sigma$. Therefore, by the Cramer rule, each μ_j satisfies

$$|\mu_j| \leqslant \frac{\sigma\hat{\delta}}{\delta}, \tag{2.9}$$

where $\hat{\delta}$ is the maximum of the absolute values of $(\sigma-1)\times(\sigma-1)$ sub-minors of $\hat{M}$. Next, by (2.4), we have

$$|c^i_{i_j}| \leqslant \frac{B}{R^{i_j}} \leqslant \begin{cases} \frac{B}{R^b} & \text{if } R \leqslant 1 \\ \frac{B}{R^j} & \text{if } R \geqslant 1. \end{cases}$$

Consequently, the length of the j-th row-vectors in $(\sigma - 1) \times (\sigma - 1)$ subminors of $\hat{M}$ does not exceed

$$\begin{cases} \frac{B\sqrt{\sigma-1}}{R^b} & \text{if } R \leqslant 1, \\ \frac{B\sqrt{\sigma-1}}{R^j} & \text{if } R \geqslant 1. \end{cases}$$

Interpreting the determinant as the volume of the span of its row-vectors, we conclude that

$$\sigma\hat{\delta} \leqslant \begin{cases} \sigma\frac{(B\sqrt{\sigma-1})^{\sigma-1}}{R^{b(\sigma-1)}} \leqslant \sigma\frac{(B\sqrt{\sigma})^{\sigma-1}}{R^{b(\sigma-1)}} & \text{if } R \leqslant 1, \\ \sigma\frac{(B\sqrt{\sigma-1})^{\sigma-1}}{R^{\frac{\sigma}{2}(\sigma-1)}} \leqslant \sigma\frac{(B\sqrt{\sigma})^{\sigma-1}}{R^{\frac{\sigma}{2}(\sigma-1)}} & \text{if } R \geqslant 1. \end{cases}$$

This bound, combined with (2.9) and Theorem 2.7, completes the proof of Proposition 2.11. □

Remark 2.12. — From the beginning of this section, we have assumed that the analytic map ψ is defined on a disc centred at the origin and that $\psi(0) = 0$. This choice is harmless since, in case ψ is defined on a ball centred at a, one can consider $\phi(z) = \psi(z+a) - \psi(a)$ and the bounds given in Theorem 2.7 and Proposition 2.11 for ϕ and the family $Q_i(w + \psi(a))$ are the same bounds for ψ and the family $Q_i(w)$ when Taylor series are considered at a instead of 0.

Remark 2.13. — A classical application of bounds given in Proposition 2.11 is for analytic plane curves $\psi(z) = (z, f(z))$ and the family $Q_i(X, Y)$ of two variables monomials of total degree at most some integer d. In this case we consider that f is given by its Taylor series at the origin and that this series converges on the disc D_R of radius R and centred at the origin, by Remark 2.12. The curve $\psi(z)$ is the standard parametrization of the graph of f. In this setting we will provide bounds for the number of zeroes of $P_d(z, f(z))$ on D_R that are uniform with respect to coefficients of polynomials P_d of degree at most d. Now note that in this situation one can only consider functions f that are bounded by 1 on D_R, since a uniform bound on the number of zeroes of $P_d(z, \frac{1}{N} f(z))$, where N bounds $f(z)$ on D_R, provides a uniform bound on the number of zeroes of $P_d(z, f(z))$. In the same way one can as well assume for simplicity that f is analytic on the unit disc, up to applying the bounds provided by Proposition 2.11 to the new function $g(z) = f(Rz)$, since a uniform bound on the zeroes of $P_d(w, g(w))$ on D_1 is a uniform bound on the zeroes of $P_d(z, f(z))$ on D_R. Of course the same rescaling effects apply in the same way for the family $X^i Y^j, i, j \leqslant d$, of two variables monomials. Nevertheless those reductions are not always possible for any family of analytic functions Q_j and for any analytic curve ψ.

Finally when one restricts to analytic functions bounded by 1 on D_1 and for the family $Q_{i,j} = X^i Y^j, i + j \leqslant d$ or for the family $X^i Y^j, i, j \leqslant d$ of two variables monomials, one can take 1 for B as bound for $Q_{i,j}(z, f(z))$ on D_1. Note that in this case, from Proposition 2.11 we deduce the following statement.

Corollary 2.14. — *Let f_λ be given as above with $R = 1$ and $B = 1$, then in the disk $D_{\frac{1}{4}}$ the maximal number $Z(f_\lambda)$ of zeroes of the family f_λ, with respect to the parameters λ, satisfies*

$$Z(f_\lambda) \leqslant 5b \log \left(4 + 2(b+1) \frac{e^{\sigma \log \sigma}}{\delta} \right).$$

Proof. — After taking $B = 1$ and $R = 1$ in the last inequality of Proposition 2.11, observe that $\sqrt{\sigma}^{\sigma+1} = e^{(\sigma+1)\frac{1}{2}\log \sigma} \leqslant e^{\sigma \log \sigma}$. □

3. Bézout bounds for transcendental analytic curves

3.1. Families of polynomials

From now on we consider the classical case of the family of polynomials with degree at most a fixed integer d. That is to say, for a given analytic function $f : (D_1, 0) \to (\mathbb{C}, 0)$

we want to bound the number of zeroes of $P_d(z, f(z))$, for P_d a two variables polynomial of degree at most d. Considering Remarks 2.12 and 2.13, for simplicity we assume that f sends 0 to 0, is given by its converging Taylor series on the unit disc D_1, and is bounded by 1 on D_1.

In fact, we consider the problem in the following (essentially equivalent) form[(3)]: give a bound for

$$\mathcal{Z}_d(f) := \max_{\substack{p_j \in \mathbb{C}[z] \\ \deg p_j \leqslant d}} \# \left\{ z \in D, \sum_{j=0}^{d} p_j(z) f^j(z) = 0 \right\}$$

for some disc $D \subset D_1$ centred at the origin. Denoting $p_j(z) = \sum_{i=0}^{d} \lambda_{j,i} z^i$, the sum $\sum_{j=0}^{d} p_j(z) f^j(z)$ has the form $\sum_{0 \leqslant i,j \leqslant d} \lambda_{j,i} z^i f^j(z)$.

With the notation of Section 2, we have to consider the family of monomials $Q_{i,j} = X^i Y^j$, $i, j \leqslant d$, and the analytic function $\psi(z) = (z, f(z))$, therefore here the number of parameters is $m = (d+1)^2$. Let us denote by $\mathcal{Q}_d$ the vector space spanned by the monomials $Q_{i,j} = X^i Y^j$, $i, j \leqslant d$. In order to guarantee $\mathcal{Z}_d(f) < \infty$ for any d as soon as some p_j is not zero, we assume that f is transcendental, that is no non-zero polynomial restricted to the graph of f vanishes identically. By Remark 2.2, the Bautin index $b = b_d$ of this family satisfies $b \geqslant m - 1 = d^2 + 2d$ and the corresponding Bautin matrix $M = M_b$ has at least $m = (d+1)^2$ lines (and exactly m columns). Furthermore the dimension s of the space L_b is 0 and therefore the dimension σ of the space generated by $v_0, \dots, v_b$ is m. The matrix M is the matrix of the linear map $\Lambda : \mathcal{Q}_d \to \mathbb{C}^{b+1}$ (in the basis of monomials) sending an element $P(X,Y) \in \mathcal{Q}_d$ to the b-jet $j_0^b(P(z, f(z))$ at the origin of the analytic map $P(z, f(z))$, and, with the notation of Definition 2.9, b is the first index of jets such that $\dim(j_0^b(\widetilde{\Lambda}(\mathcal{Q}_d))) = \dim(\widetilde{\Lambda}(\mathcal{Q}_d)) = (d+1)^2$.

