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Yang CAO & Fei XU

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STRONG APPROXIMATION WITH BRAUER–MANIN OBSTRUCTION FOR TORIC VARIETIES

by Yang CAO & Fei XU

ABSTRACT. — For smooth open toric varieties, we establish strong approximation off infinity with Brauer–Manin obstruction.

RÉSUMÉ. — Pour les variétés toriques lisses ouvertes, on établit l'approximation forte par rapport à l'obstruction de Brauer–Manin hors de infini.

1. Introduction

Strong approximation has various arithmetic application, for example to determine the existence of integral points by the local-global principle. By using Manin's idea, J.-L. Colliot-Thélène and F. Xu established strong approximation with Brauer–Manin obstruction for homogeneous spaces of semi-simple and simply connected algebraic groups in [9] to refine the classical strong approximation. Since then, a significant progress for strong approximation with Brauer–Manin obstruction has been made for various homogeneous spaces of linear algebraic groups in [1, 13, 17, 26] and families of homogeneous spaces in [5, 10]. In this paper, we study strong approximation with Brauer–Manin obstruction for open smooth toric varieties. Such varieties have been extensively studied over algebraically closed fields (see [15, 20]). However they are hard to study over number fields. For example, a smooth toric variety may not be covered by open affine toric subvarieties over a field.

Notation and terminology are standard. Let k be a number field, Ω_k the set of all primes in k and ∞_k the set of all archimedean primes in k. Write $v < \infty_k$ for $v \in \Omega_k \setminus \infty_k$. Let O_k be the ring of integers of k and $O_{k,S}$ the

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S-integers of k for a finite set S of Ω_k containing ∞_k. For each $v \in \Omega_k$, the completion of k at v is denoted by k_v and the completion of O_k at v by O_v. Write $O_v = k_v$ for $v \in \infty_k$. Let $\mathbf{A}_k$ be the ring of adeles of k and $\mathbf{A}_k^S$ the adeles of k without S-components.

For any scheme X of finite type over k, we denote

$$\begin{aligned} \mathrm{Br}(X) &= H^2_{\text{ét}}(X, \mathbb{G}_m), \\ \mathrm{Br}_1(X) &= \ker[\mathrm{Br}(X) \to \mathrm{Br}(X_{\bar{k}})], \\ \mathrm{Br}_a(X) &= \mathrm{coker}[\mathrm{Br}(k) \to \mathrm{Br}_1(X)] \end{aligned}$$

where $\mathbb{G}_m$ is the group scheme defined by the multiplicative group and $X_{\bar{k}} = X \times_k \bar{k}$ with a fixed algebraic closure $\bar{k}$ of k. We also use $\mathbb{A}^n$ to denote an affine space of dimension n. For any subset B of $\mathrm{Br}(X)$, one defines

$$X(\mathbf{A}_k)^B = \left\{ (x_v)_{v \in \Omega_k} \in X(\mathbf{A}_k) \,\middle|\, \sum_{v \in \Omega_k} \mathrm{inv}_v(\xi(x_v)) = 0, \ \forall\, \xi \in B \right\}$$

where

$$\mathrm{inv}_v : \quad \mathrm{Br}(k_v) \xrightarrow{\cong} \begin{cases} \mathbb{Q}/\mathbb{Z} & \text{if } v \text{ is finite} \\ \frac{1}{2}\mathbb{Z}/\mathbb{Z} & \text{if } v \text{ is real} \\ 0 & \text{if } v \text{ is complex} \end{cases}$$

by local class field theory. This set is a closed subset of $X(\mathbf{A}_k)$ endowed with adelic topology ([11, §4]). As discovered by Manin, class field theory implies that $X(k) \subseteq X(\mathbf{A}_k)^B$. Let π_P denote the projection from adelic points $X(\mathbf{A}_k)$ to adelic points without P-components $X(\mathbf{A}_k^P)$ for any finite subset P of Ω_k.

Definition 1.1. — *Let X be a scheme of finite type over k, and P a finite subset of Ω_k.*

1. *Suppose $X(\mathbf{A}_k) \neq \emptyset$. We say that X satisfies strong approximation off P if $X(k)$ is dense in $\pi_P(X(\mathbf{A}_k))$.*
2. *Suppose $X(\mathbf{A}_k)^{\mathrm{Br}(X)} \neq \emptyset$. We say that X satisfies strong approximation with Brauer–Manin obstruction off P if $X(k)$ is dense in $\pi_P(X(\mathbf{A}_k)^{\mathrm{Br}(X)})$.*

A toric variety over k is defined as a partial equivariant compactification of a torus over k. More precisely, a toric variety is an integral normal and separated scheme of finite type over k containing an torus T as Zariski open subset with a compatible action of T (see [12] or [15]). The main result of this paper is the following theorem.

Theorem 1.2. — *Any smooth toric variety over k satisfies strong approximation with Brauer–Manin obstruction off ∞_k.*

Comparing with Theorem 2 in [17] for tori, one also has that a smooth toric variety X satisfies strong approximation with algebraic Brauer–Manin obstruction $\mathrm{Br}_1(X)$. However, one can not force the rational point to land in some prescribed connected components at real places from our proof at this moment. As a corollary, we have:

Corollary 1.3. — *Let P be a subset of Ω_k with $P \supseteq \infty_k$. Then any smooth toric variety over k satisfies strong approximation with Brauer–Manin obstruction off P.*

We learned that D. Wei has obtained the same result in [25] under the condition $\bar{k}[X]^\times = \bar{k}^\times$. More precisely, he proves that for any smooth toric variety X satisfying $\bar{k}[X]^\times = \bar{k}^\times$, any closed subset $W \subseteq X$ with $\mathrm{codim}(W, X) \geqslant 2$, and any $v_0 \in \Omega_k$, the variety $X - W$ satisfies strong approximation with Brauer–Manin obstruction off v_0. Without the condition $\bar{k}[X]^\times = \bar{k}^\times$, this result does not hold in general (see Example 5.2).

This paper is organized as follows.

In Section 2, we study the structure of smooth toric varieties over an arbitrary field of characteristic 0. We give a structure theorem for affine smooth toric varieties (Proposition 2.4). We then define the notion of smooth toric varieties of pure divisorial type (Definition 2.5) and the notion of standard toric varieties (Definition 2.8). In any smooth toric variety, there exists a closed subvariety of codimension $\geqslant 2$, whose complement is a smooth toric variety of pure divisorial type (Proposition 2.6). We construct a morphism from a standard toric variety to a given toric variety, and prove a structure theorem for smooth toric varieties by this morphism (Proposition 2.10).

In Section 3, we extend strong approximation with Brauer–Manin obstruction off ∞_k for tori proved by Harari in [17] to a relative strong approximation with Brauer–Manin obstruction off ∞_k for tori (Proposition 3.4). We establish strong approximation off ∞_k for standard toric varieties (Corollary 3.7).

In Section 4, using the morphism constructed in Section 2, we establish the crucial step (Proposition 4.1), which gives a precise relation between the O_v-points of a given toric variety and the O_v-points of a standard toric variety for almost all place $v \in \Omega_k$. Then, by combining relative strong approximation for tori and strong approximation for standard toric varieties, we establish strong approximation with Brauer–Manin obstruction off ∞_k for smooth toric varieties of pure divisorial type (Proposition 4.3), and then for any smooth open toric varieties (Theorem 4.5).

In Section 5, we give an example (Example 5.2), which shows that the complement of a point in a toric variety may no longer satisfy strong approximation with Brauer–Manin obstruction off ∞_k. This is in contrast with the case of affine space minus a closed subscheme of codimension $\geqslant 2$ (Proposition 3.6).

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2. Structure of smooth toric varieties

Toric varieties have been extensively studied over an algebraically closed field (see [15, 20]). In this section, we study the structure of toric varieties over a field k with $\mathrm{char}(k) = 0$. Let $\bar{k}$ be an algebraic closure of k. For a torus T over k, we denote the character group of T by $T^* = \mathrm{Hom}_{\bar{k}}(T, \mathbb{G}_m)$, which is a free $\mathbb{Z}$-module of finite rank with continuous action of $\Gamma_k = \mathrm{Gal}(\bar{k}/k)$. It is well-known that these two categories are anti-equivalent (see [14, Exposé X, Proposition 1.4]).

The simplest example of toric variety is $\mathbb{A}^s \times \mathbb{G}_m^t$ containing the natural open torus $\mathbb{G}_m^{s+t}$ for some non-negative integers s and t. Such toric varieties are the building blocks of smooth toric varieties. The following lemma is due to Sumihiro in [24].

Lemma 2.1 (Sumihiro). — *Let $k = \bar{k}$. Any toric variety $(T \hookrightarrow X)$ has a finite open covering $\{U_j\}$ of X over $\bar{k}$ such that all $(T \hookrightarrow U_j)$'s are affine toric sub-varieties over $\bar{k}$. Moreover, if X is smooth, then one has isomorphisms of toric varieties over $\bar{k}$*

$$\begin{array}{ccc} T & \xrightarrow{\cong} & \mathbb{G}_m^{s_j+t_j} \\ {\scriptstyle i_T}\downarrow & & \downarrow \\ U_j & \xrightarrow[\cong]{} & \mathbb{A}^{s_j} \times_{\bar{k}} \mathbb{G}_m^{t_j} \end{array}$$

with some integers $s_j, t_j \geqslant 0$ and $s_j + t_j = \dim(T)$ for each j, where i_T is the toric embedding $T \hookrightarrow X$.

Proof. — By Lemma 8 and Corollary 2 in [24], one has a finite affine open covering $\{U_j\}$ of X over $\bar{k}$ such that all U_j's are T-stable. Since X is irreducible, one has $U_j \cap i_T(T) \neq \emptyset$. Take

$$x_0 = i_T(t_0) \in U_j(\bar{k}) \cap i_T(T(\bar{k}))$$

with $t_0 \in T(\bar{k})$ and one obtains $i_T(T(\bar{k})) = i_T(T(\bar{k})t_0) \subseteq U_j(\bar{k})$. Therefore $i_T : T \hookrightarrow U_j$ for all j by Hilbert Nullstellensatz and all U_j's are toric varieties with respect to T.

If X is smooth, all U_j's are smooth. Thus $(T \hookrightarrow U_j)$ is isomorphic to $(\mathbb{G}_m^{s_j+t_j} \hookrightarrow \mathbb{A}^{s_j} \times_{\bar{k}} \mathbb{G}_m^{t_j})$ by the criterion of smoothness for affine toric variety (see [20, Theorem 1.10]) for all j. □

Remark 2.2. — Lemma 2.1 does not hold over a general field. For example, consider the conic $x^2 - ay^2 = z^2$ inside $\mathbb{P}^2$ over $\mathbb{Q}$ with $a \notin (\mathbb{Q}^\times)^2$. This conic is a toric variety containing an open subset with $z \neq 0$ which is isomorphic to the restriction of scalar of the norm one torus

$$T = \mathrm{Res}^1_{\mathbb{Q}(\sqrt{\backepsilon})/\mathbb{Q}}(\mathbb{G}_m).$$

This toric variety has no open affine toric subvariety covering over $\mathbb{Q}$. This is because the complement of T inside this conic is a point of degree 2 over $\mathbb{Q}$. If an affine toric subvariety over $\mathbb{Q}$ contains this point, then this affine toric variety is the whole space. However, this conic is not affine.

The set of rational points of toric varieties can be covered by open affine toric sub-varieties.

Corollary 2.3. — *Let $(T \hookrightarrow X)$ be a toric variety over k. If $x \in X(k)$, there is an open affine toric subvariety $(T \hookrightarrow M)$ of $(T \hookrightarrow X)$ over k such that $x \in M(k)$.*

Proof. — For $x \in X(k)$, there is a finite Galois extension k'/k and an open affine toric variety $(T_k \times_k k' \hookrightarrow U)$ over k' such that $x \in U(k')$ by Lemma 2.1. Then

$$x \in M = \bigcap_{\sigma \in \mathrm{Gal}(k'/k)} \sigma(U)$$

and M is stable under $\mathrm{Gal}(k'/k)$. One concludes that M is defined over k by Galois descent (see [2, §6.2, Example B]) and $(T \hookrightarrow M)$ is an open affine toric variety over k by separateness of X. □

Since the number of $\bar{k}$-orbits of $T(\bar{k})$ in $X(\bar{k})$ are finite for a smooth toric variety X over k, by Galois descent, there is a smallest open affine toric subvariety containing a given rational point over k.

Proposition 2.4. — *If $(T \hookrightarrow X)$ is a smooth affine toric variety over k, then the homomorphism of Γ_k-modules*

$$\bar{k}[T]^\times/\bar{k}^\times \to \mathrm{Div}_{X_{\bar{k}}\setminus T_{\bar{k}}}(X_{\bar{k}}); \quad f \mapsto \mathrm{div}_{X_{\bar{k}}\setminus T_{\bar{k}}}(f)$$

is surjective, which induces an injective homomorphism of tori

$$\mathrm{Res}_{K/k}(\mathbb{G}_m) \to T$$

with a finite étale k-algebra K/k. Moreover,

(1) *This injective homomorphism of tori can be extended to a closed immersion of toric varieties $\mathrm{Res}_{K/k}(\mathbb{A}^1) \hookrightarrow X$.*

(2) $(\mathrm{Res}_{K/k}(\mathbb{G}_m) \hookrightarrow \mathrm{Res}_{K/k}(\mathbb{A}^1))$ *is a unique closed toric subvariety of $(T \hookrightarrow X)$ such that the quotient homomorphism*

$$\phi : T \to T_1 \quad \textit{with} \quad T_1 = T/\mathrm{Res}_{K/k}(\mathbb{G}_m)$$

can be extended to a faithful flat morphism $\phi : X \to T_1$ over k commuting with the action

$$\begin{array}{ccc} T \times_k X & \xrightarrow{\phi\times\phi} & T_1 \times_k T_1 \\ {\scriptstyle m_X}\downarrow & & \downarrow{\scriptstyle m_{T_1}} \\ X & \xrightarrow[\phi]{} & T_1 \end{array}$$

and $\phi^{-1}(1) \cong \mathrm{Res}_{K/k}(\mathbb{A}^1)$.

(3) *ϕ induces an isomorphism $\mathrm{Br}_1(T_1) \xrightarrow{\sim} \mathrm{Br}_1(X)$.*

Proof. — Since $\mathrm{Pic}(X_{\bar{k}}) = 0$ by Lemma 2.1, one has that every Weil divisor of $X_{\bar{k}}$ is principal and the following sequence

$$1 \to \bar{k}[X]^\times/\bar{k}^\times \to \bar{k}[T]^\times/\bar{k}^\times \to \mathrm{Div}_{X_{\bar{k}}\setminus T_{\bar{k}}}(X_{\bar{k}}) \to 1$$

of Γ_k-module by sending $f \mapsto \mathrm{div}_{X_{\bar{k}}\setminus T_{\bar{k}}}(f)$ is exact for any $f \in \bar{k}[T]^\times$. Since $\mathrm{Div}_{X_{\bar{k}}\setminus T_{\bar{k}}}(X_{\bar{k}})$ is a permutation Galois module, there is a finite étale k-algebra K/k such that

$$(\mathrm{Res}_{K/k}(\mathbb{G}_m))^* = \mathrm{Div}_{X_{\bar{k}}\setminus T_{\bar{k}}}(X_{\bar{k}})$$

and the induced map $\mathrm{Res}_{K/k}(\mathbb{G}_m) \to T$ is an injective homomorphism of tori.

(1). — Fix the set of generators $\{D_1, \ldots, D_s\}$ of $\mathrm{Div}_{X_{\bar{k}}\setminus T_{\bar{k}}}(X_{\bar{k}})$ such that each D_i is a primitive divisor of $X_{\bar{k}}$ for $1 \leqslant i \leqslant s$. For any $f \in \bar{k}[X]$, one has $ord_{D_i}(f) \geqslant 0$ for $1 \leqslant i \leqslant s$. This implies that the injective homomorphism of tori $\mathrm{Res}_{K/k}(\mathbb{G}_m) \to T$ can be extended to $\mathrm{Res}_{K/k}(\mathbb{A}^1) \to X$. Since D_i is principal for $1 \leqslant i \leqslant s$, one further concludes the map

$$\bar{k}[X] \to \bar{k}[\mathrm{Res}_{K/k}(\mathbb{A}^1)]$$

is surjective. Namely, the extended morphism $\mathrm{Res}_{K/k}(\mathbb{A}^1) \to X$ is a closed immersion.

(2). — By the above short exact sequence, one has that $T_1^* = \bar{k}[X]^\times/\bar{k}^\times$. Let

$$B = \{f \in \bar{k}[X]^\times : \ f(1_T) = 1\}$$

which is stable under the action of Γ_k and $\bar{k}[B]$ be the group algebra generated by B over $\bar{k}$. Then

$$\bar{k}[X]^\times \cong \bar{k}^\times \oplus B, \quad f \mapsto (f(1), f(1)^{-1}f)$$

as Γ_k-module. The $\bar{k}$-algebra isomorphism

$$\bar{k}[T_1] \cong \bar{k}[B] \quad \text{induced by} \quad B \cong \bar{k}[X]^\times/\bar{k}^\times$$

is compatible with Γ_k-action. Moreover, the natural inclusion of $\bar{k}$-algebras $\bar{k}[B] \subseteq \bar{k}[X]$ is compatible with Γ_k-action as well. This gives the morphism $X \to T_1$ over k which extends $\phi : T \to T_1$. Since ϕ is a homomorphism of tori, this implies that the diagram in (2) commutes.

Since $\phi^{-1}(1)$ contains $\mathrm{Res}_{K/k}(\mathbb{G}_m)$ and is closed, one obtains that $\phi^{-1}(1)$ contains $\mathrm{Res}_{K/k}(\mathbb{A}^1)$. By comparing the dimension, one concludes that $\phi^{-1}(1) = \mathrm{Res}_{K/k}(\mathbb{A}^1)$.

By Lemma 2.1, one has

$$X_{\bar{k}} \cong \mathbb{A}^s \times_{\bar{k}} (T_1 \times_k \bar{k}) \quad \text{and} \quad \bar{\phi} = \phi \times_k \bar{k} : \ X_{\bar{k}} \to T_1 \times_k \bar{k}$$

is the projection. Therefore $\phi : X \to T_1$ is faithfully flat.

Now we prove the uniqueness. Suppose that $(T \hookrightarrow X)$ contains another closed toric subvariety

$$(\mathrm{Res}_{K'/k}(\mathbb{G}_m) \hookrightarrow \mathrm{Res}_{K'/k}(\mathbb{A}^1))$$

with a finite étale k-algebra K'/k such that the quotient homomorphism

$$\phi' : T \to T_1' \quad \text{with} \quad T_1' = T/\mathrm{Res}_{K'/k}(\mathbb{G}_m)$$

can be extended to a morphism $\phi' : X \to T_1'$ over k satisfying $\phi'^{-1}(1) = \mathrm{Res}_{K'/k}(\mathbb{A}^1)$. In this case, ϕ' induces an injective Γ_k-homomorphism

$$\chi^* : \ T_1'^* \to \bar{k}[X]^\times/\bar{k}^\times = T_1^* \quad \text{such that} \quad T_1^*/\chi^*(T_1'^*) \text{ is torsion free}$$

and $\phi' = \chi \circ \phi$ with $T_1 \xrightarrow{\chi} T_1'$ is induced by χ^*. Since $\phi'^{-1}(1) = \mathrm{Res}_{K'/k}(\mathbb{A}^1)$, one has $\bar{k}[\phi'^{-1}(1)]^\times = \bar{k}^\times$. Since $\phi : X \to T_1$ is faithfully flat, $\phi : \phi'^{-1}(1) \to \chi^{-1}(1)$ is faithfully flat. Thus $\phi^* : \bar{k}[\chi^{-1}(1)]^\times \to \bar{k}[\phi'^{-1}(1)]^\times = \bar{k}^\times$ is injective. Since $\chi^{-1}(1) = \ker(\chi)$, one has $\bar{k}[\ker(\chi)]^\times = \bar{k}^\times$ and $\ker(\chi)$ is trivial. This implies that $T_1^* = \chi^*(T_1'^*)$ and χ is an isomorphism. One concludes that $\phi^{-1}(1) = \phi'^{-1}(1)$ and the uniqueness follows.

(3). — By the Hochschild–Serre spectral sequence (see [22, Lemma 6.3]) with $\mathrm{Pic}(X_{\bar{k}}) = \mathrm{Pic}(T_1 \times_k \bar{k}) = 0$, we have

$$Br_1(X) \cong H^2(k, \bar{k}[X]^\times) \cong H^2(k, \bar{k}[T_1]^\times) \cong Br_1(T_1)$$

induced by ϕ. □

The following kind of toric varieties is crucial for studying strong approximation.

Definition 2.5. — *A smooth toric variety $(T \hookrightarrow X)$ over k is called of pure divisorial type if the dimension of any $T(\bar{k})$-orbit of $X(\bar{k})$ is* $\dim(T)$ *or* $\dim(T) - 1$*. Equivalently, the dimension of any cone in the fan of X is strictly less than 2.*

We would like to thank J.-L. Colliot-Thélène for improving the following result.

Proposition 2.6. — *Let $(T \hookrightarrow X)$ be a smooth toric variety over k and $Y = X \setminus [(X \setminus T)_{\mathrm{sing}}]$ where $(X \setminus T)_{\mathrm{sing}}$ is the singular part of $X \setminus T$. Then $(T \hookrightarrow Y)$ is the unique open toric subvariety of $(T \hookrightarrow X)$ of pure divisorial type over k such that* $\mathrm{codim}(X \setminus Y, X) \geqslant 2$.

Proof. — Without loss of generality, one can assume that $k = \bar{k}$. By Lemma 2.1, there is a finite open affine toric subvariety covering $\{U_j\}$ with $U_j \cong \mathbb{A}^{s_j} \times_{\bar{k}} \mathbb{G}_m^{t_j}$. Since $U_j \setminus T$ is a closed subvariety of U_j defined by the equation $\prod_{i=1}^{s_j} x_i = 0$ where $x_1, \ldots, x_{s_j}$ are the coordinates of $\mathbb{A}^{s_j}$, the singular part $(U_j \setminus T)_{\mathrm{sing}}$ of $U_j \setminus T$ consists of the points of which at least two coordinates among $x_1, \ldots, x_{s_j}$ are zero. Therefore $U_j \setminus [(U_j \setminus T)_{\mathrm{sing}}]$ is an open toric subvariety of U_j of pure divisorial type with

$$\mathrm{codim}([(U_j \setminus T)_{\mathrm{sing}}], U_j) \geqslant 2.$$

Since

$$X \setminus T = \bigcup_j (U_j \setminus T) \quad \text{and} \quad (X \setminus T)_{\mathrm{sing}} = \bigcup_j (U_j \setminus T)_{\mathrm{sing}},$$

one concludes that $(T \hookrightarrow Y)$ is an open toric subvariety of $(T \hookrightarrow X)$ of pure divisorial type over k such that $\mathrm{codim}(X \setminus Y, X) \geqslant 2$.