(3) The family $\sum_{0 \leqslant i,j \leqslant d} \lambda_{j,i} z^i f^j(z)$ considered here presents the advantage to have a slightly more symmetric Bautin matrix than the family $\sum_{i+j \leqslant d} \lambda_{j,i} z^i f^j(z)$. Of course one can easily deduce the Bautin matrix of the second family from the Bautin matrix of the first family, and the problem could be considered only for the family $\sum_{i+j \leqslant d} \lambda_{j,i} z^i f^j(z)$ as well. In particular note that $Z_d(f) \leqslant \mathcal{Z}_d(f) \leqslant Z_{2d}(f)$, and therefore a polynomial bound for the maximal number of zeroes of one family gives rise to a polynomial bound for the maximal number of zeroes of the other family.

Notation 3.1. — For $i, j \in \mathbb{N}$, we denote by a_i^j the ith Taylor coefficient at the origin of the jth power f^j of f. Namely, $a_i^j = \frac{1}{i!}(f^j)^{(i)}(0)$.

With this notation, a direct computation shows that

$$M = \begin{pmatrix}
1 & 0 & \cdots & 0 & a_0^1 & 0 & \cdots & 0 & \dots & a_0^d & 0 & \cdots & 0 \\
\vdots & & & \vdots & \vdots & & & \vdots & & \vdots & & & \vdots \\
0 & & & 0 & a_{d-1}^1 & a_{d-2}^1 & & 0 & \dots & a_{d-1}^d & & & 0 \\
0 & & & 1 & a_d^1 & a_{d-1}^1 & & a_0^1 & \dots & a_d^d & & & a_0^d \\
0 & & & 0 & a_{d+1}^1 & a_d^1 & & a_1^1 & \dots & a_{d+1}^d & & & a_1^d \\
\vdots & & & \vdots & \vdots & & & \vdots & & \vdots & & & \vdots \\
0 & 0 & \cdots & 0 & a_b^1 & a_{b-1}^1 & \cdots & a_{b-d}^1 & \dots & a_b^d & a_{b-1}^d & \cdots & a_{b-d}^d
\end{pmatrix}$$

By Corollary 2.14 any non-zero minor determinant of size $m \times m$ of M will provide a bound for $\mathcal{Z}_d(f)$ on $D_{\frac{1}{4}}$, since in Corollary 2.14, δ is the maximum of all non-zero minor determinants of size $m \times m$ of M. But such a minor has to contain the first $d+1$ lines of M, as well as the last line. One sees that the absolute value Δ of any non-zero determinant of $(d^2+d) \times (d^2+d)$ minor of the following matrix

$$\tilde{M} = \begin{pmatrix}
a_{d+1}^1 & \cdots & a_1^1 & \dots & a_{d+1}^d & \cdots & a_1^d \\
\vdots & & \vdots & & \vdots & & \vdots \\
a_b^1 & \cdots & a_{b-d}^1 & \dots & a_b^d & \cdots & a_{b-d}^d
\end{pmatrix} \tag{3.1}$$

provides on $D_{\frac{1}{4}}$ the Bézout bound

$$\mathcal{Z}_d(f) \leqslant 5b \log\left(4 + 2(b+1)\frac{e^{2(d+1)^2\log(d+1)}}{\Delta}\right). \tag{3.2}$$

3.2. Bézout bound through the transcendence index

We start this subsection by the following definition of a notion of measure of the local transcendence of an analytic transcendental function.

Definition 3.2. — *For a transcendental analytic function $f : D_1 \to \mathbb{C}$ and for any $d \geqslant 1$ the d-*th transcendence index ν_d *of f is the maximal (with respect to all non-zero polynomials $P_d \in \mathcal{P}_d$) multiplicity at 0 of the function $g(z) = P_d(z, f(z))$. The non-decreasing sequence $\nu(f) = (\nu_1, \dots, \nu_d, \dots)$ is called the* transcendence sequence of f.

Remark 3.3. — The d-th transcendence index measures the maximal order of contact at the origin between the graph of f and algebraic curves of degree at most d. The higher this index is, the less f seems transcendental, since infinite ν_d means that f is algebraic.

Remark 3.4. — As already observed in Proposition 2.3, ν_d is the Bautin index of the linear family associated to f and the monomials X^iY^j of degree $i+j$ at most d. Since the number of monomials in two variables of degree at most d is $(d+1)(d+2)/2$, by Remark 2.2, $\nu_d \geqslant (d^2+3d)/2$ always.

Remark 3.5. — Using the notation b, as in the beginning of Section 3, for the Bautin index of the family $\sum_{0\leqslant j\leqslant d} p_j(z)f^j(z)$, with $\deg p_j \leqslant d$, one has by Proposition 2.3 that $b=\mu$, where μ is the maximal multiplicity, with respect to the coefficients of the polynomials p_j, of this family. Therefore one gets $\nu_d \leqslant b = \mu \leqslant \nu_{2d}$. In particular, inequality (3.2) gives the following proposition.

Proposition 3.6. — *Let $f : D_1 \to \mathbb{C}$ be a transcendental analytic function with transcendence sequence $(\nu_d)_{d\geqslant 1}$, then on $D_{\frac{1}{4}}$*

$$\mathcal{Z}_d(f) \leqslant 5\nu_{2d}\log\left(4+2(\nu_{2d}+1)\frac{e^{2(d+1)^2\log(d+1)}}{\Delta}\right),$$

where Δ is the absolute value of a certain non-zero minor determinant of size $(d^2+d)\times(d^2+d)$ of the matrix $\tilde{M}$ defined at (3.1)*.*

Now we assume that $f(z)=\sum_{k=0}^{\infty} a_k z^k$ has rational Taylor coefficients $a_k = \frac{m_k}{p_k}$, with the greatest common divisor $m_k \wedge p_k$ of m_k, p_k equal to 1 and $p_k > 0$, and let us introduce the following notation.

Notation 3.7. — For any $l \geqslant 1$, let $h_l = \max\{p_k; k=1,\dots,l\}$.

Under this assumption we can bound from below the non-zero determinant of the $(d^2+d)\times(d^2+d)$ minors of $\tilde{M}$, in terms of the transcendence index ν_{2d} and the height bound $h_{\nu_{2d}}$.

Proposition 3.8. — *For f as above and for Δ the absolute value of a non-zero minor determinant of size $(d^2+d)\times(d^2+d)$ of $\tilde{M}$ we have*

$$\Delta \geqslant h_{\nu_{2d}}^{-d^2(d+1)\nu_{2d}}.$$

Proof. — For simplicity write h for $h_{\nu_{2d}}$ and ν for ν_{2d}. Using the notation of Notation 3.1, write the coefficients a_k^1, $k\leqslant\nu$, as rational numbers having the same denominator D, this common denominator being at most h^ν. Next we write the coefficients a_i^j, for $1 \leqslant j \leqslant d,\ 1 \leqslant i \leqslant \nu$, as rational numbers having the same denominator D^d. Indeed, a_i^j are sums of the products $a_{i_1}^1 \ldots a_{i_j}^1, i_1 + \cdots + i_j = i$, with $j \leqslant d$. Therefore the determinant of a $(d^2+d) \times (d^2+d)$ minor of $\tilde{M}$ can be written as a rational number having for denominator

$$D^{d^2(d+1)} \leqslant h^{d^2(d+1)\nu}.$$

But then such a non-zero determinant cannot be smaller in absolute value than $h^{-d^2(d+1)\nu}$. □

An important special case is when there exist polynomials $R(d)$ and $S(d)$, with positive coefficients, such that

$$\nu_d \leqslant R(d),\ h_l \leqslant e^{S(l)},\ d, l \geqslant 1 \tag{3.3}$$

Under this condition we can guarantee that $\mathcal{Z}_d(f)$ grows at most polynomially in d.

Theorem 3.9. — *Assume that f has rational Taylor coefficients at the origin, and that the growth conditions* (3.3) *are satisfied. Then on $D_{\frac{1}{4}}$*

$$\mathcal{Z}_d(f) \leqslant T(d),$$

for a certain polynomial T.

Proof. — Under condition (3.3) and by Proposition 3.8, since S is an increasing function, we have

$$\frac{1}{\Delta} \leqslant h_{\nu_{2d}}^{d^2(d+1)\nu_{2d}} \leqslant e^{S(R(2d))d^2(d+1)R(2d)} = e^{U(d)}.$$

Now by Proposition 3.6, on $D_{\frac{1}{4}}$, we easily have for instance

$$\begin{aligned}\mathcal{Z}_d(f) &\leqslant 5R(2d)\log(4 + 2(R(2d)+1)e^{2(d+1)^2\log(d+1)}e^{U(d)}) \\ &\leqslant 5R(2d)\log(4R(2d)e^{2(d+1)^3}e^{U(d)}) \\ &\leqslant 10R^2(2d) + 10R(2d)(2(d+1)^3 + U(d))\end{aligned}$$ □

Remark 3.10. — Producing instances of Taylor series $f(z) = \sum_{k \geqslant 0} a_k z^k$ converging on D_1, with rational coefficients a_k having denominators bounded from above by $e^{S(k)}$, where S is a certain polynomial, is easy. Nevertheless the second assumption of Theorem 3.9, concerning the growth of the transcendence sequence of f, is more difficult to control. A polynomial bound $\nu_d(f) \leqslant R(d)$ is known for solutions of some classes of algebraic ODE's (see [6, 25, 32, 31]). We expect such a bound to hold for Taylor series produced by some natural classes of recurrence relations. In Section 3.4 we give conditions on lacunarity of the series f that allow estimates of the growth of $\nu_d(f)$. In general, we consider bounding of the growth of the transcendence sequence of f as an important open question.

If we consider polynomials $P(z,y) = p_1(z)y + p_0(z)$ of degree 1 in y, with $p_0(z), p_1(z)$ of degree d in z, we are in the framework of the classical Padé approximation. In this case the sequence of maximal multiplicities $\mu_d = \mu_d(f)$ of $g(z) = P(z, f(z)) = p_1(z)f(z) + p_0(z)$ has the following remarkable description (see, for instance, [33]): let

$$f(z) = \frac{1}{q_1(z) + \frac{1}{q_2(z)+\dots}}, \quad \deg q_l = s_l,\ l = 1, 2, \dots,$$

be a continued fraction representation of the series $f(z)$. Then $\mu_d = s_1 + s_2 + \dots + s_d$.

For polynomials $P(z,y) = p_0(z)$ of degree 0 in y the behavior of μ_d was studied in [24], in particular, it was related there to linear non-autonomous recurrence relations for the Taylor coefficients of f.

3.3. Bézout bound through the Bautin multiplicity

In the previous Section 3.2, in Proposition 3.8, some minors of the matrix $\tilde{M}$ were bounded from below, in terms of the transcendence index ν_{2d} and the height bound $h_{\nu_{2d}}$. On this base, on $D_{\frac{1}{4}}$, $\mathcal{Z}_d(f)$ was bounded from above by a polynomial in d (see Theorem 3.9), under the condition that the sequences $(\nu_d)_{d\geqslant 1}$ and $(\log h_d)_{d\geqslant 1}$ are polynomially bounded.

A special case in which the transcendence index (or thanks to the double inequality of Remark 3.5, in which the Bautin index b itself of the family $\sum_{j=0}^d p_j f^j, \deg p_j \leqslant d$) is polynomially bounded is the case that b is minimal, that is equal to d^2+2d. In this case the matrix $\tilde{M}$ of (3.1) is an invertible square matrix of size d^2+d with the same determinant as the Bautin matrix M.

Notation 3.11. — Being zero or not, let us call this $(d^2+d)\times(d^2+d)$ determinant *the Bautin determinant* of the family $\sum_{j=0}^d p_j f^j, \deg p_j \leqslant d$, and let us denote it by Δ_d.

So far the study has been done by looking at Taylor series at the origin. Let us now allow Taylor expansions of f at points z near the origin. In this situation, the Bautin matrix M, as well as its submatrix $\tilde{M}$, have entries $a_i^j(z)$ that are analytic functions in the variable z. To emphasize this dependency, we adopt the notation $\tilde{M}(z)$ and $\Delta_d(z)$ (and keep the notation

$\tilde{M}$ and Δ_d for $z = 0$). We shall assume that for each degree d the Bautin determinant $\Delta_d(z)$, as a function of z, does not vanish identically. This is in particular true when $f(z)$ is a hypertranscendental function, i.e. all the derivatives are algebraically independent. In this situation, for a generic base point z, $\Delta_d(z) \neq 0$, $\nu_d(z) \leqslant d^2 + 2d$, and therefore the transcendence index $\nu_d(z)$ is polynomially bounded. The study of Section 3.2 could then be done by shifting the origin at some generic point z, however in this translation one loses the control on the rationality of the coefficients of the Taylor expansion of f, an assumption that is necessary to formulate at some fixed point (namely the origin, for simplicity), since this assumption makes no sense at generic points z. Nevertheless, still in the case that $\Delta_d(z)$ does not vanish identically, one can use the following dichotomy:

- when $\Delta_d(0) \neq 0$, as just observed, we are in particular in the frame of Section 3.2 where ν_d is polynomially bounded (by $d^2 + 2d$),
- when $\Delta_d(0) = 0$, we can expand $\Delta_d(z)$ as a non-zero Taylor series in z at the origin and study the multiplicity of this expansion with respect to d. We will see in Theorem 3.14 that when the sequence of these multiplicities has at most a polynomial growth, then a Bézout bound for f is still possible.

Remark 3.12. — This dichotomy means that when some transversality defect for f is quantitatively well controlled (through the multiplicity of $\Delta_d(z)$), then a good zero-counting bound is possible, and for instance, in turn, a good bound for the density of rational points of bounded height in the graph of f will also be possible (see Theorem 5.2).

DEFINITION 3.13. — *For any $d \geqslant 1$, the d-*th Bautin multiplicity η_d of *f is the multiplicity at 0 of the Bautin determinant $\Delta_d(z)$, considered as an analytic function of z. The sequence $\eta(f) = (\eta_d)_{d\geqslant 1}$ is called the* Bautin multiplicity sequence of *f.*

In brief, for each $d \geqslant 1$, we can write

$$\Delta_d(z) = \alpha_d z^{\eta_d} + O(z^{\eta_d+1}), \text{ with} \alpha_d \neq 0.$$

THEOREM 3.14. — *Assume that $f : D_1 \to \mathbb{C}$ is an analytic function with rational Taylor coefficients at the origin satisfying the growth condition* (3.3) *and such that there exists a polynomial R with $\eta_d \leqslant R(d)$, for $d \geqslant 1$. Then on $D_{\frac{1}{4}}$*

$$\mathcal{Z}_d(f) \leqslant T(d),$$

for a certain polynomial T.