Suppose Z is another open toric subvariety of pure divisorial type of X. Since the dimension of $T(\bar{k})$ orbits in Z is $\dim(T)$ or $\dim(T) - 1$, one has $Z \subseteq Y$ by the above construction. If one further assumes that $\dim(X \setminus Z) <$

$\dim(T)-1$, then $X\setminus Z\subseteq X\setminus Y$ by the above construction. This implies that $Y\subseteq Z$. Therefore $Z=Y$ and the uniqueness follows. □

LEMMA 2.7. — *If $(T_i\hookrightarrow X_i)$ are smooth toric varieties over k and $(T_i\hookrightarrow Y_i)$ are the unique open toric subvarieties of pure divisorial type with $\operatorname{codim}(X_i\setminus Y_i,X_i)\geqslant 2$ for $1\leqslant i\leqslant n$ respectively, then the unique open toric subvariety $(\prod_{i=1}^n T_i\hookrightarrow Y)$ of pure divisorial type with*

$$\operatorname{codim}\left(\left(\prod_{i=1}^n X_i\right)\setminus Y,\prod_{i=1}^n X_i\right)\geqslant 2\quad \textit{in}\quad \left(\prod_{i=1}^n T_i\hookrightarrow \prod_{i=1}^n X_i\right)$$

is given by

$$Y=\bigcup_{i=1}^n (T_1\times_k\cdots\times_k T_{i-1}\times_k Y_i\times_k T_{i+1}\times_k\cdots\times_k T_n).$$

Proof. — Since

$$\dim((T_1\times_k\cdots\times_k T_n)(\bar{k})\cdot(x_1,\dots,x_n))=\sum_{i=1}^n\dim(T_i(\bar{k})\cdot x_i)$$

for any $(x_1,\dots,x_n)\in X_1(\bar{k})\times\cdots\times X_n(\bar{k})$, one obtains that

$$\dim((T_1\times_k\cdots\times_k T_n)(\bar{k})\cdot(x_1,\dots,x_n))=\dim(T_1\times_k\cdots\times_k T_n)-1$$

if and only if there is $1\leqslant i_0\leqslant n$ such that

$$\dim(T_i(\bar{k})\cdot x_i)=\begin{cases}\dim(T_i)-1 & \text{if } i=i_0,\\ \dim(T_i) & \text{otherwise.}\end{cases}$$

This implies that

$$\bigcup_{i=1}^n (T_1\times_k\cdots\times_k T_{i-1}\times_k Y_i\times_k T_{i+1}\times_k\cdots\times_k T_n)$$

is of pure divisorial type and contains all orbits of dimension $\dim(T_1\times_k\cdots\times_k T_n)$ or $\dim(T_1\times_k\cdots\times_k T_n)-1$. □

DEFINITION 2.8. — *Let d be a positive integer, and k_i/k some finite field extensions for $1\leqslant i\leqslant d$. We note $K:=\prod_{i=1}^d k_i$. A smooth toric variety $(\operatorname{Res}_{K/k}(\mathbb{G}_m)\hookrightarrow X)$ over k is called the standard toric variety with respect to K/k, if it is the unique open toric subvariety of pure divisorial type over k in*

$$(\operatorname{Res}_{K/k}(\mathbb{G}_m)\hookrightarrow\operatorname{Res}_{K/k}(\mathbb{A}^1))\quad \textit{with}\quad \operatorname{codim}(\operatorname{Res}_{K/k}(\mathbb{A}^1)\setminus X,\ \operatorname{Res}_{K/k}(\mathbb{A}^1))\geqslant 2.$$

Let X be a smooth toric variety of pure divisorial type with respect to T over k and

$$X \setminus T = \coprod_{i=1}^{d} C_i \quad \text{and} \quad U_i = X \setminus \left(\coprod_{j \neq i} C_j \right) \tag{2.1}$$

for $1 \leqslant i \leqslant d$, where the C_i's are integral closed sub-schemes of X over k with codimension one. Then U_i is an open toric subvariety of X over k for $1 \leqslant i \leqslant d$. By Lemma 2.1, one obtains that each T-orbit in X over $\bar{k}$ is smooth. Since $(X \setminus T)(\bar{k})$ is a disjoint union of $T(\bar{k})$-orbits of dimension $\dim(T) - 1$ and C_i is a sub-union among such orbits permuted by Galois action, one has that C_i is also smooth for $1 \leqslant i \leqslant d$.

Let k_i be the algebraic closure of k inside $k(C_i)$ for $1 \leqslant i \leqslant d$. There is a closed geometrically integral sub-scheme D_i over k_i such that

$$C_i \times_k \bar{k} = \coprod_{\sigma \in \Upsilon_i} \sigma(D_i) \tag{2.2}$$

where $\Upsilon_i = \Gamma_k / \Gamma_{k_i}$ is the set of all k-embedding of k_i into $\bar{k}$ for $1 \leqslant i \leqslant d$. Since Γ_{k_i} acts on $\coprod_{\tau \in \Upsilon_i, \tau \neq 1} \tau(D_i)$ stably, one concludes that $\Gamma_{\sigma(k_i)} = \sigma \Gamma_{k_i} \sigma^{-1}$ acts on $\coprod_{\tau \in \Upsilon_i, \tau \neq \sigma} \tau(D_i)$ stably for each $\sigma \in \Upsilon_i$. This implies that the scheme $\coprod_{\tau \in \Upsilon_i, \tau \neq \sigma} \tau(D_i)$ is defined over $\sigma(k_i)$ for each $\sigma \in \Upsilon_i$ by Galois descent.

For each $\sigma \in \Upsilon_i$, one defines

$$\sigma(Z_i) = (X \times_k \sigma(k_i)) \setminus \left(\left(\coprod_{\tau \in \Upsilon_i, \tau \neq \sigma} \tau(D_i) \right) \cup \left(\coprod_{j \neq i} C_j \times_k \sigma(k_j) \right) \right) \tag{2.3}$$

which is an open toric subvariety of $(T \times_k \sigma(k_i) \hookrightarrow X \times_k \sigma(k_i))$ over $\sigma(k_i)$ for $1 \leqslant i \leqslant d$. Since D_i is geometrically integral, this implies that $\sigma(Z_i)$ contains only two orbits over $\bar{k}$ for $1 \leqslant i \leqslant d$. Since $\sigma(Z_i)$ is covered by open affine toric sub-varieties over $\bar{k}$ by Lemma 2.1, the open affine toric sub-varieties which contain the closed orbit must be $\sigma(Z_i)$. This implies that $\sigma(Z_i)$ is affine and $\{\sigma(Z_i) \times_{\sigma(k_i)} \bar{k}\}_{\sigma \in \Upsilon_i}$ is a smooth open affine toric subvariety covering of $U_i \times_k \bar{k}$ for $1 \leqslant i \leqslant d$.

By Proposition 2.4 and its proof, the short exact sequence

$$1 \to \bar{k}[\sigma(Z_i)]^\times / \bar{k}^\times \xrightarrow{\phi_\sigma^*} \bar{k}[T]^\times / \bar{k}^\times \xrightarrow{\varrho_\sigma^*} \mathbb{Z}\sigma(D_i) \to 1 \tag{2.4}$$

of $\Gamma_{\sigma(k_i)}$-module given by sending f to its valuation at $\sigma(D_i)$ yields the exact sequence of tori

$$1 \to \mathbb{G}_m \xrightarrow{\varrho_\sigma} T \times_k \sigma(k_i) \xrightarrow{\phi_\sigma} T_\sigma \to 1 \tag{2.5}$$

over $\sigma(k_i)$ with $(T_\sigma)^* = \bar{k}[\sigma(Z_i)]^\times/\bar{k}^\times$ and a closed immersion of toric varieties

$$(\mathbb{G}_m \hookrightarrow \mathbb{A}^1) \xrightarrow{\varrho_\sigma} (T \times_k \sigma(k_i) \hookrightarrow \sigma(Z_i)) \tag{2.6}$$

over $\sigma(k_i)$. Moreover the morphism ϕ_σ can be extended to

$$\phi_\sigma : \sigma(Z_i) \to T_\sigma \quad \text{with} \quad \varrho_\sigma(\mathbb{A}^1) = \phi_\sigma^{-1}(1) \tag{2.7}$$

for any $\sigma \in \Upsilon_i$.

Lemma 2.9. — *With the above notation, one considers the homomorphism of* Γ_k*-modules*

$$\rho_i^* : \quad \bar{k}[T]^\times/\bar{k}^\times \to \mathrm{Div}_{(U_i\times_k\bar{k})\setminus T_{\bar{k}}}(U_i \times_k \bar{k})$$

sending f *to* $\mathrm{div}_{(U_i\times_k\bar{k})\setminus T_{\bar{k}}}(f)$ *and obtains a homomorphism* $\mathrm{Res}_{k_i/k}\,\mathbb{G}_m \xrightarrow{\rho_i} T$ *of tori over* k *for* $1 \leqslant i \leqslant d$*. If* $(\mathrm{Res}_{k_i/k}\,\mathbb{G}_m \hookrightarrow V_i)$ *is the standard toric variety with respect to* k_i/k*, then the homomorphism* ρ_i *can be extended to a morphism of toric varieties*

$$(\mathrm{Res}_{k_i/k}(\mathbb{G}_m) \hookrightarrow V_i) \xrightarrow{\rho_i} (T \hookrightarrow U_i)$$

over k *for* $1 \leqslant i \leqslant d$.

Proof. — Since

$$\rho_i^*(f) = \sum_{\sigma\in\Upsilon_i} \varrho_\sigma^*(f)$$

for any $f \in \bar{k}[T]^\times/\bar{k}^\times$ by (2.4) where Υ_i is the set of all k-embedding of k_i into $\bar{k}$, one has

$$\begin{aligned} \rho_i :\ \mathrm{Res}_{k_i/k}\,\mathbb{G}_m(\bar{k}) = (\bar{k}\otimes_k k_i)^\times = \prod_{\sigma\in\Upsilon_i} \bar{k}^\times &\to T(\bar{k}); \\ (a_\sigma)_{\sigma\in\Upsilon_i} &\mapsto \prod_{\sigma\in\Upsilon_i} \varrho_\sigma(a_\sigma) \end{aligned} \tag{2.8}$$

for $1 \leqslant i \leqslant d$. Let

$$Y_\sigma = \mathrm{Spec}(\bar{k}[x_\sigma, x_\tau, x_\tau^{-1}]_{\tau\in\Upsilon_i;\ \tau\neq\sigma}) \subset \mathrm{Res}_{k_i/k}(\mathbb{A}^1)\times_k \bar{k} = \mathrm{Spec}(\bar{k}[x_\sigma]_{\sigma\in\Upsilon_i})$$

for each $\sigma \in \Upsilon_i$. Then $\{Y_\sigma\}_{\sigma\in\Upsilon_i}$ is an open affine covering of $V_i \times_k \bar{k}$ for $1 \leqslant i \leqslant d$.

Applying (2.6) over $\bar{k}$, one obtains

$$\varrho_\sigma : \mathrm{Spec}(\bar{k}[x_\sigma]) \to \sigma(Z_i) \times_{\sigma(k_i)} \bar{k} \subseteq U_i \times_k \bar{k}$$

and ρ_i can be extended to

$$\rho_i :\ Y_\sigma \to \sigma(Z_i) \times_{\sigma(k_i)} \bar{k} \subseteq U_i \times_k \bar{k}$$

for each $\sigma \in \Upsilon_i$. Therefore ρ_i can be extended to V_i for $1 \leqslant i \leqslant d$. □

Gluing all ρ_i in Lemma 2.9 together for $1 \leqslant i \leqslant d$, one obtains the following proposition.

Proposition 2.10. — *Let $(T \hookrightarrow X)$ be a smooth toric variety of pure divisorial type over k and*

$$\rho : \quad T_0 = \mathrm{Res}_{K/k}(\mathbb{G}_m) \to T$$

be the homomorphism of tori induced by the homomorphism of Γ_k-modules

$$\rho^* : \ \bar{k}[T]^\times / \bar{k}^\times \to \mathrm{Div}_{X_{\bar{k}} \setminus T_{\bar{k}}}(X_{\bar{k}}); \quad f \mapsto \mathrm{div}_{X_{\bar{k}} \setminus T_{\bar{k}}}(f)$$

where $K = \prod_{i=1}^d k_i$ and k_i is the algebraic closure of k inside $k(C_i)$ with C_i in (2.1)*. If $T_0 = \mathrm{Res}_{K/k}(\mathbb{G}_m) \hookrightarrow V$ is the standard toric variety respect to K/k, then ρ can be extended to a morphism of toric varieties $(T_0 \hookrightarrow V) \xrightarrow{\rho} (T \hookrightarrow X)$.*

Proof. — By Lemma 2.7, one has

$$V = \bigcup_{i=1}^{d} \left(\prod_{1 \leqslant j \leqslant i-1} \mathrm{Res}_{k_j/k}(\mathbb{G}_m) \times_k V_i \times_k \prod_{i+1 \leqslant j \leqslant d} \mathrm{Res}_{k_j/k}(\mathbb{G}_m) \right)$$

where V_i is given in Lemma 2.9 for $1 \leqslant i \leqslant d$. Define

$$\begin{aligned} g_i : \prod_{1 \leqslant j \leqslant i-1} \mathrm{Res}_{k_j/k}(\mathbb{G}_m) \times_k V_i \times_k \prod_{i+1 \leqslant j \leqslant d} \mathrm{Res}_{k_j/k}(\mathbb{G}_m) & \\ \xrightarrow{\rho_1 \times \cdots \times \rho_d} T \times_k \cdots \times_k U_i \times_k \cdots \times_k T & \\ \xrightarrow{id \times \cdots \times i_{U_i} \times \cdots \times id} T \times_k \cdots \times_k X \times_k \cdots \times_k T \xrightarrow{m_X} X & \end{aligned}$$

where i_{U_i} is the open inclusion $U_i \subseteq X$ and ρ_i is given in Lemma 2.9 and m_X is the action of T for $1 \leqslant i \leqslant d$. Since $\rho^* = \bigoplus_{i=1}^d \rho_i^*$, one concludes that $g_i|_{T_0} = \rho$ for $1 \leqslant i \leqslant d$. Therefore the morphisms $\{g_i\}_{1 \leqslant i \leqslant d}$ can be glued together to obtain the required morphism. $\square$

By purity (see [4, end of p. 24]) and Lemma 2.6, one only needs to compute the Brauer groups of smooth toric varieties of pure divisorial type.

Proposition 2.11. — *One has the following exact sequence*

$$0 \to \mathrm{Br}_a(X) \to \mathrm{Br}_a(T) \xrightarrow{\rho^*} \mathrm{Br}_a(T_0)$$

for a smooth toric variety $(T \hookrightarrow X)$ of pure divisorial type over k, where ρ and T_0 are given by Proposition 2.10 and ρ^* is the induced by ρ.

Proof. — From Colliot-Thélène and Sansuc [7, §1] (see also [23, Diagram 4.15]), we have a commutative diagram with exact rows and exact columns

$$\begin{array}{ccccc}
0 & & 0 & & 0\\
\downarrow & & \downarrow & & \downarrow\\
H^2(k,\bar{k}[X]^\times/\bar{k}^\times) & \longrightarrow & H^2(k,T^*) & \xrightarrow{h_2} & H^2(k,\bar{k}[T]^\times/\bar{k}[X]^\times)\\
\downarrow & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_4}\\
Br_a(X) & \longrightarrow & Br_a(T) & \xrightarrow{h_3} & H^2(k, Div_{X_{\bar{k}}-T_{\bar{k}}}(X_{\bar{k}}))\\
\downarrow & & \downarrow & & \downarrow\\
H^1(k,Pic(X_{\bar{k}})) & \longrightarrow & H^1(k,Pic(T_{\bar{k}}))=0 & \longrightarrow & H^2(k,Pic_{X_{\bar{k}}-T_{\bar{k}}}(X_{\bar{k}})).
\end{array}$$

Since $T_0^* \cong \mathrm{Div}_{X_{\bar{k}}-T_{\bar{k}}}(X_{\bar{k}})$, the result follows from the fact that $h_3 \circ h_1 = h_4 \circ h_2$ is induced by $\rho^* : T^* \to T_0^*$. □

3. Relative strong approximation for tori

Harari proved strong approximation with Brauer–Manin obstruction off ∞_k for tori in [17]. In this section, we consider a homomorphism $T_1 \to T_2$ of tori and extend strong approximation with Brauer–Manin obstruction off ∞_k for this relative situation based on Harari's result. One can recover Harari's result when T_1 is trivial. In [13], Demarche used a similar idea for studying hyper-cohomology of complexes of two tori with finite kernel to establish strong approximation with Brauer–Manin obstruction off ∞_k for reductive groups.

DEFINITION 3.1. — *Let X be a smooth separated integral scheme of finite type over k. An integral model $\mathbf{X}$ of X over O_k (or $O_{k,S}$ for some finite subset S of Ω_k containing ∞_k) is defined to be a smooth separated integral scheme of finite type over O_k (or $O_{k,S}$) such that $\mathbf{X} \times_{O_k} k = X$ (or $\mathbf{X} \times_{O_{k,S}} k = X$).*

If T is a group of multiplicative type over k, an integral model $\mathbf{T}$ of T over O_k (or $O_{k,S}$) is defined to be an integral model of T which is a group scheme of multiplicative type over O_k (or $O_{k,S}$) extended from T.

Let X be a separated integral scheme of finite type over k and $\pi_0(X(k_v))$ be the set of connected components of $X(k_v)$ for each $v \in \infty_k$. Define

$$X(\mathbf{A}_k)_\bullet = \left[\prod_{v\in\infty_k} \pi_0(X(k_v))\right] \times X(\mathbf{A}_k^\infty)$$

and

$$X(\mathbf{A}_k)_\bullet^B = \left\{(x_v)_{v\in\Omega_k} \in X(\mathbf{A}_k)_\bullet \,\middle|\, \sum_{v\in\Omega_k} \mathrm{inv}_v(\xi(x_v)) = 0, \quad \forall\, \xi \in B\right\}$$

for any subset B of $Br_a(X)$. This is well-defined because any element in $Br_a(X)$ takes a constant value on each connected component of $X(k_v)$ for any $v \in \infty_k$ (see [1, §1.3]).

Lemma 3.2. — *Let $\psi : T_1 \to T_2$ be a homomorphism of tori. Then $\psi(T_1(k_v))$ is closed in $T_2(k_v)$ for all $v \in \Omega_k$.*

Proof. — Let T be the image of ψ. For any $v \in \Omega_k$, one has that $\psi(T_1(k_v))$ is an open subgroup of $T(k_v)$ by Corollary 1 in [21, Chapter 3]. Therefore $\psi(T_1(k_v))$ is closed in $T(k_v)$. It is clear that $T(k_v)$ is closed in $T_2(k_v)$. One concludes that $\psi(T_1(k_v))$ is closed in $T_2(k_v)$. □

Proposition 3.3. — *With the same notation as that in Lemma 3.2, one has*

$$\psi(T_1(\mathbf{A}_k)) = \left(\prod_{v\in\Omega_k} \psi(T_1(k_v))\right) \cap T_2(\mathbf{A}_k) \subseteq \prod_{v\in\Omega_k} T_2(k_v).$$

In particular, $\psi(T_1(\mathbf{A}_k))$ is closed in $T_2(\mathbf{A}_k)$.

Proof. — If ψ is surjective, one has the short exact sequence of groups of multiplicative type

$$1 \to T_0 \to T_1 \xrightarrow{\psi} T_2 \to 1$$

with $T_0 = \ker\psi$. There is a finite subset S of Ω_k containing ∞_k such that the above short exact sequence extends to

$$1 \to \mathbf{T}_0 \to \mathbf{T}_1 \xrightarrow{\psi_S} \mathbf{T}_2 \to 1$$

over $O_{k,S}$, where $\mathbf{T}_0$, $\mathbf{T}_1$ and $\mathbf{T}_2$ are integral models of T_0, T_1 and T_2 over $O_{k,S}$ respectively. For $v \notin S$, this yields exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbf{T}_0(O_v) & \longrightarrow & \mathbf{T}_1(O_v) & \xrightarrow{\psi_S} & \mathbf{T}_2(O_v) & \longrightarrow & H^1_{fppf}(O_v, \mathbf{T}_0) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & T_0(k_v) & \longrightarrow & T_1(k_v) & \xrightarrow{\psi} & T_2(k_v) & \longrightarrow & H^1(k_v, T_0). \end{array}$$

By Proposition 2.2 in [6], the natural map $H^1_{fppf}(O_v, \mathbf{T}_0) \to H^1(k_v, T_0)$ is injective. Then

$$\psi(T_1(k_v)) \cap \mathbf{T}_2(O_v) = \psi_S(\mathbf{T}_1(O_v))$$

for all $v \notin S$. Therefore

$$\psi(T_1(\mathbf{A}_k)) = \left(\prod_{v\in\Omega_k} \psi(T_1(k_v))\right) \cap T_2(\mathbf{A}_k)$$

and the subset $\psi(T_1(\mathbf{A}_k))$ is a closed subgroup of $T_2(\mathbf{A}_k)$ by Lemma 3.2.

In general, there is a closed sub-torus T of T_2 such that ψ factors through the surjective homomorphism $T_1 \to T$. By the above arguments, one has

$$\psi(T_1(\mathbf{A}_k)) = \left(\prod_{v\in\Omega_k} \psi(T_1(k_v))\right) \cap T(\mathbf{A}_k)$$

and $\psi(T_1(\mathbf{A}_k))$ is closed in $T(\mathbf{A}_k)$. Since T is a closed sub-torus of T_2, one has

$$T(\mathbf{A}_k) = \left(\prod_{v\in\Omega_k} T(k_v)\right) \cap T_2(\mathbf{A}_k)$$

and $T(\mathbf{A}_k)$ is a closed subset of $T_2(\mathbf{A}_k)$. Therefore one concludes that

$$\psi(T_1(\mathbf{A}_k)) = \left(\prod_{v\in\Omega_k} \psi(T_1(k_v))\right) \cap T_2(\mathbf{A}_k)$$

and $\psi(T_1(\mathbf{A}_k))$ is closed in $T_2(\mathbf{A}_k)$ by Lemma 3.2. □

By the functoriality of étale cohomology, one obtains an induced group homomorphism

$$\psi^*_{\mathrm{Br}} : \ \mathrm{Br}_a(T_2) \to \mathrm{Br}_a(T_1)$$

for any homomorphism $\psi : T_1 \to T_2$ of tori. For each $v \in \infty_k$, since the map ψ maps each connected component of $T_1(k_v)$ into one connected component of $T_2(k_v)$, one has

$$\psi(T_1(\mathbf{A}_k)_\bullet) \subseteq T_2(\mathbf{A}_k)_\bullet^{\ker(\psi^*_{\mathrm{Br}})}$$

by the functoriality of Brauer–Manin pairing (see [23, (5.3), p. 102]). One can extend strong approximation for tori proved by Harari in [17] to the following relative strong approximation for tori.

Proposition 3.4. — *Let $\psi : T_1 \to T_2$ be a homomorphism of tori with $Ш^1(T_1) = 0$. Then the image of $T_2(k)$ is dense in*

$$T_2(\mathbf{A}_k)_\bullet^{\ker(\psi^*_{\mathrm{Br}})} / \psi(T_1(\mathbf{A}_k)_\bullet)$$

with the quotient topology.