Proof. — The Bautin matrix of Definition 2.9 is the matrix of size $(d+1)^2$, in the base of monomials of $\mathcal{Q}_d$, of the linear map $\widehat{\Lambda} : \mathcal{Q}_d \to (\mathbb{C}\{z\})^{(d+1)^2}$ sending a polynomial to the vector of the first $(d+1)^2$ derivatives $(P(z,f(z))^{(j)}/j!$ of $P(z,f(z))$. For a given polynomial $P \in \mathcal{Q}_d$, such that the multiplicity at the origin of $P(z,f(z))$ is maximal, and therefore is the Bautin index b_d of f for the family of monomials of $\mathcal{Q}_d$, one can write $b_d = d^2+2d+r$, with $r \geqslant 0$. Then the multiplicity at the origin of the first $(d+1)^2$ derivatives of $P(z,f(z))$ is bigger than r. Now writing the Bautin matrix in a basis of $\mathcal{Q}_d$ starting with P, the elements of the first column of this matrix have multiplicity at least r. It follows that the Bautin determinant $\Delta_d(z)$ itself has multiplicity at least $r = b_d - d^2 - 2d$, and therefore $\eta_d \geqslant b_d - d^2 - 2d \geqslant \nu_d - d^2 - 2d$. As a conclusion, when η_d is polynomially bounded, ν_d is polynomially bounded as well. The existence of the polynomial T then follows from Theorem 3.9. □

Remark 3.15. — The function $\Delta_d(z)$ is a polynomial with coefficients in $\mathbb{Z}$ in the variables $a_1(z),\dots,a_{d^2+d}(z)$ and with degree $\frac{d(d+1)^2}{2}$. Indeed, the entries of $\tilde{M}$ are the functions a_i^j, with $i=1,\dots,d^2+d$, $j=1,\dots,d$, and a_i^j is a sum of products of type $a_{i_1}\dots a_{i_j}$, $i_1+\dots+i_j = i$. The Bautin multiplicity η_d therefore measures the degree of cancellation allowed by the polynomial Δ_d applied on the d^2+d first derivatives of f. This simple observation suggests to introduce, for an hypertranscendental analytic function $f : D_1 \to \mathbb{C}$, the notion of *polynomial hypertranscendence*, defined by the existence of polynomials $A, B \in \mathbb{R}[X]$ with positive coefficients, such that for any $d \in \mathbb{N}$, for any polynomial $P \in \mathbb{Z}[X_0,\dots,X_{A(d)}]$, with degree $\leqslant d$, the multiplicity of $P(\frac{f(z)}{0!}, \frac{f'(z)}{1!}, \dots, \frac{f^{(A(d))}(z)}{A(d)!})$ at the origin is bounded by $B(d)$. This notion of strong hypertranscendence, relevant by itself, is motivated here by the remark that since there exists $p \in \mathbb{N}$ such that $d^2+d \leqslant d^p$ and $\frac{d(d+1)^2}{2} \leqslant A(d^p)$, we have $\eta_d \leqslant B(d^p)$ for such a function.

3.4. Lacunary series

The aim of this section is to give concrete instances of series f satisfying polynomial growth condition for the transcendence sequence $(\nu_d)_{d\geqslant 1}$ considered in (3.3), and thus having a polynomial Bézout bound. For this we focus on lacunary series for which the computation of a bound for the transcendence sequence $(\nu_d)_{d\geqslant 1}$ is possible. This case has been considered in [21, Theorem 6.1], where a family of analytic functions with $\mathcal{Z}_d(f)$ having prescribed growth is given. Our conditions, being more flexible, improve on these earlier conditions (see Remark 3.22).

We begin with the following remark which improves on the estimates of Propositions 3.8 and 3.9, in case of lacunary series.

Remark 3.16. — Assume that the Taylor coefficients of the series f are rational numbers and denote by θ_d the amount of those non-zero coefficients among $a_0, \dots, a_{\nu_d}$. Then, with exactly the same proofs adapted to this notation, Proposition 3.8 and Theorem 3.9 may be formulated as follows: the absolute value Δ of a non-zero minor determinant of size $(d^2+d) \times (d^2+d)$ of $\tilde{M}$ satisfies $\Delta \geqslant h_{\theta_{2d}}^{-d^2(d+1)\theta_{2d}}$, and consequently when there exist polynomials R and S such that $\theta_d \leqslant R(d)$, $h_d \leqslant e^{S(d)}$, one gets $\mathcal{Z}_d(f) \leqslant T(d)\nu_{2d} \log \nu_{2d}$, for some polynomial T.

Now in order to estimate the growth of the transcendence sequence $(\nu_d)_{d\geqslant 1}$, let us assume that the lacunarity of the series f is quantitatively controlled by the following condition

$$f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}, \ a_k \neq 0 \ \text{ and for any } k \geqslant 1, \ n_{k+1} > n_k^2. \tag{3.4}$$

Note that in what follows no assumption is made on the rationality of the Taylor coefficients of f. We assume that f is analytic on D_1 and bounded there by 1.

Lemma 3.17. — *Under condition* (3.4)*, for any $l \geqslant 1$ and for any $m, j \in [0, n_{l+1} - 1]$, the series $z^m f^j(z)$ contains the non-zero monomial $(a_{l+1})^j z^{jn_{l+1}+m}$, and no other monomials with degree in $]jn_{l+1}+m, n_{l+2}[$.*

Proof. — For $j = 0$ the series $z^m f^j(z)$ is the monomial z^m and thus is of the required from. Now for $j \geqslant 1$, write $f(z)$ as the sum $f(z) = f_{l+1}(z) + \bar{f}_{l+1}(z)$, with $f_{l+1}(z) = \sum_{k=1}^{l+1} a_k z^{n_k}$, and $\bar{f}_{l+1}(z) = \sum_{k=l+2}^{\infty} a_k z^{n_k}$. The monomial $(a_{l+1})^j z^{jn_{l+1}}$ has the highest degree among the monomials of the series $f^j(z)$ which come from the terms in f_{l+1}^j. On the other hand, all other monomials in $f^j(z)$ which come from the products of terms in $f_{l+1}(z)$ and in $\bar{f}_{l+1}(z)$, have degree at least n_{l+2}. The terms of the series $x^m f^j(z)$ are the terms of $f^j(z)$ shifted by m and therefore for $m, j < n_{l+1}$ we have

$$jn_{l+1} + m \leqslant (n_{l+1} - 1)n_{l+1} + n_{l+1} - 1 = n_{l+1}^2 - 1 < n_{l+2},$$

since by condition (3.4), $n_{l+2} > n_{l+1}^2$. □

Proposition 3.18. — *Under condition* (3.4)*, for any $l \geqslant 1$ and for any d in the interval $[n_l, n_{l+1} - 1]$ we have*

$$n_{l+1} \leqslant \nu_d \leqslant n_{l+1}^2 - 1 < n_{l+2}.$$

Proof. — First of all, for $l \geqslant 1$ and $d \geqslant n_l$, we have $\nu_d \geqslant \nu_{n_l} \geqslant n_{l+1}$, since for the polynomial $P(z,y) = y - \sum_{k=1}^{l} a_k z^{n_k}$ of degree n_l, the function $P(z,f(z)) = \sum_{k=l+1}^{\infty} a_k z^{n_k}$ has multiplicity n_{l+1} at the origin.

Let now $P_d(z,y) = p_d(z)y^d + \cdots + p_1(z)y + p_0(z)$ be a polynomial of degree $d \leqslant n_{l+1} - 1$ and let us prove that the multiplicity of $P_d(z,f(z))$ at the origin is at most $n_{l+1}^2 - 1$; this will prove that $\nu_d \leqslant n_{l+1}^2 - 1 < n_{l+2}$.