Proof. — By Theorem 2 in [17] and functoriality, one has the following commutative diagram of exact sequences

$$\begin{array}{ccccccc} 0 \longrightarrow & \overline{T_1(k)} & \longrightarrow & T_1(\mathbf{A}_k)_\bullet & \longrightarrow & \mathrm{Br}_a(T_1)^D & \longrightarrow \text{Ш}^1(T_1) = 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & \overline{T_2(k)} & \longrightarrow & T_2(\mathbf{A}_k)_\bullet & \longrightarrow & \mathrm{Br}_a(T_2)^D & \end{array}$$

where $\overline{T_1(k)}$ and $\overline{T_2(k)}$ are the topological closure of $T_1(k)$ and $T_2(k)$ in $T_1(\mathbf{A}_k)_\bullet$ and $T_2(\mathbf{A}_k)_\bullet$ respectively and

$$\mathrm{Br}_a(T_i)^D = \mathrm{Hom}(\mathrm{Br}_a(T_i), \mathbb{Q}/\mathbb{Z})$$

for $i = 1, 2$. Since $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module, one has $\mathrm{Hom}(*, \mathbb{Q}/\mathbb{Z})$ is an exact functor and the sequence

$$\mathrm{Br}_a(T_1)^D \to \mathrm{Br}_a(T_2)^D \to \ker(\psi^*_{\mathrm{Br}})^D \to 0$$

is exact. Therefore the natural map

$$\overline{T_2(k)} \to T_2(\mathbf{A}_k)_\bullet^{\ker(\psi^*_{\mathrm{Br}})}/\psi(T_1(\mathbf{A}_k)_\bullet)$$

is surjective by the snake lemma. Since the topological closure of the image of $T_2(k)$ in

$$T_2(\mathbf{A}_k)_\bullet^{\ker(\psi^*_{\mathrm{Br}})}/\psi(T_1(\mathbf{A}_k)_\bullet)$$

with the quotient topology contains the image of $\overline{T_2(k)}$, one obtains the result as desired. □

Remark 3.5. — One can state Proposition 3.4 in the following equivalent version for better understanding of relative strong approximation.

If

$$\left[\left(\prod_{v\in\infty_k} a_v N_{\mathbb{C}/k_v}(T_2(\mathbb{C}))\right) \times U\right] \cap T_2(\mathbf{A}_k)^{\ker(\psi^*_{\mathrm{Br}})} \neq \emptyset,$$

for an open subset U of $T_2(\mathbf{A}_k^\infty)$ and $a_v \in T(k_v)$ with $v \in \infty_k$, then there are $x \in T_2(k)$ and $y \in T_1(\mathbf{A}_k)$ such that

$$x\psi(y) \in \left(\prod_{v\in\infty_k} a_v N_{\mathbb{C}/k_v}(T_2(\mathbb{C}))\right) \times U.$$

In order to prove our main result, we need the following useful result.

Proposition 3.6. — *Let S be a finite non-empty subset of Ω_k, and U an open subscheme of $\mathbb{A}^n$ with $\mathrm{codim}(\mathbb{A}^n \setminus U, \mathbb{A}^n) \geqslant 2$. Then U satisfies strong approximation off S.*

Proof. — Fixing the coordinates, we consider the projection

$$p:\ \mathbb{A}^n \to \mathbb{A}^1;\ (x_1,\dots,x_n)\mapsto x_1$$

of the first coordinate. It is clear that any fibre of p above rational points are k-isomorphic to $\mathbb{A}^{n-1}$. Let $Z=\mathbb{A}^n\setminus U$ and p_Z be the restriction of p to Z. We claim that the set

$$F=\{x\in\mathbb{A}^1(k):\dim(p^{-1}(x)\cap Z)=\dim(Z)\}$$

is finite. Indeed, if $p_Z(Z)$ is finite, then $p^{-1}(x)\cap Z=\emptyset$ for almost all $x\in\mathbb{A}^1(k)$ and the claim follows. Otherwise, p_Z is dominant. Then p_Z is flat over an open dense subset of $\mathbb{A}^1$ by Theorem 2.16 in [19, Chapter I, §2]. Therefore the claim follows by Remark 2.6 (b) in [19, Chapter I, §2].

Let $p_U:U\to\mathbb{A}^1$ be the restriction of p to U. Since $\dim(p^{-1}(x))>\dim(p^{-1}(x)\cap Z)$ for any $x\in\mathbb{A}^1(\bar{k})$, one has

$$p_U^{-1}(x)=p^{-1}(x)\cap U=p^{-1}(x)\setminus(p^{-1}(x)\cap Z)\neq\emptyset$$

and $p_U^{-1}(x)$ is geometrically integral. In order to apply Proposition 3.1 in [10] to p_U with an open subset $W=\mathbb{A}^1\setminus F$ of $\mathbb{A}^1$, one only needs to verify three condition (i), (ii) and (iii) there to be true. Condition (i) is clearly true. By the above claim, one obtains that

$$\operatorname{codim}(p^{-1}(x)\cap Z,p^{-1}(x))\geqslant 2$$

for all $x\in W(k)$. Condition (ii) follows from induction. Since $p^{-1}(x)(k_v)$ is Zariski dense in $p^{-1}(x)$ for any $x\in\mathbb{A}^1(k_v)$ (see Theorem 2.2 in [21, Chapter 2]), one concludes

$$p_U^{-1}(x)(k_v)=(p^{-1}(x)\cap U)(k_v)=p^{-1}(x)(k_v)\setminus(p^{-1}(x)\cap Z)(k_v)\neq\emptyset$$

for any v. This implies Condition (iii) is satisfied as well. □

Corollary 3.7. — *Let d be a positive integer, S a finite nonempty subset of Ω_k, and k_i/k some finite field extensions for $1\leqslant i\leqslant d$. We note $K:=\prod_{i=1}^d k_i$. Then the standard toric variety $(\mathrm{Res}_{K/k}(\mathbb{G}_m)\hookrightarrow X)$ satisfies strong approximation off S.*

Proof. — There exists an isomorphism $\mathrm{Res}_{K/k}(\mathbb{A}^1)\xrightarrow{\sim}\mathbb{A}^{[K:k]}$. The result holds from Proposition 3.6. □

4. Proof of main theorem

In this section, we keep the same notation as in the previous sections. Let $(T\hookrightarrow X)$ be a smooth toric variety of pure divisorial type over k.

- Fix integral models $\mathbf{X}$, $\mathbf{T}$, $\mathbf{C}_i$, $\mathbf{U}_i$ of X, T, C_i, U_i in (2.1) and $\mathbf{V}_i$ of V_i in Lemma 2.9 and $\mathbf{V}$ of V in Proposition 2.10 over $O_{k,S}$ for a finite subset $S \supset \infty_k$ in Ω_k with $1 \leqslant i \leqslant d$ respectively.
- Fix integral models $\sigma(\mathbf{Z}_i)$ of $\sigma(Z_i)$ in (2.3) and $\mathbf{T}_\sigma$ of T_σ in (2.7) over $O_{\sigma(k_i),S}$ for a finite subset $S \supset \infty_k$ in Ω_k with $1 \leqslant i \leqslant d$ and $\sigma \in \Upsilon_i$, where Υ_i is the set of all k-embedding of k_i into $\bar{k}$ and $O_{\sigma(k_i),S}$ is the integral closures of $O_{k,S}$ inside $\sigma(k_i)$.

One can always enlarge S so that

(i) The action m_X of T on X as toric variety extends to

$$m_T : \mathbf{T} \times_{O_{k,S}} \mathbf{X} \to \mathbf{X}$$

as an action of group scheme.

(ii) For $1 \leqslant i \leqslant d$, the open immersion $U_i \subseteq X$ in (2.1) can be extended to an open immersion $\mathbf{U}_i \subseteq \mathbf{X}$ over $O_{k,S}$. One obtains $\cup_{i=1}^d \mathbf{U}_i$ is an open subset of $\mathbf{X}$. Since both $\cup_{i=1}^d \mathbf{U}_i$ and $\mathbf{X}$ are integral models of X over $O_{k,S}$. By enlarging S, one concludes that $\{\mathbf{U}_i\}_{i=1}^d$ is an open covering of $\mathbf{X}$. Moreover, $\mathbf{U}_i$ is covered by

$$\mathbf{T} \xrightarrow{g_i} \mathbf{U}_i \xleftarrow{h_i} \mathbf{C}_i$$

over $O_{k,S}$, where g_i is an open immersion and h_i is the complement of g_i, which is a closed immersion. Moreover, $\mathbf{C}_i$ is smooth over $O_{k,S}$ for $1 \leqslant i \leqslant d$.

(iii) The morphism ρ in Proposition 2.10 extends to $\rho : \mathbf{V} \to \mathbf{X}$ and

$$\left\{ \prod_{1\leqslant j\leqslant i-1} \mathrm{Res}_{O_{k_j,S}/O_{k,S}}(\mathbb{G}_m) \times_{O_{k,S}} \mathbf{V}_i \times_{O_{k,S}} \prod_{i+1\leqslant j\leqslant d} \mathrm{Res}_{O_{k_j,S}/O_{k,S}}(\mathbb{G}_m) \right\}_{1\leqslant i\leqslant d}$$

is an open covering of $\mathbf{V}$.

(iv) Both morphism ϱ_σ in (2.6) and morphism ϕ_σ in (2.7) extend to

$$\varrho_\sigma : \mathbb{A}^1_{O_{\sigma(k_i),S}} \to \sigma(\mathbf{Z}_i) \quad \text{and} \quad \phi_\sigma : \sigma(\mathbf{Z}_i) \to \mathbf{T}_\sigma$$

over $O_{\sigma(k_i),S}$ for all $\sigma \in \Upsilon_i$ and $1 \leqslant i \leqslant d$. Moreover, the exact sequence in (2.5) extends to

$$1 \to \mathbb{G}_{m,O_{\sigma(k_i),S}} \to \mathbf{T} \to \mathbf{T}_\sigma \to 1$$

over $O_{\sigma(k_i),S}$ and $Im(\varrho_\sigma) = \phi_\sigma^{-1}(1_{\mathbf{T}_\sigma})$ over $O_{\sigma(k_i),S}$ for $1 \leqslant i \leqslant d$ and all $\sigma \in \Upsilon_i$.

Let $O_{\bar{k},S}$ be the integral closure of $O_{k,S}$ inside $\bar{k}$.

(v) $\mathbf{C}_i \times_{O_{k,S}} O_{\bar{k},S} = \coprod_{\sigma\in\Upsilon_i}((\sigma(\mathbf{Z}_i) \setminus \mathbf{T}) \times_{O_{\sigma(k_i),S}} O_{\bar{k},S})$ and $(\sigma(\mathbf{Z}_i) \setminus \mathbf{T}) \times_{O_{\sigma(k_i),S}} O_{\bar{k},S}$ is integral for $1 \leqslant i \leqslant d$.

(vi) The morphism ρ_i in Lemma 2.9 extends to the following commutative diagram

$$\begin{array}{ccc} \mathrm{Res}_{O_{k_i,S}/O_{k,S}}(\mathbb{G}_m) & \xrightarrow{\rho_i} & \mathbf{T} \\ \downarrow & & \downarrow \\ \mathbf{V}_i & \xrightarrow{\rho_i} & \mathbf{U}_i \end{array}$$

over $O_{k,S}$ and $\{\mathrm{Spec}(O_{\bar{k},S}[x_\sigma, x_\tau, x_\tau^{-1}]_{\tau\in\Upsilon_i;\ \tau\neq\sigma})\}_{\sigma\in\Upsilon_i}$ is an open covering of $\mathbf{V}_i \times_{O_{k,S}} O_{\bar{k},S}$ for $1 \leqslant i \leqslant d$.

The following proposition is crucial for proving our main theorem.

Proposition 4.1. — *With notation as above, one has*

$$\mathbf{X}(O_v) \cap T(k_v) = \mathbf{T}(O_v) \cdot \rho(\mathbf{V}(O_v) \cap T_0(k_v)) \subseteq X(k_v)$$

for all $v \notin S$*, where* $T_0 = \prod_{i=1}^{d} \mathrm{Res}_{k_i/k}(\mathbb{G}_m)$.

Proof. — By the above conditions (i) and (iii), one only needs to prove

$$\mathbf{X}(O_v) \cap T(k_v) \subseteq \mathbf{T}(O_v) \cdot \rho(\mathbf{V}(O_v) \cap T_0(k_v)).$$

Let $T_i = \mathrm{Res}_{k_i/k}(\mathbb{G}_m)$ for $1 \leqslant i \leqslant d$. Since

$$\rho_i(\mathbf{V}_i(O_v) \cap T_i(k_v)) \subseteq \rho(\mathbf{V}(O_v) \cap T_0(k_v))$$

by the above condition (iii) and (vi), it is sufficient to show that

$$\mathbf{U}_i(O_v) \cap T(k_v) \subseteq \mathbf{T}(O_v) \cdot \rho_i(\mathbf{V}_i(O_v) \cap T_i(k_v))$$

for each $1 \leqslant i \leqslant d$ by the above condition (ii).

Let $\alpha \in (\mathbf{U}_i(O_v) \cap T(k_v)) \setminus \mathbf{T}(O_v)$. Since α is a section

$$\alpha : \quad \mathrm{Spec}(O_v) \to \mathbf{U}_i \times_{O_{k,S}} O_v,$$

the closed point m_v of $\mathrm{Spec}(O_v)$ under α maps to $\mathbf{C}_i \times_{O_{k,S}} O_v$ by the above condition (ii). Since $\mathbf{C}_i \times_{O_{k,S}} O_v$ is smooth over O_v, there is a section

$$\beta : \quad \mathrm{Spec}(O_v) \to \mathbf{C}_i \times_{O_{k,S}} O_v$$

such that $\beta(m_v) = \alpha(m_v)$.

Fix a prime w in $\bar{k}$ above v. Extending the condition (v) to the ring of integers $O_{\bar{k}_w}$ of $\bar{k}_w$, one obtains $\sigma_\alpha \in \Upsilon_i$ such that $(\sigma_\alpha(\mathbf{Z}_i) \setminus \mathbf{T}) \times_{O_{\sigma_\alpha(k_i)}} O_{\bar{k}_w}$ is the unique connected component containing β. This implies that $\mathrm{Gal}(\bar{k}_w/k_v)$ acts on $(\sigma_\alpha(\mathbf{Z}_i) \setminus \mathbf{T}) \times_{O_{\sigma_\alpha(k_i)}} O_{\bar{k}_w}$ stably. Therefore $\sigma_\alpha(\mathbf{Z}_i)$ is defined over O_v by Galois descent and $\sigma_\alpha(D_i)$ is defined over k_v. On one hand, the Galois group $\mathrm{Gal}(\bar{k}/k)$ acts on $\{\sigma(D_i)\}_{\sigma\in\Upsilon_i}$ transitively and the stabilizer of $\sigma_\alpha(D_i)$ is $\mathrm{Gal}(\bar{k}/\sigma_\alpha(k_i))$. On the other hand, the closed subgroup $\mathrm{Gal}(\bar{k}_w/k_v)$ acts trivially on $\sigma_\alpha(D_i)$. One concludes that $\sigma_\alpha(k_i) \subseteq$

k_v and $O_{\sigma(k_i),S} \subset O_v$. Therefore all morphisms in the condition (iv) can be extended to O_v.

Since $H^1_{et}(O_v, \mathbb{G}_m) = 0$, one concludes that the homomorphism

$$\phi_{\sigma_\alpha} : \mathbf{T}(O_v) \to \mathbf{T}_{\sigma_\alpha}(O_v)$$

is surjective by the above condition (iv). There is $t \in \mathbf{T}(O_v)$ such that

$$t \cdot \alpha \in \phi^{-1}_{\sigma_\alpha}(1) = Im(\varrho_{\sigma_\alpha})$$

over O_v by the above condition (iv). This implies that there is $\gamma \in \mathbb{A}^1_{O_v}(O_v) = O_v$ such that $\varrho_{\sigma_\alpha}(\gamma) = t \cdot \alpha$. Since $\alpha \in T(k_v)$, one has that $\gamma \neq 0$. Define

$$\delta = (\delta_\sigma)_{\sigma \in \Upsilon_i} \in \mathbf{V}_i(O_{\bar{k}_v}) \subseteq \prod_{\sigma \in \Upsilon_i} O_{\bar{k}_v}$$

as follows

$$\delta_\sigma = \begin{cases} \gamma & \text{if } \sigma = \sigma_\alpha, \\ 1 & \text{otherwise.} \end{cases}$$

Since $\mathrm{Gal}(\bar{k}_v/k_v)$ acts on Υ_i but fixes σ_α, one has $\delta \in \mathbf{V}_i(O_v) \cap T_i(k_v)$ by the above condition (vi) and Galois descent. Therefore

$$\rho_i(\delta) = \varrho_{\sigma_\alpha}(\gamma) = \alpha \cdot t$$

as desired by the formula (2.8). □

The following local approximation enables us to consider $\mathbf{X}(O_v) \cap T(k_v)$ instead of $\mathbf{X}(O_v)$.

Proposition 4.2. — *Let $(T \hookrightarrow X)$ be a smooth toric variety over k_v with $v \in \Omega_k$. If $x \in X(k_v) \setminus T(k_v)$, then there is $y \in T(k_v)$ such that y is as close to x as required and*

$$\mathrm{inv}_v(\xi(x)) = \mathrm{inv}_v(\xi(y))$$

for all $\xi \in \mathrm{Br}_1(X)$.

Proof. — By Corollary 2.3, there is an open affine smooth toric subvariety M of X such that $x \in M(k_v)$. By Proposition 2.4, there are finite extensions E_i/k_v such that

$$\prod_i \mathrm{Res}_{E_i/k_v}(\mathbb{G}_m) \hookrightarrow \prod_i \mathrm{Res}_{E_i/k_v}(\mathbb{A}^1)$$

is a closed toric subvariety of $(T \hookrightarrow M)$ and the quotient homomorphism

$$\phi : T \to T_1 \quad \text{with} \quad T_1 = T/\left(\prod_i \mathrm{Res}_{E_i/k_v}(\mathbb{G}_m)\right)$$

can be extended to $\phi : M \to T_1$ and $\phi^{-1}(1) = \prod_i \mathrm{Res}_{E_i/k_v}(\mathbb{A}^1)$.

By Shapiro's Lemma and Hilbert 90, one has the map $T(k_v) \xrightarrow{\phi} T_1(k_v)$ is surjective. There is $\alpha \in T(k_v)$ such that $\phi(x) = \phi(\alpha)$. Since ϕ is T-equivariant, this implies that

$$\alpha^{-1}x \in (\phi^{-1}(1))(k_v) = \prod_i E_i.$$

Choose $z' \in \prod_i E_i^\times$ close to $\alpha^{-1}x$ such that $y = \alpha \cdot z'$ is as close to x as required.

For any $\xi \in \mathrm{Br}_1(X)$, there are $\eta \in \mathrm{Br}_1(T_1)$ such that

$$\phi^*(\eta) = \xi$$

by $\mathrm{Br}_1(X) \hookrightarrow \mathrm{Br}_1(M) \xleftarrow{\sim} \mathrm{Br}_1(T_1)$ and Proposition 2.4. Since $\phi(x) = \phi(\alpha) = \phi(\alpha z') = \phi(y)$, one has

$$\mathrm{inv}_v(\eta(\phi(x))) = \mathrm{inv}_v(\eta(\phi(y))).$$

By functoriality, this implies

$$\mathrm{inv}_v(\phi^*(\eta)(x)) = \mathrm{inv}_v(\phi^*(\eta)(y)).$$

Since

$$\mathrm{inv}_v(\phi^*(\eta)(x)) = \mathrm{inv}_v(\xi(x)) \quad \text{and} \quad \mathrm{inv}_v(\phi^*(\eta)(y)) = \mathrm{inv}_v(\xi(y)),$$

one obtains the result as desired. □

Proposition 4.3. — *If X is a smooth toric variety of pure divisorial type, then X satisfies strong approximation with Brauer–Manin obstruction off ∞_k.*

Proof. — For any non-empty open subset $\Xi \subseteq X(\mathbf{A}_k)^{\mathrm{Br}_1 X}$, there are a sufficiently large finite subset S_1 of Ω_k containing S and an open subset $W = \prod_{v\in\Omega_k} W_v$ of $X(\mathbf{A}_k)$ such that

$$\emptyset \neq W \cap X(\mathbf{A}_k)^{\mathrm{Br}_1 X} \subseteq \Xi,$$

and $W_v = \mathbf{X}(O_v)$ for all $v \notin S_1$.

Let $(x_v)_{v\in\Omega_k} \in W \cap X(\mathbf{A}_k)^{\mathrm{Br}_1 X}$. By Proposition 4.2, one can assume that $x_v \in T(k_v)$ for all $v \in \Omega_k$. Then

$$x_v \in W_v \cap T(k_v) = \mathbf{X}(O_v) \cap T(k_v) = \mathbf{T}(O_v) \cdot \rho(\mathbf{V}(O_v) \cap T_0(k_v))$$

for $v \notin S_1$ by Proposition 4.1, where $T_0 = \prod_{i=1}^d \mathrm{Res}_{k_i/k}(\mathbb{G}_m)$. Let

$$t_v \in \mathbf{T}(O_v) \quad \text{and} \quad \beta_v \in \mathbf{V}(O_v) \cap T_0(k_v)$$

such that $x_v = t_v \cdot \rho(\beta_v)$ for all $v \notin S_1$ and $t_v = x_v$ for $v \in S_1$. Then $(t_v)_{v\in\Omega_k} \in T(\mathbf{A}_k)$.

Since t_v induces a morphism $X \times_k k_v \to X \times_k k_v$ for all $v \in \Omega_k$, one has

$$\mathrm{inv}_v(\xi(x_v)) = \mathrm{inv}_v(\xi(t_v \cdot \rho(\beta_v))) = \mathrm{inv}_v((\rho^* t^* \xi)(\beta_v))$$

and

$$\mathrm{inv}_v(\xi(t_v)) = \mathrm{inv}_v((\rho^* t^* \xi)(1_{T_0}))$$

for all $\xi \in \mathrm{Br}_1(X)$. By the purity of Brauer groups (see [16, Part III, Theorem 6.1]), one has $Br_1(V \times_k k_v) = Br(k_v)$. Therefore $\mathrm{inv}_v(\xi(x_v)) = \mathrm{inv}_v(\xi(t_v))$ for all $v \in \Omega_k$.

By Proposition 2.11 and Proposition 3.4 or Remark 3.5 with $Ш^1(T_0) = 0$, there are $t \in T(k)$ and $y_{\mathbf{A}} \in T_0(\mathbf{A}_k)$ such that

$$t\rho(y_{\mathbf{A}}) \in \left(\prod_{v \in \infty_k} T(k_v) \times \prod_{v \in S_1 \setminus \infty_k} (W_v \cap T(k_v)) \times \prod_{v \notin S_1} \mathbf{T}(O_v) \right).$$

Therefore the open subset of $V(\mathbf{A}_k)$

$$\rho^{-1} \left(t^{-1} \left(\prod_{v \in \infty_k} X(k_v) \times \prod_{v \notin \infty_k} W_v \right) \right)$$

contains $y_{\mathbf{A}}$ and is not empty. Then there is

$$y \in V(k) \cap \rho^{-1} \left(t^{-1} \left(\prod_{v \in \infty_k} X(k_v) \times \prod_{v \notin \infty_k} W_v \right) \right)$$

by Corollary 3.7. This implies that

$$t \cdot \rho(y) \in \left(\prod_{v \in \infty_k} X(k_v) \times \prod_{v \notin \infty_k} W_v \right) \cap X(k)$$

as desired. □

For general smooth toric varieties, one needs to extend a part of Proposition 2.4 to integral models.