Denote by $s \leqslant d$ the highest degree of y in $P_d(z,y)$ for which the polynomial $p_s(z)$ is not identically zero, and let us write $p_s(z) = ax^r + bx^{r-1} + \ldots,\ r \leqslant d$, with $a \neq 0$. By Lemma 3.17, the summand $ax^r f^s(z)$ in $P_d(z,f(z))$ contains the monomial $v = a(a_{l+1})^s z^{sn_{l+1}+r}$. Let us show that this monomial cannot cancel with any other monomial in $P_d(z,f(z))$, because, if it is the case, since for $s, r \leqslant d < n_{l+1}$, we have

$$sn_{l+1} + r \leqslant (n_{l+1} - 1)n_{l+1} + n_{l+1} - 1 \leqslant n_{l+1}^2 < n_{l+2},$$

it will finish the proof. As just noticed, since $sn_{l+1} + r < n_{l+2}$, the monomial v can cancel only with the monomials coming from the truncated series $f_{l+1}(z) = \sum_{k=1}^{l+1} a_k z^{n_k}$ introduced in the proof of Lemma 3.17, since $f(z) - f_{l+1}(z) = \sum_{k=l+2}^{\infty} a_k z^{n_k}$. But on one hand v cannot cancel with any monomial in $p_s(z) f_{l+1}^s(z)$, and on the other hand, for any $q < s$, the monomials in $f^q(z),\ q < s$, coming from the truncated series $f_{l+1}(z)$, have degree at most qn_{l+1}. Hence the highest degree of monomials in $p_q(z) f_{l+1}^q(z)$ can be $qn_{l+1} + d < (q+1)n_{l+1} \leqslant sn_{l+1} + r$. As announced, we conclude that v cannot cancel with any other monomial in $P_d(z,f(z))$. □

Lemma 3.17 does not only let us bound the terms of the sequence ν like in Proposition 3.18, it also lets us compute some $(d+1)^2 \times (d+1)^2$ non-zero minor determinant in the Bautin matrix M (up to allowing more than $b+1$ rows in M).

Proposition 3.19. — *Under condition* (3.4)*, for any $l \geqslant 1$, for any $d \in [n_l, n_{l+1} - 1]$, there exists a $(d+1)^2 \times (d+1)^2$ minor in M (up to allowing more than $b+1$ rows in the definition of the Bautin matrix M) with non-zero determinant $\Delta = (a_{l+1})^{\frac{1}{2}d(d+1)^2}$. For $d = n_{l+1} - 1$ this determinant is the upper square $(d+1)^2 \times (d+1)^2$ minor of the Bautin matrix M.*

Proof. — Let us fix $l \geqslant 1$, and $d \in [n_l, n_{l+1} - 1]$. Then for any $j = 0, \ldots, d$ by Lemma 3.17, we have in M a lower-triangular square $(d+1) \times (d+1)$ block M_j corresponding to lines ranging from jn_{l+1} to $jn_{l+1} + d$ and columns ranging from $j(d+1)$ to $(j+1)(d+1) - 1$ in M, and M_j has for entries $(a_{l+1})^j$ on its main diagonal. Note that $dn_{l+1} + d$ may be bigger than $b+1$, so we maybe have to consider a matrix having more lines then the Bautin matrix M, which is harmless.

Now, if we drop from M all the lines which are not in M_j for some $j = 0, \dots, d$, we obtain a $(d+1)^2 \times (d+1)^2$ minor M' which is lower triangular and has the blocks M_j, $j = 0, \dots, d$, on its main diagonal. Hence its determinant Δ is $(a_{l+1})^{\frac{1}{2}d(d+1)^2}$.

For d equal to its maximal value $n_{l+1} - 1$ each line of M belongs to one block M_j, and hence M' coincides with the upper square $(d+1)^2 \times (d+1)^2$ minor of the Bautin matrix M. □

So far, in condition (3.4) we have required that the lacunarity of the series $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$ is big enough. We now require in addition that the lacunarity of f is not too big, in the following condition

(3.5) There exists $q > 2$, such that for any $k \geqslant 1$, $n_k^2 < n_{k+1} \leqslant n_k^q$.

Under this assumption we can now show that f has a polynomial Bézout bound on $D_{\frac{1}{4}}$.

Theorem 3.20. — *Under condition* (3.5), *the transcendence sequence* ν_d *grows at most polynomially in* d. *More accurately we have*

$$\nu_d(f) < d^{q^2}.$$

If, in addition, for a certain fixed $p > 0$, *and for any* $k \geqslant 1$ *we have* $|a_k| \geqslant e^{-n_k^p}$, *then on* $D_{\frac{1}{4}}$

$$\mathcal{Z}_d(f) \leqslant 10(2d)^{q^2}(1 + qd^2 + 5d^{pq+3}).$$

Proof. — Let the degree $d \geqslant 1$ be given and let l be such that $d \in [n_l, n_{l+1} - 1]$. By Proposition 3.18 we have

$$\nu_d < n_{l+2} \leqslant (n_l)^{q^2} \leqslant d^{q^2}.$$

Now by Proposition 3.19 we obtain the existence of some $(d+1)^2 \times (d+1)^2$ minor in M with non-zero determinant Δ, such that

$$\begin{aligned} |\Delta| = |a_{l+1}|^{\frac{1}{2}d(d+1)^2} &\geqslant \exp\left(-\frac{1}{2}d(d+1)^2 n_{l+1}^p\right) \\ &\geqslant \exp\left(-\frac{1}{2}d(d+1)^2 n_l^{qp}\right) \\ &\geqslant \exp\left(-\frac{1}{2}d^{qp+1}(d+1)^2\right). \end{aligned}$$

Now by Proposition 3.6,

$$\begin{aligned}\mathcal{Z}_d(f) &\leqslant 5\nu_{2d}\log\left(4+\frac{\nu_{2d}+1}{\Delta}e^{2(d+1)^3}\right)\\ &\leqslant (2d)^{q^2}5\log\left(4+((2d)^{q^2}+1)e^{\frac{1}{2}d^{qp+1}(d+1)^2+2(d+1)^3}\right).\end{aligned}$$

Since for instance, $1+(2d)^{q^2}\leqslant e^{2dq^2}$, $\frac{1}{2}d^{qp+1}(d+1)^2+2(d+1)^3\leqslant 5d^{pq+1}(d+1)^2\leqslant 10d^{pq+3}$, we have

$$\mathcal{Z}_d(f)\leqslant (2d)^{q^2}5\log\left(4+e^{2qd^2+10d^{pq+3}}\right).$$

And finally since $\log(4+e^x)\leqslant 2+x$, for $x\geqslant 0$, we obtain

$$\mathcal{Z}_d(f)\leqslant 10(2d)^{q^2}(1+qd^2+5d^{pq+3}).$$

This completes the proof of Theorem 3.20 □

Remark 3.21. — When the gaps between the degrees n_l grow faster than assumed in Theorem 3.20, that is faster than forced by condition (3.5), the asymptotic growth of the transcendence indices ν_d and of bounds on $\mathcal{Z}_d(f)$ fastens accordingly. Notice, however, that from Proposition 3.18, for a subsequence of degrees of the form $d=n_{l+1}-1$, $l\geqslant 1$, under condition (3.4) only, the multiplicity ν_d is at most d^2+2d. Under appropriate assumptions on the coefficients a_l for the above subsequence of degrees d we obtain a polynomial bound also for $\mathcal{Z}_d(f)$. This phenomenon is to compare to a similar behaviour in [22, Theorem 1.1] or [41, Theorem 0.3], [42, Theorem 1.3], where the minimal asymptotic for $\mathcal{Z}_d(f)$ is obtained on some sequence of degrees d going to infinity.