Lemma 4.4. — *Suppose an affine smooth toric variety $(T \hookrightarrow X)$ over k_v can be extended to an open immersion $\mathbf{T} \hookrightarrow \mathbf{X}$ over O_v such that $\mathbf{T}$ is a torus over O_v and $\mathbf{X}$ is an affine scheme of finite type over O_v for $v < \infty_k$. If the base change of the above open immersion fits into a commutative diagram*

$$\begin{array}{ccc} \mathbf{T} \times_{O_v} {O_v}^{ur} & \longrightarrow & \mathbf{X} \times_{O_v} O_v^{ur} \\ \downarrow \cong & & \downarrow \cong \\ \mathbb{G}_{m,O_v^{ur}}^{s+t} & \longrightarrow & \mathbb{A}^s_{O_v^{ur}} \times_{O_v^{ur}} \mathbb{G}^t_{m,O_v^{ur}} \end{array}$$

over O_v^{ur}, where O_v^{ur} is the ring of integers of the maximal unramified extension k_v^{ur} of k_v such that the left vertical arrow is an isomorphism of group schemes over O_v^{ur}, then one has the following commutative diagram

$$\begin{array}{ccc}\prod_{i=1}^h \mathrm{Res}_{O_{k_i}/O_v}(\mathbb{G}_{m,O_{k_i}}) & \xrightarrow{\iota} & \mathbf{T} \\ \downarrow & & \downarrow \\ \prod_{i=1}^h \mathrm{Res}_{O_{k_i}/O_v}(\mathbb{A}^1_{O_{k_i}}) & \longrightarrow & \mathbf{X}\end{array}$$

where the horizontal arrows are closed immersions and the vertical arrows are open immersions and O_{k_i}'s are the rings of integers of finite unramified extensions k_i/k_v for $1 \leqslant i \leqslant h$. Moreover ι is a homomorphism of O_v-tori and the quotient map $\phi : \mathbf{T} \to \mathrm{coker}(\iota)$ can be extended to $\phi : \mathbf{X} \to \mathrm{coker}(\iota)$ such that

$$\phi^{-1}(1) = \prod_{i=1}^h \mathrm{Res}_{O_{k_i}/O_v}(\mathbb{A}^1_{O_{k_i}})$$

over O_v.

Proof. — Since

$$\mathrm{Pic}(\mathbf{X} \times_{O_v} O_v^{ur}) = \mathrm{Pic}(\mathbb{A}^s_{O_v^{ur}} \times_{O_v^{ur}} \mathbb{G}^t_{m,O_v^{ur}}) = 0,$$

one has the following short exact sequence

$$1 \to O_v^{ur}[\mathbf{X}]^\times/O_v^{ur\times} \xrightarrow{\phi^*} O_v^{ur}[\mathbf{T}]^\times/O_v^{ur\times} \\ \xrightarrow{\iota^*} \mathrm{Div}_{(\mathbf{X}\times_{O_v} O_v^{ur})\setminus(\mathbf{T}\times_{O_v} O_v^{ur})}(\mathbf{X} \times_{O_v} O_v^{ur}) \to 1$$

of $\mathrm{Gal}(k_v^{ur}/k_v)$-module by sending $f \mapsto \mathrm{div}_{(\mathbf{X}\times_{O_v} O_v^{ur})\setminus(\mathbf{T}\times_{O_v} O_v^{ur})}(f)$ for any $f \in O_v^{ur}[\mathbf{T}]^\times$. By Theorem 1.2 and Theorem 3.1 in [14, Exposé VIII], one obtains an exact sequence of affine group schemes

$$1 \to \prod_{i=1}^h \mathrm{Res}_{O_{k_i}/O_v}(\mathbb{G}_{m,O_{k_i}}) \xrightarrow{\iota} \mathbf{T} \xrightarrow{\phi} \mathrm{coker}(\iota) \to 1$$

over O_v where O_{k_i}'s are the rings of integers of the finite unramified extensions k_i/k_v for $1 \leqslant i \leqslant h$, and where $\mathrm{coker}(\iota)$ is a torus over O_v with

$$\mathrm{Hom}_{O_v^{ur}}(\mathrm{coker}(\iota) \times_{O_v} O_v^{ur}, \mathbb{G}_{m,O_v^{ur}}) = O_v^{ur}[\mathbf{X}]^\times/O_v^{ur\times}$$

as $\mathrm{Gal}(k_v^{ur}/k_v)$-module. Let

$$\mathbf{B} = \{f \in O_v^{ur}[\mathbf{X}]^\times \mid f(1_{\mathbf{T}}) = 1\}$$

which is stable under the action of $\mathrm{Gal}(k_v^{ur}/k_v)$. Then $O_v^{ur}[\mathbf{X}]^\times = O_v^{ur\times} \oplus \mathbf{B}$ as $\mathrm{Gal}(k_v^{ur}/k_v)$-module and

$$\mathrm{coker}(\iota) \times_{O_v} O_v^{ur} \cong \mathrm{Spec}(O_v^{ur}[\mathbf{B}]) \quad \text{induced by} \quad \mathbf{B} \cong O_v^{ur}[\mathbf{X}]^\times / O_v^{ur\times}$$

is compatible with $\mathrm{Gal}(k_v^{ur}/k_v)$-action by Theorem 1.2 in [14, Exposé VIII]. Moreover, the natural inclusion of O_v^{ur}-algebras $O_v^{ur}[\mathbf{B}] \subseteq O_v^{ur}[\mathbf{X}]$ which is also compatible with $\mathrm{Gal}(k_v^{ur}/k_v)$-action gives the extension $\mathbf{X} \xrightarrow{\phi} \mathrm{coker}(\iota)$ of $\mathbf{T} \xrightarrow{\phi} \mathrm{coker}(\iota)$ over O_v.

Write

$$\mathbf{T} \times_{O_v} O_v^{ur} = \mathrm{Spec}(O_v^{ur}[x_1, x_1^{-1}, \dots, x_s, x_s^{-1}, y_1, y_1^{-1}, \dots, y_t, y_t^{-1}])$$

and

$$\mathbf{X} \times_{O_v} O_v^{ur} = \mathrm{Spec}(O_v^{ur}[x_1, \dots, x_s, y_1, y_1^{-1}, \dots, y_t, y_t^{-1}])$$

such that $x_i(1_{\mathbf{T}}) = y_j(1_{\mathbf{T}}) = 1$ for $1 \leqslant i \leqslant s$ and $1 \leqslant j \leqslant t$ by the given diagram. Then

$$\mathrm{coker}(\iota) \times_{O_v} O_v^{ur} = \mathrm{Spec}(O_v^{ur}[y_1, y_1^{-1}, \dots, y_t, y_t^{-1}])$$

and

$$\phi^{ur} = \phi \times_{O_v} O_v^{ur} : \quad \mathbf{X} \times_{O_v} O_v^{ur} \to \mathrm{coker}(\iota) \times_{O_v} O_v^{ur}$$

is the projection and

$$\phi^{-1}(1) \times_{O_v} O_v^{ur} = (\phi^{ur})^{-1}(1) = \mathrm{Spec}(O_v^{ur}[x_1, \dots, x_s]).$$

Since

$$\mathrm{div}_{(\mathbf{X}\times_{O_v} O_v^{ur})\setminus(\mathbf{T}\times_{O_v} O_v^{ur})}(x_i) = \mathrm{div}_{\mathbf{X}\times_{O_v} O_v^{ur}}(x_i)$$

and the action of $\mathrm{Gal}(k_v^{ur}/k_v)$ on $\{\mathrm{div}_{\mathbf{X}\times_{O_v} O_v^{ur}}(x_i)\}_{i=1}^s$ is same as the action on the coordinates $\{x_i\}_{i=1}^s$ by smoothness of $\mathbf{X} \times_{O_v} O_v^{ur}$ and the normalization of x_i for $1 \leqslant i \leqslant s$, one concludes that

$$\phi^{-1}(1) = \prod_{i=1}^h \mathrm{Res}_{O_{k_i}/O_v}(\mathbb{A}^1_{O_{k_i}})$$

as required. □

Theorem 4.5. — *Any smooth toric variety satisfies strong approximation with Brauer–Manin obstruction off ∞_k.*

Proof. — Let $(T \hookrightarrow X)$ be a smooth toric variety over k and $\mathfrak{F}$ be the set of all open affine toric sub-varieties over $\bar{k}$. Since there are only finitely many $T(\bar{k})$-orbits in $X(\bar{k})$, one gets $\mathfrak{F}$ is finite. Moreover if A and B are in $\mathfrak{F}$, then $A \cap B \in \mathfrak{F}$ and $\sigma(A) \in \mathfrak{F}$ for any $\sigma \in \Gamma_k$ by the separateness of X over k. Let k'/k be a finite Galois extension such that $T \times_k k' \cong \mathbb{G}_m^n$ and

U is defined over k' and $U \cong \mathbb{A}^{s_U} \times \mathbb{G}_m^{t_U}$ with non-negative integers s_U and t_U over k' for all $U \in \mathfrak{F}$.

By Proposition 2.6, there is a unique open toric subvariety $Y \subset X$ of pure divisorial type over k such that $dim(X \setminus Y) < \dim(T) - 1$. Let S be a finite subset of Ω_k containing ∞_k and $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{T}$ be the integral model of X, Y and T over $O_{k,S}$ respectively such that

(1) Every prime $v \notin S$ is unramified in k'/k.
(2) The open immersion $T \hookrightarrow X$ and the action $T \times_k X \xrightarrow{m_X} X$ extend to

$$\mathbf{T} \hookrightarrow \mathbf{X} \quad \text{and} \quad \mathbf{T} \times_{O_{k,S}} \mathbf{X} \xrightarrow{m_{\mathbf{X}}} \mathbf{X}$$

over $O_{k,S}$.
(3) The open immersion $T \hookrightarrow Y$ and the action $T \times_k Y \xrightarrow{m_Y} Y$ extend to

$$\mathbf{T} \hookrightarrow \mathbf{Y} \quad \text{and} \quad \mathbf{T} \times_{O_{k,S}} \mathbf{Y} \xrightarrow{m_{\mathbf{Y}}} \mathbf{Y}$$

over $O_{k,S}$.
(4) The open immersion $Y \hookrightarrow X$ extends to $\mathbf{Y} \hookrightarrow \mathbf{X}$ over $O_{k,S}$.
(5) Let $\underline{\mathfrak{F}}$ be the set of an integral model $\mathbf{U}$ over the integral closure $O_{k',S}$ of $O_{k,S}$ in k' with an open immersion $\mathbf{U} \hookrightarrow \mathbf{X} \times_{O_{k,S}} O_{k',S}$ over $O_{k',S}$ which extends $U \hookrightarrow X \times_k k'$ over k' for each element $U \in \mathfrak{F}$ such that

$$\mathbf{A} \cap \mathbf{B} \in \underline{\mathfrak{F}} \quad \text{and} \quad \sigma(\mathbf{A}) \in \underline{\mathfrak{F}}$$

whenever $\mathbf{A}, \mathbf{B} \in \underline{\mathfrak{F}}$ and $\sigma \in \mathrm{Gal}(k'/k)$. Moreover

$$\mathbf{T} \times_{O_{k,S}} O_{k',S} \cong \mathbb{G}^n_{m,O_{k',S}} \quad \text{and} \quad \mathbf{X} \times_{O_{k,S}} O_{k',S} = \bigcup_{\mathbf{U} \in \underline{\mathfrak{F}}} \mathbf{U}$$

with $\mathbf{U} \cong \mathbb{A}^{s_U}_{O_{k',S}} \times \mathbb{G}^{t_U}_{m,O_{k',S}}$ over $O_{k',S}$ for each $\mathbf{U} \in \underline{\mathfrak{F}}$.

Let $W = \prod_{v \in \Omega_k} W_v$ be an open subset of $X(\mathbf{A}_k)$ and S_1 be a finite subset of Ω_k containing S such that

$$(x_v)_{v \in \Omega_k} \in W \cap X(\mathbf{A}_k)^{\mathrm{Br}_a X} \quad \text{and} \quad W_v = \mathbf{X}(O_v)$$

for all $v \notin S_1$.

For $v \in S_1$, we can assume that $x_v \in T(k_v) \cap W_v \subseteq Y(k_v) \cap W_v$ by Proposition 4.2.

For $v \notin S_1$, we can assume that $x_v \in \mathbf{T}(O_v)$. Indeed, since $x_v \in \mathbf{X}(O_v)$, there is $\mathbf{U} \in \underline{\mathfrak{F}}$ in the above condition (5) such that $x_v \in \mathbf{U}(O_{k'_w})$ for a prime $w|v$ in k', where $O_{k'_w}$ is the ring of integers of k'_w. By the above condition (5)

$$\bigcap_{\sigma\in\mathrm{Gal}(k'_w/k_v)} \sigma(\mathbf{U}) \in \underline{\mathfrak{F}}$$

and there is an affine scheme $\mathbf{M}_v$ over O_v such that

$$\mathbf{M}_v \times_{O_v} O_{k'_w} = \left(\bigcap_{\sigma\in\mathrm{Gal}(k'_w/k_v)} \sigma(\mathbf{U})\right) \times_{O_{k',S}} O_{k'_w}$$

with $x_v \in \mathbf{M}_v(O_v)$ by Galois descent. Since

$$\bigcap_{\sigma\in\mathrm{Gal}(k'_w/k_v)} \sigma(\mathbf{U}) \times_{O_{k',S}} k' \in \mathfrak{F}$$

and every element in $\mathfrak{F}$ is a smooth affine open toric subvariety of $X_{\bar{k}}$, one obtains that $\mathbf{M}_v \times_{O_v} k_v$ is a smooth affine toric variety. Moreover

$$(\mathbf{T}_v = \mathbf{T} \times_{O_{k,S}} O_v \hookrightarrow \mathbf{M}_v) \quad \text{extends} \quad (T_v = T \times_k k_v \hookrightarrow \mathbf{M}_v \times_{O_v} k_v).$$

By the above condition (1) and (5), one can apply Lemma 4.4 to obtain a surjective homomorphism of O_v-tori $\mathbf{T}_v \xrightarrow{\phi} \mathbf{T}_1$ for some O_v-torus $\mathbf{T}_1$ such that

$$\ker(\phi) = \prod_{i=1}^{h} \mathrm{Res}_{O_{k_i}/O_v}(\mathbb{G}_{m,O_{k_i}})$$

where O_{k_i}'s are the rings of integers of finite unramified extensions k_i/k_v for $1 \leqslant i \leqslant h$. Moreover, this map ϕ can be extended to a morphism $\mathbf{M}_v \xrightarrow{\phi} \mathbf{T}_1$. Since $H^1_{et}(O_v, \ker(\phi)) = 0$, one has $\mathbf{T}_v(O_v) \xrightarrow{\phi} \mathbf{T}_1(O_v)$ is surjective by étale cohomology. If $x_v \notin \mathbf{T}(O_v)$, there is $t_v \in \mathbf{T}_v(O_v)$ such that $\phi(x_v) = \phi(t_v)$. By Proposition 2.4 or the proof of Proposition 4.2, one has

$$\mathrm{inv}_v(\xi(x_v)) = \mathrm{inv}_v(\xi(t_v))$$

for all $\xi \in \mathrm{Br}_1(X)$. Therefore one can replace x_v with t_v if necessary.

Therefore one can assume

$$(x_v)_{v\in\Omega_k} \in \left[\prod_{v\in S_1} (W_v \cap Y(k_v)) \times \prod_{v\notin S_1} \mathbf{Y}(O_v)\right] \cap Y(\mathbf{A}_k)^{\mathrm{Br}_a(Y)}$$

by the above condition (3) and $\mathrm{Br}_a(X) \cong \mathrm{Br}_a(Y)$ induced by open immersion. By Proposition 4.3, there is $y \in Y(k) \subseteq X(k)$ such that

$$y \in \prod_{v\in S_1} (W_v \cap Y(k_v)) \times \prod_{v\notin S_1} \mathbf{Y}(O_v) \subseteq \prod_{v\in\infty} X(k_v) \times \prod_{v\notin\infty_k} W_v$$

by the above condition (4) as desired. □

5. An example

At the end of [18], Harari and Voloch constructed an open curve which does not satisfy strong approximation with Brauer–Manin obstruction. However their counter-example is not geometrically rational. Colliot-Thélène and Wittenberg gave an open rational surface over $\mathbb{Q}$ ([8, Example 5.10]) which does not satisfy strong approximation with Brauer–Manin obstruction. Here we provide another such open rational surface. We explain that the complement of a point in a toric variety may no longer satisfy strong approximation with Brauer–Manin obstruction. We also show that strong approximation with Brauer–Manin obstruction is not stable under finite extensions of the ground field.

Before giving the explicit example, we have the following lemma.

LEMMA 5.1. — *Let $f: X \to Y$ be a morphism of schemes over a number field k such that the induced map $f^*: \mathrm{Br}(Y) \to \mathrm{Br}(X)$ is surjective. If $Y(k)$ is discrete in $Y(\mathbf{A}_k^S)$ and X satisfies strong approximation with Brauer–Manin obstruction off S for some finite subset S of Ω_k, then any fiber $f^{-1}(y)$ satisfies strong approximation off S for $y \in Y(k)$.*

Proof. — Since $Y(k)$ is discrete in $Y(\mathbf{A}_k^S)$, there is an open subset U_y of $Y(\mathbf{A}_k^S)$ such that

$$Y(k) \cap U_y = \{y\}$$

for each $y \in Y(k)$. Let

$$(x_v)_{v \notin S} \in W \subseteq f^{-1}(y)(\mathbf{A}_k^S)$$

be a non-empty open subset. Since $f^{-1}(y)$ is a closed sub-scheme of X, there is an open subset W_1 of $X(\mathbf{A}_k^S)$ such that $W = W_1 \cap [f^{-1}(y)(\mathbf{A}_k^S)]$. Let $x_v \in f^{-1}(y)(k_v)$ for $v \in S$. Then

$$(x_v)_{v \in \Omega_k} \in \left[\prod_{v \in S} X(k_v) \times (W_1 \cap f^{-1}(U_y))\right] \cap X(\mathbf{A}_k)^{\mathrm{Br}(X)} \neq \emptyset$$

by the surjection of $f^*: \mathrm{Br}(Y) \to \mathrm{Br}(X)$ and the functoriality of Brauer–Manin pairing. Since X satisfies strong approximation with Brauer–Manin obstruction off S, there is $x \in X(k)$ such that $x \in W_1 \cap f^{-1}(U_y)$. This implies that $f(x) \in U_y$ and $f(x) = y$. Therefore $x \in W$ as desired. □

Example 5.2. — Let $X = (\mathbb{A}^1 \times_k \mathbb{G}_m) \setminus \{(0,1)\}$ over a number field k.

(1) If $k = \mathbb{Q}$ or an imaginary quadratic field, then X does not satisfy strong approximation with Brauer–Manin obstruction off ∞_k.

(2) Otherwise X satisfies strong approximation with Brauer–Manin obstruction off ∞_k.

Proof.

(1). — If $k = \mathbb{Q}$ or an imaginary quadratic field, one takes $Y = \mathbb{G}_m$ and the morphism $f : X \to Y$ by restriction of the projection map $\mathbb{A}^1 \times_k \mathbb{G}_m \to \mathbb{G}_m$ to X. Since $O_k^\times$ is finite, one has $Y(k)$ is discrete in $Y(\mathbf{A}_k^\infty)$. The morphism f induces an isomorphism

$$f^* : \mathrm{Br}(Y) = \mathrm{Br}(\mathbb{G}_m) \xrightarrow{\cong} \mathrm{Br}(\mathbb{A}^1 \times \mathbb{G}_m) = \mathrm{Br}(X)$$

by homotopy invariance of etale cohomology (see [19, Chapter VI, §4, Corollary 4.20]) and the standard Kummer sequence argument. Suppose X satisfies strong approximation with Brauer–Manin obstruction off ∞_k. Then all fibers $f^{-1}(y)$ satisfy strong approximation off ∞_k by Lemma 5.1. However $f^{-1}(1) \cong \mathbb{G}_m$ does not satisfy strong approximation off ∞_k. A contradiction is derived.

(2). — Let $W = \prod_{v\in\Omega_k} W_v$ be an open subset in $X(\mathbf{A}_k)$ with $(x_v)_{v\in\Omega_k} \in W \cap X(\mathbf{A}_k)^{\mathrm{Br}_1(X)}$. There is a finite subset S of Ω_k containing ∞_k such that

$$\begin{cases} x_v \in U_v \times V_v \subseteq W_v \subseteq (k_v^\times \times k_v \setminus \{(1,0)\}) & \text{for } v \in S, \\ x_v \in W_v = \mathbf{X}(O_v) = (O_v^\times \times O_v^\times) \cup ((O_v^\times \setminus (1+\pi_v O_v)) \times O_v) & \text{for } v \notin S, \end{cases}$$

where U_v and V_v are the open subsets of $k_v^\times$ and k_v respectively for $v \in S$ and π_v is the uniformizer of k_v for $v \notin S$. Consider two projection

$$p : \mathbb{G}_m \times_k \mathbb{A}^1 \to \mathbb{G}_m \quad \text{and} \quad q : \mathbb{G}_m \times_k \mathbb{A}^1 \to \mathbb{A}^1.$$

If k is neither $\mathbb{Q}$ nor an imaginary quadratic field, then $O_k^\times$ is infinite. Therefore $k^\times$ is not discrete in $\mathbb{G}_m(\mathbf{A}_k^\infty)$.

Since $k^\times$ is dense in $\mathrm{Pr}_\infty(\mathbb{G}_m(\mathbf{A}_k)^{\mathrm{Br}_a(\mathbb{G}_m)})$, one concludes that $k^\times \setminus \{1\}$ is also dense in $\mathrm{Pr}_\infty(\mathbb{G}_m(\mathbf{A}_k)^{\mathrm{Br}_a(\mathbb{G}_m)})$. By the functoriality of Brauer–Manin pairing, one has $p((x_v)) \in \mathbb{G}_m(\mathbf{A}_k)^{\mathrm{Br}_a(\mathbb{G}_m)}$. Choose an open subset $\prod_{v\in\Omega_k} M_v$ of $\mathbb{G}_m(\mathbf{A}_k)$ containing $p((x_v)_{v\in\Omega_k})$ such that

$$\begin{cases} M_v = k_v^\times & v \in \infty_k, \\ M_v = U_v & v \in S \setminus \infty_k, \\ M_v = O_v^\times & v \notin S. \end{cases}$$

There is $b \in k^\times \setminus \{1\}$ such that $b \in \prod_{v\in\Omega_k} M_v$. Let S_1 be a finite subset of Ω_k containing S such that $b - 1 \in O_v^\times$ for all $v \notin S_1$. Choose an open subset $\prod_{v\in\Omega_k} N_v$ of $\mathbf{A}_k$

$$\begin{cases} N_v = k_v & v \in \infty_k, \\ N_v = V_v & v \in S, \\ N_v = O_v^\times & v \in S_1 \setminus S, \\ N_v = O_v & v \notin S_1. \end{cases}$$

Then there is $c \in k^\times$ such that $c \in \prod_{v\in\Omega_k} N_v$ by strong approximation for $\mathbb{A}^1$. Then $(b,c) \in W$ as desired. □

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Yang CAO
School of Mathematical Sciences,
Capital Normal University,
105 Xisanhuanbeilu,
100048 Beijing, China
yangcao1988@gmail.com

Fei XU
School of Mathematical Sciences,
Capital Normal University,
105 Xisanhuanbeilu,
100048 Beijing, China
xufei@math.ac.cn