Remark 3.22. — In [21, Corollary 6.2], for $\alpha\geqslant 3$, the lacunarity condition,

$$n_{k+1}=n_k^{\alpha-1},\quad a_k=e^{-n_k\log n_k} \tag{3.6}$$

is considered as a condition implying a polynomial Bézout bound for f (with prescribed order of asymptotic in $[\alpha-1,\alpha]$ as explained in the introduction). Here our condition (3.5) and our assumption $|a_k|\geqslant e^{-n_k^p}$, for some $p>0$, in Theorem 3.20, are somewhat more flexible as conditions giving a polynomial Bézout bound for f.

3.5. Rational Taylor coefficients via recurrence relations

In this section we study recurrence relations for the Taylor coefficients of the series $f(z)=\sum_{k=1}^{\infty}a_kz^k$, assuming that the starting coefficients are rational numbers. In case the recurrence relation is linear (with polynomial coefficients in $1/k$) it turns out that the a_k's satisfy the bound $h_l \leqslant e^{S(l)},\ l \geqslant 1$ of condition (3.3), one of the two hypotheses required in Theorems 3.9 and 3.14. Furthermore in case f is a solution of an algebraic differential equation with polynomial coefficients, the other hypothesis required in Theorem 3.9, namely a bound that is polynomial in d for the transcendence index ν_d, is automatically satisfied (see Remark 3.24 and Theorem 3.25).

Let

$$Q(k, u_1, u_2, \dots, u_r) = \sum_{|\beta| \leqslant d_1} p_\beta(k) u^\beta$$

be a polynomial of degree d_1 in the variables $u_1, u_2, \dots, u_r$ and with coefficients $p_\beta(k) = \sum_{i=0}^{d_2} c_{\beta,i} (\frac{1}{k})^i$ being polynomials in $\frac{1}{k}$ of degree d_2. We consider a polynomial recurrence relation of length r of the form

$$a_{k+1} = Q(k, a_k, a_{k-1}, \dots, a_{k-r+1}). \tag{3.7}$$

We assume that the coefficients $c_{\beta,i}$ are rational numbers, as well as the initial terms $a_0, a_1, \dots, a_{r-1}$ of the sequence $a = (a_0, a_1, \dots, a_{r-1}, a_r, \dots)$. For $k \geqslant r-1$, let D_k denote the common denominator of $a_0, \dots, a_k$, when those rational numbers are written in their irreducible form. We also denote by L_1 (respectively, L_2) the common denominator of all the coefficients $c_{\beta,i}$, $i = 0, \dots, d_2$, $|\beta| \leqslant d_1$ (respectively, the common denominator of all the initial given terms $a_0, a_1, \dots, a_{r-1}$) again when those rational numbers are written in their irreducible form. Note that $D_{r-1} = L_2$.

Proposition 3.23. — *With the notation above, for any $k \geqslant r-1$,*

$$D_k \leqslant e^{M d_1^{k-r+1} k \log k},$$

where

$$M = \max\left(\frac{\log L_2}{(r-1)\log(r-1)},\ d_2 + \frac{\log L_1}{\log 2} \right).$$

Proof. — The products of a_j entering Q in (3.7) can be written with denominator $D_k^{d_1}$. Therefore, the next term a_{k+1}, and hence all the terms $a_0, \dots, a_k, a_{k+1}$, can be written with the common denominator $\tilde{D}_{k+1} = L_1 k^{d_2} D_k^{d_1}$. Now we prove by induction that

$$D_k \leqslant e^{M d_1^{k-r+1} k \log k}. \tag{3.8}$$

For $k = r-1$ we have $D_{r-1} = L_2$, and (3.8) is satisfied by the choice of M. Assuming that the required inequality is satisfied for a certain $k \geqslant r-1$, we now prove it for $k+1$. We have

$$D_{k+1} \leqslant \tilde{D}_{k+1} = L_1 k^{d_2} D_k^{d_1} \leqslant L_1 k^{d_2} e^{M d_1^{k+1-r+1} k \log k}$$
$$= e^{M d_1^{k+1-r+1} k \log k + \log L_1 + d_2 \log k}$$
$$= e^{M d_1^{k+1-r+1} (k \log k + \frac{\log L_1 + d_2 \log k}{M d_1^{k+1-r+1}})}.$$

By the choice of M the last expression does not exceed

$$e^{M d_1^{k+1-r+1}(k \log k + \log k)} < e^{M d_1^{k+1-r+1}(k+1)\log(k+1)}.$$

This completes the proof of Proposition 3.23. □

Notice that for $d_1 > 1$ the denominators grow as a double exponent, i.e. faster than an exponent of a polynomial in k. The trivial example of recurrence relation $a_{k+1} = a_k^2$, i.e. $a_k = a_0^{2^k}$, shows that this growth indeed happens in recurrence relations of the form (3.7).

Remark 3.24. — However, in the special case of linear recurrence relations of the form (3.7), we have $d_1 = 1$, and the bound of Proposition 3.23 takes the form $D_k \leqslant e^{Mk \log k}$. This special case includes Poincaré-type recurrence relations, which are satisfied by the Taylor coefficients of solutions $f(z)$ of linear differential equations with polynomial coefficients. In the more general case where f satisfies an algebraic differential equation $f^{(d)} = Q(z, f(z), \dots, f^{(d-1)})$, where Q is some given polynomial in $\mathbb{Q}[X_1, \dots, X_d]$, iteration of derivation of each member of this equation leads to equations of type $f^{(k)}(z) = Q_k(z, f(z), \dots, f^{(d-1)})$, where Q_k is a polynomial in $\mathbb{Q}[X_1, \dots, X_d]$ with controlled height of its coefficients and controlled degree with respect to d. Studying these derivations and using some results of [7] one obtains also in this case the bound on the height of $a_k = f^{(k)}(0)$ required by our growth condition (3.3). Therefore, combining well-known bounds on the transcendence sequences $(\nu_d)_{d\geqslant 1}$ of solutions of differential equations with polynomial coefficients, that turn out to be polynomially bounded in d (see [6, 25, 31, 32]), our Proposition 3.23 and Theorem 3.9 above, we immediately obtain the following statement, a result, which was recently proved (among others results in this direction and by others methods) in [7, Corollary 4, Theorem 6] (note that in [7], no assumption on the rationality of initial conditions is required).

Theorem 3.25. — *Let f be an analytic function, defined on the unit disc, that is solution of an algebraic differential equation with rational coefficients and initial conditions. Then there exists a polynomial T such that on $D_{\frac{1}{4}}$, $\mathcal{Z}_d(f) \leqslant T(d)$.*

4. Bautin determinant for random series

In this section, we discuss the behaviour of the Bautin determinant for random Taylor coefficients. We prove that for any $p \in]0,1[$, there exists a set E_p of probability p, such that for any series $f \in E_p$, the corresponding Bautin determinant, as a function of d, is bounded from below by $e^{U_p(d)}$, for a certain polynomial U_p. In case $\Delta_d \neq 0$, the Bautin index of the family $\sum_{j=0}^{d} p_j f^j$ is d^2+2d, and consequently the transcendence index ν_d is bounded by d^2+2d, and thus the growth condition on the transcendence indices sequence in Theorem 3.9 is fulfilled. It follows that for any $p \in]0,1[$, for Taylor coefficients in a set of probability p, $\mathcal{Z}_d(f)$ is polynomially bounded in d.

Let us fix some integer $d \geqslant 1$ and let us start by the following remark.

Remark 4.1. — As already noticed in Remark 3.15, as a polynomial in the Taylor coefficients of the series f, the Bautin determinant (of size d^2+d), still denoted Δ_d, is a polynomial in the variables $a_1, \ldots, a_{d^2+d}$ and with degree $\frac{d(d+1)^2}{2}$.

For $d \geqslant 1$, following this remark, and to be more general, we will consider instead of the polynomial Δ_d of arity (number of variables) d^2+d and degree $d(d+1)^2/2$, any polynomial with arity and degree polynomially bounded in d.

Let $I = [-1,1] \subset \mathbb{R}$. We consider the unit n-dimensional cubes $I^n \in \mathbb{R}^n$, $n \geqslant 1$, and the infinite dimensional unit cube $I^\infty = \varprojlim_{n\in\mathbb{N}} I^n$ that comes with its standard projections $\pi_n : I^\infty \to I^n$. Let us denote by μ_n the probability Lebesgue measure on I^n, for any $n \geqslant 1$. For any $n \geqslant 1$ and any measurable set $G \subset I^n$ denote by $\tilde{G} \subset I^\infty$ the cylinder $\pi_n^{-1}(G)$ over G. The probability measure μ on I^∞ is defined by setting $\mu(E) = \sum_{i=1}^{\infty} \mu_{n_i}(G_{n_i})$, for any subset $E \subset I^\infty$ that can be expressed as a disjoint union of cylinders $\tilde{G}_{n_i}$, with G_{n_i} a μ_{n_i}-measurable in I^{n_i}.

We identify the sequences $(a_k)_{k\geqslant 0} \in I^\infty$ with the analytic functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$, this series converging at least in the interior of D_1. For a polynomial Q of arity m and for $f = (a_k)_{k\geqslant 0} \in I^\infty$ we denote $Q(a_0, \ldots, a_{m-1}) = Q(\pi_m(f))$ by $Q(f)$.

Let finally $(Q_d)_{d\geqslant 1}$ be a sequence of polynomials Q_d, of arity m_d and degree q_d, and let us assume

$$|Q_d|_{I^{m_d}} = \max\{|Q_d(x)|; x \in I^{m_d}\} \geqslant 1. \tag{4.1}$$

Note that we also have $|\Delta_d|_{I^{d^2+d}} \geqslant 1$, since $\Delta_d = 1$ when for instance $a_{d+1}^1 = 1$ and $a_i^j = 0$ for $i = 1, \ldots, d^2+d$, $j = 1, \ldots, d$, $i \neq d+1$, $j \neq 1$.

Theorem 4.2. — *With the above notation, for any $p \in]0,1[$, there exists a set $E_p \subset I^\infty$ of measure p, such that for any $f \in E_p$, and for any $d \geqslant 1$,*

$$|Q_d(f)| \geqslant \left(\frac{3(1-p)}{2\pi^2 d^2 m_d}\right)^{q_d}.$$

In particular, for q_d, m_d satisfying $q_d \leqslant d^{\kappa_1}, m_d \leqslant d^{\kappa_2}$, for some $\kappa_1, \kappa_2 \geqslant 0$, we have, with probability at least p:

$$|Q_d(f)| \geqslant e^{-(\gamma_p+\kappa_2)d^{\kappa_1+1}},$$

where γ_p goes to $+\infty$ as p goes to 1.

Proof. — Let $p \in]0,1[$, $d \geqslant 1$, and $\theta_d = \frac{6(1-p)}{\pi^2 d^2}$. We define the real number ε_d as the maximum of the numbers ε such that the set

$$V_d = \{u \in I^{m_d},\ |Q_d(u)| \leqslant \varepsilon\}$$

satisfies $\mu_{m_d}(V_d) \leqslant \theta_d$. Now for $V = \bigcup_{d=0}^\infty V_d \subset I^\infty$ and $E_p = I^\infty \setminus V$ we have

$$\mu(V) \leqslant \sum_{d=0}^{\infty} \theta_d = 1-p, \text{ and thus } \mu(E_p) \geqslant p.$$

Clearly, for any $f \in E_p$, for any $d \geqslant 1$, we have $|Q_d(f)| \geqslant \varepsilon_d$. It remains to estimate the numbers ε_d. We use for this purpose the following multivariate Remez inequality proved in [18] (see also [14]).

> *Let Z be a measurable subset of I^n. Then for every real polynomial P in n variables and of degree d,*

$$|P|_{I^n} < \left(\frac{4n}{\lambda}\right)^d |P|_Z, \tag{4.2}$$

> *where $\lambda = \mu_n(Z)$.*

Applying inequality (4.2) to $Z = V_d$ and $P = Q_d$, by our assumption (4.1) we get

$$1 \leqslant |Q_d|_{I^{q_d}} < \left(\frac{4m_d}{\mu_{m_d}(V_d)}\right)^{q_d} \varepsilon_d,$$

or equivalently $\mu_{m_d}(V_d) \leqslant 4m_d\varepsilon_d^{\frac{1}{q_d}}$. In particular, for $\varepsilon_d = \left(\frac{\theta_d}{4m_d}\right)^{q_d} = \left(\frac{3(1-p)}{2\pi^2 d^2 m_d}\right)^{q_d}$, we have $\mu_{m_d}(V_d) \leqslant \theta_d$. This completes the proof of the first inequality of Theorem 4.2. Substituting into this inequality $q_d = d^{\kappa_1}, m_d = d^{\kappa_2}$, we obtain for any $f \in E_p$ and for any $d \geqslant 1$

$$\begin{aligned}|Q_d(f)| \geqslant \left(\frac{3(1-p)}{2\pi^2}\right)^{d^{\kappa_1}} d^{-2-\kappa_2 d^{\kappa_1}} &= e^{-c_p d^{\kappa_1}} e^{-(2+\kappa_2 d^{\kappa_1})\log d} \\ \geqslant e^{-c_p d^{\kappa_1}} e^{-(2+\kappa_2 d^{\kappa_1})d} \geqslant e^{-(2+c_p+\kappa_2)d^{\kappa_1+1}} &= e^{-(\gamma_p+\kappa_2)d^{\kappa_1+1}},\end{aligned}$$

where $c_p = \log\left(\frac{2\pi^2}{3(1-p)}\right) > 0$, $\gamma_p = 2 + c_p$. □

We apply Theorem 4.2 to the case where $Q_d = \Delta_d$, then as a consequence of Theorem 3.9, we obtain the following statement.

Corollary 4.3. — *With the above notation, for any $p \in]0,1[$, there exists a set $E_p \subset I^\infty$ of measure p, such that for any $f \in E_p$, on $D_{\frac{1}{4}}$, $\mathcal{Z}_d(f) \leqslant C_p d^8$, where $C_p \to +\infty$ as $p \to 1$. Or, in other words, with probability 1, random series satisfy polynomial Bézout bounds (with degree at most 8).*

5. Analytic functions with few rational points in their graph

We start this section by the following definition.

Definition 5.1. — *Let $x = (x_1, \dots, x_n) \in \mathbb{Q}^n$. The height of x is the integer $\max\{|a_i|, |b_i|; i = 1, \dots, n\}$, where $x_i = a_i/b_i$ with $a_i, b_i \in \mathbb{Z}$, $a_i \wedge b_i = 1$, $i = 1, \dots, n$.*

Explicit bounds on the number $\#X(\mathbb{Q}, T)$ of rational points $X(\mathbb{Q}, T)$ of height at most T, in some given set $X \subset \mathbb{R}^n$, are usually related to Bézout bounds satisfied by X. Let us assume for instance that X is a transcendental set definable in some o-minimal structure expanding the real field, and to simplify, of dimension 1. Then following [36], that generalizes the classical by now method of [10], one knows that $X(\mathbb{Q}, T)$ is contained in a certain number $H_{X,T,d}$ of hypersurfaces of $\mathbb{R}^n$ of degree d, this number being of the form $C_{X,d}T^{\tau_d}$, with $\tau_d \to 0$ when $d \to \infty$. It follows that since the definable set X satisfies a Bézout bound (see [10, Theorem 1], [36])

$$\forall\, \epsilon > 0,\ \exists\, C_{X,\epsilon},\ \forall\, T \geqslant 1,\ \#X(\mathbb{Q}, T) \leqslant C_{X,\epsilon} T^{\epsilon}. \tag{5.1}$$

Now in case the curve X is given by a system of convenient parametrizations (as mild parametrizations defined in [35], or slow parametrizations defined in [23]), or more simply, in case X is the graph Γ_f of some transcendental analytic function f on a compact interval of $\mathbb{R}$, a computation shows that the constant $C_{\Gamma_f,d}$ is polynomially bounded in d and that $T^{\tau_{\lfloor \log T \rfloor}}$ is a constant K independent of T (see [35, Proposition 2.4], [23, Proposition 2.18]). Therefore, for some polynomial Q, on gets

$$\#X(\mathbb{Q}, T) \leqslant Z_{\lfloor \log T \rfloor}(f) K Q(\lfloor \log T \rfloor). \tag{5.2}$$

Moreover, in this situation when f has a Bézout bound polynomial in d, one obtains the following improvement of the general bound (5.1)

$$\exists\, \beta,\ \exists\, \alpha > 0,\ \forall\, T \geqslant 1,\ \#X(\mathbb{Q}, T) \leqslant \beta \log^{\alpha} T. \tag{5.3}$$

Recently several results appeared, establishing in different cases bounds for $\#X(\mathbb{Q}, T)$ as in (5.3), some of them proving the existence of convenient parametrizations for certain families of sets X with respect to log-bounds as in (5.3), the others proving polynomial Bézout bounds in some particular cases (see, among these results, [4, 5, 9, 7, 8, 13, 12, 19, 23, 28, 29, 30]). In the same spirit we give hereafter direct Diophantine applications of the polynomial Bézout bounds obtained in previous sections of the paper.

Let f be an analytic function converging on D_1 (on D_8 for condition 3 of Theorem 5.2) and let us denote by Γ_f its graph over $D_{\frac{1}{4}}$. As a consequence of Theorems 3.9, 3.14, 3.20 and 3.25, and Corollary 4.3, one has

Theorem 5.2. — *Assume that one of the following conditions is satisfied*

1. *The Taylor coefficients of f at the origin are rational and the growth conditions* (3.3) *are satisfied,*
2. *f has rational Taylor coefficients at the origin satisfying the second growth condition of* (3.3) *on denominators and η_d is polynomially bounded,*
3. *$f(z) = \sum_{k\geqslant 1} a_k z^{n_k}$, the lacunarity condition* (3.5) *is satisfied and for some $p > 0$, for any $k \geqslant 1$, $|a_k| \geqslant e^{-n_k^p}$,*
4. *f is a solution of an algebraic differential equation with rational coefficients and initial conditions,*
5. *f is a random series, in the sense of Section 4.*

Then there exist $\alpha, \beta > 0$ such that

$$\#\Gamma_f(\mathbb{Q}, T) \leqslant \beta \log^{\alpha} T.$$

Remark 5.3. — Not only for functions definable in some o-minimal structures, but also for analytic functions, the asymptotics of (5.1) is sharp, since, for instance by [34, Example 7.5], [41] or [42], there exist functions analytic on a neighbourhood of a compact interval having asymptotically as many as possible rational points of height at most T in their graph with respect to (5.1). For instance, for any $\epsilon \in]0,1[$, more than $\frac{1}{2}e^{2\log^{1-\epsilon} T}$ points, for an infinite sequence of heights T. In consequence one cannot expect polynomial Bézout bounds in all degree d for these analytic functions, since by (5.2) one has

$$\frac{1}{2}e^{2\lfloor \log T\rfloor^{1-\epsilon}} \leqslant \frac{1}{2}e^{2\log^{1-\epsilon} T} \leqslant Z_{\lfloor \log T\rfloor}(f)C_{\Gamma_f,\lfloor \log T\rfloor}T^{\tau_{\lfloor \log T\rfloor}}.$$

And thus for any $\zeta \in]0,1[$, there exists a sequence of degrees d going to infinity and such that

$$\mathcal{Z}_d(f) \geqslant Z_d(f) \geqslant e^{d^\zeta}. \tag{5.4}$$

As a consequence of (5.4), the condition that $\Delta_d \geqslant e^{-U(d)}$ for some positive polynomial U, may not be satisfied for particular analytic functions f. Indeed, when $\Delta_d \neq 0$, the transcendence index ν_d is polynomially bounded in d and in case $\Delta_d \geqslant e^{-U(d)}$, by Proposition 3.6

$$\begin{aligned}\mathcal{Z}_d(f) \leqslant 5\nu_{2d}\log\left(4+(\nu_{2d}+1)\frac{e^{2(d+1)^3}}{\Delta}\right)\\ \leqslant 5\nu_{2d}[\log(5)+\log(\nu_{2d}+1)+2(d+1)^3+U(d)].\end{aligned}$$

Remark 5.4. — The condition $|a_k| \geqslant e^{-n_k^p}$ of Theorem 5.2 (4) allows in particular order 0 for the lacunary series $f(z) = \sum_{k\geqslant 1} a_k z^{n_k}$ when f is an entire function, since the order of f is given by $\limsup_{n\to\infty} -\frac{n_k \log n_k}{\log |a_k|}$ (see [27, Theorem 14.1.1]), contrariwise to [13, Theorem 1.1] and [22, Section 7] where order 0 is not allowed. Furthermore the conditions of Theorem 5.2 allow to consider analytic functions that are not entire.

Remark 5.5. — Statement (5) of Theorem 5.2 can be seen as a consequence of [12, Theorem 2.7], that in fact shows that in case $f(0)$ has a convenient transcendence measure, then the set $\#\Gamma_f(\mathbb{Q},T)$ satisfies the conclusion of Theorem 5.2, and on the other hand the set of real numbers having this convenient transcendence measure is a full set of $\mathbb{R}$.

Remark 5.6. — Using the estimates of [23, Theorem 2.20] (see also [35, proof of Theorem 1.5]) and Corollary 4.3, on deduces that for random series, with probability one, the exponent α in the bound of Theorem 5.2 may be chosen as 8.

Remark 5.7. — The conditions 1 to 4 on f in Theorem 5.2 are natural with the aim of showing that there are few rational points in Γ_f, since combinations of finer conditions are considered in order to obtain more remarkable Diophantine properties for Γ_f. For instance, in Siegel–Shidlovskii's theorem a combination of conditions comparable to conditions (1) and (4) of Theorem 5.2, among others, imply that there is at most one rational point in Γ_f. More accurately, let f be a E-function, that is a Taylor series

$\sum_{k=0}^{\infty} a_k z^k$ with, for simplicity, rational coefficients a_k satisfying for any $\varepsilon > 0$

$$|a_k| = O(k^{k(\varepsilon-1)}) \text{ and } |q_k| = O(k^{\varepsilon k}),$$

where q_k is a common denominator for $a_0, a_1, 2!a_2, \ldots, k!a_k$. Assuming moreover that f is solution of a linear differential equation of order n with coefficients in $\mathbb{Q}[z]$, such that $f, f', \ldots, f^{(n-1)}$ are algebraically independent over $\mathbb{C}(z)$, then for any algebraic number $z_0 \neq 0$, the numbers $f(z_0), \ldots, f^{(n-1)}(z_0)$ are algebraically independent over $\mathbb{Q}$, and in particular $f(z_0)$ is transcendental (see for instance [32, Theorem 2.1] or [40, Theorem 3, p. 123]).

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Manuscrit reçu le 23 septembre 2016,
révisé le 2 novembre 2017,
accepté le 13 décembre 2017.

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